Balanced trees and Heap structure

Data Structures and Algorithms

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Overview

1. **AVL Tree**
   - AVL Tree Concepts
   - AVL Balance
   - AVL Tree Operations

2. **Splay Tree**

3. **Multiway Trees**

4. **B-Trees**

AVL Tree Concepts

AVL Tree

Definition

AVL Tree is:

- A Binary Search Tree,
- in which the heights of the left and right subtrees of the root differ by at most 1, and
- the left and right subtrees are again AVL trees.

Discovered by G.M.Adel'son-Vel'skii and E.M.Landis in 1962.

AVL Tree is a Binary Search Tree that is balanced tree.

AVL Tree

A binary tree is an AVL Tree if

- Each node satisfies BST property: key of the node is greater than the key of each node in its left subtree and is smaller than or equals to the key of each node in its right subtree.
- Each node satisfies balanced tree property: the difference between the heights of the left subtree and right subtree of the node does not exceed one.

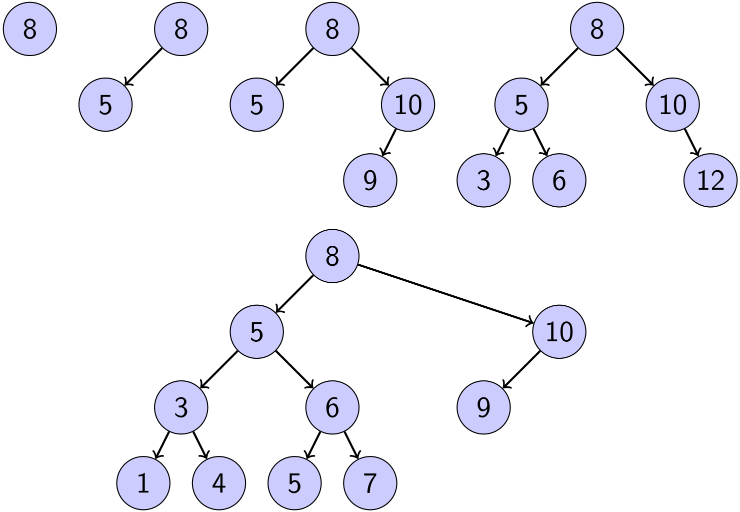
AVL Tree

Balance factor

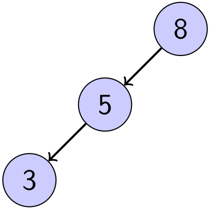
- left_higher (LH): $H_L = H_R + 1$
- equal_height (EH): $H_L = H_R$
- right_higher (RH): $H_R = H_L + 1$

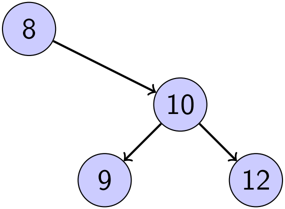
(H_L, H_R: the heights of left and right subtrees)

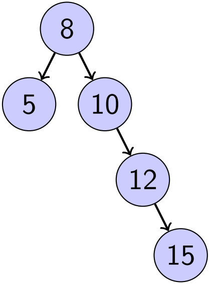
AVL Trees

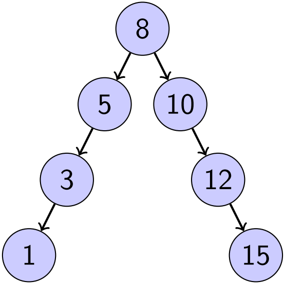


Non-AVL Trees









Why AVL Trees?

- When data elements are inserted in a BST in sorted order: 1, 2, 3, ...
  BST becomes a degenerate tree.
  Search operation takes O(n), which is inefficient.

- It is possible that after a number of insert and delete operations, a binary tree may become unbalanced and increase in height.

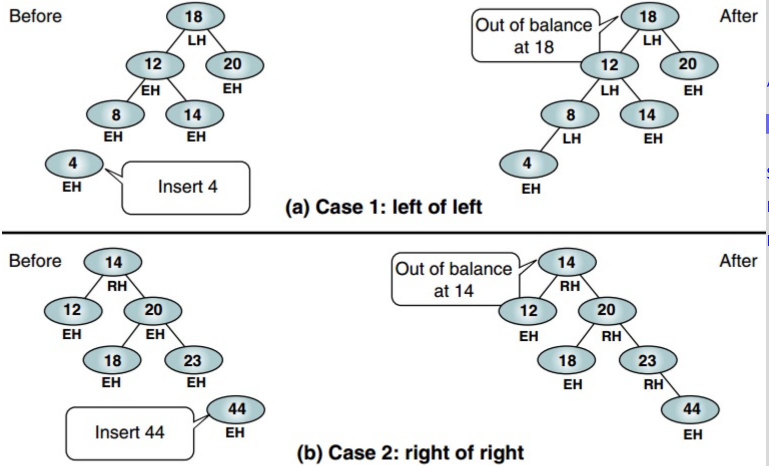
- AVL trees ensure that the complexity of search is $O(log_2 n)$.

