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Extended Bridge to CS
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Homework 8
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Question 7

Solve the following questions from the Discrete Math zyBooks:

1. **Exercise 6.1.5**

   A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?
   $|S| = C(52, 5) = \binom{52}{5}$

   (b) What is the probability that the hand is three of a kind? A three of a kind has three cards of the same rank. The other two cards do not have the same rank as each other and do not have the same rank as the three with the same rank.
   rank of the 3 cards: n = 13 total ranks; r = 1 (same) rank $\rightarrow C(13, 1) = \binom{13}{1}$
   suit of the 3 cards: n = 4 total suits; r = 3 suits $\rightarrow C(4, 3) = \binom{4}{3}$
   rank of the 2 cards: $n = 13-1 = 12$ total ranks remaining; $r = 2$ ranks $\rightarrow C(12, 2) = \binom{12}{2}$
   suit of 4th card: n = 4 total suits; r = 1 suit $\rightarrow C(4, 1) = \binom{4}{1}$
   suit of 5th card: n = 4 total suits; r = 1 suit $\rightarrow C(4, 1) = \binom{4}{1}$
   p(E)=$\frac{\binom{13}{3}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = \mathbf{\frac{88}{4165}}$

   (c) What is the probability that all 5 cards have the same suit?
   rank of the 5 cards: n = 13 total ranks; r = 5 ranks $\rightarrow C(13, 5) = \binom{13}{5}$
   suit of the 5 cards: n = 4 total suits; r = 1 (same) suit $\rightarrow C(4, 1) = \binom{4}{1}$
   p(E)=$\frac{\binom{13}{5}\binom{4}{1}}{\binom{52}{5}} = \mathbf{\frac{33}{16660}}$

   (d) What is the probability that the hand is a two of a kind? A two of a kind has two cards of the same rank (called a pair). Among the remaining three cards, not the pair, no two have the same rank and none of them have the same rank as the pair.
   rank of the pair: n = 13 total ranks; r = 1 rank $\rightarrow C(13, 1) = \binom{13}{1}$
   suit of the pair: n = 4 total suits; r = 2 suits $\rightarrow C(4, 2) = \binom{4}{2}$
   rank of the last 3 cards: $n = 13-1 = 12$ remaining ranks; r = 3 ranks $\rightarrow C(12, 3) = \binom{12}{3}$
   suit of 3rd card: n = 4 total suits; r = 1 suit $\rightarrow C(4, 1) = \binom{4}{1}$
   suit of 4th card: n = 4 total suits; r = 1 suit $\rightarrow C(4, 1) = \binom{4}{1}$

suit of 5th card: n = 4 total suits; r = 1 suit $\rightarrow C(4,1) = \binom{4}{1}$

p(E)=$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{}5} = \mathbf{\frac{352}{833}}$

2. **Exercise 6.2.4**

A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

$|S| = C(52,5) = \binom{52}{5}$

(a) The hand has at least one club.

$|\overline{E}| = n = 52 - 13 = 39$ non-club cards; r = 5 cards $\rightarrow C(39,5) = \binom{39}{5}$

$p(E) = 1 - \frac{\binom{39}{5}}{\binom{52}{5}} = \mathbf{\frac{7411}{9520}}$

(b) The hand has at least two cards with the same rank.

rank of the 5 cards: n = 13 ranks; r = 5 (different) ranks $\rightarrow C(13,5) = \binom{13}{5}$

suit of the 5 cards: n = 4 total suits; r = 1 suit; k = 5 cards $\rightarrow C(4,1)^5 = \binom{4}{1}^5$

p(E)=$1 - \frac{\binom{13}{5}\binom{4}{1}^5}{\binom{52}{5}} = \mathbf{\frac{2053}{4165}}$

(c) The hand has exactly one club or exactly one spade.

