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Extended Bridge to CS
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Homework 7
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Question 3

1. Exercise 8.2.2
(b) $f(n) = n^3 + 3n^2 + 4$. Prove that f is $\Theta(n^3)$.
Claim: $f(n) = n^3 + 3n^2 + 4 = \Theta(n^3)$
Proof: We shall prove that $n^3 + 3n^2 + 4 = O(n^3)$ and $n^3 + 3n^2 + 4 = \Omega(n^3)$.

Claim: $n^3 + 3n^2 + 4 = O(n^3)$
Proof: Let $c = 8$ and $n_0 = 1$. We shall show that for any $n \geq 1$,
$n^3 + 3n^2 + 4 \leq 8 \cdot n^3$.
1.) Since $n \geq 1, n \leq n^3$, so that:

$$n^3 \leq n^3$$
$$3n^2 \leq 3n^3$$
$$4 \leq 4n^3$$

2.) Add these inequalities to get:
$n^3 + 3n^2 + 4 \leq n^3 + 3n^3 + 4n^3 = 8n^3$
3.) Therefore, $n^3 + 3n^2 + 4 \leq 8n^3$, which means that $n^3 + 3n^2 + 4 = O(n^3)$. ■

Claim: $n^3 + 3n^2 + 4 = \Omega(n^3)$
Proof: Let $c = 1$ and $n_0 = 1$. We shall prove that for any $n \geq 1$,
$n^3 + 3n^2 + 4 \geq 1 \cdot n^3$.
1.) Drop every term except for n^3 to get:
$n^3 + 3n^2 + 4 \geq n^3$.
2.) Therefore, $n^3 + 3n^2 + 4 \geq 1 \cdot n^3$, which means that $n^3 + 3n^2 + 4 = \Omega(n^3)$. ■

Since $n^3+3n^2+4 = O(n^3)$ and $n^3+3n^2+4 = \Omega(n^3)$, by definition, $n^3+3n^2+4 = \Theta(n^3)$. ■

2. Exercise 8.3.5
(a) Describe in English how the sequence of numbers is changed by the algorithm. (Hint: try the algorithm out on a small list of positive and negative numbers with p=0)

This algorithm rearranges the sequence of numbers of n length with numbers less than p on the left and numbers greater than p on the right. The first innermost while-loop searches through the sequence to find the left-most number less than p, using i as a pointer. The second innermost while-loop searches through the sequence for the right-most number greater than or equal to p, using j as a pointer. The numbers that correspond to the positions of i and j are then swapped. This process is repeated by the outer-most while-loop as long as i is less than j.

(b) What is the total number of times that the lines $i := i + 1$ or $j := j - 1$ are executed on a sequence of length n? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the two lines are executed.

The lines $i := i + 1$ and $j := j - 1$ are executed based on the numbers within the sequence and how they relate to p, in addition to the length of the sequence. $i := i + 1$ would execute more if many of the numbers in the sequence are less than p, while $j := j - 1$ would execute fewer times, and visa versa if there are more numbers in the sequence that are greater than p.

(c) What is the total number of times that the swap operation is executed? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times that the swap is executed.

The swap operation is also executed based on the numbers within the sequence and how they relate to p, in addition to the length of the sequence. The amount of swaps is dependent on how many numbers are on the wrong side of p, which is if numbers less than p are not on the left and if numbers greater than p are not on the right.

(d) Give an asymptotic lower bound for the time complexity of the algorithm. Is it important to consider the worst-case input in determining an asymptotic lower bound (using Ω on the time complexity of the algorithm? (Hint: argue that the number of swaps is at most the number of times that i is incremented or j is decremented).

An asymptotic lower bound for the time complexity of the algorithm could be $\Omega(n)$ if the swap operation is executed at most $\frac{n}{2}$ times as each swap moves two numbers. It is not important to consider the worst-case input because the algorithm will still take the $\Omega(n)$ even when given the input for the best-case.

(e) Give a matching upper bound (using O-notation) for the time complexity of the algorithm.

The matching upper bound for the time complexity of the algorithm would be $O(n)$.