AVL Balance

Balancing Trees

- When we insert a node into a tree or delete a node from a tree, the resulting tree may be unbalanced.
  $\rightarrow$ rebalance the tree.
- Four unbalanced tree cases:
  - left of left: a subtree of a tree that is left high has also become left high;
  - right of right: a subtree of a tree that is right high has also become right high;
  - right of left: a subtree of a tree that is left high has become right high;
  - left of right: a subtree of a tree that is

Unbalanced tree cases



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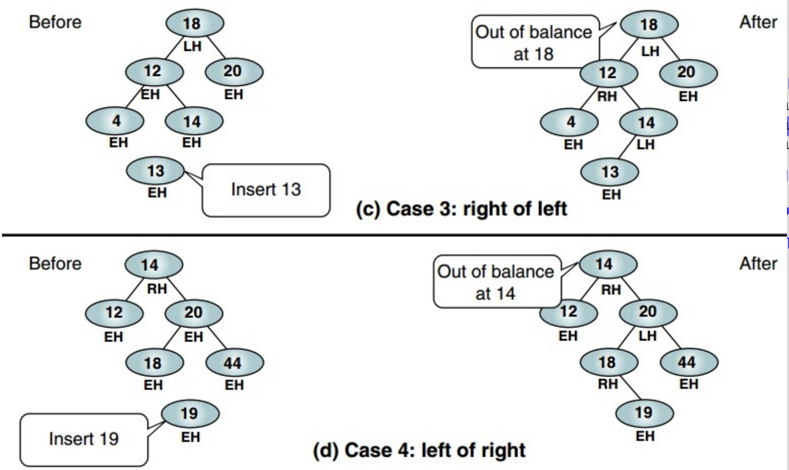
Unbalanced tree cases



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Rotate Right

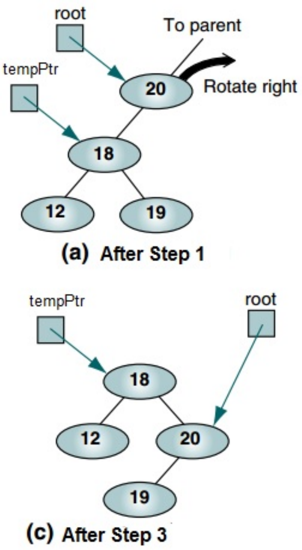
```
Algorithm rotateRight(ref root <pointer>)
Exchanges pointers to rotate the tree right.

Pre: root is pointer to tree to be rotated
Post: node rotated and root updated

tempPtr = root->left
root->left = tempPtr->right
tempPtr->right = root
Return tempPtr

End rotateRight
```

Rotate Right



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Rotate Left

Algorithm rotateLeft(ref root <pointer>)
Exchanges pointers to rotate the tree left.

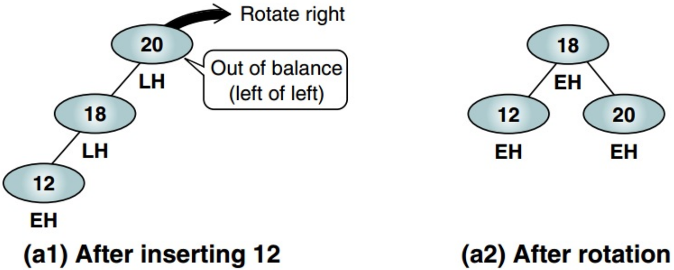
Pre: `root` is pointer to tree to be rotated
Post: node rotated and root updated

tempPtr = root->right
root->right = tempPtr->left
tempPtr->left = root
Return tempPtr

End rotateLeft

Balancing Trees - Case 1: Left of Left

Out of balance condition created by a left high subtree of a left high tree
$\rightarrow$ balance the tree by rotating the out of balance node to the right.



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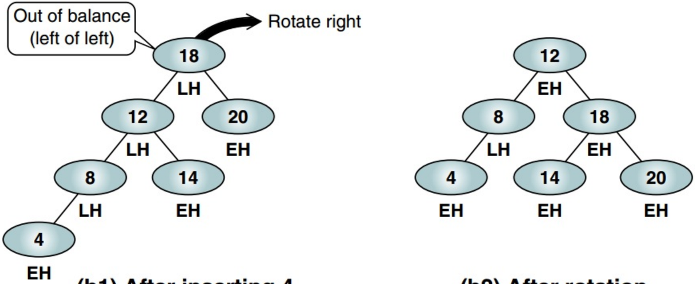
Balancing Trees - Case 1: Left of Left



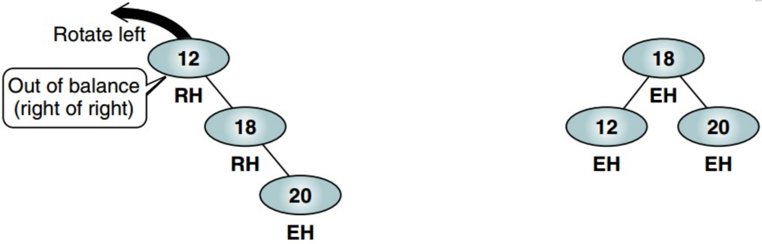
(b1) After inserting 4

(b2) After rotation

(Source: Data Structures - A Pseudocode Approach with C++)

Balancing Trees - Case 2: Right of Right

Out of balance condition created by a right high subtree of a right high tree
→ balance the tree by rotating the out of balance node to the left.