C = hand with exactly one club card; no spade cards

one club card: n = 13 club cards; r = 1 club card $\rightarrow C(13,1) = \binom{13}{1}$

non-club and non-spade cards: $n = 52-13-13 = 26$ cards; r = 4 cards $\rightarrow C(26,4) = \binom{26}{4}$

S = hand with exactly one spade card; no spade cards

one spade card: n = 13 spade cards; r = 1 spade card $\rightarrow C(13,1) = \binom{13}{1}$

non-club and non-spade cards: $n = 52-13-13 = 26$ cards; r = 4 cards $\rightarrow C(26,4) = \binom{26}{4}$

$C \cup S$ = hand with exactly one club card and exactly one spade card

one club card: n = 13 club cards; r = 1 club card $\rightarrow C(13,1) = \binom{13}{1}$

one spade card: n = 13 spade cards; r = 1 spade card $\rightarrow C(13,1) = \binom{13}{1}$

non-club and non-spade cards: $n = 52-13-13 = 26$ cards; r = 3 cards $\rightarrow C(26,3) = \binom{26}{3}$

p(E)=$\frac{\binom{13}{1}\binom{26}{4}+\binom{13}{1}\binom{26}{4}+\binom{13}{1}\binom{13}{1}\binom{26}{3}}{\binom{52}{5}} = \mathbf{\frac{65}{204}}$

(d) The hand has at least one club or at least one spade.

$\overline{E}$ = does not have clubs and does not have spades

$|\overline{E}|$: $n = 52 - 13 - 13 = 26$ cards; r = 5 cards $\rightarrow C(26,5) = \binom{26}{5}$

$p(E) = 1 - \frac{\binom{26}{5}}{\binom{52}{5}} = \mathbf{\frac{9743}{9996}}$

Question 8

Solve the following questions from the Discrete Math zyBooks:

1. **Exercise 6.3.2**

The letters {a, b, c, d, e, f, g} are put in a random order. Each permutation is equally likely.

$|S| = \binom{7}{1} \cdot \binom{6}{1} \cdot \binom{5}{1} \cdot \binom{4}{1} \cdot \binom{3}{1} \cdot \binom{2}{1} \cdot \binom{1}{1} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7!$

(a) Calculate the probability of each individual event. That is calculate p(A), p(B), and p(C).

A: The letter b falls in the middle (with three before it and three after it)

3 letters before b: $\binom{6}{1}\binom{5}{1}\binom{4}{1} = 6 \cdot 5 \cdot 4$

3 letters after b: $\binom{3}{1}\binom{2}{1}\binom{1}{1} = 3 \cdot 2 \cdot 1$

b in the middle: $\binom{1}{1} = 1$

$|A| = 1 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 1 \cdot 6! = 6!$

$p(A) = \frac{6!}{7!} = \mathbf{\frac{1}{7}}$

B: The letter c appears to the right of b, although c is not necessarily immediately to the right of b.

choosing c and b: $\binom{7}{2} = 21$

c to the right of b: $\frac{21}{2} \approx 10$

rest of letters: $\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

$|B| = 10 \cdot 5!$

$p(B) = \frac{10 \cdot 5!}{7!} = \mathbf{\frac{1}{2}}$

C: The letters "def" occur together in that order.

arrange 5 units {a, b, c, def, g}: $\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} == 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

$|C| = 5!$

$p(C) = \frac{5!}{7!} = \mathbf{\frac{1}{42}}$

(b) What is $p(A|C)$?