Question 4

Solve the following questions from the Discrete Math zyBooks:

1. Exercise 5.1.2
   Compute the number of passwords that satisfy the given constraints
   digits, D: $|D| = 10$ letters, L: $|L| = 26$ special characters, C: $|C| = 4$

   (b) Stings of length 7, 8, or 9. Characters can be special characters, digits, or letters.
   $|C \cup D \cup L| = |C| + |D| + |L| = 4 + 10 + 26 = 40$
   string of $7 = 40^7$ string of $8 = 40^8$ string of $9 = 40^9$
   Number of passwords $= \mathbf{40^7} + \mathbf{40^8} + \mathbf{40^9}$

   (c) Strings of length 7, 8, 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.
   first character: $|C \cup D| = |C| + |D| = 4 + 10 = 14$
   rest of the characters: $|C \cup D \cup L| = |C| + |D| + |L| = 4 + 10 + 26 = 40$
   string of $7 = 14 \cdot 40^{7-1} = 14 \cdot 40^6$
   string of $8 = 14 \cdot 40^{8-1} = 14 \cdot 40^7$
   string of $9 = 14 \cdot 40^{9-1} = 14 \cdot 40^8$
   number of passwords $= \mathbf{14} \cdot (\mathbf{40^6} + \mathbf{40^7} + \mathbf{40^8})$
2. Exercise 5.3.2
   (a) How many strings are there over the set $\{a, b, c\}$ that have length 10 in which no two consecutive characters are the same?
   $S = \{a, b, c\} \rightarrow |S| = 3$
   first character: $|S| = 3$ rest of the characters: $|S| - 1 = 3 - 1 = 2$
   string of $10 = 3 \cdot 2^{10-1} = \mathbf{3} \cdot \mathbf{2^9}$
3. Exercise 5.3.3
   License plate numbers in a certain state consist of seven characters. The first character is a digit (0 through 9). The next four characters are capital letters (A through Z) and the last two characters are digits. Therefore, a license plate number in this state can be any string of the form:
   Digit-Letter-Letter-Letter-Letter-Digit-Digit

   (b) How many license plate numbers are possible if no digit appears more than once?
   first digit: $|D| = 10$ (4) Letters: $|L|^4 = 26^4$ second digit: $|D| - 1 = 10 - 1 = 9$
   third digit: $|D| - 2 = 10 - 2 = 8$ number of license plate numbers: $\mathbf{10} \cdot \mathbf{26^4} \cdot \mathbf{9} \cdot \mathbf{8}$

(c) How many license plate numbers are possible if no digit or letter appears more than once?

first digit: $|D| = 10$ first letter: $|L| = 26$ second letter: $|L| - 1 = 26 - 1 = 25$
third letter: $|L| - 2 = 26 - 2 = 24$ fourth letter: $|L| - 3 = 26 - 3 = 23$
second digit: $|D| - 1 = 10 - 1 = 9$ third digit: $|D| - 2 = 10 - 2 = 8$
number of license plate numbers: $\mathbf{10 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 9 \cdot 8}$

4. Exercise 5.2.3

Let $B = \{0, 1\}$. B^n is the set of binary strings with n bits. Define the set E_n to be the set of binary strings with n bits that have an even number of 1's. Note that zero is an even number, so a string with zero 1's (i.e., a string that is all 0's) has an even number of 1's.

(a) Show a bijection between B^9 and E_{10}. Explain why your function is a bijection.
A function $f : B^9 \to E_{10}$, such that $x \in B^9$ and $y \in E_{10}$. f(x) is obtained by adding a 1 to x if the string has an odd number of 1's and adding a 0 if the string has an even number of 1's. This function would be one-to-one and onto, which is a bijection by definition.

(b) What is $|E_{10}|$?
Since there is a bijection between B^9 and E_{10}, B^9 and E_{10} would have the same cardinality. $|B^9| = |B|^9 = \mathbf{2^9} = |E_{10}|$.

Question 5

Solve the following questions from the Discrete Math zyBooks:

1. Exercise 5.4.2
   At a certain university in the U.S., all phone numbers are 7 digits long and start with either 824 or 825.

   (a) How many different phone numbers are possible?
   first digit: $|\{8\}| = 1$ second digit: $|\{2\}| = 1$ third digit: $|\{4, 5\}| = 2$
   rest of digits: $|D|^{7-3} = 10^4$ number of possible numbers: $1 \cdot 1 \cdot 2 \cdot 10^4 = \mathbf{2 \cdot 10^4}$

   (b) How many different phone numbers are there in which the last four digits are all different?
   first (3) digits: $1 \cdot 1 \cdot 2 = 2$ fourth digit: $|D| = 10$ fifth digit: $|D| - 1 = 10 - 1 = 9$
   sixth digit: $|D| - 2 = 10 - 2 = 8$ seventh digit: $|D| = 10 - 3 = 7$
   number of possible numbers: $\mathbf{2 \cdot 10 \cdot 9 \cdot 8 \cdot 7}$

2. Exercise 5.5.3
   How many 10-bit strings are there subject to each of the following restrictions? $\rightarrow n = 10$