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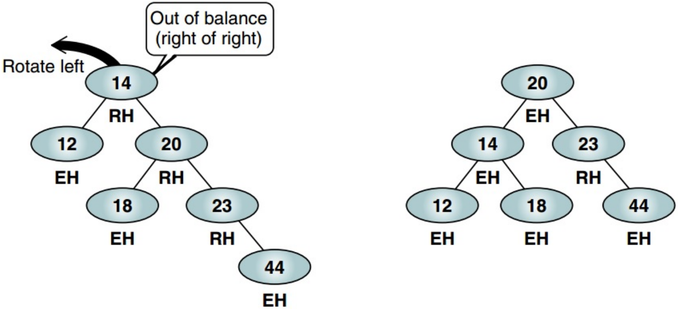
Balancing Trees - Case 2: Right of Right



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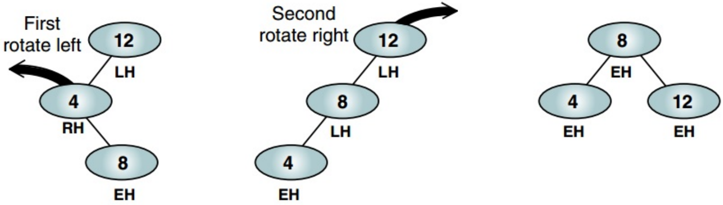
Balancing Trees - Case 3: Right of Left

Out of balance condition created by a right high subtree of a left high tree
→ balance the tree by two steps:

1. rotating the left subtree to the left;
2. rotating the root to the right.



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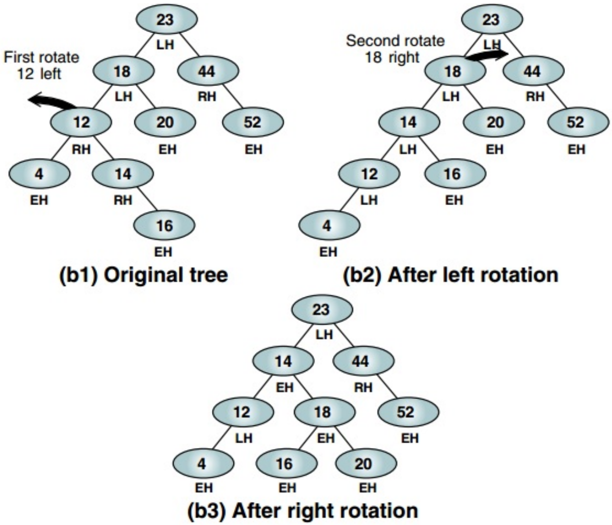
Balancing Trees - Case 3: Right of Left

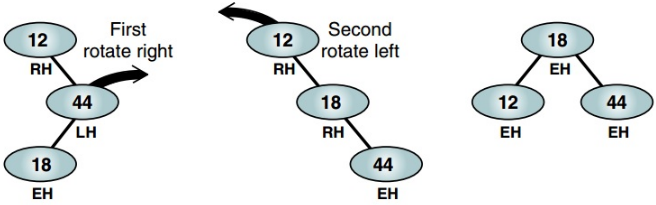


(Source: Data Structures - A Pseudocode Approach with C++)

Balancing Trees - Case 4: Left of Right

Out of balance condition created by a left high subtree of a right high tree
→ balance the tree by two steps:

1. rotating the right subtree to the right;
2. rotating the root to the left.



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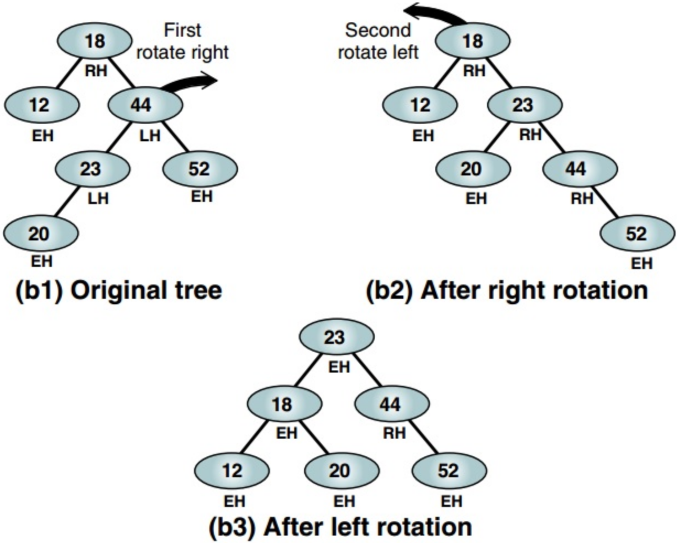
Balancing Trees - Case 4: Left of Right



(Source: Data Structures - A Pseudocode Approach with C++)

AVL Tree Operations

AVL Tree Structure

```
node
  data <dataType>
  left <pointer>
  right <pointer>
  balance <balance_factor>
end node
```

```
avlTree
  root <pointer>
end avlTree
```

```
// General dataTye:
dataType
  key <keyType>
  field1 <...>
  field2 <...>
  ...
  fieldn <...>
end dataType
```

Note: Array is not suitable for AVL Tree.

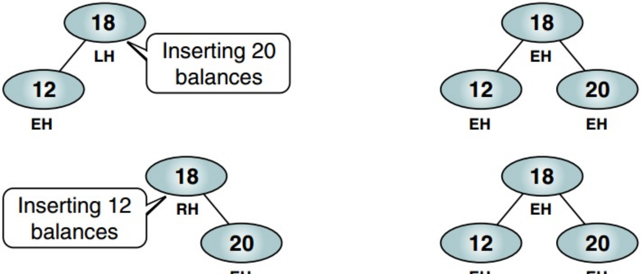
AVL Tree Operations

- Search and retrieval are the same for any binary tree.
- AVL Insert
- AVL Delete

AVL Insert

- Insert can make an out of balance condition.