2 letters before b: $\binom{4}{1}\binom{3}{1} = 4 \cdot 3$

b in the middle: $\binom{1}{1} = 1$

2 letters after b: $\binom{2}{1}\binom{1}{1} = 2 \cdot 1$

$|A \cap C| = 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 1 \cdot 4! = 4!$; $|C| = 5!$

$p(A|C) = \frac{4!}{5!} = \frac{24}{120} = \mathbf{\frac{1}{5}}$

(c) What is $p(B|C)$?
choosing c and b: $\binom{5}{2} = 10$
rest of the letters: $\binom{3}{1}\binom{2}{1}\binom{1}{1} = 3 \cdot 2 \cdot 1 = 3!$
$|B \cap C| = 10 \cdot 3! = 10 \cdot 6 = 60$; $|C| = 5! = 120$
$p(B|C) = \frac{60}{120} = \mathbf{\frac{1}{2}}$

(d) What is $p(A|B)$?
b in position 4: $\binom{1}{1} = 1$
possible positions for c: $\binom{3}{1} = 3$
rest of the letters: $\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} == 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$
$p(A \cap B) = \frac{1 \cdot 3 \cdot 5!}{7!} = \frac{1}{14}$; p(B)=$\frac{1}{2}$
$P(A|B) = \frac{\frac{1}{14}}{\frac{1}{2}} = \mathbf{\frac{1}{7}}$

(e) Which pairs of events among A, B, and C are independent?
$p(A|C) = \frac{1}{5} \neq \frac{1}{7} = p(A) \rightarrow$ not independent
$p(B|C) = \frac{1}{2} = \frac{1}{2} = p(B) \rightarrow$ independent
$p(A|B) = \frac{1}{7} = \frac{1}{7} = p(A) \rightarrow$ independent

2. **Exercise 6.3.6**
A biased coin is flipped 10 times. In a single flip of the coin, the probability of heads is $\frac{1}{3}$ and the probability of tails is $\frac{2}{3}$. The outcomes of the coin flips are mutually independent. What is the probability of each event?

(b) The first 5 flips come up heads. The last 5 flips come up tails $\rightarrow (\frac{1}{3})^5(\frac{2}{3})^5$
(c) The first flip comes up heads. The rest of the flips come up tails $\rightarrow (\frac{1}{3})^1(\frac{2}{3})^9$

3. **Exercise 6.4.2**
(a) Assume that you have two dice, one of which is fair, and the other is biased toward landing on six, so that 0.25 of the time it lands on six, and 0.15 of the time it lands on each of 1,2,3,4 and 5. You choose a die at random, and roll it six times, getting the values 4,3,6,6,5,5. What is the probability that the die you chose is the fair die? The outcomes of the rolls are mutually independent.
F = fair die $\overline{F}$ = biased die X={4,3,6,6,5,5}
p(F)=$\frac{1}{2}$ $p\overline{F} = \frac{1}{2}$ $p(X|F) = (\frac{1}{6})^6$ $p(X|\overline{F}) = (0.15)^4(0.25)^2$
$p(F|X) = \frac{(\frac{1}{6})^6(\frac{1}{2})}{(\frac{1}{6})^6(\frac{1}{2})+(0.15)^4(0.25)^2(\frac{1}{2})} \approx \mathbf{0.40}$

Question 9

Solve the following questions from the Discrete Math zyBooks:

1. **Exercise 6.5.2**
   A hand of 5 cards is dealt from a perfectly shuffled deck of playing cards. Let the random variable A denote the number of aces in the hand.

   (a) What is the range of A?
   4 aces in a deck $\rightarrow$ {0,1,2,3,4}

   (b) Give the distribution over the random variable A.
   $|S| = C(52, 5)$
   0 ace cards: $n = 52 - 4$ ace cards $= 48$ cards; r = 5 cards $\binom{48}{5} = C(48, 5)$
   1 ace cards: n = 48 cards; $r = 5 - 1$ ace card = 4 cards $\binom{48}{4} = C(48, 4)$
   2 ace cards: n = 48 cards; $r = 5 - 2$ ace cards = 3 cards $\binom{48}{3} = C(48, 3)$
   3 ace cards: n = 48 cards; $r = 5 - 3$ ace cards = 2 cards $\binom{48}{2} = C(48, 2)$
   4 ace cards: n = 48 cards; $r = 5 - 4$ ace cards = 1 card $\binom{48}{1} = C(48, 1)$
   Distribution: $(0,\frac{C(48,5)}{C(52,5)})$,$(1,\frac{C(48,4)}{C(52,5)})$,$(2,\frac{C(48,3)}{C(52,5)})$,$(3,\frac{C(48,2)}{C(52,5)})$,$(4,\frac{C(48,1)}{C(52,5)})$