   (a) No restrictions: $\mathbf{2^{10}}$
   (b) The string starts with 001: $1 \cdot 1 \cdot 1 \cdot 210 - 3 = \mathbf{2^7}$
   (c) The string start with 001 or 10: $2^7 + (1 \cdot 1 \cdot 2^{10-2}) = \mathbf{2^7 + 2^8}$
   (d) The first two bits are the same as the last two bits: $2^{10-2} \cdot 1 \cdot 1 = \mathbf{2^8}$
   (e) The string has exactly six 0's: $C(10, 6) = \binom{10}{6} = \frac{10!}{(10-6)!6!} = \mathbf{210}$
   (f) The string has exactly six 0's and the first bit is 1:
   $1 \cdot C(9, 6) = C(9, 6) = \binom{9}{6} = \frac{9!}{(9-6)!6!} = \mathbf{84}$
   (g) There is exactly one 1 in the first half and exactly three 1's in the second half:
   first half: $C(5, 1) = \binom{5}{1} = \frac{5!}{(5-1)!1!} = \mathbf{5}$
   second half: $C(5, 3) = \binom{5}{3} = \frac{5!}{(5-3)!3!} = \mathbf{10}$
   number of strings: $5 \cdot 10 = \mathbf{50}$

3. Exercise 5.5.5
   (a) There are 30 boys and 35 girls that try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make his selection?
   girls: $n = 35$ $r = 10 \rightarrow \binom{35}{10}$
   boys: $n = 30$ $r = 10 \rightarrow \binom{30}{10}$
   Number of ways to make a selection: $\boldsymbol{\binom{35}{10} \cdot \binom{30}{10}}$

4. Exercise 5.5.8

This question refers to a standard deck of playing cards. If you are unfamiliar with playing cards, there is an explanation in "Probability of an event" section under the heading "Standard playing cards." A five-card hand is just a subset of 5 cards from a deck of 52 cards.

(c) How many five-card hands are made entirely of hearts and diamonds?
$n = 26$ heart and diamond cards and $r = 5$ cards $\to C(26,5) = \boldsymbol{\binom{26}{5}}$

(d)How many five-card hands have four cards of the same rank?
rank of the 4 cards: $n = 13$ ranks $r = 1$ (same) rank $\to C(13,1) = \binom{13}{1} = \frac{13!}{(13-1)!1!} = 13$
suits of the 4 cards: $n = 4$ suits $r = 4$ suits $\to C(4,4) = \binom{4}{4} = \frac{4!}{(4-4)!4!} = 1$
rank of the 5th card: $n = 12$ ranks $r = 1$ rank $\to C(12,1) = \binom{12}{1} = \frac{12!}{(12-1)!1!} = 12$
suit of the 5th card: $n = 4$ suits $r = 1$ suit $\to C(4,1) = \binom{4}{1} = \frac{4!}{(4-1)!1!} = 4$
number of five-card hands: $13 \cdot 1 \cdot 12 \cdot 4 = \mathbf{624}$

(e) A "full house" is a five-card hand that has two cards of the same rank and three cards of the same rank. How many five-card hands contain a full house?
rank of the 2 cards: $n = 13$ ranks $r = 1$ (same) rank $\to C(13,1) = \binom{13}{1} = \frac{13!}{(13-1)!1!} = 13$
suits of the 2 cards: $n = 4$ suits $r = 2$ suits $\to C(4,2) = \binom{4}{2} = \frac{4!}{(4-2)!2!} = 6$
rank of the 3 cards: $n = 12$ ranks $r = 1$ (same) rank $\to C(12,1) = \binom{12}{1} = \frac{12!}{(12-1)!1!} = 12$
suit of the 3 cards: $n = 4$ suits $r = 3$ suit $\to C(4,3) = \binom{4}{3} = \frac{4!}{(4-3!3!} = 4$
number of five-card hands: $13 \cdot 6 \cdot 12 \cdot 4 = \mathbf{3744}$

(f) How many five-card hands do not have any two cards of the same rank?
5 different ranks: $n = 13$ ranks $r = 5$ (different) rank $\to C(13,5) = \binom{13}{5}$
suits of the 5 cards: $|R|^5 = 4^5$
number of five-card hands: $\boldsymbol{\binom{13}{5} \cdot 4^5}$

5. Exercise 5.6.6

The country has two major political parties, the Democrats and the Republicans. Suppose that the national senate consists of 100 members, 44 of which are Democrats and 5 of which are Republicans.

