Hochiminh city University of Technology
Faculty of Computer Science and Engineering

COMPUTER GRAPHICS

CHAPTER 05:

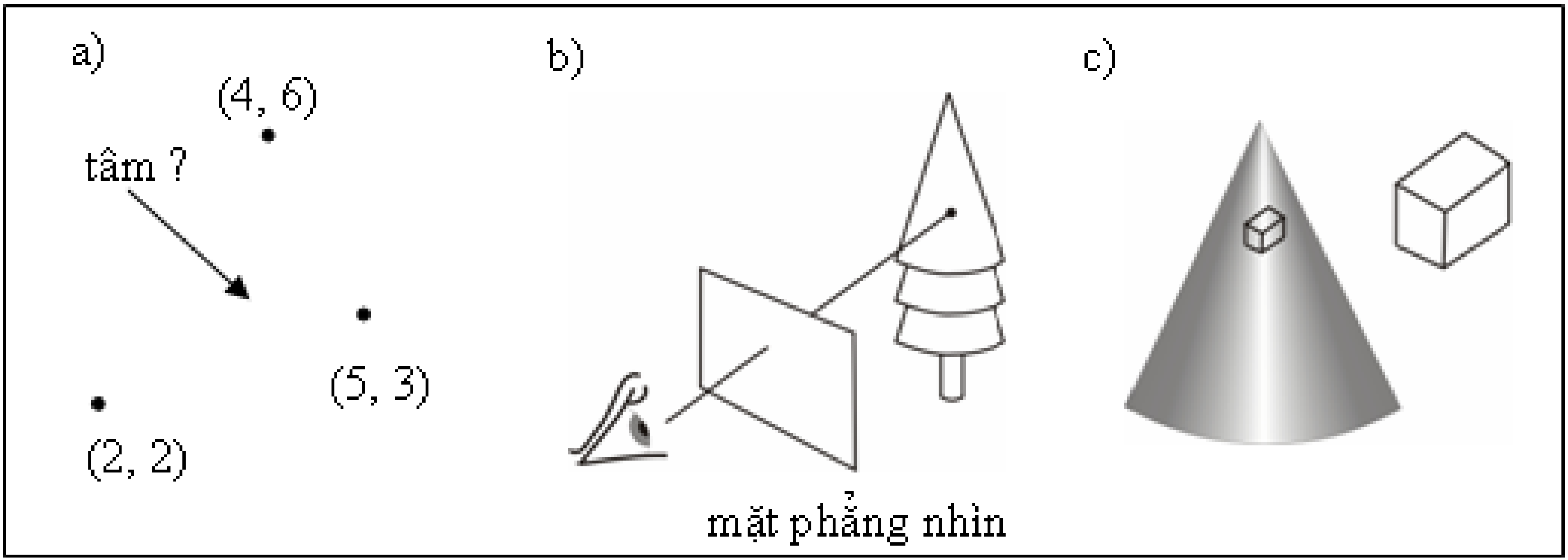
Vector in Computer Graphics

OUTLINE

- Vector
- Dot product
- Cross Product
- Scalars
- Points
- Affine Sums
- Parametric Form
- Line
- Plane
- Some Example
- Representation

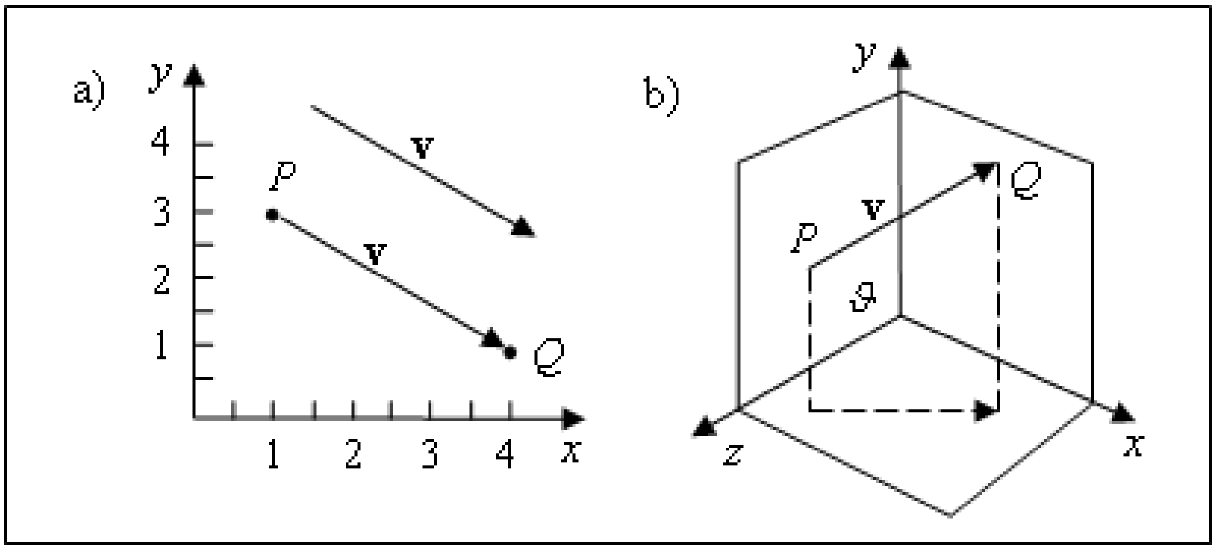
Vector

- Why vector important?
  - Remove Hidden-face.
  - Normal vector
  - Three basic elements in geometry: scalar, point, vector



Vector

- Physical definition: a vector is a quantity with two attributes: 1)**Direction; 2)Magnitude**
- Examples include: 1)**Force; 2)Velocity**
- Vectors Lack Position: These vectors are identical
  - **Same length and magnitude**



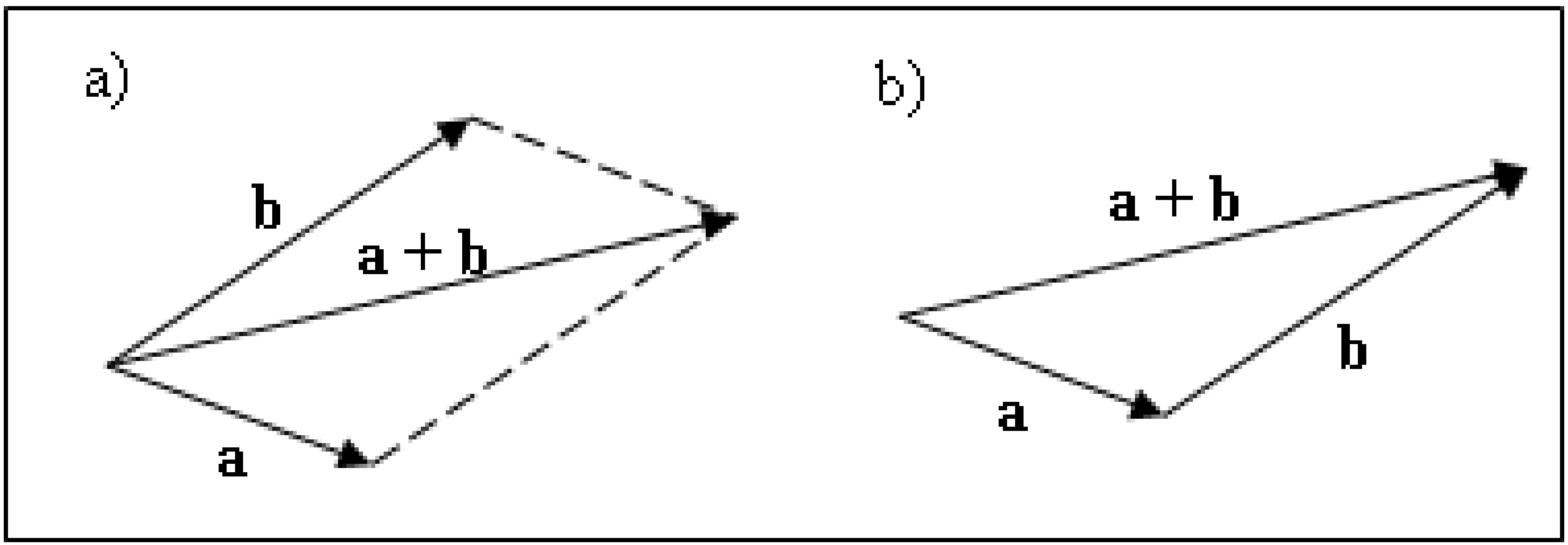
Vector

- A = (1, 2, 9), B = (4, 6, 3)
- AB = (4 – 1, 6 - 2, 3 - 9) = (3, 4, -6)
- BA = (1 – 4, 2 – 6, 9 – 3) = (-3, -4, 6)

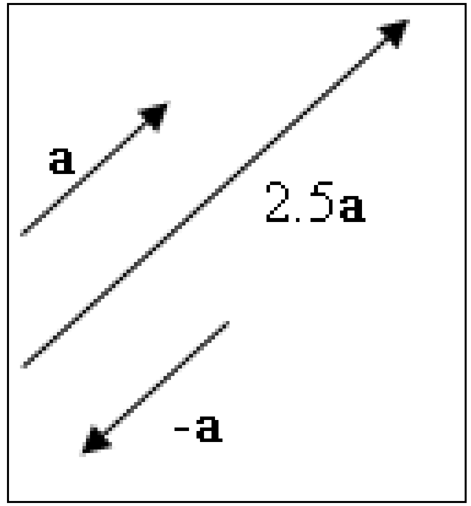
Vector

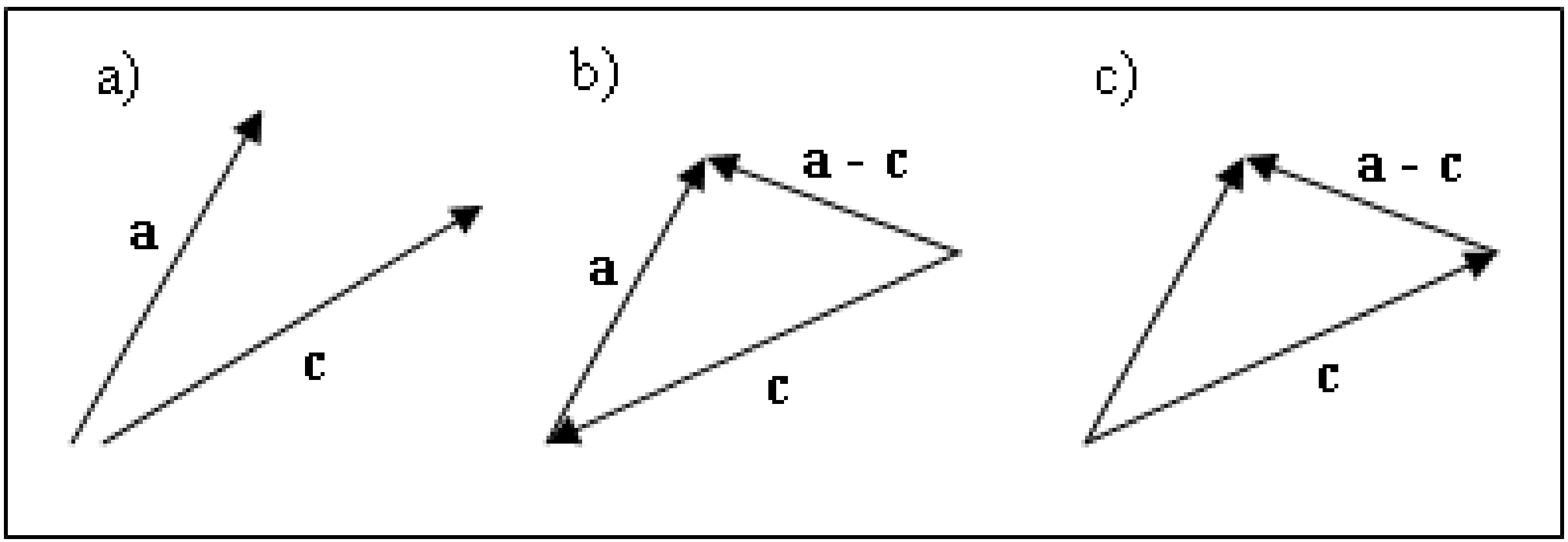
a = (2, 5, 6), **b** = (-2, 7, 1)

- Addition: **a** + **b** = (0, 12, 7)
- Scalar-vector multiplication: 6**a** = (12, 30, 39)
- Subtraction: **a** - **b** = **a** + (-**b**) = (4, -2, 5)



Vector





Vector

- **Magnitude:** $$|\mathbf{w}| = \sqrt{w_1^2 + w_2^2 + \ldots + w_n^2}$$

- **Unit vector:** $$\mathbf{u}_a = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Dot product

