PSQT Q&A

1 Define Axioms of Probability.

2 Define Empirical Probability.

3 Define Geometric Probability.

4 Define Conditional Probability.

5 Define Baye’s Theorem Statement.

6 What are the properties of Classical probability of an Event A?

7 State and prove Baye's Theorem for Probability.

8 State and prove Theorem for Total Probability.

9 If A and B are any two arbitrary events of the same sample space then prove that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

10 A box contains 2 white and 4 black balls. Another box contains 5 white and 7 black balls. A ball is transferred from the box A to box B. Then a ball is drawn from the box B. Find the probability that it is white.

11 The probability that a student Mr. Anand passed mathematics is 2/3 , the probability that he passes statistics is 4/9, if the probability of passing at least one subject is 4/5. What is the probability that Mr. Anand will pass both the subjects?

12 A factory produces its entire output with three machines. Machine I, II, III produce 50%, 30% and 20% of the output but 4%, 2% and 4% of their outputs are defective respectively. What Fraction of the total output are defective?

13 Box A contains 5 red and 3 white marbles and box B contains 2 red and 6 white marbles. If a marble is drawn from each box, what is the probability that they are both of the same color?

14 The probability that the students A, B, C solve a problem are 1/2 , 3/4 , 1/4 respectively. What is the Probability that the problem is solved?

15 A class consists of 6 girls and 10 boys. If a committee of 3 is chosen at random from the class, find the probability that (i) 3 boys are selected (ii) exactly 2 girls are selected.

16 In a shooting competition, Mr. X can shoot at the bulls eye 4 times out of 5 Shots and Mr. Y can shoot 5 times out of 6 and Mr. Z can shoot 3 times out of 4. Find the probability that the target will be hit at least twice.

17 A bag contains 3 white and 5 red balls. Three balls are drawn after shaking the bag. The odds against these balls being red is______

18 In a sample of 446 cars, stopped at a road block, only 67 of the drivers had their seat belts fastened. Estimate the probability that a driver stopped on that road, will have his or her seat belt

fastened.

19 A card is drawn from a well shuffled pack of cards. What is the probability that it is either a Spade or a King?

20 A fair die is tossed twice. Find the probability of getting a 4, 5 or 6 on the first toss and 1, 2, 3 or 4 on the second toss.

21 Define random variables with examples.

22 Determine the binomial distribution for mean 4 and variance 3.
23 Negative Binomial Distribution.

24 Uniform Exponential Distributions.

25 Write the properties of Correlation.

26 Find the relation between Correlation and Regression.

27 Write the Regression Lines of X on Y and Y on X.

28 What do you mean by Test for goodness of fit ? Explain

29 Define chi square test and write any two properties.

30 Curve fitting

31 What is meant by Standard Error

32 Properties of good estimator

33 Unbiasedness

34 Write a short note on (i) Statistic (ii) Standard Error (iii) Interval Estimation

35 Find maximum likelihood estimator for Poisson distribution parameter.

36 A short note on Unbiasedness and interval estimation.

37 Distinguish between point and interval estimation

38 Explain null hypothesis and critical region.

39 null and alternative hypothesis

40 testing of equality of variances

41 Write the properties of F-test

42 What is meant by Queue/Service discipline

43 Write any two characteristics of queueing models.

44 mean inter arrival time

1. Axioms of Probability

Kolmogorov's axioms:

(i) **Non-negativity:** $P(A) \geq 0$ for any event A.

(ii) **Normalization:** $P(S) = 1$, where S is the sample space.

(iii) **Additivity:** If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B).$$

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2. Empirical Probability

The probability of an event based on actual experiments or observations.

$$P(E) = \frac{\text{Number of times event } E \text{ occurs}}{\text{Total number of trials}}.$$

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3. Geometric Probability

Probability based on length, area, or volume:

$$P(E) = \frac{\text{Favorable measure (length/area/volume)}}{\text{Total measure of sample space}}.$$

4. Conditional Probability

The probability of event A given that event B has already occurred:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0.$$

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5. Bayes' Theorem (Statement)

If $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive events and A is any event with $P(A) > 0$, then

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{n} P(B_j)P(A|B_j)}.$$

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6. Properties of Classical Probability (for event A)

(i) $0 \le P(A) \le 1$

(ii) $P(S) = 1$

(iii) $P(\phi) = 0$

(iv) $P(A^c) = 1 - P(A)$

(v) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

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7. Bayes' Theorem (Proof)

We know:

$$P(B_i|A) = \frac{P(B_i \cap A)}{P(A)}.$$

Also, $P(B_i \cap A) = P(B_i)P(A|B_i)$.
And, $P(A) = \sum_{j=1}^{n} P(B_j)P(A|B_j)$.
Substituting,

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{n} P(B_j)P(A|B_j)}.$$

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8. Theorem of Total Probability (Statement & Proof)

If $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive events, then for any event A:

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A|B_i).$$

Proof:

Since $B_1, B_2, \ldots, B_n$ partition the sample space,

$$A = \bigcup_{i=1}^{n} (A \cap B_i).$$

By additivity,

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(B_i)P(A|B_i).$$

9. Theorem: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

From set theory:

$$A \cup B = A + B - (A \cap B).$$

Thus,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

(Proved)

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10. Problem (Two Boxes – Transfer and Draw)

Box A: 2 white, 4 black.

Box B: 5 white, 7 black.

Step 1: Case 1 – White ball transferred from A to B.

Probability = $\frac{2}{6} = \frac{1}{3}$.

Then Box B has: 6 white, 7 black → Probability of drawing white = $\frac{6}{13}$.

Step 2: Case 2 – Black ball transferred from A to B.

Probability = $\frac{4}{6} = \frac{2}{3}$.

Then Box B has: 5 white, 8 black → Probability of drawing white = $\frac{5}{13}$.

Step 3: Total probability

$$P(\text{white}) = \frac{1}{3} \cdot \frac{6}{13} + \frac{2}{3} \cdot \frac{5}{13} = \frac{6}{39} + \frac{10}{39} = \frac{16}{39}.$$

11. Probability both subjects

Given $P(M)=\frac{2}{3},\ P(S)=\frac{4}{9},\ P(M\cup S)=\frac{4}{5}$.

$$P(M\cap S)=P(M)+P(S)-P(M\cup S)=\frac{2}{3}+\frac{4}{9}-\frac{4}{5}=\frac{14}{45}.$$

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12. Fraction defective (total output)

$$P(\text{def})=0.5(0.04)+0.3(0.02)+0.2(0.04)=0.034=3.4\%.$$

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13. Same color from two boxes

A: $5R, 3W$;; B: $2R, 6W$.

$$P=\frac{5}{8}\cdot\frac{2}{8}+\frac{3}{8}\cdot\frac{6}{8}=\frac{5}{32}+\frac{9}{32}=\frac{7}{16}.$$

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14. Problem is solved (by at least one)

$P_A=\frac{1}{2},\ P_B=\frac{3}{4},\ P_C=\frac{1}{4}$ (independent).

$$P(\text{none})=\tfrac{1}{2}\cdot\tfrac{1}{4}\cdot\tfrac{3}{4}=\tfrac{3}{32}\ \Rightarrow\ P(\text{solved})=1-\tfrac{3}{32}=\tfrac{29}{32}.$$

15. Committee of 3 from 6 girls, 10 boys (total 16)

$$\binom{16}{3}=560.$$

(i) 3 boys: $\frac{\binom{10}{3}}{560}=\frac{120}{560}=\frac{3}{14}$.

(ii) Exactly 2 girls: $\frac{\binom{6}{2}\binom{10}{1}}{560}=\frac{150}{560}=\frac{15}{56}$.

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16. At least two hits (X, Y, Z shoot once)

$p_X=\frac{4}{5},\ p_Y=\frac{5}{6},\ p_Z=\frac{3}{4}$.

$$P(\text{exactly }2)=\frac{4}{5}\frac{5}{6}\frac{1}{4}+\frac{4}{5}\frac{3}{4}\frac{1}{6}+\frac{5}{6}\frac{3}{4}\frac{1}{5}=\frac{1}{6}+\frac{1}{10}+\frac{1}{8}=\frac{47}{120}.$$

$$P(\text{exactly }3)=\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{3}{4}=\frac{1}{2}.$$

So $P(\geq 2)=\frac{47}{120}+\frac{1}{2}=\frac{107}{120}$.

