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Extended Bridge to CS
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Homework 11
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Question 5

(a) Use mathematical induction to prove that for any positive integer, n, 3 divides $n^3 + 2n$ (leaving no remainder).
Hint: you may want to use the formula: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Theorem, P(n): For every positive integer, n, 3 evenly divides $n^3 + 2n$.
Proof: By induction on n.
Base Case: $n = 1 \Rightarrow (1)^3 + 2(1) = 1 + 2 = 3$
Since 3 evenly divides 3, P(1) is true.
Inductive Step:
Suppose that for a positive integer, k, 3 evenly divides $k^3 + 2k$.
We shall prove that 3 evenly divides $(k + 1)^3 + 2(k + 1)$.

1. By the inductive hypothesis, 3 evenly divides $k^3 + 2k$, which means that $k^3 + 2k = 3m$ for some integer m. We must show that $(k + 1)^3 + 2(k + 1)$ can also be expressed as 3 times an integer.
2. $(k + 1)^3 + 2(k + 1)$ can be simplified using algebra and the hint provided:

$$\begin{aligned}(k + 1)^3 + 2(k + 1) &= (k^3 + 3k^2(1) + 3k(1)^2 + (1)^3) + (2k + 2)\\ &= k^3 + 3k^2 + 3k + 2k + 3\\ &= (k^3 + 2k) + (3k^2 + 3k + 3)\end{aligned}$$

3. By the inductive hypothesis, this is equivalent to $3m + (3k^2 + 3k + 3)$.
4. This can be further simplified to $3m + 3(k^2 + k + 1)$.
5. Since m and k are integers, $3(k^2 + k + 1)$ can be expressed as 3j for some integer j, and $3m + 3(k^2 + k + 1)$ can be rewritten as $3m + 3j$.
6. This can be further simplified to $3(m + j)$.
7. Since m and j are integers, $m + j$ would also be an integer. Therefore, $(k + 1)^3 + 2(k + 1)$ can be expressed as 3 times an integer. This means that $(k + 1)^3 + 2(k + 1)$ can be evenly divided by 3 and P(k+1) is true. ■

(b) Use strong induction to prove that any positive integer, n, ($n \geq 2$) can be written as a product of primes.

Theorem, P(n): For every $n \geq 2$, n can be written as a product of primes.
Proof: By strong induction on n.
Base Case: $n = 2$
Since 2 is already a prime number, it is already a product of the prime number, 2.
Inductive Step:
Suppose that for some integer $k \geq 2$, any integer, j, in the range from 2 to k, can be expressed as a product of prime numbers.
We shall prove that $k + 1$ can also be expressed as a product of prime numbers.

1. If $k + 1$ is prime, then it is already a product of the prime number, $k + 1$.
2. If $k + 1$ is not a prime number, it can be expressed as a product of two integers, a and b, where both $a \geq 2$ and $b \geq 2$. This would be written as $k + 1 = a \cdot b$. We must show that both $a \leq k$ and $b \leq k$ in order to apply the inductive hypothesis.
3. $k + 1 = a \cdot b$ can be rewritten as $a = \frac{k+1}{b}$ and $b = \frac{k+1}{a}$.
4. Since $a \geq 2$ and $b \geq 2$, $a = \frac{k+1}{b} < k + 1$ and $b = \frac{k+1}{a} \leq k + 1$.
5. Since a and b both fall in the range from 2 to k, the inductive hypothesis can be applied to express a and b as products of prime numbers: $a = p_1 \cdot p_2 \cdot \ldots \cdot p_m$ and $b = q_1 \cdot q_2 \cdot \ldots \cdot q_j$.
6. $k+1$ can then be expressed as a product of primes $k+1 = a \cdot b = (p_1 \cdot p_2 \cdot \ldots \cdot p_m) \cdot (q_1 \cdot q_2 \cdot \ldots \cdot q_j)$. Therefore, P(k+1) is true. ■

Question 6

Solve the following questions from the Discrete Math zyBook:

1. Exercise 7.4.1
   Define P(n) to be the assertion that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Verify that P(3) is true.
$P(3) \Rightarrow n = 3$

$$\begin{aligned} 1^2 + 2^2 + 3^2 &= \frac{(3)(3+1)(2(3)+1)}{6} \\ 1+4+9 &= \frac{(3)(4)(7)}{6} \\ 14 &= \frac{84}{6} \\ 14 &= 14 \quad \checkmark \end{aligned}$$

(b) Express P(k).

$$P(k) \Rightarrow \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

(c) Express P(k+1).

$$\begin{aligned} P(k+1) \Rightarrow \sum_{j=1}^{k+1} j^2 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

(d) In an induction proof that for every positive integer, n, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$, what must be proven in the base case?
In the base case, you need to prove that P(1) is true, where $n = 1$.