- Definition: The dot product d of two *n-dimensional* vectors $\mathbf{v} = (v_1, v_2, ..., v_n)$ and $\mathbf{w} = (w_1, w_2, ..., w_n)$ is denoted as $\mathbf{v} \bullet \mathbf{w}$ and has the value:

$$d = \mathbf{v} \bullet \mathbf{w} = \sum_{i=1}^{n} v_i w_i$$

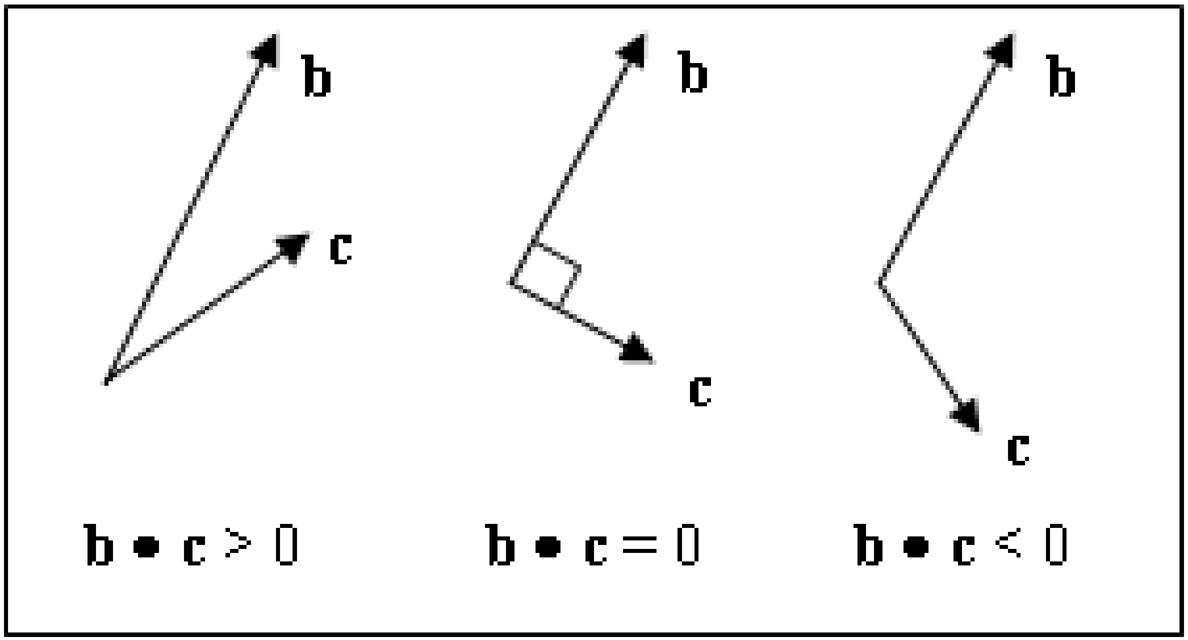
- Properties
  - Symmetry: $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$
  - Linearity: $(\mathbf{a} + \mathbf{c}) \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{b} + \mathbf{c} \bullet \mathbf{b}$
  - Homogeneity: $(s\mathbf{a}) \bullet \mathbf{b} = s(\mathbf{a} \bullet \mathbf{b})$
  - $|\mathbf{b}|^2 = \mathbf{b} \bullet \mathbf{b}$

Dot product

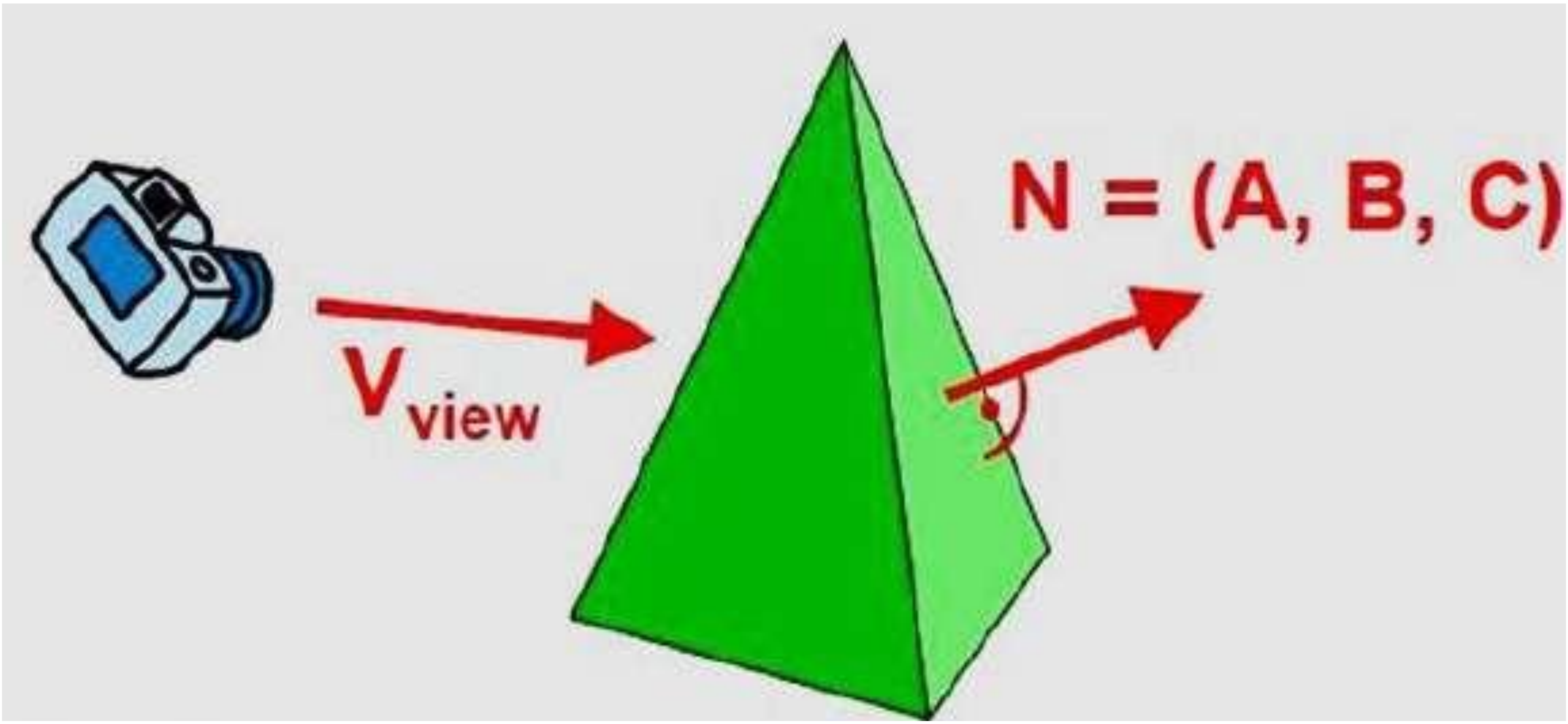
- The angle between two vectors

  b•c = |**b**||**c**| cos(θ)

$$\cos(\theta) = \mathrm{u_b} \bullet \mathrm{u_c}$$

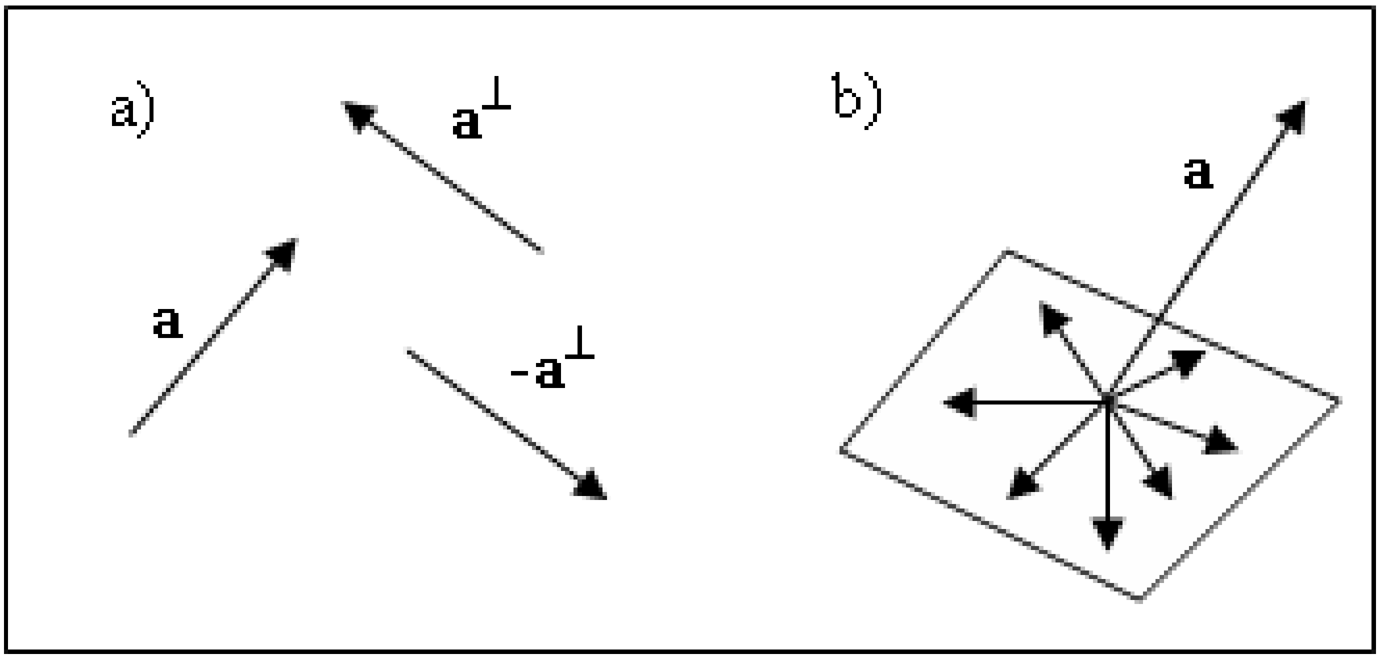


Dot product



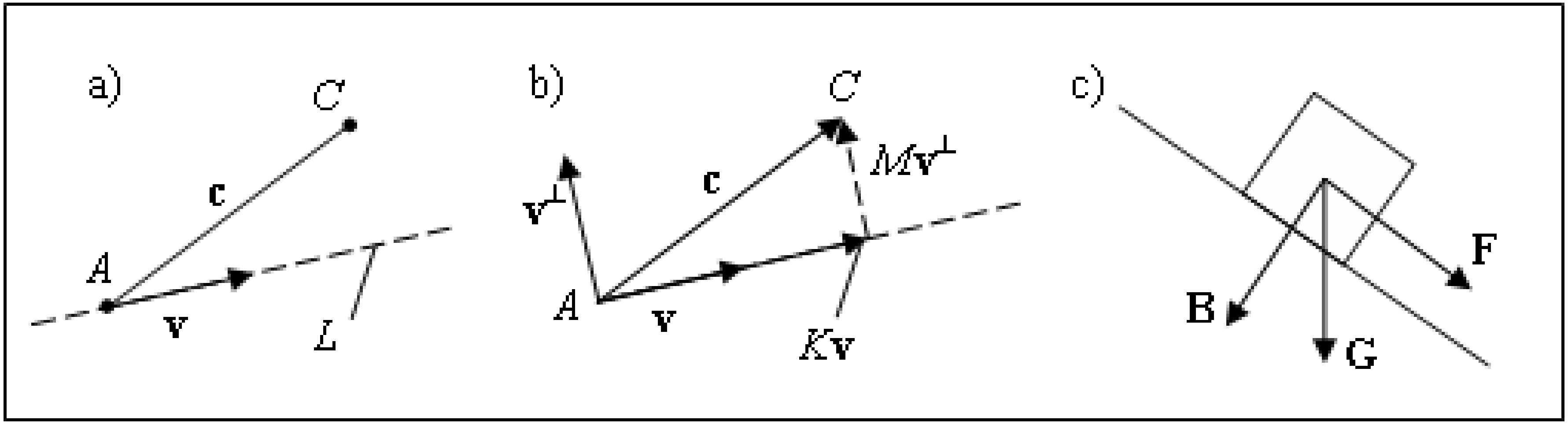
Dot product

- The 2D "perp" vector
  - **Suppose a** = (*ax*, *ay*), then **a** ⊥ = (-*ay*, *ax*) is the counterclockwise perpendicular to **a**.



Dot product

- Orthogonal Projections and the Distance from a point to a line



$\mathbf{c} = K\mathbf{v} + M\mathbf{v}^{\perp}$ $\quad$ (K, M = ?)

Dot product

- $\mathbf{c} = K\mathbf{v} + M\mathbf{v}^{\perp}$ $(K, M = ?)$

  $(cx, cy) = K(vx, vy) + M(vx^{\perp}, vy^{\perp})$

$$\begin{cases} cx = K*vx + M*vx^{\perp} \\ cy = K*vy + M*vy^{\perp} \end{cases}$$

Dot Product

- $\mathbf{c} = K\mathbf{v} + M\mathbf{v}^{\perp}$ $(K, M = ?)$

$\mathbf{c}\bullet\mathbf{v} = K\mathbf{v}\bullet\mathbf{v} + M\mathbf{v}^{\perp}\bullet\mathbf{v} \rightarrow$ $$K = \frac{\mathbf{c}\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}$$

$\mathbf{c}\bullet\mathbf{v}^{\perp} = K\mathbf{v}^{\perp}\bullet\mathbf{v} + M\mathbf{v}^{\perp}\bullet\mathbf{v}^{\perp} \rightarrow$ $$M = \frac{\mathbf{c}\bullet\mathbf{v}^{\perp}}{\mathbf{v}^{\perp}\bullet\mathbf{v}^{\perp}}$$

$$\mathbf{c} = \left(\frac{\mathbf{v}\bullet\mathbf{c}}{|\mathbf{v}|^2}\right)\mathbf{v} + \left(\frac{\mathbf{v}^{\perp}\bullet\mathbf{c}}{|\mathbf{v}|^2}\right)\mathbf{v}^{\perp}$$

$$\mathbf{distance} = \left|\frac{\mathbf{v}^{\perp}\bullet\mathbf{c}}{|\mathbf{v}|^2}\mathbf{v}^{\perp}\right| = \frac{|\mathbf{v}^{\perp}\bullet\mathbf{c}|}{|\mathbf{v}|}$$