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17. In a bag with 3 white and 5 red balls, 3 balls are drawn (without replacement). Find the odds against all being red.

- Total ways:

$$\binom{8}{3} = 56$$

- Favorable (all 3 red):

$$\binom{5}{3} = 10$$

- Probability (all red):

$$P(\text{all red}) = \frac{10}{56} = \frac{5}{28}.$$

- Probability (not all red):

$$P(\text{not all red}) = 1 - \frac{5}{28} = \frac{23}{28}.$$

- Odds against all red =

$$\frac{P(\text{not all red})}{P(\text{all red})} = \frac{23/28}{5/28} = \frac{23}{5}.$$

✅ **Final Answer:** Odds against all red = 23 : 5

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18. Empirical seat-belt probability

$$\hat{p} = \frac{67}{446} \approx 0.150 \text{ (about } 15.0\%\text{)}.$$

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19. Spade or King from 52-card deck

$$\frac{13 + 4 - 1}{52} = \frac{16}{52} = \frac{4}{13}.$$

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20. Die twice: (4/5/6) then (1/2/3/4)

$$P = \frac{3}{6} \cdot \frac{4}{6} = \frac{12}{36} = \frac{1}{3}.$$

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21. Define random variables (with examples)

A **random variable** X is a function that assigns a real number to each outcome in a sample space of a random experiment.

- **Discrete random variable:** takes countable values.
  Example: $X =$ number of heads in 3 coin tosses ($\{0, 1, 2, 3\}$).
- **Continuous random variable:** takes values from an interval (uncountable).
  Example: $Y =$ lifetime (in hours) of a bulb, which can be any nonnegative real number.

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22. Determine the binomial distribution for mean 4 and variance 3

Binomial parameters n, p satisfy

$$\text{mean} = np = 4, \qquad \text{var} = np(1-p) = 3.$$

From these,

$$4(1-p) = 3 \implies 1 - p = \tfrac{3}{4} \implies p = \tfrac{1}{4}.$$

Then $n = \dfrac{4}{p} = 16$.
So $X \sim \text{Binomial}(n = 16, p = \frac{1}{4})$ with PMF

$$P(X = k) = \binom{16}{k} \left(\tfrac{1}{4}\right)^k \left(\tfrac{3}{4}\right)^{16-k}, \quad k = 0, 1, \ldots, 16.$$

23. Negative Binomial Distribution

Definition (one common form): The negative binomial gives the probability that the r-th success occurs on the k-th trial (with independent trials, success probability p). Its PMF:

$$P(K = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \qquad k = r, r+1, \ldots$$

Mean and variance: $\mathbb{E}[K] = \dfrac{r}{p}$, $\text{Var}(K) = \dfrac{r(1-p)}{p^2}$.

(There are equivalent parameterizations that count number of failures before r-th success.)

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24. Uniform and Exponential Distributions

Uniform distribution

- **Discrete uniform:** equal probability on a finite set, e.g. fair die: $P(X = i) = 1/6$ for $i = 1, \ldots, 6$.
- **Continuous uniform** $U(a, b)$: density $f(x) = \dfrac{1}{b-a}$ for $x \in [a, b]$. Mean $(a+b)/2$, variance $(b - a)^2/12$.

Exponential distribution

- Continuous distribution for nonnegative values; parameter $\lambda > 0$. Density:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

Mean $1/\lambda$, variance $1/\lambda^2$. Memoryless property: $P(X > s + t \mid X > s) = P(X > t)$.

25. Properties of Correlation (Pearson correlation coefficient r)

1. **Range:** $-1 \le r \le 1$.
2. **Symmetry:** $r_{XY} = r_{YX}$.
3. **Unitless:** dimensionless measure (scale-invariant).
4. **Sign:** $r > 0$ indicates positive linear association; $r < 0$ negative.
5. **Extremes:** $r = 1$ or -1 ⇔ perfect linear relationship.
6. **Zero** r implies no *linear* correlation but does not imply independence (only if joint distribution is special, e.g., bivariate normal then zero ⇒ independence).
7. **Invariant to linear scaling:** if $X' = aX + b$ and $Y' = cY + d$ with $ac > 0$, then $r_{X'Y'} = r_{XY}$ (sign may flip if $ac < 0$).

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26. Relation between Correlation and Regression

For two variables X, Y with sample standard deviations s_X, s_Y and Pearson correlation r, the slopes of regression lines relate to r by

$$\text{slope of } (Y \text{ on } X) = b_{Y|X} = r\frac{s_Y}{s_X}, \qquad \text{slope of } (X \text{ on } Y) = b_{X|Y} = r\frac{s_X}{s_Y}.$$

Thus $b_{Y|X} \cdot b_{X|Y} = r^2$. The magnitude of r measures how well either regression predicts the other linearly.

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27. Regression lines of X on Y and Y on X

Let sample means $\bar{X}, \bar{Y}$, covariance s_{XY}, and variances s_X^2, s_Y^2.

- **Regression line of Y on X:**

$$\widehat{Y} = a_{Y|X} + b_{Y|X}X, \quad b_{Y|X} = \frac{s_{XY}}{s_X^2}, \quad a_{Y|X} = \bar{Y} - b_{Y|X}\bar{X}.$$

- **Regression line of X on Y:**

$$\widehat{X} = a_{X|Y} + b_{X|Y}Y, \quad b_{X|Y} = \frac{s_{XY}}{s_Y^2}, \quad a_{X|Y} = \bar{X} - b_{X|Y}\bar{Y}.$$

(Using population parameters replace sample estimates by $\mathrm{Cov}(X, Y),\ \mathrm{Var}(X), \mathrm{Var}(Y)$.)

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28. Test for goodness of fit — what it means (brief)

A **goodness-of-fit test** checks whether observed data are consistent with a specified probability distribution (the null hypothesis). You compare observed frequencies to expected frequencies under the hypothesized distribution and use a test statistic to decide whether discrepancies are too large to be attributed to chance.

Commonly used test: **Chi-square goodness-of-fit** (for categorical/discretized data).

29. Define chi-square test and two properties

Chi-square test (χ^2): For categorical data in k classes, test statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i},$$

where O_i are observed counts and E_i expected counts under H_0. Under H_0 (with large-sample conditions) χ^2 approximately follows a chi-square distribution with $k - c - 1$ degrees of freedom (where c is the number of parameters estimated from data).

Two properties:

1. **Additivity:** If $U \sim \chi^2_m$ and $V \sim \chi^2_n$ independent, then $U + V \sim \chi^2_{m+n}$.
2. **Non-negativity & skewness:** χ^2 values are nonnegative and the distribution is right-skewed; as df increase it becomes more symmetric.

30. Curve fitting (short note)

Curve fitting is the process of finding a curve (function) that best fits a set of observed data points. Main methods:

- **Least Squares (LS) method:** choose parameters to minimize the sum of squared residuals. For a linear model $y = \beta_0 + \beta_1 x$ LS gives closed-form normal equations. For polynomial or multiple regression LS extends naturally.
- **Nonlinear curve fitting:** for models nonlinear in parameters (e.g., exponential, logistic), use iterative numerical optimization (Gauss–Newton, Levenberg–Marquardt).
- **Model selection & diagnostics:** check residuals, R^2 (coefficient of determination), adjusted R^2, standard error of estimate; avoid overfitting by using simpler models or penalized criteria (AIC/BIC, cross-validation).
- **Transformation approach:** sometimes transform variables (log, reciprocal) to linearize relationships and apply linear LS.

31. **What is meant by Standard Error**
    Standard error (SE) of an estimator is the standard deviation of its sampling distribution.
    If $\hat{\theta}$ is an estimator of parameter θ, then

$$\mathrm{SE}(\hat{\theta}) = \sqrt{\mathrm{Var}(\hat{\theta})}.$$

Example: The SE of the sample mean $\bar{X}$ (from i.i.d. samples with population variance σ^2) is $\sigma/\sqrt{n}$. When σ is unknown it is estimated by $s/\sqrt{n}$.