2. **Exercise 6.6.1**
   (a) Two student council representatives are chosen at random from a group of 7 girls and 3 boys. Let G be the random variable denoting the number of girls chosen. What is E[G]?
   $|S|$: n = 10 students; r = 2 students $\binom{10}{2} = 45$ G={0,1,2}
   0 girls, 2 boys: $n_2 = 3$ boys; $r_2 = 2$ boys $\rightarrow \binom{3}{2} = 3$
   1 girl, 1 boy: $n_1 = 7$ girls; $r_1 = 1$ girl $\rightarrow \binom{7}{1} = 7$ $n_2 = 3$ boys; $r_2 = 1$ boy $\rightarrow \binom{3}{1} = 3$
   2 girls, 0 boys: $n_1 = 7$ girls; $r_1 = 2$ girls $\rightarrow \binom{7}{12} = 21$
   E[G]=$(0 \cdot \frac{3}{45}) + (1 \cdot \frac{7 \cdot 3}{45}) + (2 \cdot \frac{21}{45}) = 0 + \frac{21}{45} + \frac{42}{45} = \mathbf{\frac{7}{5}}$

3. **Exercise 6.6.4**
(a) A fair die is rolled once. Let X be the random variable that denotes the square of the number that shows up on the die. What is E[X]?
X={0,1,4,9,16.25,36} p(each number)=$\frac{1}{6}$

$$\begin{aligned}E[X] &= (0\cdot\frac{1}{6}) + (1\cdot\frac{1}{6}) + (4\cdot\frac{1}{6}) + (9\cdot\frac{1}{6}) + (16\cdot\frac{1}{6}) + (25\cdot\frac{1}{6}) + (36\cdot\frac{1}{6})\\ &= 0 + \frac{1}{6} + \frac{4}{6} + \frac{9}{6} + \frac{16}{6} + \frac{25}{6} + \frac{36}{6}\\ &= \mathbf{\frac{91}{6}}\end{aligned}$$

(b) A fair coin is tossed three times. Let Y be the random variable that denotes the square of the number of heads. What is E[Y]?
HHH: $Y = 3^2 = 9$ HHT: $Y = 2^2 = 4$ HTH: $Y = 2^2 = 4$ HTT: $Y = 1^2 = 1$
THH: $Y = 2^2 = 4$ THT: $Y = 1^2 = 1$ TTH: $Y = 1^2 = 1$ TTT: $Y = 0^2 = 0$
Y = {9,4,1,0} p(each outcome) = $\frac{1}{8}$

$$\begin{aligned}E[X] &= (9\cdot\frac{1}{8}) + (4\cdot\frac{3}{8}) + (1\cdot\frac{3}{8}) + (0\cdot\frac{1}{8})\\ &= \frac{9}{8} + \frac{12}{8} + \frac{3}{8} + 0\\ &= \mathbf{3}\end{aligned}$$

4. **Exercise 6.7.4**
(a) A class of 10 students hang up their coats when they arrive at school. Just before recess, the teacher hands one coat selected at random to each child. What is the expected number of children who get his or her own coat?
$p(A_1) = \frac{1}{10}$ $E[A_1] = (1\cdot\frac{1}{10}) + (1\cdot\frac{1}{10}) = \frac{1}{10}$ $E[A] = 10\cdot\frac{1}{10} = \mathbf{1}$

Question 10

Solve the following questions from the Discrete Math zyBooks:

1. **Exercise 6.8.1**

   The probability that a circuit board produced by a particular manufacturer has a defect is 1%. You can assume that errors are independent, so the event that one circuit board has a defect is independent of whether a different circuit board has a defect.