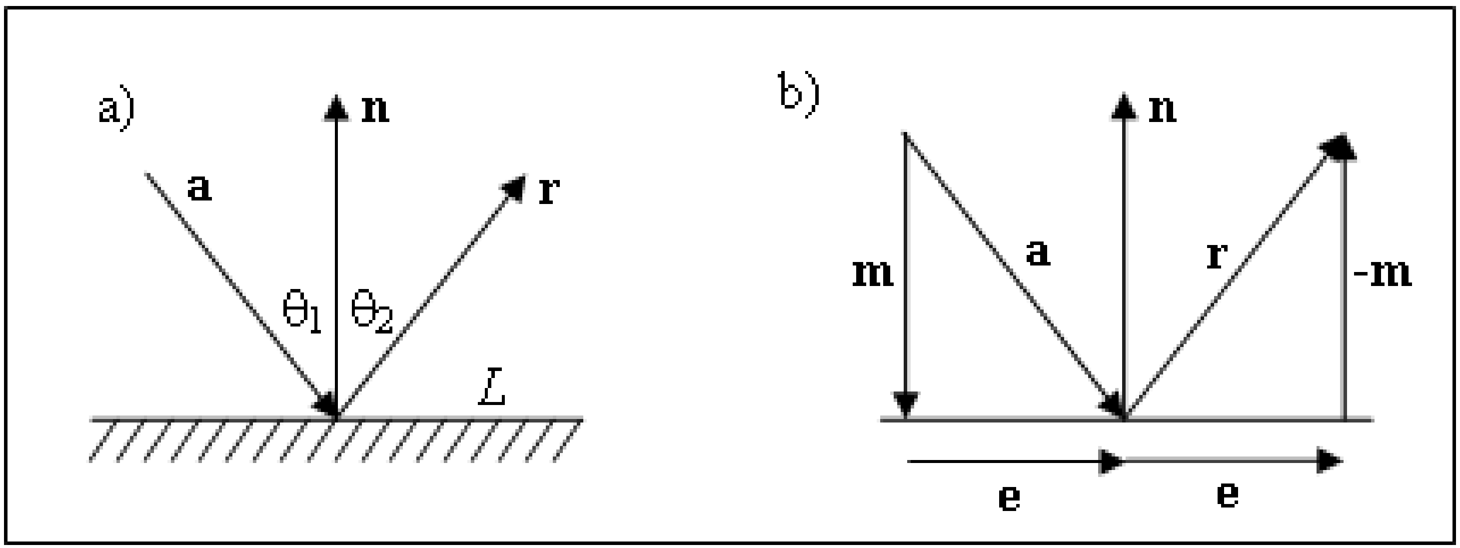
Dot product

- Reflection

r = **e** – **m**, **e** = **a** - **m** → **r** = **a** - 2**m**

$$\mathbf{m} = \frac{\mathbf{a} \bullet \mathbf{n}}{|\mathbf{n}|^2}\mathbf{n} = (\mathbf{a} \bullet \mathbf{u_n})\mathbf{u_n}$$

$$\mathbf{r} = \mathbf{a} - 2(\mathbf{a} \bullet \mathbf{u_n})\mathbf{u}_n$$



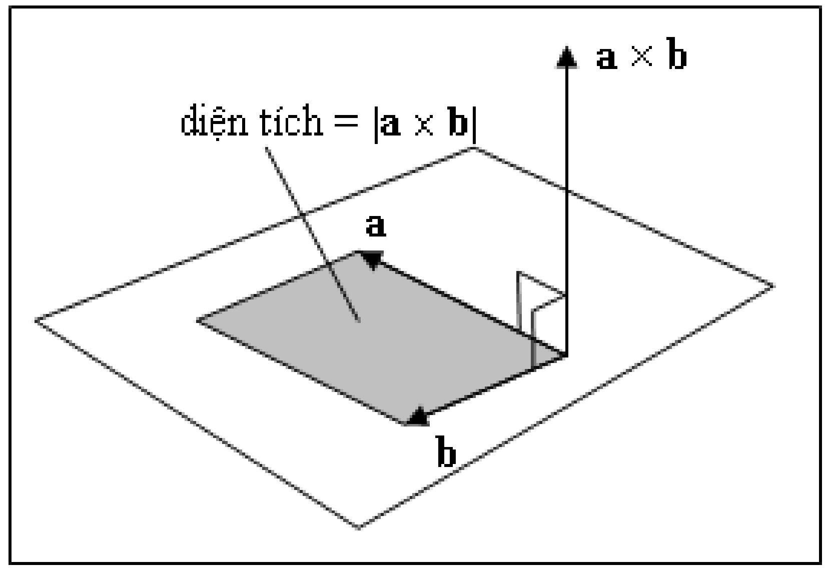
Cross Product

- The cross product of two vector is a vector
- Only for 3-dimensional vector
- Suppose **a** = (a_x, a_y, a_z) and **b** = (b_x, b_y, b_z), then the cross product of a and b is

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

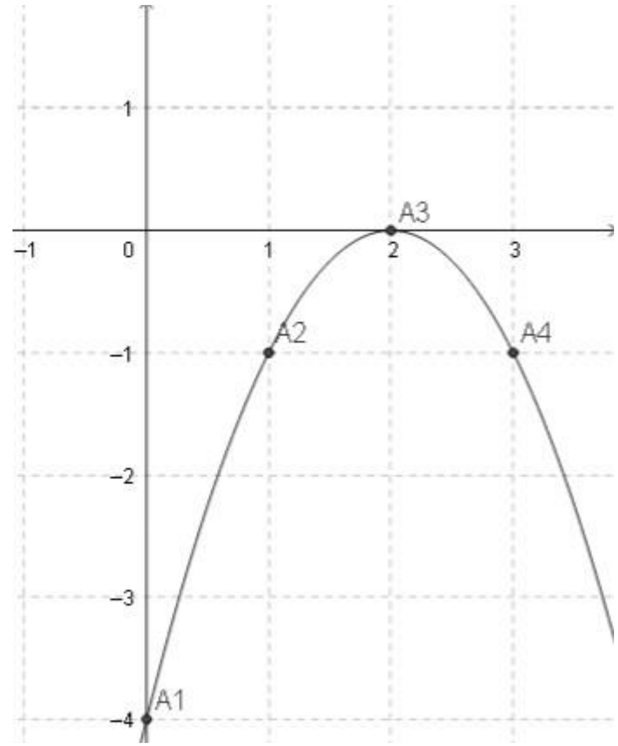
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$$



Parametric Form

- Explicit

$y = -x^2 + 4x - 4$



- How to draw

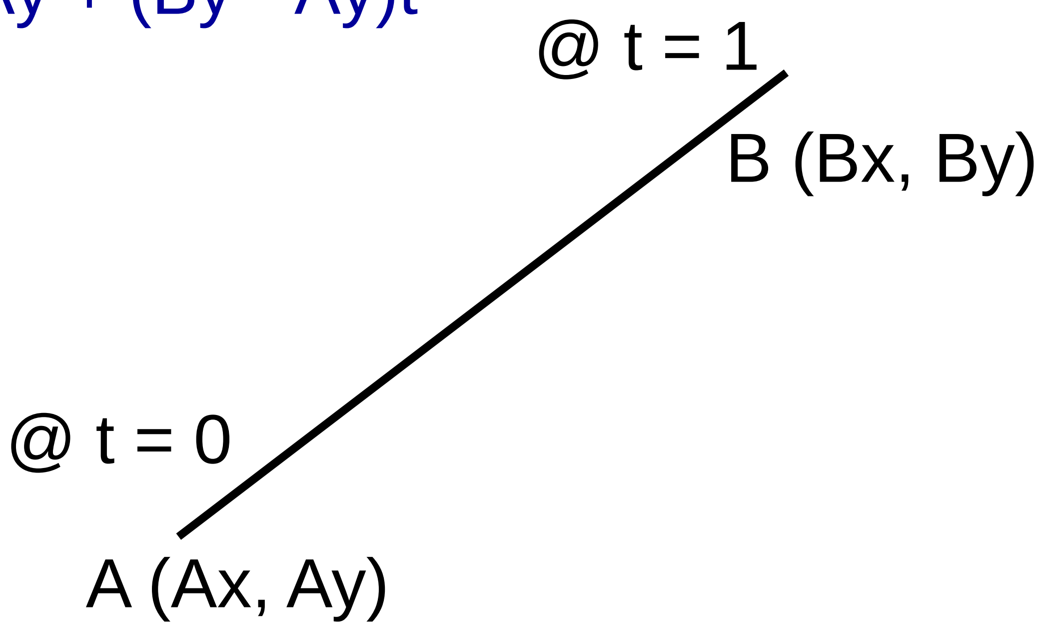
$x^2 + y^2 = 1$

Parametric Form

- Parametric form
  - Ex 1: a straight line passes through A and B. Choose a parametric form that visit A at t = 0, visit B at t = 1.

$$x(t) = Ax + (Bx - Ax)t$$

$$y(t) = Ay + (By - Ay)t$$



Parametric Form

- Parametric form

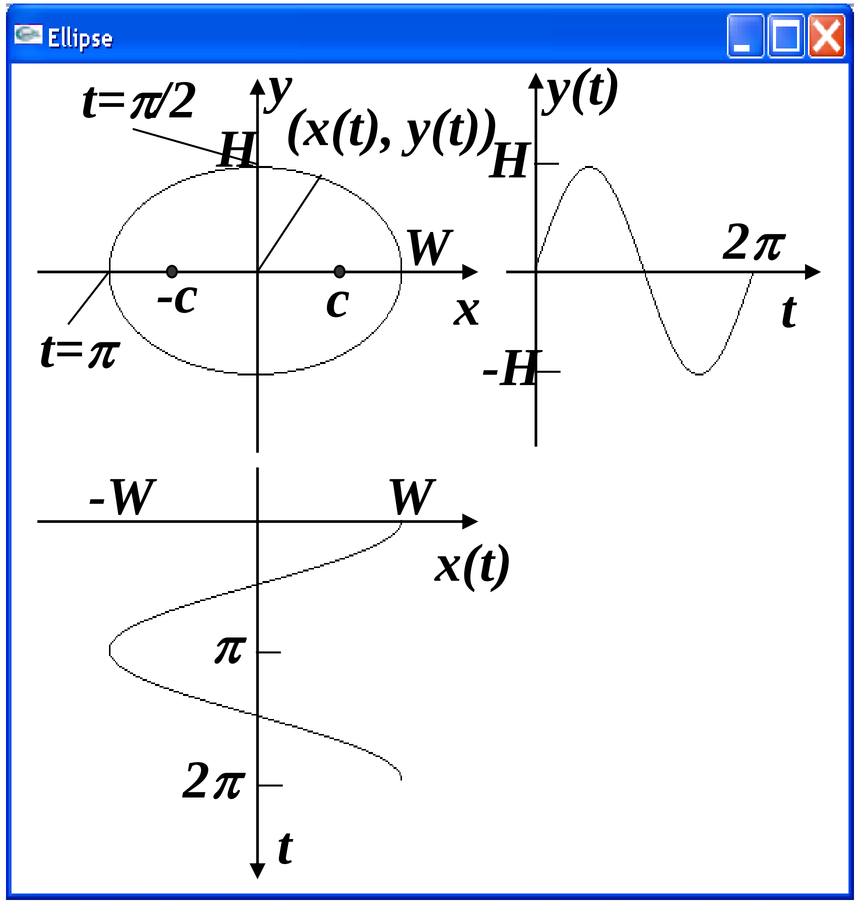
  $(x/W)^2 + (y/H)^2 = 1$

EX 2: Ellipse with radius W and H

$x(t) = W\cos(t)$

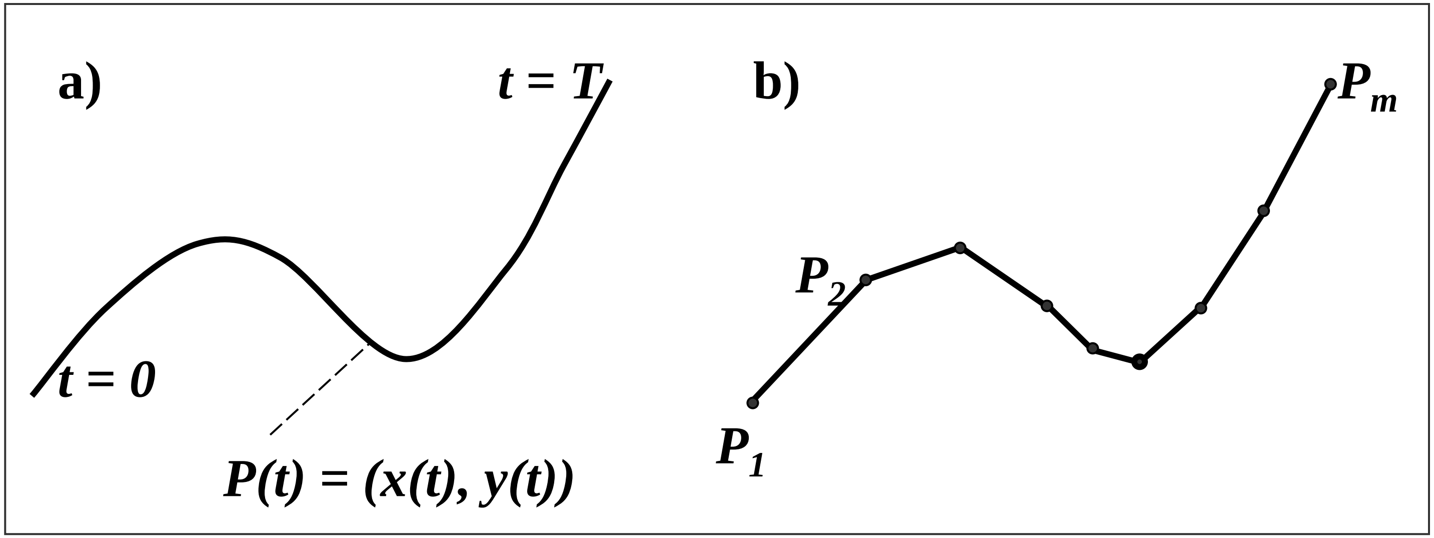
$y(t) = H\sin(t)$

với $(0 \le t \le 2\pi)$



Parametric Form

- Draw parametric form curve




```
//draw the curve (x(t), t(t)) using
//the array t[0], ..., t[n-1] of "sample-times"
glBegin(GL_LINE_STRIP);
      for(int i=0;i<n;i++)
            glVertex2f(x(t[i]), y(t[i]));
glEnd() ;
```