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32. **Properties of a good estimator**
    A desirable estimator should ideally have these properties:

- **Unbiasedness:** $E[\hat{\theta}] = \theta$.
- **Consistency:** $\hat{\theta} \xrightarrow{p} \theta$ as $n \to \infty$.
- **Efficiency:** Among unbiased estimators, it has the smallest variance (or attains the Cramér–Rao lower bound).
- **Sufficiency:** Uses all information about θ contained in the sample (a sufficient statistic).
- **Asymptotic normality:** Sampling distribution tends to normal for large n.
  (Practical estimators trade off between these properties.)

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33. **Unbiasedness (definition & short note)**
    An estimator $\hat{\theta}$ is **unbiased** for parameter θ if $E[\hat{\theta}] = \theta$ for all parameter values in the model.
    If $E[\hat{\theta}] \neq \theta$, the difference $\mathrm{Bias}(\hat{\theta}) = E[\hat{\theta}] - \theta$ measures systematic error.
    Unbiasedness alone doesn't guarantee a good estimator—variance also matters.

34. **Short note on (i) Statistic (ii) Standard Error (iii) Interval Estimation**

(i) **Statistic:** A function of observed sample data (no unknown parameters). Examples: sample mean $\bar{X}$, sample variance s^2. Statistics are random variables whose distributions we study.

(ii) **Standard Error:** See Q31 — the standard deviation of a statistic's sampling distribution; used to quantify estimator precision and to form confidence intervals and hypothesis tests.

(iii) **Interval Estimation:** Instead of a point estimate, provide a range (confidence interval) that, with specified confidence level $1-\alpha$ (e.g., 95%), will contain the true parameter in repeated sampling. For example, a $100(1-\alpha)\%$ CI for a population mean (with known σ) is

$$\bar{X} \pm z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

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35. **Find maximum likelihood estimator for Poisson distribution parameter**
    If $X_1, \dots, X_n$ are i.i.d. $\mathrm{Poisson}(\lambda)$ with PMF $P(X=k) = e^{-\lambda}\lambda^k/k!$, the log-likelihood is

$$\ell(\lambda) = -n\lambda + \left(\sum_i X_i\right)\log\lambda + \mathrm{const}.$$

Differentiate and set to zero:

$$\frac{d\ell}{d\lambda} = -n + \frac{\sum_i X_i}{\lambda} = 0 \implies \hat{\lambda}_{\mathrm{MLE}} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

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36. **Short note on Unbiasedness and Interval Estimation**

- **Unbiasedness:** (See Q33) Important property but not sufficient; unbiased estimators with huge variance can be worse than slightly biased but low-variance ones.
- **Interval estimation:** Builds on estimator ± margin of error (SE × critical value). Confidence intervals provide both a point estimate and a measure of uncertainty and are generally preferred for practical inference.

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37. **Distinguish between point and interval estimation**

- **Point estimation:** Produces a single best guess $\hat{\theta}$ for parameter θ. Easy but gives no measure of uncertainty.
- **Interval estimation:** Provides a range (L, U) with a confidence level (e.g., 95%) that quantifies uncertainty. More informative for decision making.

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38. **Explain null hypothesis and critical region**

- **Null hypothesis (H_0):** the statement we assume true for the purpose of testing (e.g., $H_0 : \mu = \mu_0$).
- **Critical region (rejection region):** Set of sample outcomes where we reject H_0. It is chosen so that the Type I error probability $P(\text{reject } H_0 \mid H_0)$ equals the significance level α. If the test statistic falls into the critical region, we reject H_0.

39. **Null and alternative hypothesis**

- **Null hypothesis (H_0):** baseline claim (no effect, equality).
- **Alternative hypothesis (H_1 or H_a):** claim we want evidence for (inequality, difference, effect). Can be one-sided ($>$ or $<$) or two-sided ($\neq$). The test's design (critical region) depends on whether H_1 is one- or two-sided.

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40. **Testing of equality of variances**

To test $H_0 : \sigma_1^2 = \sigma_2^2$ vs alternatives, use the **F-test** statistic

$$F = \frac{S_1^2}{S_2^2},$$

where S_1^2, S_2^2 are sample variances (typically put the larger variance in numerator depending on the alternative). Under normality and H_0, $F \sim F_{n_1-1,\, n_2-1}$. Compare observed F to critical values from the F-distribution for chosen α.

(If normality is questionable, consider Levene's test or the Brown–Forsythe test as robust alternatives.)

41. **Write the properties of F-test (F-distribution properties used in F-test)**

- **Support:** $F \geq 0$ (nonnegative).
- **Parameters:** Two degrees of freedom (d_1, d_2) corresponding to numerator and denominator.
- **Right-skewed:** Especially for small df; becomes more symmetric as df increase.
- **Relation to chi-square:** If $U \sim \chi^2_{d_1}$ and $V \sim \chi^2_{d_2}$ independent, then $F = (U/d_1)/(V/d_2) \sim F_{d_1,d_2}$.
- **Used for comparing variances and in ANOVA:** Ratio structure makes it natural for testing equality of variances or comparing mean models (ANOVA).
- **Non-symmetric critical regions:** Typically one-sided (right-tail) tests for variance ratios >1; two-tailed tests require reciprocal consideration or placing larger variance in numerator.

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42. **What is meant by Queue / Service discipline**

Queue discipline (service discipline) is the rule that determines the order in which waiting customers/jobs are served. Common disciplines:

- **FCFS (First-Come, First-Served)** — standard queue order.
- **LCFS (Last-Come, First-Served)** — newest served first (stack behavior).
- **Priority:** some customers served before others based on priority classes.
- **Random order / Processor sharing:** service capacity shared equally.

43. Write any two characteristics of queueing models

Two key characteristics are:

1. **Arrival process:** Describes how customers/jobs arrive (e.g., Poisson arrivals with rate λ).
2. **Service mechanism:** Includes service time distribution (e.g., exponential with mean $1/\mu$), number of servers, and the queue discipline (order of service).

(Other characteristics: system capacity, source population size, and performance measures such as average waiting time, queue length, and utilization.)

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44. Mean inter-arrival time

The **mean inter-arrival time** is the expected time between two successive arrivals.
If arrivals follow a Poisson process with average rate λ arrivals per unit time, then

$$\text{Mean inter-arrival time} = \frac{1}{\lambda}.$$

Example: If $\lambda = 4$ arrivals per hour, mean inter-arrival time = $1/4$ hour = **15 minutes.**

LONG ANSWER TYPE

Long Answer Type Questions :

1. *Consider the probability density functions*

   $f(x) = 0.5(x+1) \quad -1 \leq x \leq 1$

   $= 0 \quad$ *elsewhere* *Calculate mean and median.*

2. *Given the joint density* $f(x, y) = x + y \quad 0 < x < 1, 0 < y < 1$

   $= 0 \quad$ *elsewhere*

   Find the conditional density of Y given X = x.

3. *Prove that if two discrete random variables X and Y have the same PGFs, then they must have the same distributions and PMF's.*

4. *Define (i) joint distribution (ii) marginal distribution (iii) conditional distribution . Let the random variables X and Y have the joint PDF given by*

   $f(x, y) = e^{-(x+y)} \quad x,y \geq 0$

   $0 \quad$ *otherwise*

   Find the marginal distribution of X. Are the random variables independent.

5. *Define PDF of a discrete random variable and derive the expressions for its expectation and variance in terms of PDF.*

6. *If X and Y are discrete random variables and K is a constant. Prove that*

   (i) $E(X + K) = E(X) + K$

   (ii) $E(X + Y) = E(X) + E(Y)$

7. *Let X denote the number of heads in a single toss of 4 fair coins. Determine*

   (i) $P(X < 2)$ (ii) $P(1 < X \leq 3)$

Probability Distributions

8. *Derive mean and variance of Poisson Distribution*
9. *Derive Moment Generating Function of Exponential Distribution*
10. *If X has an exponential distribution with mean 2, find* $P[X < 1 / X < 2]$
11. *If X follows an exponential distribution with parameter 1, find the distribution of* $Y = X/(1+X)$
12. *Define Binomial Distribution and derive its Mean and Variance*
13. *Define Exponential Distribution , State and prove the memory loss property of the Exponential Distribution*
14. *Derive normal Distribution as limiting case of Binomial Distribution, clearly stating the conditions involved.*
15. *The marks obtained in mathematics by 1000 students is normally distributed with mean 78% and standard deviation 11% Determine : (i) How many students got marks above 90% (ii) What was the highest mark obtained by the lowest 10% of the students? (iii) within what limits did the middle of 90% of the students lie?*

.... *Multivariate Analysis*

16. *Calculate the coefficient of rank correlation.*

X	85	60	73	40	90
Y	93	75	65	50	80

17. *Describe the chi-square test of independence of two attributes*
18. *Calculate Karl Pearson's coefficient of correlation from the following data and*

interpret its value.