- Otherwise, some inserts can make an automatic balancing.



AVL Insert Algorithm

Algorithm AVLInsert(ref root <pointer>, val newPtr <pointer>, ref taller <boolean>)

Using recursion, insert a node into an AVL tree.

Pre: `root` is a pointer to first node in AVL tree/subtree

`newPtr` is a pointer to new node to be inserted

Post: `taller` is a Boolean: `true` indicating the subtree height has increased, `false` indicating same height

Return `root` returned recursively up the

AVL Insert Algorithm

```
// Insert at root
if root null then
    root = newPtr
    taller = true
    return root
end
```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AVL Insert Algorithm

```
if newPtr->data.key < root->data.key then
    root->left = AVLInsert(root->left, newPtr, taller)
    // Left subtree is taller
    if taller then
        if root is LH then
            root = leftBalance(root, taller)
        else if root is EH then
            root->balance = LH
        else
            root->balance = EH
            taller = false
        end
```

AVL Insert Algorithm

```
else
    root->right = AVLInsert(root->right, newPtr, taller)
    // Right subtree is taller
    if taller then
        if root is LH then
            root->balance = EH
            taller = false
        else if root is EH then
            root->balance = RH
        else
            root = rightBalance(root, taller)
        end
    end
end
return root
End AVLInsert
```

AVL Left Balance Algorithm

Algorithm leftBalance(ref root <pointer>, ref taller <boolean>)
This algorithm is entered when the left subtree is higher than the right subtree.

Pre: `root` is a pointer to the root of the [sub]tree
`taller` is `true`
Post: `root` has been updated (if necessary)
`taller` has been updated

AVL Left Balance Algorithm

```
leftTree = root->left
// Case 1: Left of left. Single rotation right.
if leftTree is LH then
    root = rotateRight(root)
    root->balance = EH
    leftTree->balance = EH
    taller = false
```


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AVL Left Balance Algorithm

```
// Case 2: Right of Left. Double rotation required.
else
    rightTree = leftTree->right
    if rightTree->balance = LH then
        root->balance = RH
        leftTree->balance = EH
    else if rightTree->balance = EH then
        leftTree->balance = EH
    else
        root->balance = EH
        leftTree->balance = LH
    end
    rightTree->balance = EH
    root->left = rotateLeft(leftTree)
    root = rotateRight(root)
    taller = false
end
return root
```

AVL Right Balance Algorithm

Algorithm rightBalance(ref root <pointer>, ref taller <boolean>)

This algorithm is entered when the right subtree is higher than the left subtree.

Pre: `root` is a pointer to the root of the [sub]tree

`taller` is `true`

Post: `root` has been updated (if necessary)

`taller` has been updated

AVL Right Balance Algorithm

```
rightTree = root->right
// Case 1: Right of right. Single rotation left.
if rightTree is RH then
    root = rotateLeft(root)
    root->balance = EH
    rightTree->balance = EH
    taller = false
```

AVL Right Balance Algorithm

```
// Case 2: Left of Right. Double rotation required.
else
    leftTree = rightTree->left
    if leftTree->balance = RH then
        root->balance = LH
        rightTree->balance = EH
    else if leftTree->balance = EH then
        rightTree->balance = EH
    else
        root->balance = EH
        rightTree->balance = RH
    end
    leftTree->balance = EH
    root->right = rotateRight(rightTree)
    root = rotateLeft(root)
    taller = false
end
return root
```

AVL Delete Algorithm

The AVL delete follows the basic logic of the binary search tree delete with the addition of the logic to balance the tree. As with the insert logic, the balancing occurs as we back out of the tree.

Algorithm AVLDelete(ref root <pointer>, val deleteKey <key>, ref shorter <boolean>, ref success <boolean>)

This algorithm deletes a node from an AVL tree and rebalances if necessary.

Pre: `root` is a pointer to the root of the [sub]tree

`deleteKey` is the key of node to be deleted

Post: node deleted if found, tree unchanged if not found

`shorter` is `true` if subtree is shorter

`success` is `true` if deleted, `false` if not found

Return pointer to root of (potential) new subtree

AVL Delete Algorithm

```
if tree null then
    shorter = false
    success = false
    return null
end
if deleteKey < root->data.key then
    root->left = AVLDelete(root->left, deleteKey,
      shorter, success)
    if shorter then
        root = deleteRightBalance(root, shorter)
    end
else if deleteKey > root->data.key then
    root->right = AVLDelete(root->right,
      deleteKey, shorter, success)
    if shorter then
        root = deleteLeftBalance(root, shorter)
    end
```

AVL Delete Algorithm

```
// Delete node found – test for leaf node
else
    deleteNode = root
    if no right subtree then
        newRoot = root->left
        success = true
        shorter = true
        recycle(deleteNode)
        return newRoot
    else if no left subtree then
        newRoot = root->right
        success = true
        shorter = true
        recycle(deleteNode)
        return newRoot
```

AVL Delete Algorithm

```
else
    // ... // Delete node has two subtrees
    else
        exchPtr = root->left
        while exchPtr->right not null do
            exchPtr = exchPtr->right
        end
        root->data = exchPtr->data
        root->left = AVLDelete(root->left,
         exchPtr->data.key, shorter, success)
        if shorter then
            root = deleteRightBalance(root,
             shorter)
        end
    end
end
Return root
End AVLDelete
```