Parametric Form

Superellipse

- Implicit form

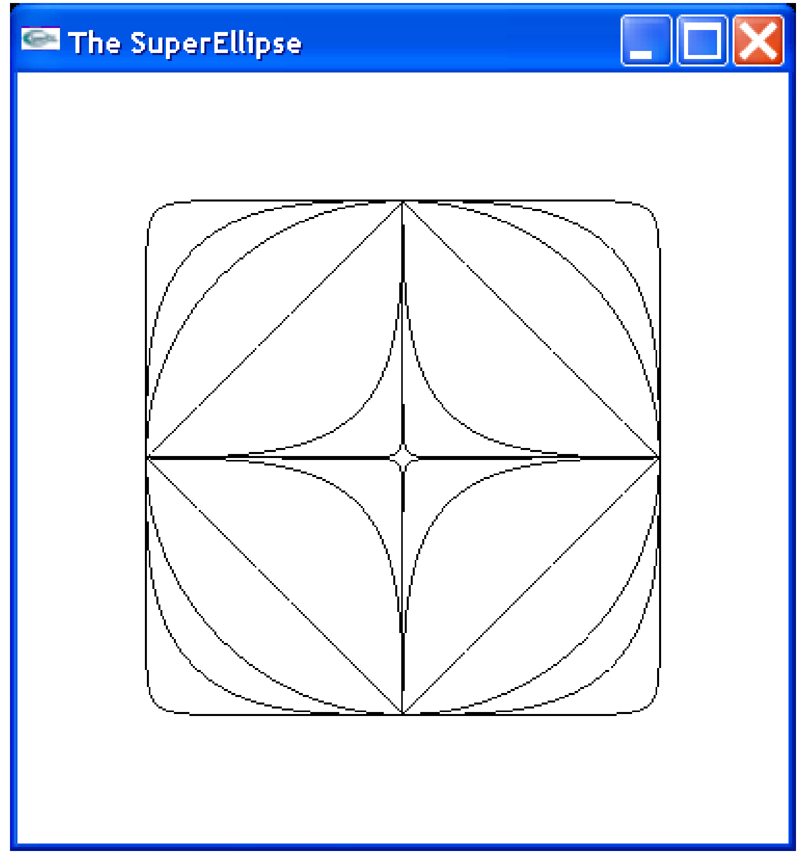
$$\left(\frac{x}{W}\right)^n + \left(\frac{y}{H}\right)^n = 1$$

- Parametric form

$$x(t) = W\cos(t)\left|\cos^{2/n-1}(t)\right|$$

$$y(t) = H\sin(t)\left|\sin^{2/n-1}(t)\right|$$

- n = 2m/(2n+1)
- n < 1 inward
- n > 1 outward
- n = 1 square



Parametric Form

❑ Superhyperbola

– Parametric form

$$x(t) = W\sec(t)\left|\sec^{2/n-1}(t)\right|$$

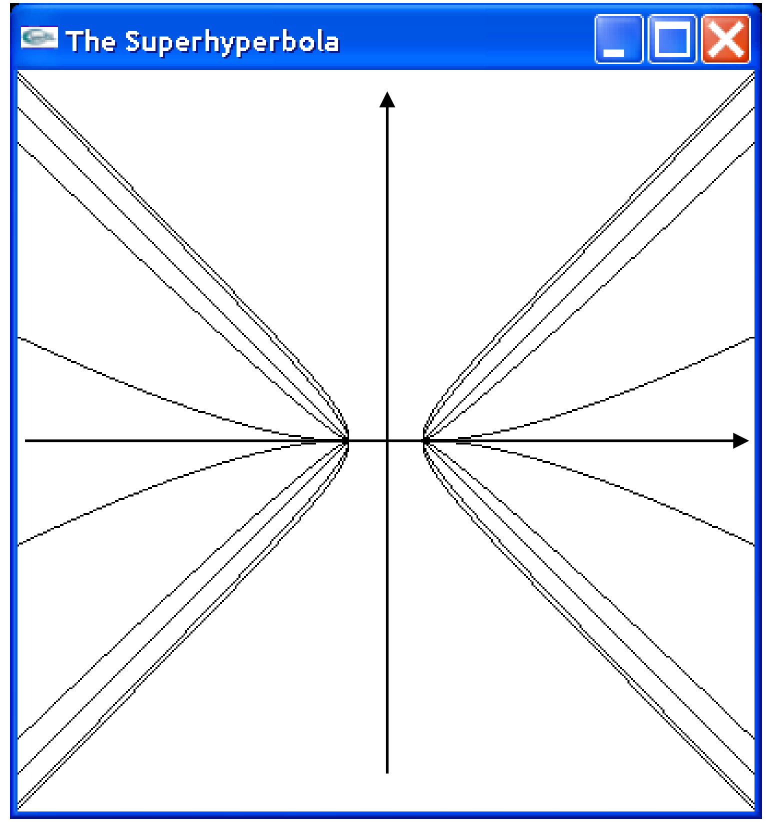
$$y(t) = H\tan(t)\left|\tan^{2/n-1}(t)\right|$$

– n = 2m/(2n+1)

– n < 1 inward

– n > 1 outward

– n = 1 line



Parametric Form

- **3D curves**

  $P(t) = (x(t), y(t), z(t))$

Helix

$x(t) = \cos(t)$

$y(t) = \sin(t)$

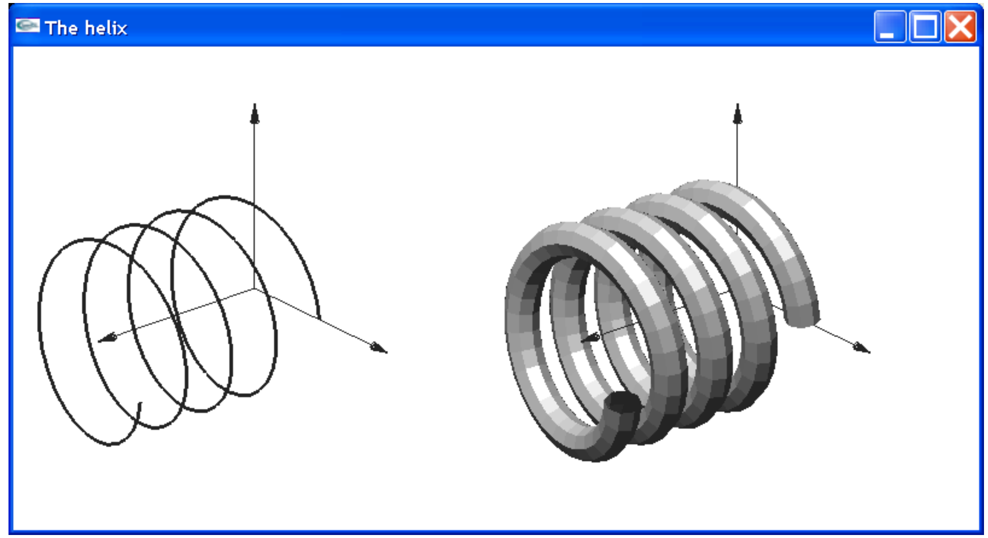
$z(t) = bt$

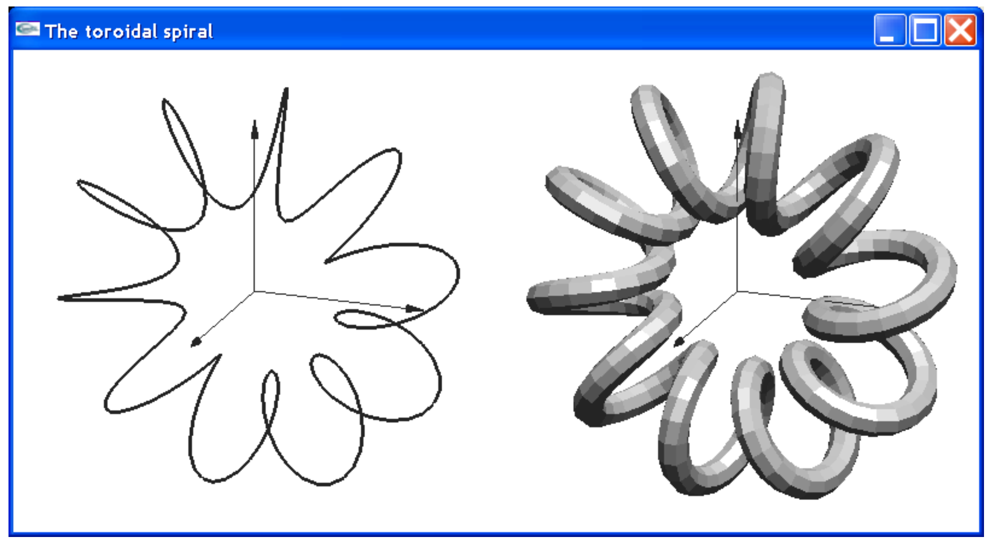
Toroidal spiral

$x(t) = (a\sin(ct) + b)\cos(t),$

$y(t) = (a\sin(ct) + b)\sin(t),$

$z(t) = a\cos(ct)$

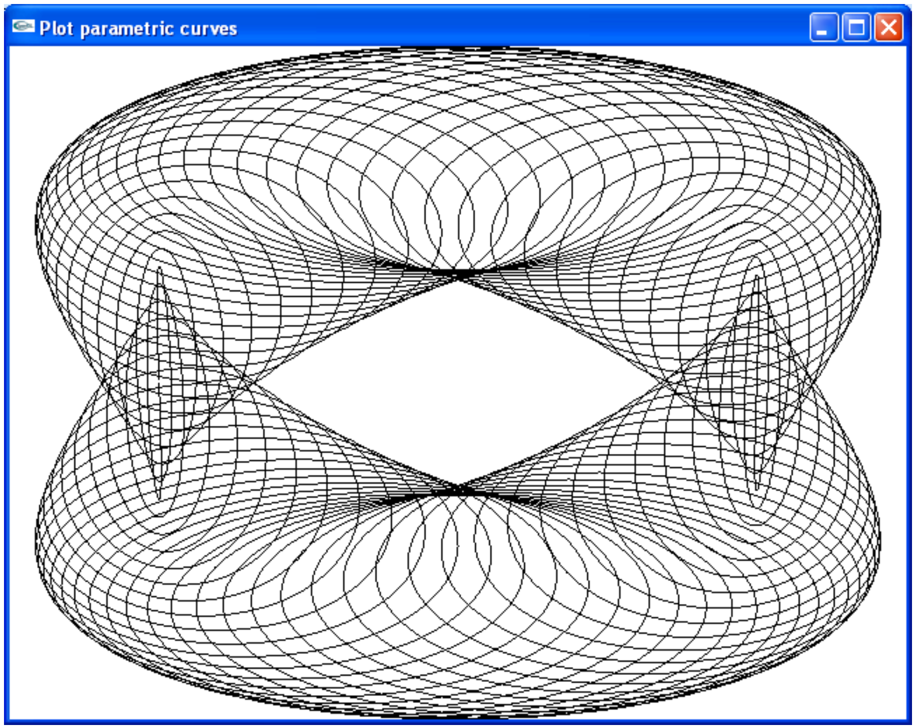




Parametric Form

x = cos(t) - cos(80*t)*sin(t);

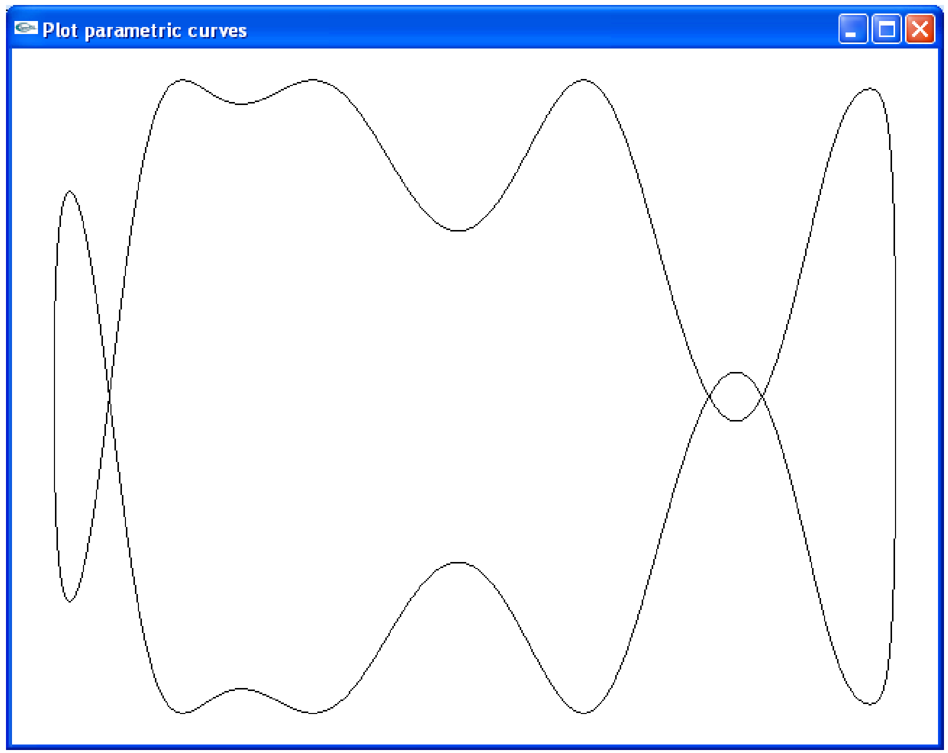
y = 2.0*sin(t) - sin(80*t);



Parametric Form

x = cos(t);