Roll.no. of students	*1*	*2*	*3*	*4*	*5*
Marks in accountancy	*48*	*35*	*17*	*23*	*47*
Marks in math	*45*	*20*	*40*	*25*	*45*

19. A short note on Test of correlation coefficient

20. Define Spearman's correlation rank coefficient. A consumer panel tests nine bands of microwave ovens for overall quality. the ranks assigned by the panel and the suggested retail prices are as follows :

Manufacturer	*Panel rating*	*Suggested price $*
A	*6*	*480*
B	*9*	*395*
C	*2*	*575*
D	*8*	*550*
E	*5*	*510*
F	*1*	*545*
G	*7*	*400*
H	*4*	*465*
I	*3*	*420*

is there a significant relationship between the quality and the price of a microwave oven? Use a level of 0.05% level of significance.

21. 10 observations on price x and supply y the following data was obtained. ∑ x = 130 , ∑ y = 220 , ∑ x^2 = 2288 , ∑ y^2 = 5506 and ∑ xy = 3467 find (i) Coefficient of correlation (ii) the regression line of y on x

Estimation ………………………………………………………… Testing of Hypothesis & Sample Tests

22. Write down the general procedure for testing of Hypothesis

23. In a sample of 1000 people of Maharashtra, 540 are rice eaters and the rest are wheat eaters. Can we assume that both rice and wheat are equally popular in this state at 1% level of significance ?

24. The height of 10 males of a given locality are found to be 70, 67, 62, 68, 61, 68, 70, 64, 64, 66 inches. Is it reasonable to believe that the average height is greater than 64 inches at 5% level of significance ?

25. Define the sampling distribution and explain its importance in statistical inference

26. Explain the small sample tests for the equality of two means , starting the assumptions if any.

27. *Write about null hypothesis and testing of null hypothesis with suitable examples*
28. *In a random sample of 60 workers the average time taken by them to get to work is 33.8 minutes with a std. deviation of 6.1 minutes, can we reject the null hypothesis μ = 32.6 minutes in favors of alternate hypothesis μ > 32.6 at α=0.025 level of significance.*
29. *Describe the F-test for the equality of variances of two populations*
30. *A group of boys and girls were given an intelligence test. The mean score, S.D's and number in each group are as follows :*

	Boys	*Girls*
Mean	*124*	*121*
S.D	*12*	*10*
Sample size	*18*	*14*

Is the mean score of boys significant different from that of girls ?

Queuing Theory

31. *A toll gate is operated on a freeway where cars arrive according to a Poisson distribution with same frequency of 1.2 cars per minute. The time of completing payment follows an exponential distribution with mean of 20 seconds. Find (i) The idle time of the counter (ii) Average no. of cars in the system (iii) Average no. of cars in the queue (iv) Average time that a car spends in the system (v) Average time that a car spends in the queue*
32. *a brief note on M/M/C model*
33. *Describe kendall's notation for queuing systems . what is a queue discipline and explain the various types of queue discipline with examples.*
34. *Explain the M/M/C : ∞/FIFO queuing model and derive the following expressions. (i) Average no. of customers in the queue (ii) Average waiting time of customers in the queue*
35. *In a tool crib manned by a single assistant, operators arrive at the tool crib at a rate of 10 per hour. Each operator needs 3 minutes on an average to be served. Find out loss of production due to time lost in waiting for an operator in shift of 8 hours if the rate of production is 100 units per shift.*
36. *Explain the operating characteristics of queue system in detail*

1) For the density $f(x) = \frac{1}{2}(x+1)$ on $[-1,1]$ (and 0 elsewhere)

Mean

$$\mu = \int_{-1}^{1} x\, f(x)\, dx = \tfrac{1}{2}\int_{-1}^{1} x(x+1)\, dx = \tfrac{1}{2}\int_{-1}^{1} (x^2+x)\, dx.$$

Compute integrals:
$$\int_{-1}^{1} x^2\, dx = \frac{2}{3}, \quad \int_{-1}^{1} x\, dx = 0.$$
Hence

$$\mu = \tfrac{1}{2}\cdot\frac{2}{3} = \boxed{\tfrac{1}{3}}.$$

Median m — solve $F(m) = 1/2$ where $F(x) = \int_{-1}^{x} \frac{1}{2}(t+1)\, dt$ for $-1 \le x \le 1$.

Compute CDF on $[-1,1]$:

$$F(x) = \tfrac{1}{2}\Big(\frac{x^2}{2} + x - \big(\frac{1}{2} - 1\big)\Big) = \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4}.$$

Set $F(m) = 1/2$:

$$\frac{m^2}{4} + \frac{m}{2} + \frac{1}{4} = \frac{1}{2} \implies m^2 + 2m - 1 = 0.$$

Solve:

$$m = -1 \pm \sqrt{2}.$$

Only the root in $[-1,1]$ is $m = -1+\sqrt{2}$.
So median $\boxed{m = -1+\sqrt{2} \approx 0.4142.}$

2) Joint density $f(x,y) = x+y$ on $0<x<1,\ 0<y<1$. Find conditional density of Y given $X = x$.

First the marginal of X:

$$f_X(x) = \int_0^1 (x+y)\, dy = x + \frac{1}{2}, \qquad 0<x<1.$$

Then the conditional density

$$f_{Y\mid X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}}, \qquad 0<y<1,$$

and $f_{Y\mid X}(y \mid x) = 0$ elsewhere. (Check: for fixed $x \in (0,1)$, $\int_0^1 \frac{x+y}{x+1/2}\, dy = 1$.)

---

3) If two discrete random variables have the same PGFs, they have the same distribution/PMF — proof sketch

Let X be integer-valued with PMF $p_k = P(X=k)$. Its probability generating function (PGF) is

$$G_X(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} p_k s^k, \qquad |s|<1.$$

If two discrete r.v.'s X and Y have PGFs $G_X(s) = G_Y(s)$ for all s in an open interval around 0, then the two power series are identical. By uniqueness of power series expansions (coefficients of equal power series are equal), we must have $p_k^{(X)} = p_k^{(Y)}$ for every k. Hence the PMFs (and therefore the distributions) coincide.

4) Definitions + given joint PDF $f(x,y) = e^{-(x+y)}$ for $x, y \ge 0$

Definitions (brief):

- **Joint distribution / joint PDF:** $f_{X,Y}(x,y)$ gives density for pair (X,Y); $P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$.
- **Marginal distribution / marginal PDF:** $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$ and similarly for $f_Y(y)$.
- **Conditional distribution / conditional PDF:** $f_{Y|X}(y \mid x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$ when $f_X(x) > 0$.

Marginal of X:

$$f_X(x) = \int_0^{\infty} e^{-(x+y)}\,dy = e^{-x}\int_0^{\infty} e^{-y}\,dy = e^{-x}, \qquad x \ge 0.$$

So $f_X(x) = e^{-x}$ $(x \ge 0)$ (an $\mathrm{Exp}(1)$ PDF). Similarly $f_Y(y) = e^{-y}$ $(y \ge 0)$.

Independence check: Since $f_{X,Y}(x,y) = e^{-(x+y)} = e^{-x}e^{-y} = f_X(x)f_Y(y)$ for all $x, y \ge 0$, X and Y are independent.