Delete Right Balance

```
Algorithm deleteRightBalance(ref root <pointer>, ref shorter <boolean>)
The (sub)tree is shorter after a deletion on the left branch. Adjust the balance factors and if necessary balance the tree by rotating left.

Pre: tree is shorter
Post: balance factors updated and balance restored
root updated
shorter updated

if root LH then
    root->balance = EH
else if root EH then
    root->balance = RH
    shorter = false
```

Delete Right Balance

```
else
    rightTree = root->right
    if rightTree LH then
        leftTree = rightTree->left
        if leftTree LH then
            rightTree->balance = RH
            root->balance = EH
        else if leftTree EH then
            root->balance = LH
            rightTree->balance = EH
        else
            root->balance = LH
            rightTree->balance = EH
        end
        leftTree->balance = EH
        root->right = rotateRight(rightTree)
        root = rotateLeft(root)
```

Delete Right Balance

```
else
    // ...

    else
        if rightTree not EH then
            root->balance = EH
            rightTree->balance = EH
        else
            root->balance = RH
            rightTree->balance = LH
            shorter = false
        end
        root = rotateLeft(root)
    end
end
return root
End deleteRightBalance
```

Delete Left Balance

```
Algorithm deleteLeftBalance(ref root <pointer>,
  ref shorter <boolean>)
The (sub)tree is shorter after a deletion on the
  right branch. Adjust the balance factors and if
  necessary balance the tree by rotating right.

Pre: tree is shorter
Post: balance factors updated and balance restored
root updated
shorter updated

if root RH then
  |   root->balance = EH
else if root EH then
  |   root->balance = LH
  |   shorter = false
```

Delete Left Balance

```
else
    leftTree = root->left
    if leftTree RH then
        rightTree = leftTree->right
        if rightTree RH then
            leftTree->balance = LH
            root->balance = EH
        else if rightTree EH then
            root->balance = RH
            leftTree->balance = EH
        else
            root->balance = RH
            leftTree->balance = EH
        end
        rightTree->balance = EH
        root->left = rotateLeft(leftTree)
        root = rotateRight(root)
```

Delete Left Balance

```
else
    // ...

    else
        if leftTree not EH then
            root->balance = EH
            leftTree->balance = EH
        else
            root->balance = LH
            leftTree->balance = RH
            shorter = false
        end
        root = rotateRight(root)
    end
end
return root
End deleteLeftBalance
```

Outcomes

- **L.O.3.1** - Depict the following concepts: binary tree, complete binary tree, balanced binary tree, AVL tree, multi-way tree, etc.
- **L.O.3.2** - Describe the strorage structure for tree structures using pseudocode.
- **L.O.3.3** - List necessary methods supplied for tree structures, and describe them using pseudocode.
- **L.O.3.4** - Identify the importance of "blanced" feature in tree structures and give examples to demonstate it.
- **L.O.3.5** - Identiy cases in which AVL tree and B-tree are unblanced, and demonstrate methods to resolve all the cases step-by-step using figures.

Outcomes

- **L.O.3.6** - Implement binary tree and AVL tree using C/C++.
- **L.O.3.7** - Use binary tree and AVL tree to solve problems in real-life, especially related to searching techniques.
- **L.O.3.8** - Analyze the complexity and develop experiment (program) to evaluate methods supplied for tree structures.
- **L.O.8.4** - Develop recursive implementations for methods supplied for the following structures: list, tree, heap, searching, and graphs.
- **L.O.1.2** - Analyze algorithms and use Big-O notation to characterize the computational complexity of algorithms composed by using the following control structures: sequence, branching, and iteration (not recursion).

Contents

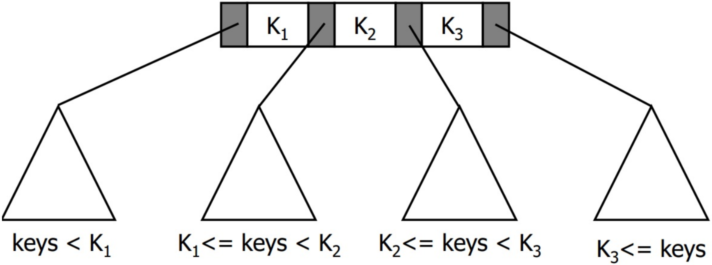
Multiway Trees

Multiway Trees

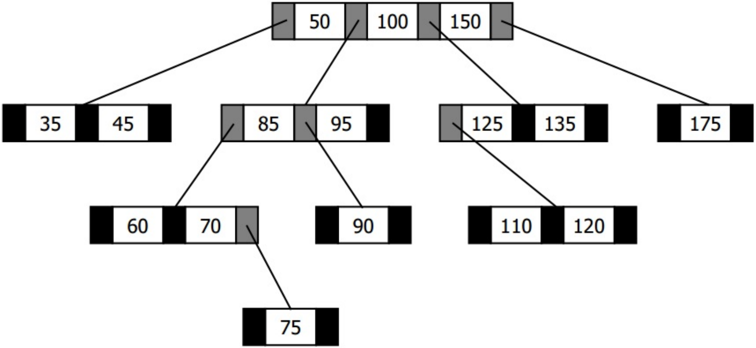
Tree whose outdegree is not restricted to 2 while retaining the general properties of binary search trees.

