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Extended Bridge to CS
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Homework 5
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Question 3
Solve the following questions from the Discrete Math zyBooks:

1. Exercise 4.1.3
   (b) $f(x) = \frac{1}{x^2-4}$
   $f(2) = \frac{1}{(2)^2-4} = \frac{1}{4-4} = \frac{1}{0} =$ undefined
   $f(-2) = \frac{1}{(-2)^2-4} = \frac{1}{4-4} = \frac{1}{0} =$ undefined
   This is not a function since -2 and 2 would not be mapped to a target.

   (c) $f(x) = \sqrt{x^2}$
   This is a function, where the range = $\{0\} \cup \mathbb{R}^+$**.**

2. Exercise 4.1.5
   (b) Let $A = \{2, 3, 4, 5\}$
   $f : A \to \mathbb{Z}$, such that $f(x) = x^2$
   $f(2) = (2)^2 = 4$ $\quad$ $f(3) = (3)^2 = 9$ $\quad$ $f(4) = (4)^2 = 16$ $\quad$ $f(5) = (5)^2 = 25$
   Range $= \{\mathbf{4, 9, 16, 25}\}$

   (d) $f : \{0, 1\}^5 \to \mathbb{Z}$. For $x \in \{0, 1\}^5, f(x)$ is the number of 1's that occur in x. For example, $f(01101) = 3$, because there are three 1's in the string "01101".
   Range $= \{\mathbf{0, 1, 2, 3, 4, 5}\}$

   (h) Let $A = \{1, 2, 3\}$
   $f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (y, x)$
   $A \times A = A^2 \Rightarrow \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
   Range $= \{\mathbf{(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), (1, 3), (2, 3), (3, 3)}\}$

   (i) Let $A = \{1, 2, 3\}$
   $f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (x, y + 1)$
   $A \times A = A^2 \Rightarrow \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
   Range $= \{\mathbf{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}\}$

(l) Let $A = \{1, 2, 3\}$
$f : P(A) \to P(A)$. For $X \subseteq A, f(X) = X - \{1\}$
$P(A) \Rightarrow \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
Range $= \{\emptyset, \{\mathbf{2}\}, \{\mathbf{3}\}, \{\mathbf{2}, \mathbf{3}\}\}$

Question 4

Solve the following questions from the Discrete Math zyBooks:

1. Exercise 4.2.2

   (c) $h : \mathbb{Z} \to \mathbb{Z}$. $h(x) = x^3 \Rightarrow$ **one-to-one, but not onto**
   Not all integers are perfect cubes, therefore, the function is not onto. (ex: There is no integer where $f(x) = 18$.)
   All integers of the domain are mapped to different targets, therefore, the function is one-to-one.

   (g) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(x, y) = (x + 1, 2y) \Rightarrow$ **one-to-one, but not onto**
   No integer pair is mapped to any target where 2y equals an odd number, as this would make y equal to a fraction or decimal number, which is not an element of the integer set $\mathbb{Z}$. (ex: There is no integer pair where f(x,y)=(2,1).) Therefore, the function is not onto.
   All integers of the domain are mapped to different targets, therefore, the function is one-to-one.

   (k) $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, $f(x, y) = 2^x + y \Rightarrow$ **neither onto nor one-to-one**
   There is no positive integer pair where $f(x, y) = 1$, since x and y would have to be equal to 0, which is not a positive integer. Therefore, the function is not onto.
   Positive integer pairs $f(1, 3)$ and $f(2, 1)$ both map to 5, therefore, the function is not one-to-one.

2. Exercise 4.2.4

   (b) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$. $\Rightarrow$ **neither onto nor one-to-one**
   There is no input string mapped to 000, therefore, the function is not onto.
   $f(000)$ and $f(100)$ are both mapped to 100, therefore, the function is not one-to-one.

   (c) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example $f(011) = 100$. $\Rightarrow$ **both onto and one-to-one**
   All targets are mapped to an input string, therefore, the function is onto.
   All input strings are mapped to different targets, therefore, the function is one-to-one.

   (d) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, $f(100) = 1001$. $\Rightarrow$ **one-to-one, but not onto**

There is no input string mapped to 0101, therefore, the function is not onto.
All input strings are mapped to different targets, therefore, the function is one-to-one.

(g) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $B = \{1\}$. $f : P(A) \rightarrow P(A)$, For $X \subseteq A, f(x) = X - B$. Recall that for a finite set $A, P(A)$ denotes the power set of A which is the set of all subsets of $A \Rightarrow$ **neither onto nor one-to-one**
There is no domain element that is mapped to $\{1\}$, therefore, the function is not onto.
Both $f(\{1, 2, 3\})$ and $f(\{2, 3\})$ are mapped to $\{2, 3\}$, therefore, the function is not one-to-one.