   (a) What is the probability that out of 100 circuit boards made exactly 2 have defects?
   k = 2 defects   n = 100 circuit boards   p = 1% = 0.01   q = 1 - 0.01 = 0.99
   b(2;100,0.01)=$\binom{100}{2}(0.01)^2(0.99)^{100-2} = \boldsymbol{\binom{100}{2}(0.01)^2(0.99)^{98}}$

   (b) What is the probability that out of 100 circuit boards made at least 2 have defects?
   0 defects: k = 0
   b(0;100,0.01)=$\binom{100}{0}(0.01)^0(0.99)^{100-0} = (0.99)^{100}$
   1 defects: k = 1
   b(1;100,0.01)=$\binom{100}{1}(0.01)^1(0.99)^{100-1} = \binom{100}{1}(0.01)^1(0.99)^{99} = 1 \cdot (0.99)^{99} = (0.99)^{99}$
   at least 2 defects: $\boldsymbol{1 - (0.99)^{100} - (0.99)^{99}}$

   (c) What is the expected number of circuit boards with defects out of the 100 made?
   X = # of defects   n = 100 circuit boards   E[X]=$100 \cdot \frac{1}{100} = \mathbf{1}$

   (d) Now suppose that the circuit boards are made in batches of two. Either both circuit boards in a batch have a defect or they are both free of defects. The probability that a batch has a defect is 1%. What is the probability that out of 100 circuit boards (50 batches) at least 2 have defects? What is the expected number of circuit boards with defects out of the 100 made? How do your answers compared to the situation in which each circuit board is made separately?
   0 defects: k = 0; n = 50
   b(0;50,0.01)=$\binom{50}{0}(0.01)^0(0.99)^{50-0} = (0.99)^{50}$
   1 defects: k = 1; n = 50
   b(1;50,0.01)=$\binom{50}{1}(0.01)^1(0.99)^{50-1} = 50(0.01)(0.99)^{49}$
   at least 2 defects: $\boldsymbol{1 - (0.99)^{50} - 50(0.01)(0.99)^{49}}$
   expected # of defects: 2 batches $\rightarrow c = 2$
   E[2X]=2E[X]=$2 \cdot 50 \cdot 0.01 = \mathbf{1}$

2. **Exercise 6.8.3**
A gambler has a coin which is either fair (equal probability heads or tails) or is biased with a probability of heads equal to 0.3. Without knowing which coin he is using, you ask him to flip the coin 10 times. If the number of heads is at least 4, you conclude that the coin is fair. If the number of heads is less than 4, you conclude that the coin is biased.
(b) What is the probability you reach an incorrect conclusion if the coin is fair?
0 heads: k = 0 n = 10 flips p = 0.3 q = 1 - 0.03 = 0.7
b(0;10,0.3)=$\binom{10}{0}(0.3)^0(0.7)^{10-0} = (0.7)^{10}$
1 head: k = 1 n = 10 flips p = 0.3 q = 1 - 0.03 = 0.7
b(1;10,0.3)=$\binom{10}{1}(0.3)^1(0.7)^{10-1} = 10(0.3)(0.7)^9$
2 heads: k = 2 n = 10 flips p = 0.3 q = 1 - 0.03 = 0.7
b(2;10,0.3)=$\binom{10}{2}(0.3)^2(0.7)^{10-2} = \binom{10}{2}(0.3)^2(0.7)^8$
3 heads: k = 3 n = 10 flips p = 0.3 q = 1 - 0.03 = 0.7
b(3;10,0.3)=$\binom{10}{3}(0.3)^3(0.7)^{10-3} = \binom{10}{3}(0.3)^3(0.7)^7$
incorrect conclusion: $\mathbf{1 - (0.7)^{10} - (10(0.3)(0.7)^9) - (\binom{10}{2}(0.3)^2(0.7)^8) - (\binom{10}{3}(0.3)^3(0.7)^7)}$