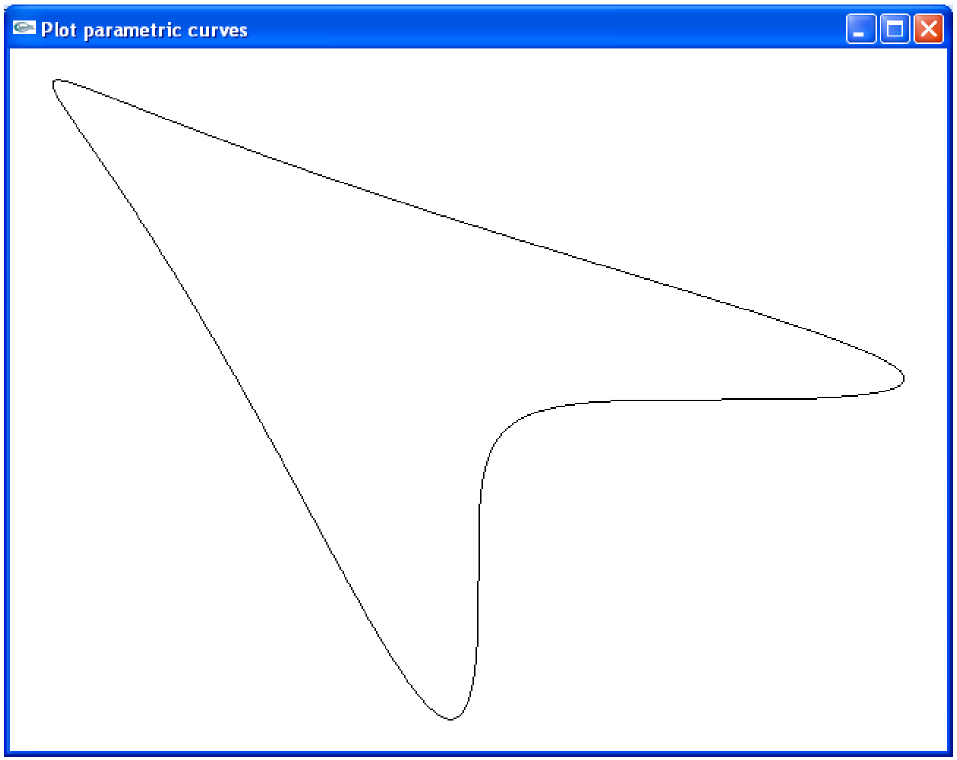
y = sin(t + sin(5.0*t));



Parametric Form

x = sin(t+sin(t));

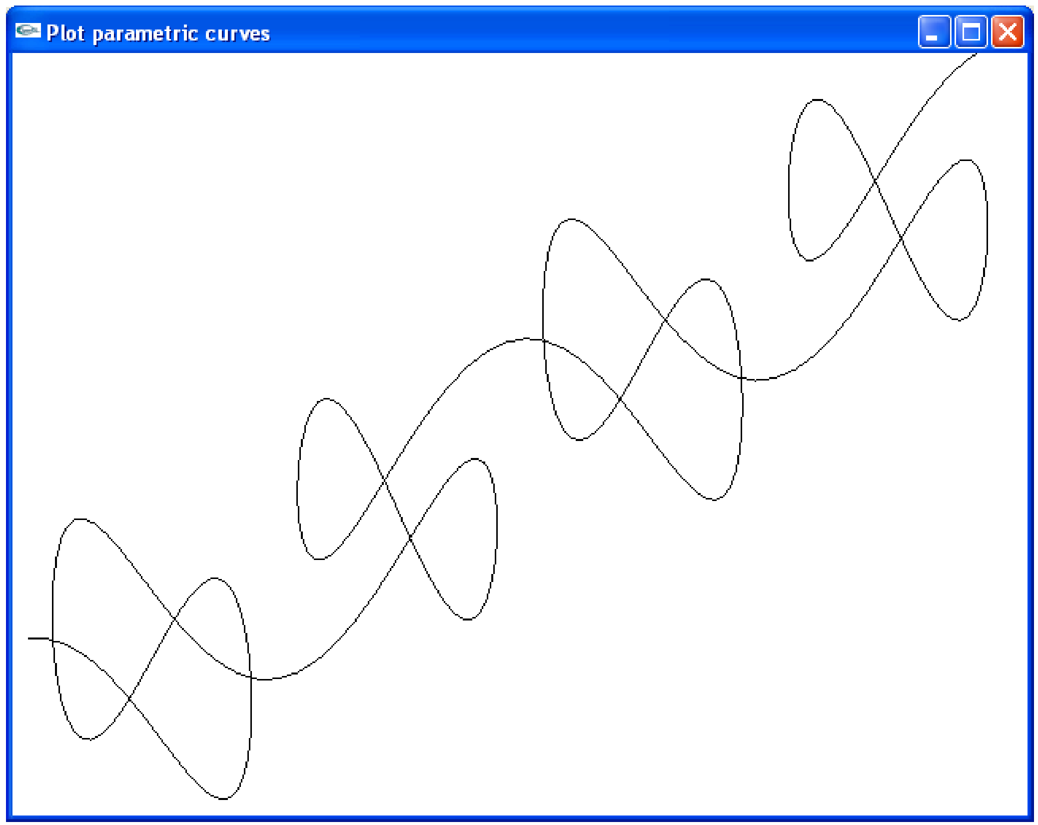
y = cos(t + cos(t));



Parametric Form

x = t + 2.0*sin(2.0*t);

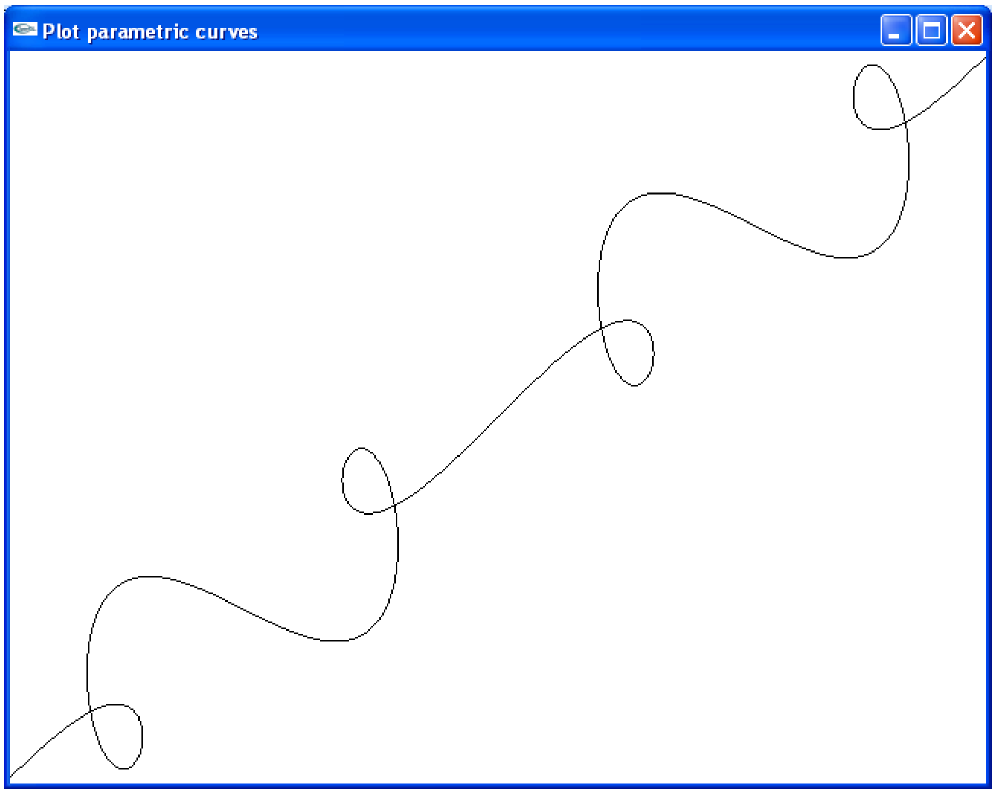
y = t + 2.0*cos(5.0*t);



Parametric Form

$x = t + \sin(2.0*t);$

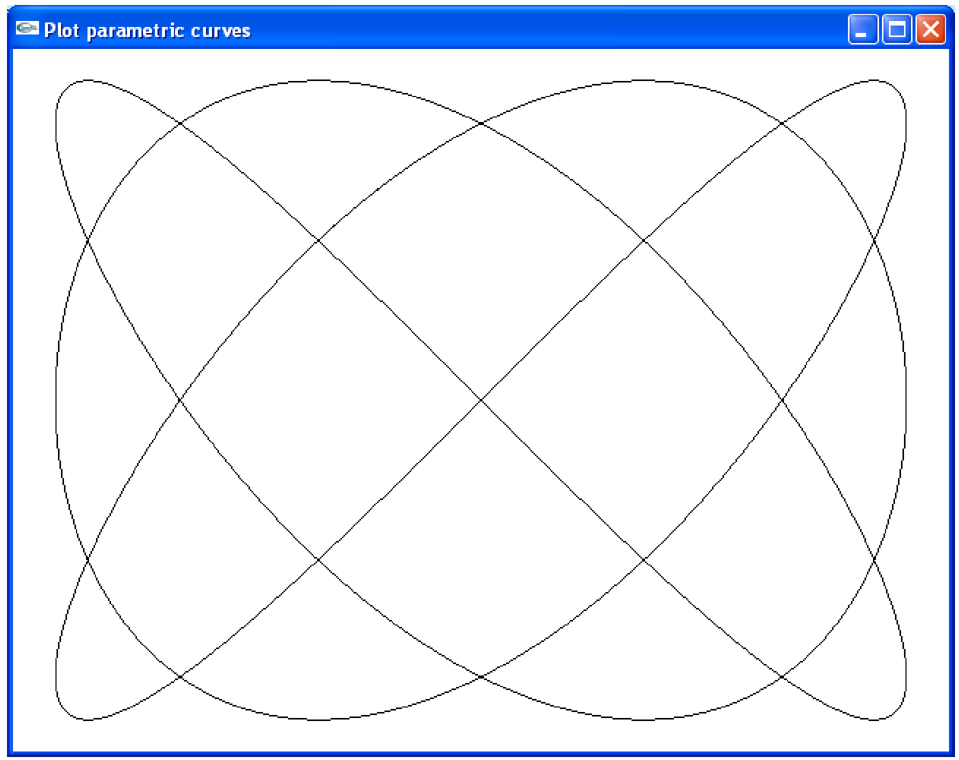
$y = t + \sin(3.0*t);$



Parametric Form

x = sin(3.0*t);

y = sin(4.0*t);

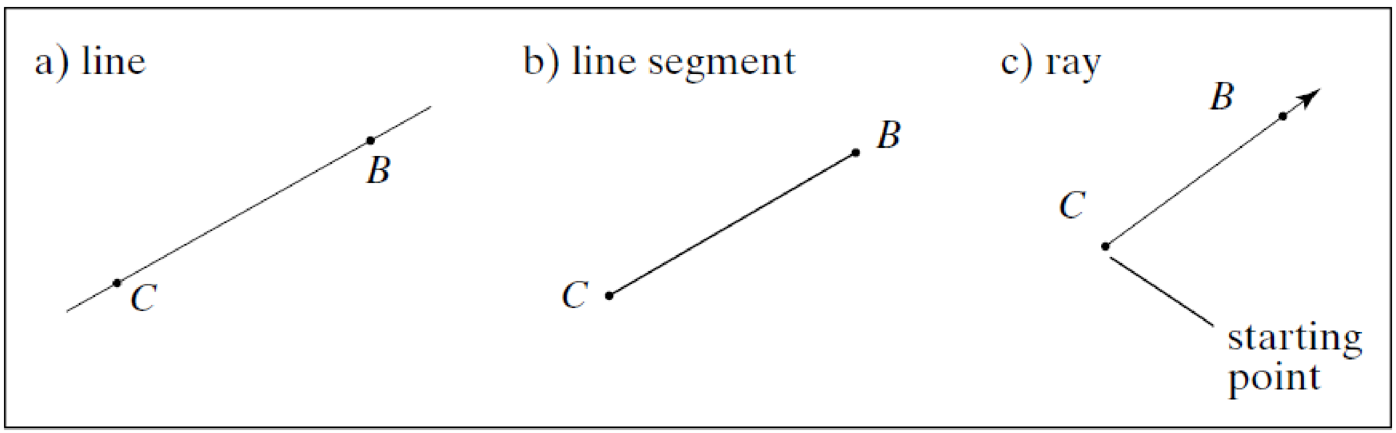


Line

$y = 5x + 3$ (1)

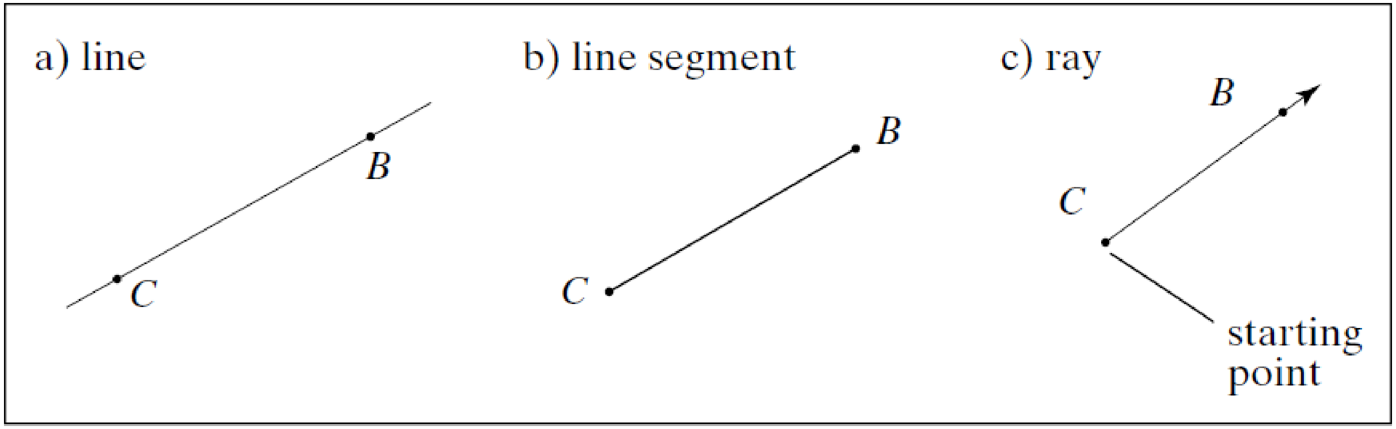
$x = 1$ (2)

$$\frac{x-3}{1} = \frac{y+2}{6} = \frac{z-3}{-2}$$ (3)



