NYU, Tandon School of Engineering
Extended Bridge to CS
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Homework 6
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Question 5

Use the definition of Θ in order to show the following:

1. $5n^3 + 2n^2 + 3n = \Theta(n^3)$
   Claim: $5n^3 + 2n^2 + 3n = \Theta(n^3)$
   Proof: We shall prove that $5n^3 + 2n^2 + 3n = O(n^3)$ and $5n^3 + 2n^2 + 3n = \Omega(n^3)$.

   Claim: $5n^3 + 2n^2 + 3n = O(n^3)$
   Proof: Let $c = 10$ and $n_0 = 1$. We shall prove show that for any $n \geq 1$,
   $5n^3 + 2n^2 + 3n \leq 10 \cdot n^3$.
   1.) Since $n \geq 1, n \leq n^3$, so that:

   $$5n^3 \leq 5n^3$$
   $$2n^2 \leq 2n^3$$
   $$3n \leq 3n^3$$

   2.) Add these inequalities to get:
   $5n^3 + 2n^2 + 3n \leq 5n^3 + 2n^3 + 3n^3 = 10n^3$
   3.) Therefore, $5n^3 + 2n^2 + 3n \leq 10n^3$, which means that $5n^3 + 2n^2 + 3n = O(n^3)$. ■

   Claim: $5n^3 + 2n^2 + 3n = \Omega(n^3)$
   Proof: Let $c = 5$ and $n_0 = 1$. We shall prove that for any $n \geq 1$,
   $5n^3 + 2n^2 + 3n \geq 5 \cdot n^3$.
   1.) Drop every term except for $5n^3$ to get: $5n^3 + 2n^2 + 3n \geq 5n^3$.
   2.) Therefore, $5n^3 + 2n^2 + 3n \geq 5 \cdot n^3$, which means that $5n^3 + 2n^2 + 3n = \Omega(n^3)$. ■

   Since $5n^3 + 2n^2 + 3n = O(n^3)$ and $5n^3 + 2n^2 + 3n = \Omega(n^3)$, by definition, $5n^3 + 2n^2 + 3n = \Theta(n^3)$. ■

2. $\sqrt{7n^2+2n-8} = \Theta(n)$
Claim: $\sqrt{7n^2+2n-8} = \Theta(n)$
Proof: We shall prove that $\sqrt{7n^2+2n-8} = O(n)$ and $\sqrt{7n^2+2n-8} = \Omega(n)$.

Claim: $\sqrt{7n^2+2n-8} = O(n)$
Proof: Let $c = 3$ and $n_0 = 1$. We shall prove show that for any $n \geq 1$,
$\sqrt{7n^2+2n-8} \leq 3 \cdot n$.
1.) Since $n \geq 1, n \leq n^2$, so that:

$$7n^2 \leq 7n^2$$
$$2n \leq 2n^2$$
$$-8 \leq 0$$

2.) Add these inequalities to get:
$7n^2+2n-8 \leq 7n^2+2n^2 = 9n^2$
3.) Take the square root of both sides to get:
$\sqrt{7n^2+2n-8} \leq \sqrt{9n^2} = 3n$
4.) Therefore, $\sqrt{7n^2+2n-8} \leq 3 \cdot n$, which means that $\sqrt{7n^2+2n-8} = O(n)$. ■

Claim: $\sqrt{7n^2+2n-8} = \Omega(n)$
Proof: Let $c = \sqrt{7}$ and $n_0 = 4$. We shall prove that for any $n \geq 1$,
$\sqrt{7n^2+2n-8} \geq \sqrt{7} \cdot n$.
1.) Drop every term except for $\sqrt{7n^2}$ to get: $\sqrt{7n^2+2n-8} \geq \sqrt{7n^2} = n\sqrt{7}$. This expression is only true if and only if $2n-8 \geq 0$.
2.) Solve for n in $2n-8 \geq 0$ to get $n \geq 4$.
3.) Therefore, $\sqrt{7n^2+2n-8} \geq \sqrt{7} \cdot n$, which means that $\sqrt{7n^2+2n-8} = \Omega(n)$. ■

Since $\sqrt{7n^2+2n-8} = O(n)$ and $\sqrt{7n^2+2n-8} = \Omega(n)$, by definition, $\sqrt{7n^2+2n-8} = \Theta(n)$. ■