5) PDF of a *discrete* random variable (more correctly: PMF) and formulas for expectation & variance

PMF (probability mass function): For a discrete random variable X taking values in a countable set $\mathcal{X}$,

$$p_X(x) = P(X = x), \qquad x \in \mathcal{X},$$

with $\sum_{x\in\mathcal{X}} p_X(x) = 1$ and $p_X(x) \ge 0$.

Expectation (mean):

$$\mathbb{E}[X] = \sum_{x\in\mathcal{X}} x\,p_X(x),$$

provided the sum converges absolutely.

Second moment and variance:

$$\mathbb{E}[X^2] = \sum_{x\in\mathcal{X}} x^2 p_X(x),$$

$$\mathrm{Var}(X) = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big] = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2.$$

(Those formulas follow by linearity of expectation and the definition of variance.)

6. Linearity of Expectation

Let X, Y be discrete random variables, and K a constant.

(i) $E(X+K)=E(X)+K$

By definition of expectation:

$$E(X+K)=\sum_x (x+K)\,P(X=x).$$

Split the sum:

$$E(X+K)=\sum_x xP(X=x)+\sum_x KP(X=x).$$

$\sum_x xP(X=x)=E(X)$, and $\sum_x P(X=x)=1$. So:

$$E(X+K)=E(X)+K.$$

(ii) $E(X+Y)=E(X)+E(Y)$

By definition:

$$E(X+Y)=\sum_{x,y}(x+y)P(X=x,Y=y).$$

Split:

$$E(X+Y)=\sum_{x,y}xP(X=x,Y=y)+\sum_{x,y}yP(X=x,Y=y).$$

First term:

$$\sum_{x,y}xP(X=x,Y=y)=\sum_x x\Big(\sum_y P(X=x,Y=y)\Big)=\sum_x xP(X=x)=E(X).$$

Second term similarly gives $E(Y)$.

So:

$$E(X+Y)=E(X)+E(Y).$$

✅ Proved.

7. Tossing 4 fair coins

Let X = number of heads in 4 fair coin tosses.
Then $X \sim \mathrm{Binomial}(n = 4, p = \frac{1}{2})$.

PMF:

$$P(X = k) = \binom{4}{k} \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{2}\right)^{4-k} = \frac{\binom{4}{k}}{16}, \quad k = 0, 1, 2, 3, 4.$$

(i) $P(X < 2)$

That means $X = 0$ or $X = 1$.

$$P(X = 0) = \frac{\binom{4}{0}}{16} = \frac{1}{16},$$
$$P(X = 1) = \frac{\binom{4}{1}}{16} = \frac{4}{16}.$$

So:

$$P(X < 2) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}.$$

(ii) $P(1 < X \leq 3)$

That means $X = 2$ or $X = 3$.

$$P(X = 2) = \frac{\binom{4}{2}}{16} = \frac{6}{16},$$
$$P(X = 3) = \frac{\binom{4}{3}}{16} = \frac{4}{16}.$$

So:

$$P(1 < X \leq 3) = \frac{6}{16} + \frac{4}{16} = \frac{10}{16} = \frac{5}{8}.$$

8. Mean and variance of the Poisson(λ) distribution

If $X \sim \text{Poisson}(\lambda)$, the PMF is

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \dots$$

Use the probability-generating function or MGF. The MGF is

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = \exp\big(\lambda(e^t - 1)\big).$$

Differentiate $M_X(t)$ and evaluate at $t = 0$:

- Mean:

$$\mathbb{E}[X] = M_X'(0) = \lambda e^0 = \lambda.$$

- Second moment:

$$M_X''(t) = \lambda e^t\big(\lambda e^t + 1\big) \quad \Rightarrow \quad \mathbb{E}[X^2] = M_X''(0) = \lambda(\lambda + 1).$$

- Variance:

$$\text{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2 = \lambda(\lambda + 1) - \lambda^2 = \lambda.$$

Result: $\mathbb{E}[X] = \text{Var}(X) = \lambda.$

9. Moment generating function (MGF) of Exponential(λ)

If $X \sim \text{Exp}(\lambda)$ with PDF $f_X(x) = \lambda e^{-\lambda x},\ x \geq 0$, then for $t < \lambda$,

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx = \lambda \int_0^{\infty} e^{-(\lambda - t)x}\, dx = \lambda \cdot \frac{1}{\lambda - t} = \frac{\lambda}{\lambda - t},$$

valid for $t < \lambda$ (integral diverges for $t \geq \lambda$).

---

10. If $X \sim \text{Exp}$ with mean 2, find $P[X < 1 \mid X < 2]$.

Mean $= 2 \Rightarrow$ rate parameter $\lambda = 1/2$. For exponential distribution the CDF is $F(x) = 1 - e^{-\lambda x}$.

Conditional probability:

$$P[X < 1 \mid X < 2] = \frac{P(X < 1 \text{ and } X < 2)}{P(X < 2)} = \frac{P(X < 1)}{P(X < 2)} = \frac{1 - e^{-\lambda \cdot 1}}{1 - e^{-\lambda \cdot 2}}.$$

With $\lambda = \frac{1}{2}$,

$$P = \frac{1 - e^{-1/2}}{1 - e^{-1}}.$$

11. If $X \sim \mathrm{Exp}(1)$, distribution of $Y = \dfrac{X}{1+X}$

Mapping: $Y = g(X) = \dfrac{X}{1+X}$. For $X \geq 0$, $Y \in [0,1)$. Invert:

$$X = \frac{y}{1-y}, \qquad 0 \leq y < 1,$$

and Jacobian

$$\frac{dX}{dy} = \frac{1}{(1-y)^2}.$$

PDF transformation:

$$f_Y(y) = f_X\big(x(y)\big)\left|\frac{dx}{dy}\right| = e^{-x} \cdot \frac{1}{(1-y)^2} \quad \text{with } x = \frac{y}{1-y}.$$

Thus, for $0 < y < 1$,

$$\boxed{f_Y(y) = \frac{1}{(1-y)^2}\exp\Big(-\frac{y}{1-y}\Big)}$$

and $f_Y(y) = 0$ elsewhere.

(One can check $\int_0^1 f_Y(y)\,dy = 1$ by substitution.)

12. Binomial distribution — definition and mean & variance

Definition: $X \sim \mathrm{Bin}(n,p)$ if X counts successes in n independent Bernoulli trials each with success probability p. PMF:

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

Derivation of mean and variance (standard):

Write $X = \sum_{i=1}^n I_i$ where I_i are iid Bernoulli(p). Then

$$\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[I_i] = np,$$

and because the I_i are independent,

$$\mathrm{Var}(X) = \sum_{i=1}^n \mathrm{Var}(I_i) = n \cdot p(1-p) = np(1-p).$$

13. Exponential distribution: definition and memoryless property (proof)

Definition. $X \sim \mathrm{Exp}(\lambda)$ has PDF $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. The survival function is $S(x) = P(X > x) = e^{-\lambda x}$.

Memoryless (memory loss) property. For $s, t \geq 0$,

$$P(X > s + t \mid X > s) = \frac{P(X > s + t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t).$$

Thus the conditional distribution of the remaining life given survival to time s is the same as the original distribution — the exponential is the unique continuous distribution with this property (discrete analogue: geometric).

14. Normal distribution as limiting case of Binomial (De Moivre–Laplace / CLT statement)

Let $X_n \sim \mathrm{Bin}(n, p)$ with mean $\mu_n = np$ and variance $\sigma_n^2 = np(1-p) = npq$. Define the standardized variable

$$Z_n = \frac{X_n - np}{\sqrt{npq}}.$$

Central Limit Theorem / De Moivre–Laplace: as $n \to \infty$,

$$Z_n \xrightarrow{d} N(0, 1),$$

i.e. for any fixed $a < b$,

$$P\big(a \leq Z_n \leq b\big) \to \Phi(b) - \Phi(a),$$

where Φ is the standard normal CDF.

Conditions: $n \to \infty$ with fixed $0 < p < 1$ (equivalently both np and $n(1-p)$ →∞). A more precise local form uses Stirling's formula to show

$$P(X_n = k) \approx \frac{1}{\sqrt{2\pi npq}} \exp\Big(-\frac{(k-np)^2}{2npq}\Big)$$

for k near np.