Line

- Line, line segment, ray



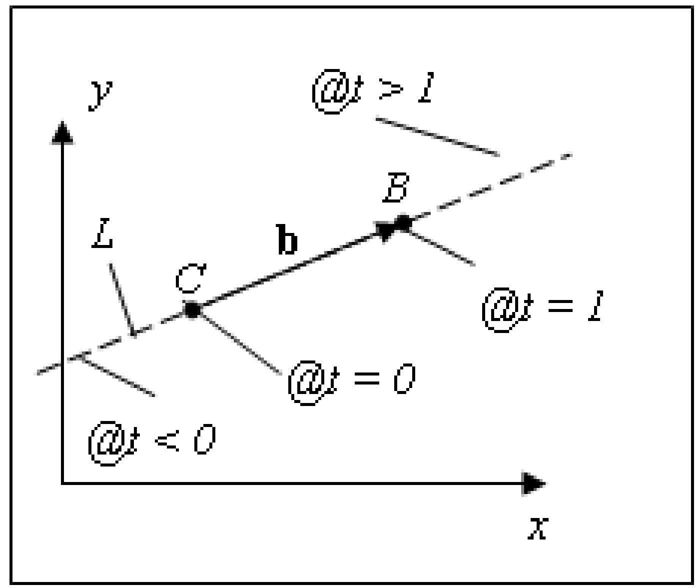
- Parametric form

  $L(t) = C + \mathbf{b}t$

  Line segment, $0 \leq t \leq 1$

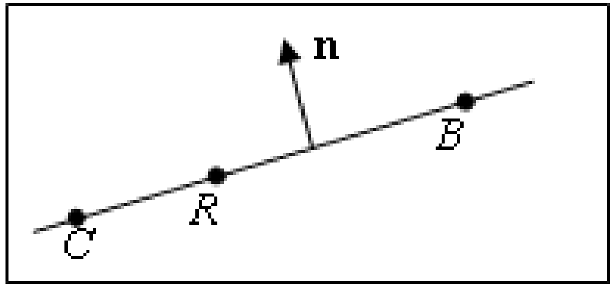
  Ray, $0 \leq t \leq \infty$

  Line, $-\infty \leq t \leq \infty$

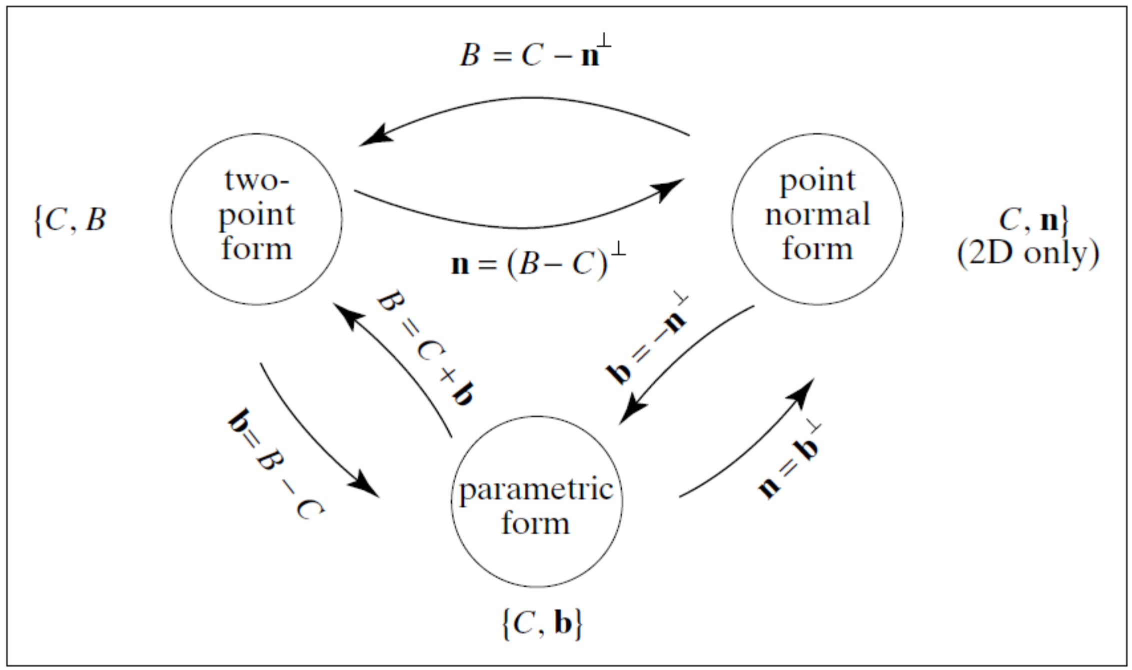


Line

- Point-normal form: $\mathbf{n} \bullet (R - C) = 0$



- Conversion



Plane

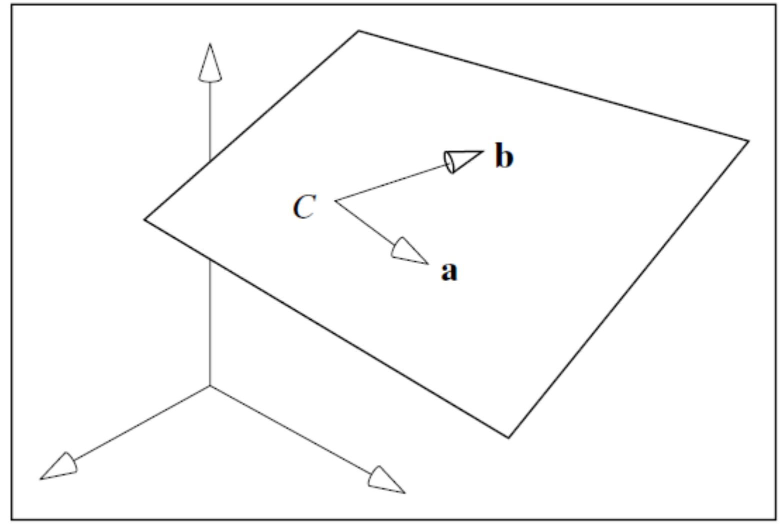
- Parametric form:

  $P(s, t) = C + s\mathbf{a} + t\mathbf{b}$

- Point-normal form:

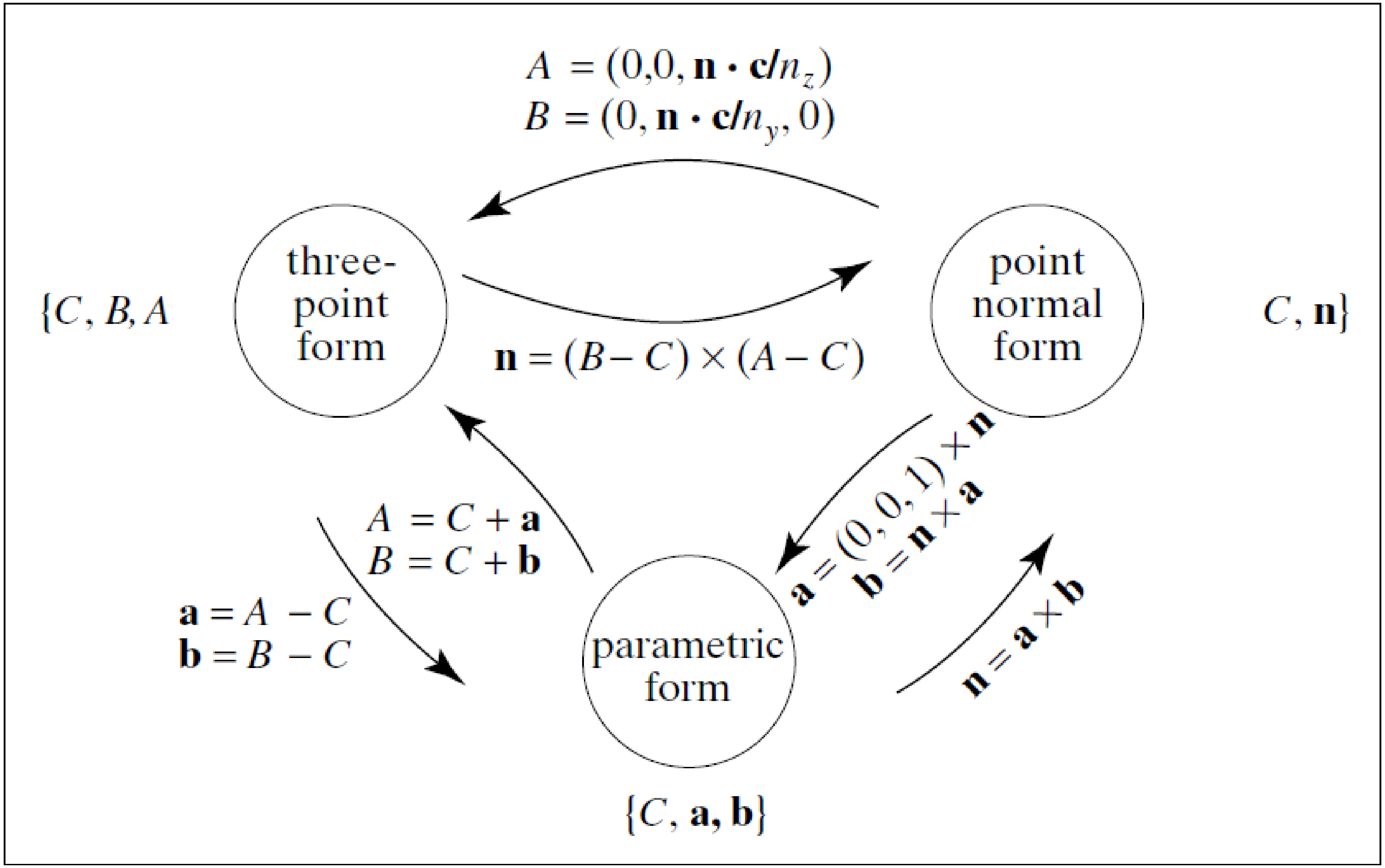
  $\mathbf{n}\bullet(R - C) = 0$

  $n = a \times b$



Plane

❑ Conversion



Some Examples

- Intersection of two line segment

$AB(t) = A + \mathbf{b}t$; $CD(u) = C + \mathbf{d}u$

Find t and u such as $A + \mathbf{b}t = C + \mathbf{d}u$

$\mathbf{b}t = \mathbf{c} + \mathbf{d}u$ với $\mathbf{c} = C - A$

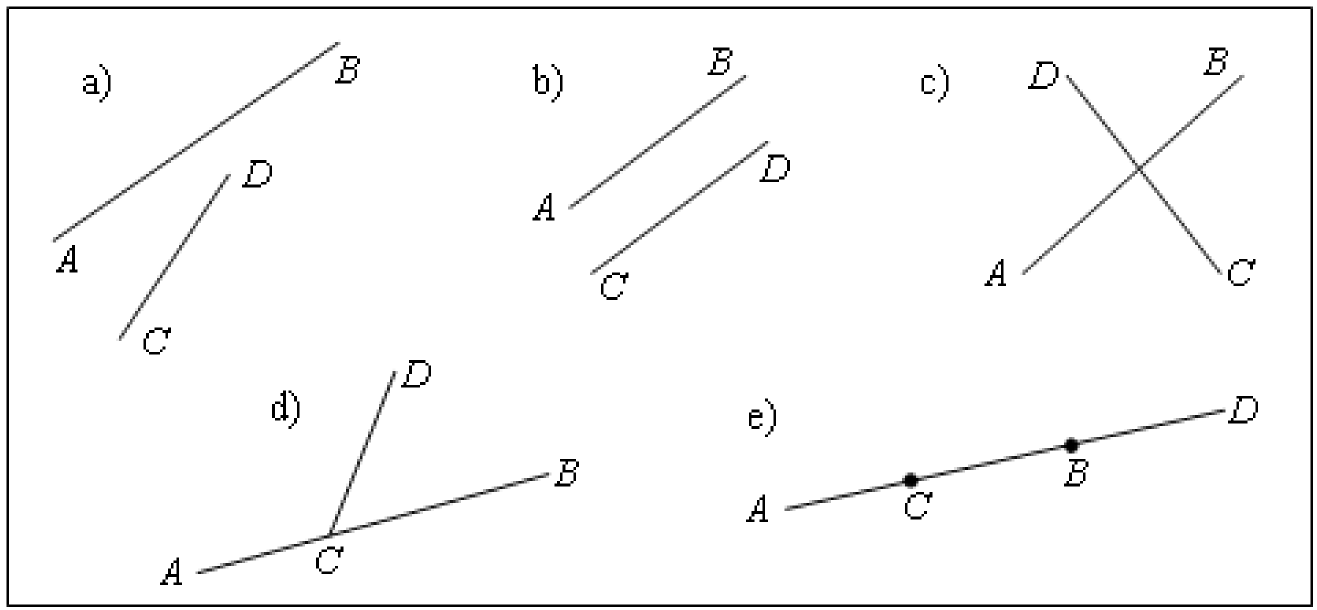
$\mathbf{d}^{\perp} \bullet \mathbf{b}t = \mathbf{d}^{\perp} \bullet \mathbf{c}$

✓ $\mathbf{d}^{\perp} \bullet \mathbf{b} \neq 0.$

$$t = \frac{\mathbf{d}^{\perp} \bullet \mathbf{c}}{\mathbf{d}^{\perp} \bullet \mathbf{b}}$$