M-Way Search Trees

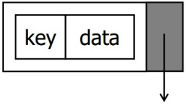
- Each node has `m - 1` data entries and `m` subtree pointers.
- The key values in a subtree such that:
  - $\geq$ the key of the left data entry
  - $<$ the key of the right data entry.



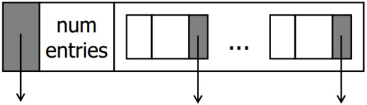
M-Way Search Trees



M-Way Node Structure



entry
    key <key type>
    data <data type>
    rightPtr <pointer>
end entry



node
    firstPtr <pointer>
    numEntries <integer>
    entries <array[1 .. m-1] of entry>
end node

B-Trees

B-Trees

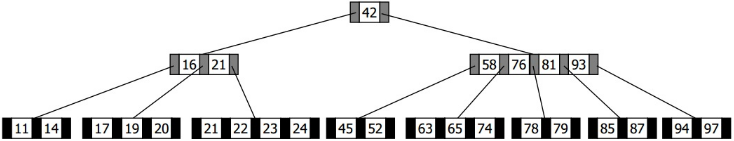
- M-way trees are unbalanced.
- Bayer, R. & McCreight, E. (1970) created B-Trees.

B-Trees

A B-tree is an m-way tree with the following additional properties ($m \geq 3$):

- The root is either a leaf or has at least 2 subtrees.
- All other nodes have at least $\lceil m/2 \rceil - 1$ entries.
- All leaf nodes are at the same level.

B-Trees

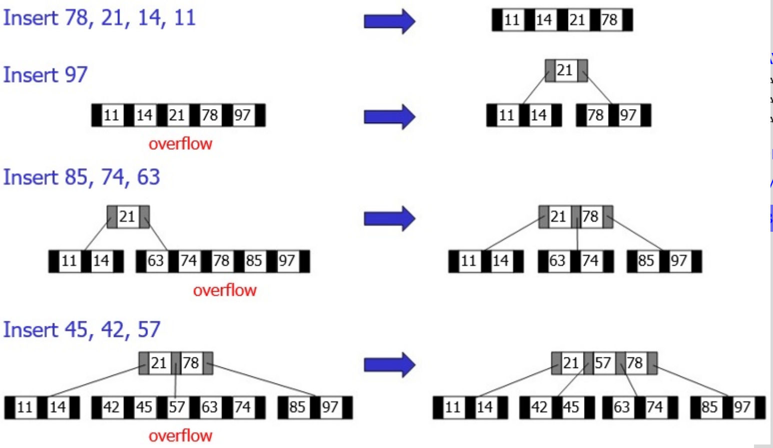


Hình: $m = 5$

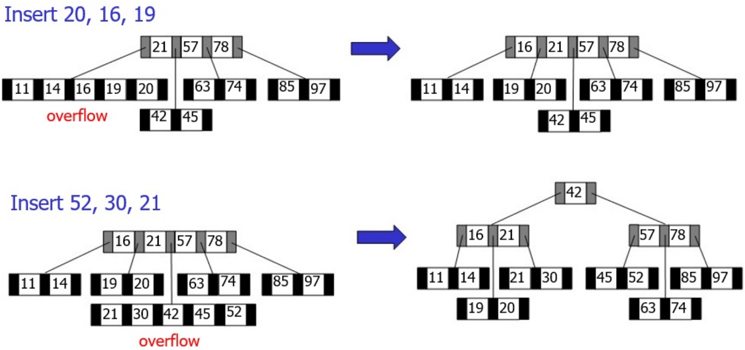
B-Tree Insertion

- Insert the new entry into a leaf node.
- If the leaf node is overflow, then split it and insert its median entry into its parent.

B-Tree Insertion



B-Tree Insertion



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B-Tree Insertion

```
Algorithm BTreeInsert(ref root <pointer>, val data <record>)
Inserts data into B-tree. Equal keys placed on right branch.
Pre: root is a pointer to the B-tree. May be null.
Post: data inserted
Return pointer to B-tree root.
taller = insertNode(root, data, upEntry)
if taller then
    // Tree has grown. Create new root.
    allocate(newPtr)
    newPtr−>entries[1] = upEntry
    newPtr−>firstPtr = root
    newPtr−>numEntries = 1
    root = newPtr
end
return root
End BTreeInsert
```

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B-Tree Insertion

Algorithm insertNode (ref root <pointer>, val data <record>, ref upEntry <entry>)

Recursively searches tree to locate leaf for data. If node overflow, inserts median key's data into parent.

Pre: `root` is a pointer to tree or subtree. May be `null`.

Post: `data` inserted

`upEntry` is overflow entry to be inserted into parent.