15. Normal computations — marks of 1000 students

Given: scores ~ Normal with mean $\mu = 78\%$ and standard deviation $\sigma = 11\%$. Total students = 1000.

Let Z be standard normal.

(i) How many students got marks above 90%?

Compute

$$z = \frac{90 - 78}{11} = \frac{12}{11} \approx 1.0909.$$

Probability a student scores $> 90\%$ is $P(Z > 1.0909) \approx 0.137656$. So out of 1000 students,

$$\text{approx } 1000 \times 0.137656 \approx \boxed{138 \text{ students (approximately).}}$$

(ii) Highest mark obtained by the lowest 10% of students (10th percentile)

10th percentile $z_{0.10} \approx -1.28155$. Convert to raw score:

$$x_{0.10} = \mu + z_{0.10}\sigma = 78 + (-1.28155)(11) \approx 63.90\%.$$

So the highest mark among the lowest 10% is about $\boxed{63.9\%}$.

(iii) Limits that contain the middle 90% of students

Middle 90% corresponds to 5th and 95th percentiles. $z_{0.05} \approx -1.64485$, $z_{0.95} \approx +1.64485$. Convert:

$$\text{lower} = 78 + (-1.64485)(11) \approx 59.91\%,$$

$$\text{upper} = 78 + 1.64485(11) \approx 96.09\%.$$

So the middle 90% lie approximately between $\boxed{59.9\% \text{ and } 96.1\%}$.

16. Coefficient of Rank Correlation (Spearman's ρ)

We have:

$$X : 85,\ 60,\ 73,\ 40,\ 90$$

$$Y : 93,\ 75,\ 65,\ 50,\ 80$$

Step 1. Rank the data (highest = rank 1 or lowest = rank 1, both valid as long as consistent). Let's use highest = rank 1.

- For X:
  90 (1), 85 (2), 73 (3), 60 (4), 40 (5).
- For Y:
  93 (1), 80 (2), 75 (3), 65 (4), 50 (5).

Step 2. Compute differences in ranks (d) and d^2.

X	RankX	Y	RankY	d = RankX − RankY	d²
85	2	93	1	1	1
60	4	75	3	1	1
73	3	65	4	-1	1
40	5	50	5	0	0
90	1	80	2	-1	1

$\sum d^2 = 4.$

Step 3. Formula:

$$\rho = 1 - \frac{6\sum d^2}{n(n^2-1)} = 1 - \frac{6(4)}{5(25-1)} = 1 - \frac{24}{120} = 0.80$$

✅ **Answer:** Spearman's rank correlation = $\mathbf{0.80}$ (strong positive).

17. Chi-square test of independence of two attributes (Short Note)

The **chi-square test of independence** is used to determine whether two categorical attributes (variables) are independent.

- **Null hypothesis H_0:** The two attributes are independent.
- **Alternative hypothesis H_1:** The two attributes are not independent.
- **Test statistic:**

$$\chi^2 = \sum \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

where O_{ij} = observed frequency, $E_{ij} = \frac{\text{row total} \times \text{column total}}{\text{grand total}}$.

- **Degrees of freedom:** $(r-1)(c-1)$, where r = no. of rows, c = no. of columns.
- **Decision:** If $\chi^2 > \chi^2_{\alpha,(r-1)(c-1)}$, reject H_0.

Conclusion: It helps decide whether two categorical factors are associated or independent.

18. Karl Pearson's correlation coefficient

Data:

$$X : 48, 35, 17, 23, 47 \quad Y : 45, 20, 40, 25, 45$$

Step 1. Compute means.
$\bar{X} = \frac{48+35+17+23+47}{5} = 34.$
$\bar{Y} = \frac{45+20+40+25+45}{5} = 35.$

Step 2. Compute deviations and products.

X	Y	x = X-34	y = Y-35	x²	y²	xy
48	45	14	10	196	100	140
35	20	1	-15	1	225	-15
17	40	-17	5	289	25	-85
23	25	-11	-10	121	100	110
47	45	13	10	169	100	130

Totals:
$\sum x^2 = 776, \sum y^2 = 550, \sum xy = 280.$

Step 3. Formula:

$$r = \frac{\sum xy}{\sqrt{\sum x^2 \cdot \sum y^2}} = \frac{280}{\sqrt{776 \cdot 550}} \approx \frac{280}{653.8} = 0.428$$

19. Short Note: Test of Correlation Coefficient

- Used to test whether a sample correlation coefficient r significantly differs from zero (i.e., if there is a linear relationship between two variables in the population).
- **Null hypothesis** H_0: $\rho = 0$ (no correlation).
- **Test statistic:**

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}, \quad \text{df} = n - 2$$

- Compare t with critical value from Student's t table.
- If $|t| > t_{\alpha/2, n-2}$, reject H_0.

20. Spearman's correlation rank coefficient (with test)

We are given panel ranks (quality) and prices. Need to rank prices first.

Step 1. Arrange prices and assign ranks (lowest price = rank 1, highest = rank 9).

Manufacturer	Panel Rank (X)	Price	Price Rank (Y)
A	6	480	5
B	9	395	1
C	2	575	9
D	8	550	8
E	5	510	6
F	1	545	7
G	7	400	2
H	4	465	4
I	3	420	3

Step 2. Compute differences (d).

Panel Rank (X)	Price Rank (Y)	d = X−Y	d²
6	5	1	1
9	1	8	64
2	9	-7	49
8	8	0	0
5	6	-1	1
1	7	-6	36
7	2	5	25
4	4	0	0
3	3	0	0

$\sum d^2 = 176.$

Step 3. Compute ρ.

$$\rho = 1 - \frac{6 \cdot 176}{9(81-1)} = 1 - \frac{1056}{720} = -0.467$$

Step 4. Test significance at α=0.05.
Test statistic:

$$t = \frac{\rho\sqrt{n-2}}{\sqrt{1-\rho^2}} = \frac{-0.467 \cdot \sqrt{7}}{\sqrt{1-0.218}} \approx -1.37$$

Critical $t_{0.05,7} \approx 2.365$. Since $|t| < 2.365$, not significant.

✅ Conclusion: No significant correlation between price and panel rating.

21. Regression and Correlation from summary data

Given:

$$\sum x = 130, \quad \sum y = 220, \quad \sum x^2 = 2288, \quad \sum y^2 = 5506, \quad \sum xy = 3467, \quad n = 10$$

Step 1. Compute means.
$\bar{x} = 130/10 = 13, \quad \bar{y} = 220/10 = 22.$

Step 2. Compute S_{xx}, S_{yy}, S_{xy}.

$$S_{xx} = \sum x^2 - \frac{(\sum x)^2}{n} = 2288 - \frac{130^2}{10} = 2288 - 1690 = 598$$

$$S_{yy} = \sum y^2 - \frac{(\sum y)^2}{n} = 5506 - \frac{220^2}{10} = 5506 - 4840 = 666$$

$$S_{xy} = \sum xy - \frac{\sum x \sum y}{n} = 3467 - \frac{130 \cdot 220}{10} = 3467 - 2860 = 607$$

Step 3. Correlation coefficient.

$$r = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}} = \frac{607}{\sqrt{598 \cdot 666}}$$

Denominator ≈ $\sqrt{398,268}$ ≈ 631.4.

$$r = 607/631.4 \approx 0.962$$

Step 4. Regression line of y on x.
Equation: $y - \bar{y} = b_{yx}(x - \bar{x})$,
where $b_{yx} = S_{xy}/S_{xx} = 607/598 \approx 1.015$.

So regression line:

$$y - 22 = 1.015(x - 13) \quad \Rightarrow \quad y \approx 1.015x + 8.80$$

22. General procedure for testing a hypothesis (step-by-step)

1. **State the hypotheses.**
   - Null hypothesis H_0: the baseline claim (e.g., parameter = a specific value).
   - Alternative H_1: what you want to test (one-sided or two-sided).
2. **Choose significance level** α (common choices 0.10, 0.05, 0.01).
3. **Decide the test statistic** appropriate to the data and hypotheses (e.g. z, t, χ^2, F, etc.).
4. **Derive the sampling distribution** of the test statistic under H_0 (or rely on asymptotics).
5. **Compute the observed value** of the test statistic from the sample.
6. **Find the critical region** (or compute p-value) corresponding to α.
7. **Decision rule:**
   - If the test statistic falls in the critical region (or p-value $\le \alpha$), **reject** H_0.
   - Otherwise **do not reject** H_0.
8. **State conclusion in context**, and mention the possibility of Type I/II errors.

