GCT561 Scientific concept and thinking
Name: Chaelin Kim
Student ID: 20205096

HW2: The Monty Hall Problem

You're playing a game show. You are shown three closed doors, behind one of which sits your dream car. If you guess which door, you win the car. The game is played as follows. You choose one door, then the host (who knows where the car is) opens a door with nothing behind it.

(a) You choose door #2. Then the host opens #1, of course with nothing behind it. Prove that it makes sense at this point to change your choice to #3.

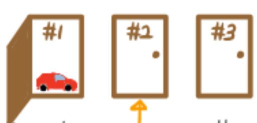
H_i : The car is behind the door #i.
D_1 : The host opens #1, there is nothing behind.

① $$P(H_2|D_1) = \frac{P(H_2 \cap D_1)}{P(D_1)} = \frac{P(D_1|H_2)P(H_2)}{\sum_{i=1}^{3} P(D_1|H_i)P(H_i)} \quad (\because \text{Bayes' theorem})$$

- $P(H_1) = P(H_2) = P(H_3) = \frac{1}{3}$
- $P(D_1|H_i)$

Assume that the participant choose the door #2

· $P(D_1|H_1)$



The host only can open the door #3 and cannot open the door #1

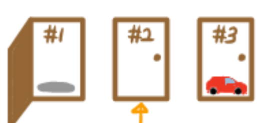
$\therefore P(D_1|H_1) = 0$

· $P(D_1|H_2)$

The host can open the door #1 or #3

$\therefore P(D_1|H_2) = \frac{1}{2}$

· $P(D_1|H_3)$



The host only can open the door #1,

$\therefore P(D_1|H_3) = 1$

$$\therefore P(H_2|D_1) = \frac{P(D_1|H_2)P(H_2)}{\sum_{i=1}^{3} P(D_1|H_i)P(H_i)} = \frac{\frac{1}{2}\cdot\frac{1}{3}}{0\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} + 1\cdot\frac{1}{3}} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

$$\boxed{P(H_2|D_1) = \frac{1}{3}}$$

② $$P(H_3|D_1) = \frac{P(D_1|H_3)P(H_3)}{\sum_{i=1}^{3} P(D_1|H_i)P(H_i)} = \frac{1\cdot\frac{1}{3}}{0\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} + 1\cdot\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

$$\boxed{P(H_3|D_1) = \frac{2}{3}}$$

$\Rightarrow \underbrace{P(H_3|D_1)}_{\frac{2}{3}} > \underbrace{P(H_2|D_1)}_{\frac{1}{3}}$

$\therefore$ the participant should change the choice to #3 to win the car.

(b) You choose door #2. Before the host has a chance to approach a door to open it, there is an earthquake which causes door #1 to open. When the door is opened by the earthquake, there is no car behind it. Prove that it does not matter whether you keep your choice of door #2.

H_i : The car is behind the door #i.

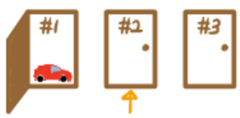
D_1' : The #1 is opened by the earthquake, there is nothing behind.

① $P(H_2|D_1') = \dfrac{P(D_1'|H_2)P(H_2)}{\sum_{i=1}^{3} P(D_1'|H_i)P(H_i)}$ (∵ Bayes' theorem)

- $P(H_1) = P(H_2) = P(H_3) = \frac{1}{3}$
- $P(D_1'|H_i)$

Assume that the participant choose the door #2

· $P(D_1'|H_1)$



The event D_1' means that there is nothing behind when the #1 is opened by the earthquake,

∴ $P(D_1'|H_1) = 0$

· $P(D_1'|H_2)$

When #1 is opened by the earthquake, there must be nothing behind #1

∴ $P(D_1'|H_2) = 1$

· $P(D_1'|H_3)$

Same as $P(D_1'|H_2)$, ∴ $P(D_1'|H_3) = 1$

$$∴ P(H_2|D_1') = \frac{P(D_1'|H_2)P(H_2)}{\sum_{i=1}^{3} P(D_1'|H_i)P(H_i)} = \frac{1 \cdot \frac{1}{3}}{0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$\boxed{P(H_2|D_1') = \frac{1}{2}}$

② $$P(H_3|D_1') = \frac{P(D_1'|H_3)P(H_3)}{\sum_{i=1}^{3} P(D_1'|H_i)P(H_i)} = \frac{1 \cdot \frac{1}{3}}{0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$\boxed{P(H_3|D_1') = \frac{1}{2}}$

$\Rightarrow \underbrace{P(H_3|D_1')}_{\frac{1}{2}} = \underbrace{P(H_2|D_1')}_{\frac{1}{2}}$

∴ It does no matter whether the participant keep the choice of door #2