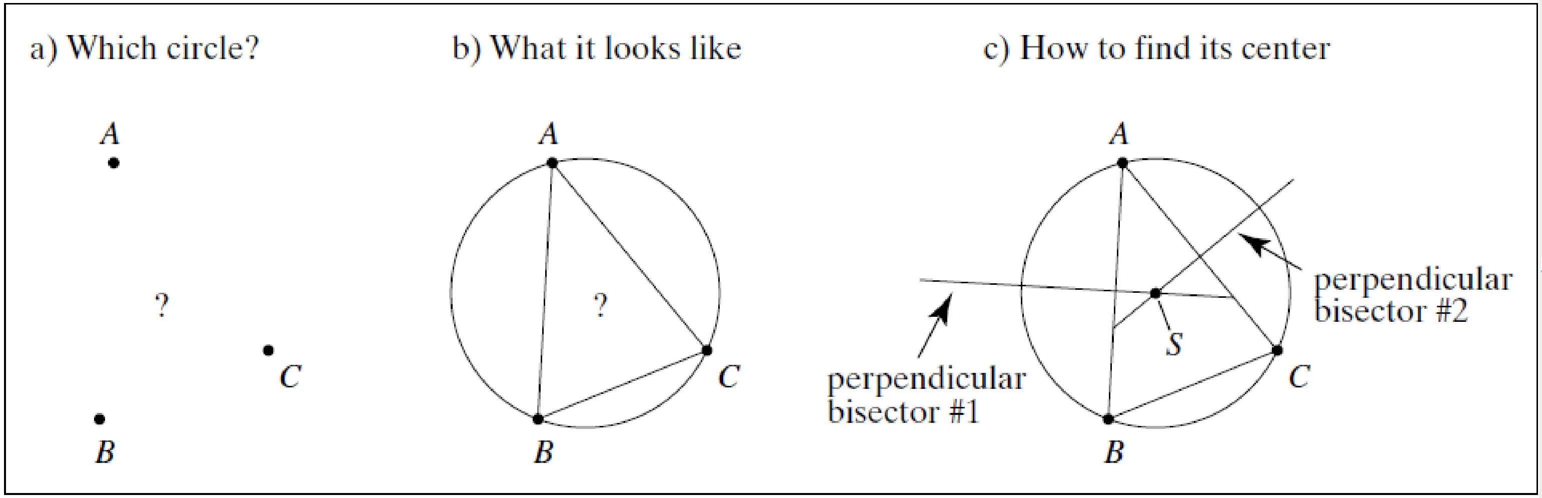
$$u = \frac{\mathbf{b}^{\perp} \bullet \mathbf{c}}{\mathbf{d}^{\perp} \bullet \mathbf{b}}$$

✓ $\mathbf{d}^{\perp} \bullet \mathbf{b} = 0$



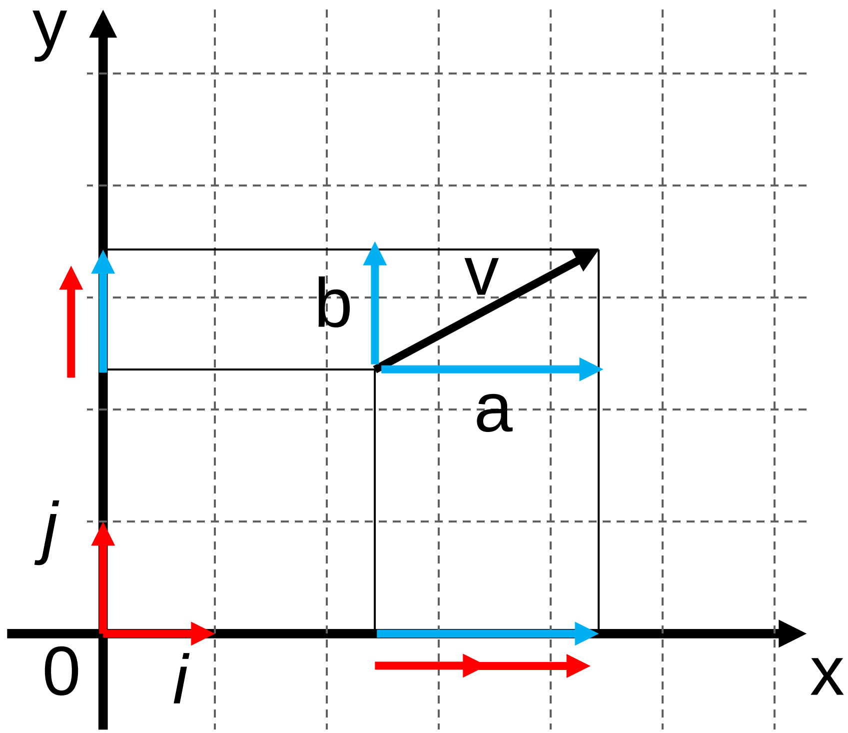
Some Examples

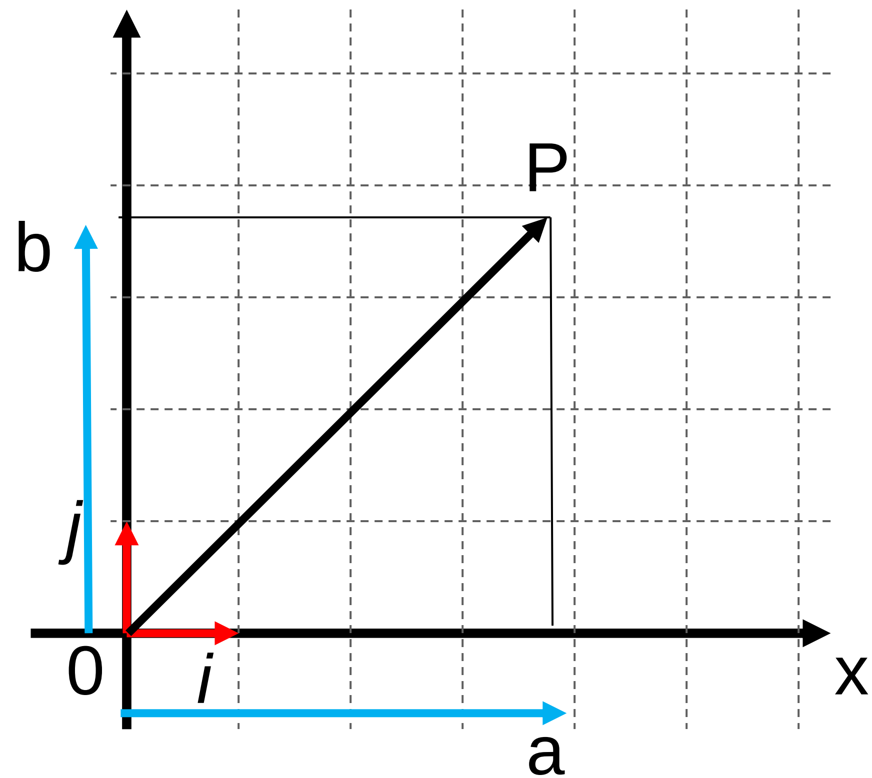
- The Circle through 3 points



- Perpendicular bisector $L(t) = \frac{1}{2}(A+B) + (B-A)^{\perp} t$
- $\mathbf{a} = B - A$; $\mathbf{b} = C - B$; $\mathbf{c} = A - C$;
- Perp. bisector AB: $A + \mathbf{a}/2 + \mathbf{a}^{\perp} t$; AC: $A - \mathbf{c}/2 + \mathbf{c}^{\perp} u$
- $\mathbf{a}^{\perp} t = \mathbf{b}/2 + \mathbf{c}^{\perp} u \rightarrow$ $t = \frac{1}{2} \frac{\mathbf{b} \bullet \mathbf{c}}{\mathbf{a}^{\perp} \bullet \mathbf{c}}$ $S = A + \frac{1}{2}\left(\mathbf{a} + \frac{\mathbf{b} \bullet \mathbf{c}}{\mathbf{a}^{\perp} \bullet \mathbf{c}} \mathbf{a}^{\perp}\right)$

Representation

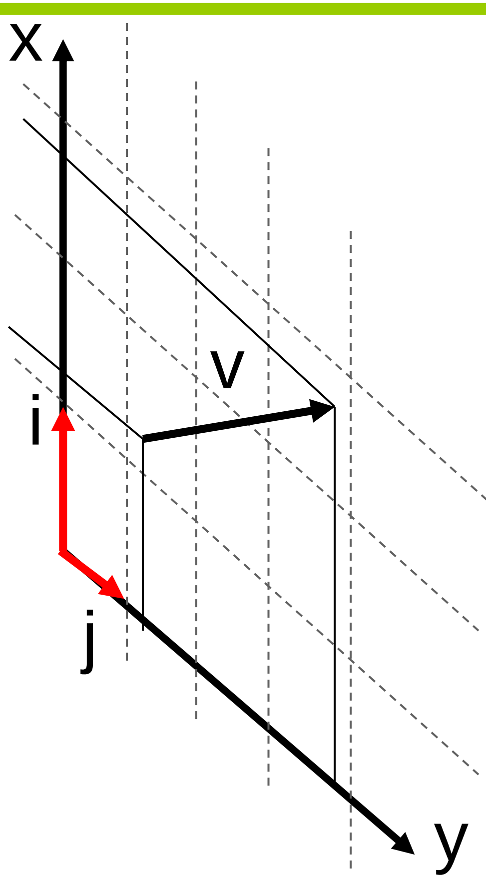


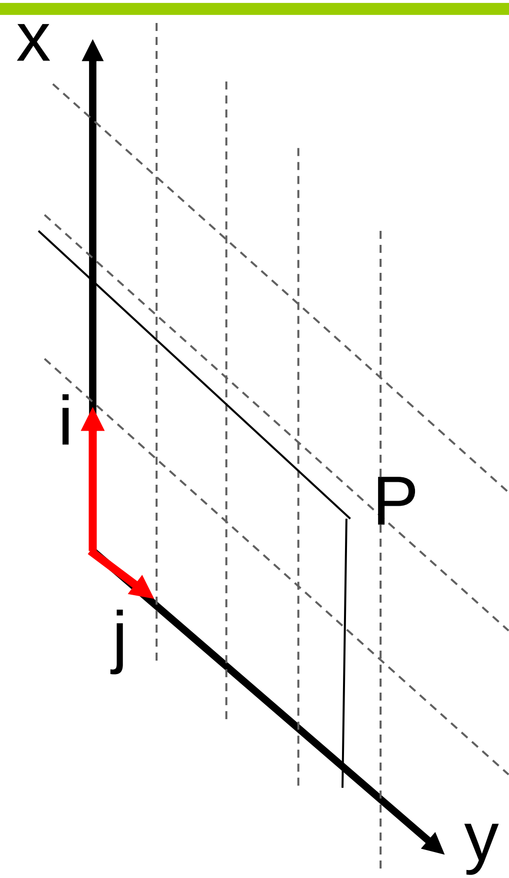


v = a + b = 2.1*i* + 1.2*j* → v = (2.1, 1.2)

P = O + OP = O + a + b = O + 3.9*i* + 3.7*j* → P = (3.9, 3.7)

Representation





v = 1.6i + 2.7j → v = (1.6, 2.7)

P = (1.9, 3.6)

Representation

- Frames
  - Frame determined by (P_0, v_1, v_2, v_3)
  - Within this frame, every vector can be written as

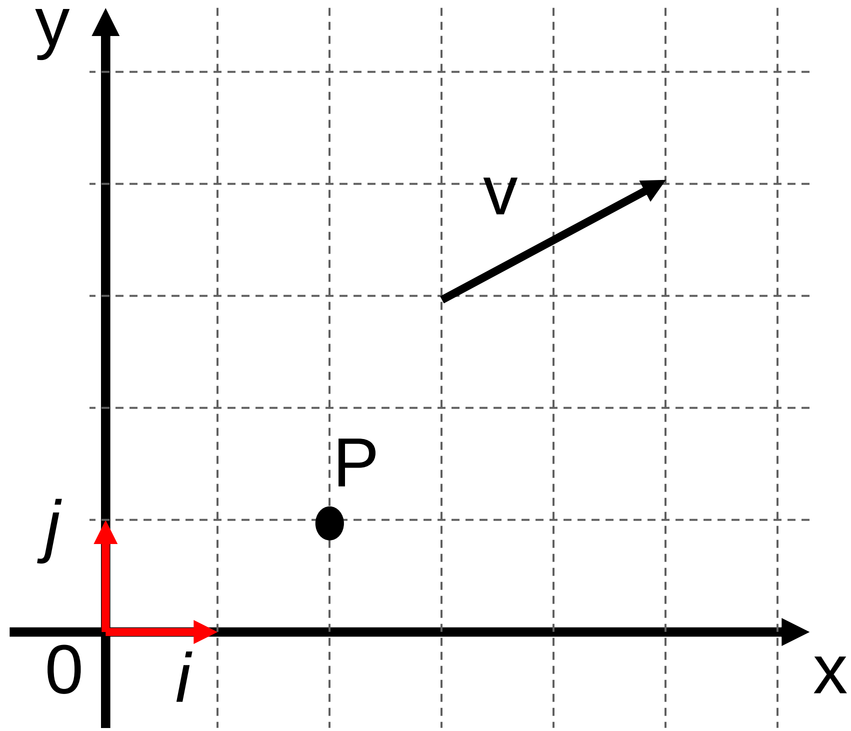
$$v=\alpha_1 v_1+\alpha_2 v_2+\ldots.+\alpha_n v_n$$