(e) In an inductive proof that for every positive integer, n, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$, what must be proven in the inductive step?
In the inductive step, you need to prove that for all positive integers, k, P(k) implies that P(k+1) is true.

(f) What would be the inductive hypothesis in the inductive step from your previous answer?
The inductive hypothesis will be to assume that P(k) is true for some positive integer, k, where $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$, and we will prove P(k+1), such that $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$.

(g) Prove by induction that for any positive integer, n, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

Theorem, P(n): For any positive integer, n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof: By induction on n.
Base Case: $n = 1$

$$\begin{aligned} 1^2 &= \frac{(1)(1+1)(2(1)+1)}{6} \\ 1 &= \frac{(1)(2)(3)}{6} \\ 1 &= \frac{6}{6} \\ 1 &= 1 \quad \checkmark \text{ P(1) is true.} \end{aligned}$$

Inductive Step:
Suppose that for some positive integer, k, $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$.
We shall prove that for $k+1$, $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$.

1. Start with the left side of the equation:

$$\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2$$

2. By the inductive hypothesis, this is equivalent to:

$$\sum_{j=1}^{k+1} j^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

3. Simplify the equation further using algebra:

$$\begin{aligned}\sum_{j=1}^{k+1} j^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\ &= \frac{(k+1)(k(2k+1)+6(k+1))}{6} \\ &= \frac{(k+1)(2k^2+k+6k+6)}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6}\end{aligned}$$

4. Since $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$, then P(k+1) is true. ■

2. Exercise 7.4.3
   Prove each of the following statements using mathematical induction.
   Hint: you may want to use the following fact: $\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$.
   (c) Prove that for $n \geq 1$,

$$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$$

Theorem, P(n): For $n \geq 1$,

$$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$$

Proof: By induction on n.
Base Case: $n = 1$

$$\begin{aligned}\sum_{j=1}^{1} \frac{1}{j^2} &= 2 - \frac{1}{1} \\ \frac{1}{1^2} &= 2 - 1 \\ \frac{1}{1} &= 1 \\ 1 &= 1 \quad \checkmark \text{ P(1) is true.}\end{aligned}$$

Inductive Step:
Suppose that for some integer $k \geq 1$, $\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$.
We shall prove that for $k+1$, $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$.

1. Start by using algebra to simplify the left side of the inequality:

$$\sum_{j=1}^{k+1} \frac{1}{j^2} = \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2}$$

2. By the inductive hypothesis, this is equivalent to:

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

3. Algebra and the hint provided can be used to show that:

$$\begin{aligned} 2 - \frac{1}{k} + \frac{1}{(k+1)^2} &\leq 2 - \frac{1}{k+1} \\ 2 - \frac{1}{k} + \frac{1}{(k+1)^2} &\leq 2 - \frac{1}{k} + (\frac{1}{k} - \frac{1}{k+1}) \\ 2 - \frac{1}{k} + \frac{1}{(k+1)^2} &\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)} \quad \checkmark \end{aligned}$$

4. Therefore, P(k+1) is true. ■

3. Exercise 7.5.1
Prove each of the following statements using mathematical induction.
(a) Prove that for any positive integer, n, 4 evenly divides $3^{2n} - 1$.

Theorem, P(n): For any positive integer, n, 4 evenly divides $3^{2n} - 1$.
Proof: By induction on n.
Base Case: $n = 1 \Rightarrow 3^{2(1)} - 1 = 3^2 - 1 = 9 - 1 = 8$
Since 4 evenly divides 8, P(1) is true.
Inductive Step:
Suppose that for a positive integer, k, 4 evenly divides $3^{2k} - 1$.
We shall prove that 4 evenly divides $3^{2(k+1)} - 1$.

1. By inductive hypothesis, 4 evenly divides $3^{2k} - 1$, which means that $3^{2k} - 1 = 4m$ for some integer m. We must show that $3^{2(k+1)} - 1$ can also be expressed as 4 times an integer.
2. By adding 1 to both sides of $3^{2k} - 1 = 4m$, we get $3^{2k} = 4m + 1$, which is equivalent to the statement in the inductive hypothesis.
3. Using algebra, $3^{2(k+1)} - 1 = 3^{2k+2} - 1 = 9 \cdot 3^{2k} - 1$.
4. By the inductive hypothesis, this is equivalent to $9 \cdot (4m + 1) - 1$.
5. Using algebra, this can be simplified to $36m + 8$, which can then be rewritten as $4(9m + 2)$.
6. Since m is an integer, $4(9m+2)$ can be expressed as $4j$ for some integer, j. Therefore, $3^{2(k+1)} - 1$ can be expressed as 4 times an integer, which means $3^{2(k+1)} - 1$ can be evenly divided by 4 and P(k+1) is true. ■