Return tree taller <boolean>.

if *root null* **then**

    upEntry.data = data

    upEntry.rightPtr = null

    taller = true

B-Tree Insertion

```
else
    entryNdx = searchNode(root, data.key)
    if entryNdx > 0 then
        subTree = root−>entries[entryNdx].rightPtr
    else
        subTree = root−>firstPtr
    end
    taller = insertNode(subTree, data, upEntry)
    if taller then
        if node full then
            splitNode(root, entryNdx, upEntry)
            taller = true
        else
            insertEntry(root, entryNdx, upEntry)
            taller = false
            root−>numEntries = root−>numEntries + 1
        end
    end
end
return taller
```

B-Tree Insertion

```
Algorithm searchNode(val nodePtr <pointer>, val target <key>)
Search B-tree node for data entry containing key <= target.
Pre: nodePtr is pointer to non-null node.
target is key to be located.
Return index to entry with key <= target.
0 if key < first entry in node

if target < nodePtr−>entry[1].data.key then
    walker = 0
else
    walker = nodePtr−>numEntries
    while target < nodePtr−>entries[walker].data.key do
        walker = walker - 1
    end
end
return walker
```

B-Tree Insertion

Algorithm splitNode(val node <pointer>, val entryNdx <index>, ref upEntry <entry>)
Node has overflowed. Split node. No duplicate keys allowed.
Pre: `node` is pointer to node that overflowed.
`entryNdx` contains index location of parent.
`upEntry` contains entry being inserted into split node.
Post: `upEntry` now contains entry to be inserted into parent.

minEntries = minimum number of entries
allocate (rightPtr)
// Build right subtree node
if *entryNdx* <= *minEntries* **then**
    fromNdx = minEntries + 1
else

B-Tree Insertion

```
else
    fromNdx = minEntries + 2
end
toNdx = 1
rightPtr->numEntries = node->numEntries –
  fromNdx + 1
while fromNdx <= node->numEntries do
    rightPtr->entries[toNdx] =
      node->entries[fromNdx]
    fromNdx = fromNdx + 1
    toNdx = toNdx + 1
end
node->numEntries =
  node->numEntries−rightPtr->numEntries
if entryNdx <= minEntries then
    insertEntry(node, entryNdx, upEntry)
else
```

B-Tree Insertion

```
else
    insertEntry(rightPtr, entryNdx−minEntries,
      upEntry)
    node->numEntries = node->numEntries− 1
    rightPtr->numEntries = rightPtr->numEntries
      + 1
end
// Build entry for parent
medianNdx = minEntries + 1
upEntry.data = node->entries[medianNdx].data
upEntry.rightPtr = rightPtr
rightPtr->firstPtr =
  node->entries[medianNdx].rightPtr
return
End splitNode
```

B-Tree Insertion

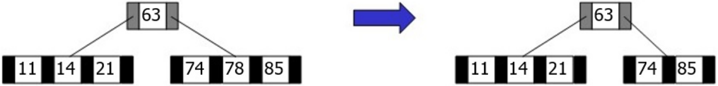
```
Algorithm insertEntry(val node <pointer>, val entryNdx <index>, val newEntry <entry>)
Inserts one entry into a node by shifting nodes to make room.
Pre: node is pointer to node to contain data.
entryNdx is index to location for new data.
newEntry contains data to be inserted.
Post: data has been inserted in sequence.

shifter = node->numEntries + 1
while shifter > entryNdx + 1 do
    node->entries[shifter] = node->entries[shifter - 1]
    shifter = shifter - 1
end
node->entries[shifter] = newEntry
node->numEntries = node->numEntries + 1
return
```

B-Tree Deletion

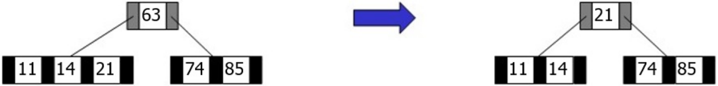
- It must take place at a leaf node.
- If the data to be deleted are not in a leaf node, then replace that entry by the largest entry on its left subtree.

B-Tree Deletion

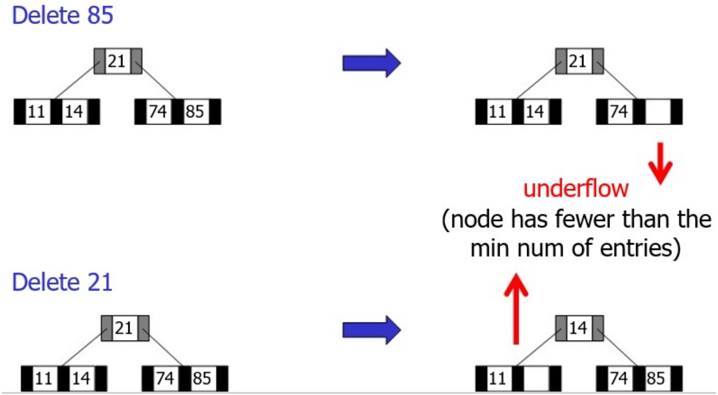
Delete 78



Delete 63



B-Tree Deletion

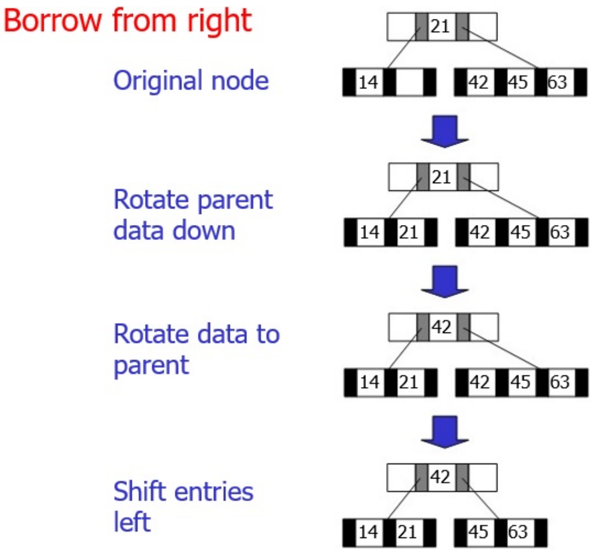


Reflow

For each node to have sufficient number of entries:

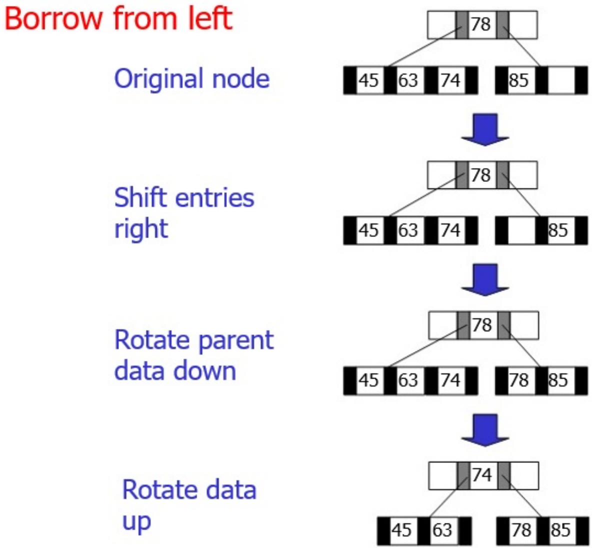
- Balance: shift data among nodes.
- Combine: join data from nodes.

Balance



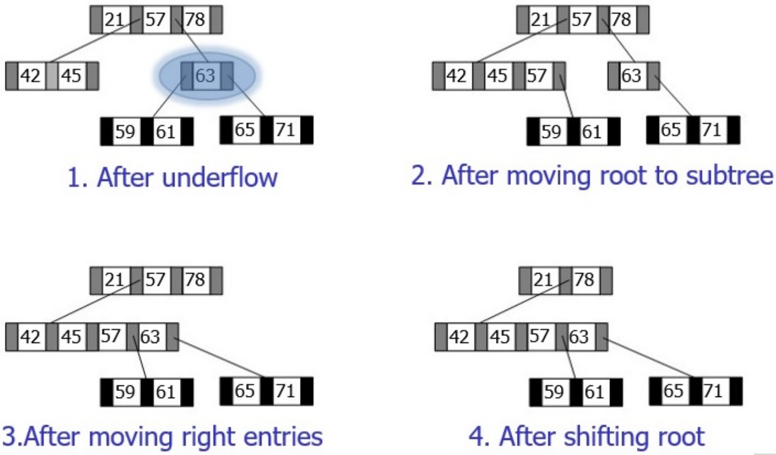
Borrow from right
Original node
Rotate parent data down
Rotate data to parent
Shift entries left
21
14
42
45
63
21
14
21
42
45
63
42
14
21
42
45
63
42
14
21
45
63

Balance

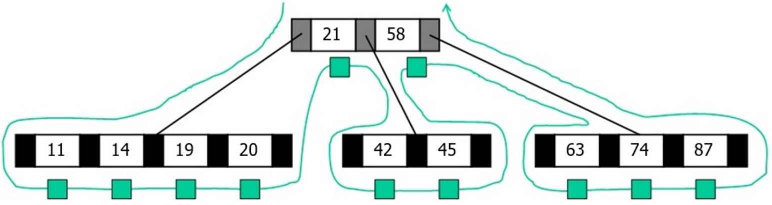


Borrow from left
Original node
Shift entries right
Rotate parent data down
Rotate data up
78
45
63
74
85
78
45
63
74
85
78
45
63
74
78
85
74
45
63
78
85

Combine


1. After underflow
2. After moving root to subtree
3.After moving right entries
4. After shifting root

B-Tree Traversal



B-Tree Traversal

```
Algorithm BTreeTraversal (val root <pointer>)
Processes tree using inorder traversal.
Pre: root is pointer to B-Tree.
Post: Every entry has been processed in order.
scanCount = 0
ptr = root−>firstPtr
while scanCount <= root−>numEntries do
    if ptr not null then
        BTreeTraversal(ptr)
    end
    scanCount = scanCount + 1
    if scanCount <= root−>numEntries then
        process (root−>entries[scanCount].data)
        ptr = root−>entries[scanCount].rightPtr
    end
end
return
End BTreeTraversal
```

B-Tree Search

Algorithm BTreeSearch(val root <pointer>, val target <key>, ref node <pointer>, ref entryNo <index>)

Recursively searches a B-tree for the target key.

Pre: `root` is pointer to a tree or subtree

`target` is the data to be located

Post:

if found – –

node is pointer to located node

entryNo is entry within node

if not found – –

node is null and entryNo is zero

Return found <boolean>

B-Tree Search

```
if target < first entry then
    return BTreeSearch (root−>firstPtr, target, node,
      entryNo)
else
    entryNo = root−>numEntries
    while target < root−>entries[entryNo].data.key
      do
        entryNo = entryNo - 1
    end
    if target = root−>entries[entryNo].data.key then
        found = true
        node = root
    else
        return BTreeSearch
          (root−>entries[entryNo].rightPtr, target,
          node, entryNo)
    end
end
```

B-Tree Variations

- B*Tree: the minimum number of (used) entries is two thirds.
- B+Tree:
  - Each data entry must be represented at the leaf level.
  - Each leaf node has one additional pointer to move to the next leaf node.

THANK YOU.