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23. Test whether rice and wheat equally popular (n=1000, rice = 540) at $\alpha = 0.01$

- $n = 1000, \ \hat{p} = 540/1000 = 0.54.$
- $H_0 : p = 0.5$ (equal popularity) vs $H_1 : p \neq 0.5$ (two-sided).

Test statistic (large-sample z):

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{0.54 - 0.5}{\sqrt{0.25/1000}} = \frac{0.04}{0.015811} \approx 2.5298.$$

Critical value for two-tailed $\alpha = 0.01$: $z_{0.005} \approx \pm 2.5758.$

$|z| = 2.53 < 2.5758$ → **fail to reject** H_0 at 1% level.

Conclusion: At the 1% level, the sample does **not** provide sufficient evidence to say rice and wheat are *not* equally popular. We may continue to assume equal popularity (the difference is not significant at 1%, though it is close).

24. Test whether mean height > 64 inches (n=10) at $\alpha = 0.05$

Data: $70, 67, 62, 68, 61, 68, 70, 64, 64, 66.$

Compute sample mean and SD:

- $\bar{x} = 66.0$
- sample standard deviation $s \approx 3.1623$
- $n = 10$

Test: $H_0 : \mu = 64$ vs $H_1 : \mu > 64$ (one-sided). Use t-test (small sample, assume normality).

Test statistic:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{66 - 64}{3.1623/\sqrt{10}} = \frac{2}{3.1623/3.1623} = 2.000.$$

Degrees of freedom $= n - 1 = 9$. Critical $t_{0.05,9} \approx 1.833$ (one-sided).

Since $t = 2.00 > 1.833$, **reject** H_0.

Conclusion: At the 5% level it is reasonable to conclude the average height is **greater than 64 inches**.

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25. Sampling distribution — definition & importance

Definition: The sampling distribution of a statistic (e.g., $\bar{X}$, S^2, $\hat{p}$) is the probability distribution of that statistic across all possible samples of a fixed size from the population.

Importance in inference:

- It allows us to compute the variability (standard error) of the statistic.
- Basis for confidence intervals and hypothesis tests (we use the sampling distribution under H_0 to determine critical values and p-values).
- Central Limit Theorem gives approximate normal sampling distributions for many statistics when sample size is large, enabling approximate inference.

26. Small-sample tests for equality of two means — assumptions & procedures

Problem: Test $H_0 : \mu_1 = \mu_2$.

Common assumptions (small sample):

- Samples are independent.
- Each population is normally distributed.
- Either variances are equal (pooled test) or not (Welch's test).

(A) Pooled t-test (equal variances assumed):

- Estimate pooled variance:

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}.$$

- Test statistic:

$$t = \frac{\bar{X}_1 - \bar{X}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}, \qquad \mathrm{df} = n_1 + n_2 - 2.$$

- Use two-sided or one-sided critical values as per H_1.

(B) Welch's t-test (unequal variances):

- Test statistic:

$$t = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{S_1^2/n_1 + S_2^2/n_2}}.$$

- Degrees of freedom (Welch–Satterthwaite approx):

$$\nu \approx \frac{(S_1^2/n_1 + S_2^2/n_2)^2}{\frac{(S_1^2/n_1)^2}{n_1-1} + \frac{(S_2^2/n_2)^2}{n_2-1}}.$$

- Compare $|t|$ to $t_{\alpha/2,\nu}$.

Choice: If variances seem equal (or small samples with assumption reasonable), pooled test; else use Welch's test (more robust).

27. Null hypothesis & testing (short with example)

- **Null hypothesis** H_0: baseline statement assumed true until evidence suggests otherwise. Example: $H_0 : \mu = 10$.
- **Alternative** H_1: contradicts H_0 (e.g., $\mu > 10$, $\mu < 10$, or $\mu \neq 10$).
- **Testing process:** choose α, compute test statistic from sample, compare with critical value or compute p-value; conclude to reject or not reject H_0.
- **Example:** Testing whether a coin is fair: $H_0 : p = 0.5$ vs $H_1 : p \neq 0.5$. If a sample of flips has extreme enough proportion of heads, reject H_0.

Remember: "Fail to reject" H_0 is not proof H_0 is true; it indicates insufficient evidence against it.

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28. Test: mean commute time — $n = 60,\ \bar{x} = 33.8,\ s = 6.1$; test $H_0 : \mu = 32.6$ vs $H_1 : \mu > 32.6$, $\alpha = 0.025$

Compute:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{33.8 - 32.6}{6.1/\sqrt{60}} = \frac{1.2}{6.1/7.74597} \approx \frac{1.2}{0.7878} \approx 1.524.$$

Degrees of freedom $= 59$. One-sided critical $t_{0.025,59} \approx 1.671$.

Since $t = 1.524 < 1.671$, **fail to reject** H_0 at $\alpha = 0.025$.

Approximate one-sided p-value ≈ 0.064 (>0.025).

Conclusion: There is **not** strong enough evidence at the 2.5% level to conclude the true mean is greater than 32.6 minutes.

29. F-test for equality of variances (description)

Purpose: Test $H_0 : \sigma_1^2 = \sigma_2^2$ vs alternatives (two-sided or one-sided).

Assumptions:

- Both populations are normally distributed.
- Samples are independent.

Test statistic:

$$F = \frac{S_1^2}{S_2^2},$$

where by convention S_1^2 is the variance from the sample put in the numerator (often the larger sample variance to make two-tailed region easier).

Distribution under H_0: $F \sim F_{\nu_1,\nu_2}$ with $\nu_1 = n_1 - 1,\ \nu_2 = n_2 - 1$.

Decision:

- For two-sided α: reject H_0 if $F < F^{\text{lower}}_{\alpha/2,\nu_1,\nu_2}$ or $F > F_{1-\alpha/2,\nu_1,\nu_2}$.
- Alternatively, put larger variance in numerator and compare to upper critical value.

Note: The F-test is sensitive to departures from normality; use Levene's or Brown–Forsythe tests if normality is doubtful.

30. Are mean scores of boys and girls significantly different?

Given:

- Boys: $\bar{x}_1 = 124,\ s_1 = 12,\ n_1 = 18.$
- Girls: $\bar{x}_2 = 121,\ s_2 = 10,\ n_2 = 14.$

We test $H_0 : \mu_1 = \mu_2$ vs $H_1 : \mu_1 \neq \mu_2$ (two-sided). Use Welch's t (do not assume equal variances unless tested and justified).

Compute:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_1^2/n_1 + s_2^2/n_2}} = \frac{124 - 121}{\sqrt{144/18 + 100/14}} = \frac{3}{\sqrt{8 + 7.142857}} = \frac{3}{\sqrt{15.142857}} \approx 0.771.$$

Degrees of freedom (Welch approximation):

$$\nu \approx \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{(s_1^4/(n_1^2(n_1 - 1))) + (s_2^4/(n_2^2(n_2 - 1)))} \approx 29.8 \text{ (approx)}.$$

Two-sided critical $t_{0.025,\nu\approx 30} \approx 2.042.$

Since $|t| = 0.771 < 2.042$, fail to reject H_0.

Approximate two-sided p-value ≈ 0.45.

Conclusion: There is **no significant difference** between the mean scores of boys and girls at the 5% level.

31. M/M/1 queue — numerical answers

Given: arrivals Poisson with rate $\lambda = 1.2$ cars/min. Service times exponential with mean 20 seconds $= 20/60 = \frac{1}{3}$ min, so service rate $\mu = 3$ cars/min. This is an M/M/1 queue. Define traffic intensity

$$\rho = \frac{\lambda}{\mu} = \frac{1.2}{3} = 0.4.$$

Standard M/M/1 steady-state formulas (valid when $\rho < 1$):

$$P_0 = 1 - \rho, \qquad L = \frac{\rho}{1-\rho}, \qquad L_q = \frac{\rho^2}{1-\rho},$$
$$W = \frac{L}{\lambda}, \qquad W_q = \frac{L_q}{\lambda}.$$

Compute:

(i) **Idle time of the counter** = $P_0 = 1 - \rho = 1 - 0.4 = 0.6$.
→ The server is idle 60% of the time.