  - Every point can be written as

$$P = P_0+\beta_1 v_1+\beta_2 v_2+\ldots.+\beta_n v_n$$

Representation

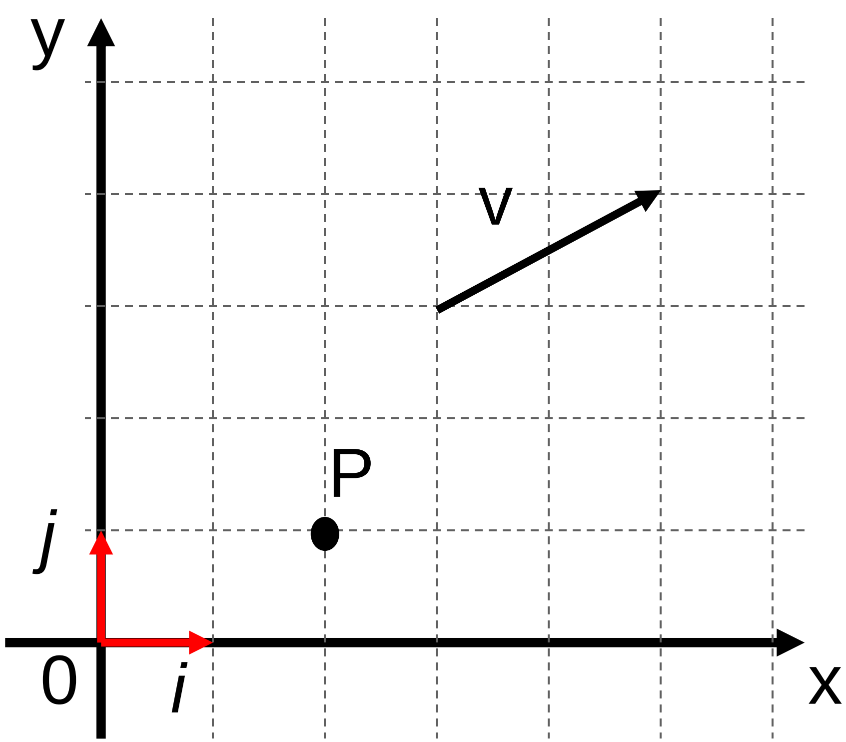
- Confusing Points and Vector



v = (2, 1); P = (2, 1)

Representation

- Confusing Points and Vector



v = 2*i + 1*j =
 = 2*i + 1*j + 0*O
 = (2, 1, 0)

P = 2*i + 1*j + O
 = 2*i + 1*j + 1*O
 = (2, 1, 1)

Representation

- A Single Representation

If we define $0 \cdot P = \mathbf{0}$ and $1 \cdot P = P$ then we can write

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = [\alpha_1\ \alpha_2\ \alpha_3\ 0]\ [v_1\ v_2\ v_3\ P_0]^T$$

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = [\beta_1\ \beta_2\ \beta_3\ 1]\ [v_1\ v_2\ v_3\ P_0]^T$$

Thus we obtain the four-dimensional *homogeneous coordinate* representation

$$\mathbf{v} = [\alpha_1\ \alpha_2\ \alpha_3\ 0]^T$$

$$\mathbf{p} = [\beta_1\ \beta_2\ \beta_3\ 1]^T$$

Representation

- Homogeneous Coordinates

  The homogeneous coordinates form for a three dimensional point [x y z] is given as

  $$\mathbf{p} = [x' \ y' \ z' \ w]^T = [wx \ wy \ wz \ w]^T$$

  We return to a three dimensional point (for $w \neq 0$) by

  $x \leftarrow x'/w$

  $y \leftarrow y'/w$

  $z \leftarrow z'/w$

  If $w=0$, the representation is that of a vector

  Note that homogeneous coordinates replaces points in three dimensions by lines through the origin in four dimensions

  For w=1, the representation of a point is [x y z 1]

Representation

- Change of Coordinate Systems
  - Consider two representations of a the same vector with respect to two different bases. The representations are

$$\mathbf{a}=[\alpha_1\ \alpha_2\ \ \alpha_3\ ]$$
$$\mathbf{b}=[\beta_1\ \ \beta_2\ \ \beta_3]$$

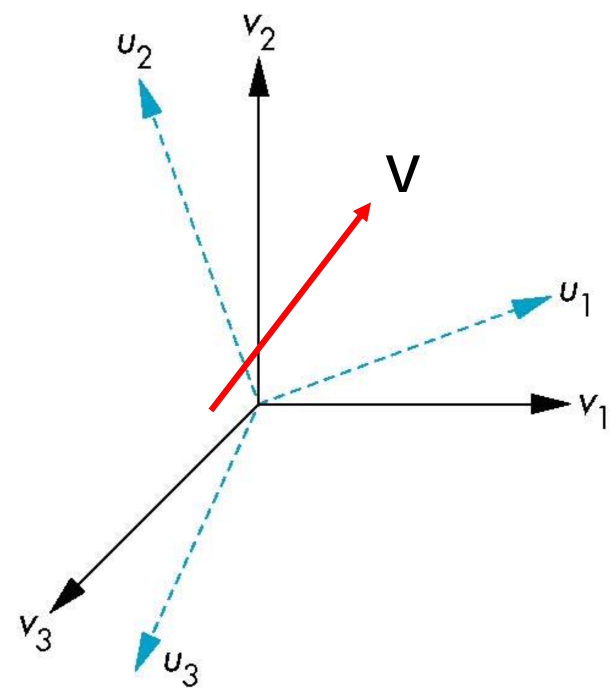
where

$$\text{v}=\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3=[\alpha_1\ \alpha_2\ \alpha_3]\ [v_1\ v_2\ v_3]^T$$
$$=\beta_1 u_1+\beta_2 u_2+\beta_3 u_3=[\beta_1\ \beta_2\ \beta_3]\ [u_1\ u_2\ u_3]^T$$

Representation

- Change of Coordinate Systems
  - Each of the basis vectors, u1,u2, u3, are vectors that can be represented in terms of the first basis

$$u_1 = \gamma_{11}v_1+\gamma_{12}v_2+\gamma_{13}v_3$$
$$u_2 = \gamma_{21}v_1+\gamma_{22}v_2+\gamma_{23}v_3$$
$$u_3 = \gamma_{31}v_1+\gamma_{32}v_2+\gamma_{33}v_3$$



Representation

- Change of Coordinate Systems

  The coefficients define a 3 x 3 matrix

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

  and the bases can be related by

$$\mathbf{a} = \mathbf{M}^T \mathbf{b}$$

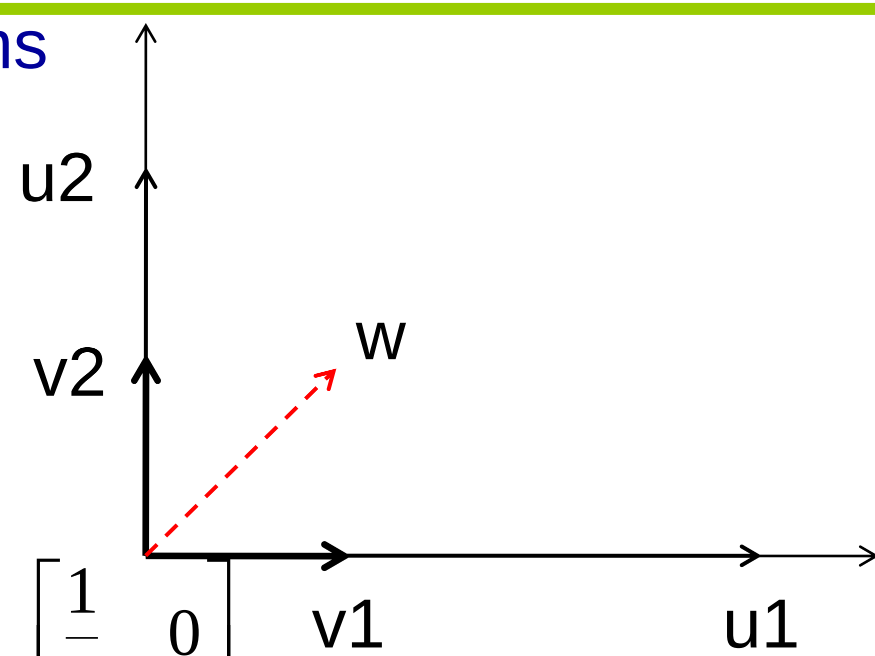
Representation

- Change of Coordinate Systems

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u_1 = 3v_1$$

$$u_2 = 2v_2$$



$$M = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}; M^T = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}; \left(M^T\right)^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; b = Ta = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \end{bmatrix}$$

Representation

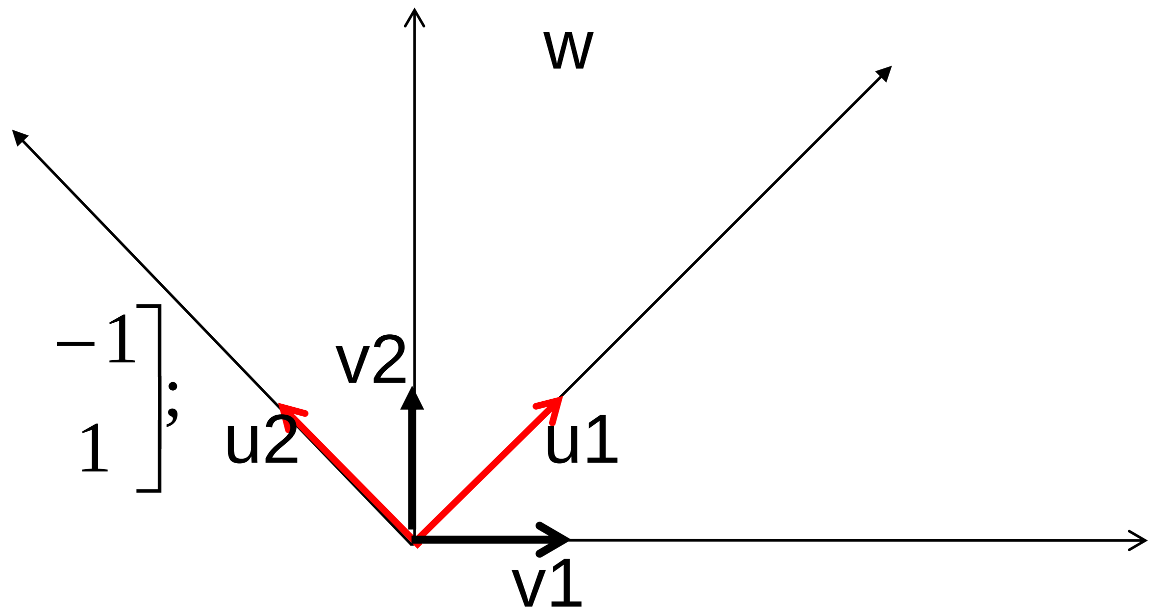
Change of Coordinate Systems

$$u_1 = v_1 + v_2$$

$$u_2 = -v_1 + v_2$$

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; M^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix};$$

$$\left(M^T\right)^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$



$$b = \left(M^T\right)^{-1} a = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

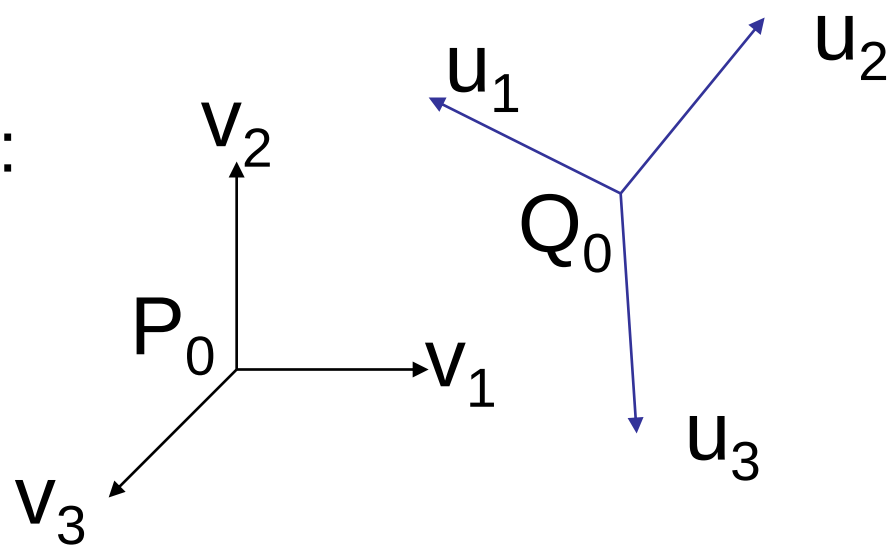
Representation

- Change of Frames
  - We can apply a similar process in homogeneous coordinates to the representations of both points and vectors

Consider two frames:

(P_0, v_1, v_2, v_3)

(Q_0, u_1, u_2, u_3)



  - Any point or vector can be represented in either frame
  - We can represent Q_0, u_1, u_2, u_3 in terms of P_0, v_1, v_2, v_3

Representation

Change of Frames

Extending what we did with change of bases

$$u_1 = \gamma_{11}v_1+\gamma_{12}v_2+\gamma_{13}v_3$$
$$u_2 = \gamma_{21}v_1+\gamma_{22}v_2+\gamma_{23}v_3$$
$$u_3 = \gamma_{31}v_1+\gamma_{32}v_2+\gamma_{33}v_3$$
$$Q_0 = \gamma_{41}v_1+\gamma_{42}v_2+\gamma_{43}v_3 +\gamma_{44}P_0$$

defining a 4 x 4 matrix

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$

Representation

- Change of Frames

  Within the two frames any point or vector has a representation of the same form

  $\mathbf{a}=[\alpha_1\ \alpha_2\ \ \alpha_3\ \alpha_4\,]$ in the first frame
  $\mathbf{b}=[\beta_1\ \beta_2\ \ \beta_3\ \beta_4\,]$ in the second frame

  where $\alpha_4 = \beta_4 = 1$ for points and $\alpha_4 = \beta_4 = 0$ for vectors and

$$\mathbf{a}=\mathbf{M}^T\mathbf{b}$$

  The matrix $\mathbf{M}$ is 4 x 4 and specifies an affine transformation in homogeneous coordinates