(a) How many ways are there to select a committee of 10 senate members with the same number of Democrats and Republicans?
Democrats: $n = 44$ $r = 10/2 = 5 \to C(44, 5) = \binom{44}{5}$
Republicans: $n = 56$ $r = 10/2 = 5 \to C(56, 5) = \binom{56}{5}$
number of ways to select a committee: $\boldsymbol{\binom{44}{5} \cdot \binom{56}{5}}$

(b) Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?
Democrat speaker: $n = 44$ $r = 1 \to C(44, 1) = \binom{44}{1} = 44$
Democrat vice speaker: $n = 44 - 1 = 43$ $r = 1 \to C(43, 1) = \binom{43}{1} = 43$
Republican speaker: $n = 56$ $r = 1 \to C(56, 1) = \binom{56}{1} = 56$
Republican vice speaker: $n = 56 - 1 = 55$ $r = 1 \to C(55, 1) = \binom{55}{1} = 55$
number of ways to select: $\mathbf{44 \cdot 43 \cdot 56 \cdot 55}$

Question 6

Solve the following questions from the Discrete Math zyBooks:

1. Exercise 5.7.2
   A 5-card hand is drawn from a deck of standard playing cards.

   (a) How many 5-card hands have at least one club?
   total 5-card hand: $n = 52$ cards $r = 5$ cards $\rightarrow C(52, 5) = \binom{52}{5}$
   total 5-card with no club: $n = 52 - 13 = 39$ cards $r = 5$ cards $\rightarrow C(39, 5) = \binom{39}{5}$
   number of 5-card hands: $\boldsymbol{\binom{52}{5} \cdot \binom{39}{5}}$

   (b) How many 5-card hands have at least two cards with the same rank?
   total 5-card hand: $n = 52$ cards $r = 5$ cards $\rightarrow C(52, 5) = \binom{52}{5}$
   5 different ranks: $n = 13$ ranks $r = 5$ (different) ranks $\rightarrow C(13, 5) = \binom{13}{5}$
   suits of the cards: $|R|^5 = 4^5$
   number of 5-card hands: $\boldsymbol{\binom{56}{5} - (\binom{13}{5} \cdot 4^5)}$
2. Exercise 5.8.4
   How many ways are there to deal hands from a standard playing deck to four players if:

   (a) Each player gets exactly 13 cards.
   all 20 comic books given independently to 5 kids $\rightarrow \mathbf{5^{20}}$

   (b) Each player gets seven cards and the rest of the cards remain in the deck?
   1rst kid: $C(20, 4) = \binom{20}{4} = \frac{20!}{(20-4)!4!} = \frac{20!}{16!4!}$
   2nd kid: $C(16, 4) = \binom{16}{4} = \frac{16!}{(16-4)!4!} = \frac{16!}{12!4!}$
   3rd kid: $C(12, 4) = \binom{12}{4} = \frac{12!}{(12-4)!4!} = \frac{12!}{8!4!}$
   4th kid: $C(8, 4) = \binom{8}{4} = \frac{8!}{(8-4)!4!} = \frac{8!}{4!4!}$
   5th kid: $C(4, 4) = \binom{4}{4} = \frac{4!}{(4-4)!4!} = \frac{4!}{0!4!}$
   $\frac{20!}{\cancel{16!}4!} \cdot \frac{\cancel{16!}}{\cancel{12!}4!} \cdot \frac{\cancel{12!}}{\cancel{8!}4!} \cdot \frac{\cancel{8!}}{\cancel{4!}4!} \cdot \frac{\cancel{4!}}{4!} = \mathbf{\frac{20!}{4!4!4!4!4!}}$

Question 7

How many one-to-one functions are there from a set with five elements to sets with the following number of elements?
$|S_1| = 5$

(a) $|S_2| = 4$
A function is one-to-one if all elements of S_1 are mapped to different elements of S_2. For a function to be well-defined, all elements of S_1 must be mapped to an element of S_2. The only way for this to happen is for at least two elements of S_1 to be mapped to the same element of S_2. Therefore, there will be no one-to-one function from S_1 to S_2.

(b) $|S| = 5$
1rst element of S_1: $|S_1| = 5$
2nd element of S_1: $|S_1| - 1 = 5 - 1 = 4$
3rd element of S_1: $|S_1| - 2 = 5 - 2 = 3$
4th element of S_1: $|S_1| - 3 = 5 - 3 = 2$
5th element of S_1: $|S_1| - 4 = 5 - 4 = 1$
number of functions: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \mathbf{120}$

(c) $|S| = 6$
1rst element of S_1: $|S_1| = 6$
2nd element of S_1: $|S_1| - 1 = 6 - 1 = 5$
3rd element of S_1: $|S_1| - 2 = 6 - 2 = 4$
4th element of S_1: $|S_1| - 3 = 6 - 3 = 3$
5th element of S_1: $|S_1| - 4 = 6 - 4 = 2$
number of functions: $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = \mathbf{720}$

(d) $|S| = 7$
1rst element of S_1: $|S_1| = 7$
2nd element of S_1: $|S_1| - 1 = 7 - 1 = 6$
3rd element of S_1: $|S_1| - 2 = 7 - 2 = 5$
4th element of S_1: $|S_1| - 3 = 7 - 3 = 4$
5th element of S_1: $|S_1| - 4 = 7 - 4 = 3$
number of functions: $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = \mathbf{2520}$