3. Give an example of a function from the set of integers to the set of positive integers that is:
(a) one-to-one, but not onto
$f : \mathbb{Z} \rightarrow \mathbb{Z}^+$, $f(x) \begin{cases} 3x & \text{if } x > 0 \\ -3x + 1 & \text{if } x \leq 0 \end{cases}$

(b) onto, but not one-to-one
$f : \mathbb{Z} \rightarrow \mathbb{Z}^+$, $f(x) = |x|$

(c) one-to-one and onto
$f : \mathbb{Z} \rightarrow \mathbb{Z}^+$, $f(x) \begin{cases} 2x & \text{if } x > 0 \\ -2x + 1 & \text{if } x \leq 0 \end{cases}$

(d) neither one-to-one nor onto
$f : \mathbb{Z} \rightarrow \mathbb{Z}^+$, $f(x) = x^2 + 2$

Question 5

Solve the following questions from the Discrete Math zyBooks:

1. Exercise 4.3.2

   (c) $f : \mathbb{R} \to \mathbb{R}$. $f(x) = 2x + 3 \Rightarrow \mathbf{f^{-1}(y) = \frac{y-3}{2}}$
   The function is a bijection since it is both onto and one-to-one and, therefore, has a well-defined inverse.

   (d) Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, $f : P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. For $X \subseteq A$, $f(X) = |X|$. Recall that for a finite set A, $P(A)$ denotes the power set of A which is a set of all subsets of A. $\Rightarrow$ $\mathbf{f^{-1}(y)}$ **is not a well-defined function.**
   The function is not a bijection since it is not one-to-one (some subsets have the same cardinality). Therefore, the function does not have a well-defined inverse.

   (g) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$. $\Rightarrow \mathbf{f^{-1} = f}$
   The function is a bijection since it is both onto and one-to-one and, therefore, has a well-defined inverse. The inverse would simply be undoing the reversing of the bits.

   (i) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(x, y) = (x + 5, y - 2) \Rightarrow \mathbf{f^{-1}(x, y) = (x - 5, y + 2)}$
   The function is a bijection since it is both onto and one-to-one and, therefore, has a well-defined inverse.

2. Exercise 4.4.8

   (c) $f \circ h \Rightarrow \mathbf{2x^2 + 5}$

$$f(h(x)) = 2(x^2 + 1) + 3$$
$$2x^2 + 2 + 3$$
$$2x^2 + 5$$

   (d) $h \circ f \Rightarrow \mathbf{4x^2 + 12x + 10}$

$$h(f(x)) = (2x + 3)^2 + 1$$
$$4x^2 + 12x + 9 + 1$$
$$4x^2 + 12x + 10$$

3. Exercise 4.4.2
   (b) $(f \circ h)(52) \Rightarrow \mathbf{121}$

$$f(h(x)) = (\lceil \frac{x}{5} \rceil)^2$$
$$f(h(52)) = (\lceil \frac{52}{5} \rceil)^2$$
$$(11)^2$$
$$121$$

   (c) $(g \circ h \circ f)(4) \Rightarrow \mathbf{16}$

$$g(h(f(x)))$$
$$f(4) = 4^2 = 16$$
$$h(16) = \lceil \frac{16}{5} \rceil = 4$$
$$g(4) = 2^4 = 16$$

   (d) Give a mathematical expression for $h \circ f \Rightarrow \mathbf{h(f(x))} = \lceil \frac{\mathbf{x^2}}{\mathbf{5}} \rceil$

4. Exercise 4.4.6
   (c) What is $(h \circ f)(010)$? $\Rightarrow \mathbf{111}$

$$h(f(x))$$
$$f(010) = 110$$
$$h(110) = 111$$

   (d) What is the range of $h \circ f$? $\Rightarrow \{\mathbf{101}, \mathbf{111}\}$
   $\{0,1\}^3 = \{000, 010, 100, 001, 011, 101, 111\}$
   Range of $f = \{100, 110, 101, 111\}$

   (e) What is the range of $g \circ f$? $\Rightarrow \{\mathbf{001}, \mathbf{011}, \mathbf{101}, \mathbf{111}\}$
   $\{0,1\}^3 = \{000, 010, 100, 001, 011, 101, 111\}$
   Range of $f = \{100, 110, 101, 111\}$

5. Exercise 4.4.4
(c) Is it possible that f is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is ”yes,” give a specific example for f and g.
This is not possible because function f must be solved first before the values are plugged into function g. Being that function f is not one-to-one, some of the elements of set X are mapped to the same element of set Y. Since function $g \circ f$ can be written as $g \circ f : X \to Z$, these elements of set X would also map to the same element of set Z, making function $g \circ f$ also not one-to-one.

(d) Is it possible that g is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is ”yes,” give a specific example for f and g.
This may be possible if only one of the elements mapped to elements of set Z that make function g not one-to-one is mapped to an element of set X through function f (function f must be one-to-one, but not onto). The elements of set Y that are mapped to elements of set X through function f must not be mapped to the same elements of set Z through function $g \circ f$.

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
x_1 & \rightarrow & y_1 & \rightarrow & z_1 \\
x_2 & \rightarrow & y_2 & \rightarrow & z_2 \\
x_3 & \rightarrow & y_3 & \rightarrow & z_3 \\
 & & y_4 & \nearrow &
\end{array}$$