(ii) **Average number of cars in the system**

$$L = \frac{0.4}{1-0.4} = \frac{0.4}{0.6} = \frac{2}{3} \approx 0.6667 \text{ cars.}$$

(iii) **Average number of cars in the queue**

$$L_q = \frac{0.4^2}{1-0.4} = \frac{0.16}{0.6} \approx 0.2667 \text{ cars.}$$

(iv) **Average time a car spends in the system**

$$W = \frac{L}{\lambda} = \frac{2/3}{1.2} = \frac{0.6667}{1.2} \approx 0.5556 \text{ min} = 33.33 \text{ seconds.}$$

(v) **Average time a car spends waiting in queue**

$$W_q = \frac{L_q}{\lambda} = \frac{0.2667}{1.2} \approx 0.2222 \text{ min} = 13.33 \text{ seconds.}$$

(Quick check — service time mean is $1/\mu = 1/3$ min ≈ 20 s; $W = W_q + 1/\mu$ ⇒ 13.33 s + 20 s = 33.33 s,

32. Brief note on the M/M/c model

M/M/c denotes a multi-server Markovian queue with:

- **M**: Poisson arrivals (rate λ),
- **M**: exponential service times (rate μ per server),
- **c**: number of identical parallel servers,
- usually ∞ capacity and FCFS discipline by default.

Key quantities:

- offered load $a = \lambda/\mu$. Server utilization per server $\rho = a/c$. Stability requires $\rho < 1$.
- Probability of zero customers P_0:

$$P_0 = \left(\sum_{n=0}^{c-1} \frac{a^n}{n!} + \frac{a^c}{c!(1-\rho)} \right)^{-1}.$$

- Average number in queue (Erlang-C formula):

$$L_q = P_0 \frac{a^c}{c!} \frac{\rho}{(1-\rho)^2}.$$

- Average waiting time in queue:

$$W_q = \frac{L_q}{\lambda}, \qquad W = W_q + \frac{1}{\mu}.$$

Applications: call centres, server farms, bank tellers — anywhere several identical servers serve arriving jobs/customers.

33. Kendall's notation and queue disciplines

Kendall's notation: $A/B/c/K/m/D$ — short form usually written $A/B/c$, where:

- A: arrival process (e.g., M = Markov/Poisson, D = deterministic, G = general),
- B: service time distribution (M, D, G, ...),
- c: number of servers,
- optional K: system capacity (∞ if omitted), m: source population size, D: queue discipline (e.g., FIFO). Common shorthand: **M/M/1, M/M/c, M/G/1**, etc.

Queue discipline (service discipline): rule deciding order of service. Examples:

- **FCFS / FIFO (First-Come, First-Served):** customers served in arrival order. Common in banks, toll booths.
- **LCFS (Last-Come, First-Served):** newest arrival served first (stack). Useful in some preemptive systems.
- **Priority queueing:** customers have priority classes; higher priority served first (e.g., emergency patients triage).
- **Processor Sharing (PS):** service capacity shared among all present customers (fair-share), used as an approximation for time-sharing computer systems.
- **Random order service:** next customer chosen uniformly at random.
  Each discipline affects performance measures (waiting times by class, fairness).

34. M/M/c : ∞ / FIFO model — derivation of L_q and W_q

Model: Poisson arrivals λ, exponential service μ per server, c identical servers, infinite waiting room, FIFO.

Define $a = \lambda/\mu$ (offered load), $\rho = \dfrac{a}{c} = \dfrac{\lambda}{c\mu}$ (utilization per server). Assume $\rho < 1$.

Step 1 — P_0 (probability system empty):

$$P_0 = \left(\sum_{n=0}^{c-1} \frac{a^n}{n!} + \frac{a^c}{c!} \cdot \frac{1}{1-\rho} \right)^{-1}.$$

Step 2 — Erlang C and L_q: the probability that an arriving customer has to wait (Erlang C) is

$$P(\text{wait}) = \frac{a^c}{c!} \frac{1}{(1-\rho)} P_0.$$

Average number waiting in queue:

$$\boxed{L_q = P_0 \frac{a^c}{c!} \frac{\rho}{(1-\rho)^2}}.$$

This formula follows from birth–death steady-state probabilities and summing expected queue length for states $n \geq c$.

Step 3 — Waiting times: by Little's law for queue,

$$W_q = \frac{L_q}{\lambda}, \qquad W = W_q + \frac{1}{\mu}.$$

These are the standard results (Erlang-C formulas) used for capacity planning (choose c to achieve target $P(\text{wait})$ or W_q).

35. Tool crib problem — compute loss of production

Given: single assistant (M/M/1). Arrival rate $\lambda = 10$ per hour. Average service time per operator = 3 minutes = 0.05 hr ⇒ $\mu = 1/0.05 = 20$ per hour. Shift length = 8 hours. Production rate = 100 units/shift ⇒ $100/8 = 12.5$ units/hour.

Compute utilization: $\rho = \lambda/\mu = 10/20 = 0.5$.

For M/M/1:

$$L_q = \frac{\rho^2}{1-\rho} = \frac{0.5^2}{0.5} = 0.5 \text{ customers (on average in queue)}.$$

Average waiting time in queue per operator:

$$W_q = \frac{L_q}{\lambda} = \frac{0.5}{10} = 0.05 \text{ hour} = 3 \text{ minutes}.$$

Total arrivals in shift = $\lambda \times 8 = 80$ operators. Total time lost in waiting = $80 \times W_q = 80 \times 0.05 = 4$ hours (aggregate across all operators).

Production per hour = 12.5 units/hr. So production lost due to waiting-time = (lost hours)×(units/hr) = $4 \times 12.5 = 50$ units per shift.

Answer: 50 units of production lost per 8-hour shift due to operators' waiting time.

(Interpretation: you could also present loss as fraction of output = 50/100 = 50% of production — but that seems large because waiting aggregates across many operators; the per-operator average wait is 3 minutes, but many operators sum to 4 hours total waiting.)

36. Operating characteristics of a queueing system (detailed)

Operating (performance) characteristics typically used to evaluate queueing systems:

Input parameters

- Arrival rate λ, arrival process (Poisson, etc.).
- Service rate μ, service time distribution.
- Number of servers c.
- System capacity (finite K or infinite).
- Population source (finite or infinite).
- Service discipline (FCFS, priority, etc.).

Key performance measures (steady-state)

- **Utilization** ρ: fraction of time servers are busy. For single server $\rho = \lambda/\mu$.
- **Idle probability** P_0: probability system is empty.
- L: average number in system (queue + in service).
- L_q: average number waiting in queue (excluding those in service).
- W: average time in system (sojourn time).
- W_q: average waiting time in queue.
- **Throughput:** actual completion rate ($\approx\lambda$ in stable open systems).
- **Probability of delay (or blocking):** $P(\text{wait})$ (Erlang-C) or blocking probability for finite-capacity systems (Erlang-B).
- **Tail probabilities:** $P(N > n)$ probability of more than n customers in system.
- **Distributional characteristics:** e.g., exponential vs heavy-tailed service times affect variability, waiting time distribution shape.
- **Performance by class:** in priority systems, average delay per priority class.

Relations & laws

- **Little's Law** (universal, if steady-state): $L = \lambda W$ and $L_q = \lambda W_q$.
- For birth–death queues, closed-form steady-state probabilities often exist.

Design/operational considerations

- Trade-offs: capacity (servers) vs waiting cost. Adding servers reduces W_q and blocking but increases server cost.
- Variability impact: higher variance in arrivals/services increases delays even if mean rates unchanged.
- Choice of discipline: affects fairness and class-specific delays (priority helps high-priority jobs but may starve low priority).
- Robustness: some tests (e.g., F-test) are sensitive to assumptions; similarly queue metrics assume steady-state and certain distributions — check validity.

The above solved answers are just for reference only.

If it contain any error please contact allubalaji12@gmail.com /Rectify using online resources