GCT561 Scientific concept and thinking
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HW3: Majority Rule

1. Consider the symmetric channel.

(a) In class we saw that, when P(s=0) = P(s=1) = 1/2. The simple majority rule is the best option when the encoding process involved duplicating each bit three times.

- **Calculate the resulting error probability.**

⇒ Error rate

① 2 bits flipped : ${}_3C_2 f^2(1-f) = 3f^2(1-f)$

e.g.) 000 → 011 : $(1-f)f^2$; 101 : $f(1-f)f$; 110 : $f^2(1-f)$ → $3f^2(1-f)$

② 3 bits flipped : ${}_3C_3 f^3(1-f)^0 = f^3$

→ ① + ② $= 3f^2(1-f) + f^3$

$= 3f^2 - 2f^3$

→ $f = 0.1$,

$P_{err} = 3\cdot(0.1)^2 - 2(0.1)^3$

$= 0.03 - 0.002$

$= 0.028$

- **Calculate the resulting error when each bit is duplicated five times.**

⇒ Error rate

① 3 bits flipped : ${}_5C_3 f^3(1-f)^2 = 10f^3(1-f)^2 = 10f^3 - 20f^4 + 10f^5$

② 4 bits flipped : ${}_5C_4 f^4(1-f)^1 = 5f^4(1-f) = 5f^4 - 5f^5$

③ 5 bits flipped : ${}_5C_5 f^5(1-f)^0 = f^5$

→ ① + ② + ③ $= 10f^3 - 15f^4 + 6f^5$

→ $f = 0.1$,

$P_{err} = 10\cdot(0.1)^3 - 15(0.1)^4 + 6(0.1)^5$

$= 0.00856$

- **Can you generalize it when each bit is duplicated 2n+1 times (n is a positive integer)?**

n : # of 1 in received code r

N : length of the code r

⇒ Likelihood ratio : $\left(\frac{1-f}{f}\right)^{N-2n}$

⇒ Error rate : $\sum_{n=\frac{N+1}{2}}^{N} \binom{N}{n} f^n (1-f)^{N-n}$

e.g.) $f = 0.1$

$\sum_{n=\frac{N+1}{2}}^{N} \binom{N}{n} (0.1)^n (0.9)^{N-n}$

(b) **Assume that P(s=0) = 2/3 and P(s=1) = 1/3. With each bit duplicated three times in the encoding phase, what is the error rate when one takes the majority rule?**

$P(s=0)=\frac{2}{3},\ P(s=1)=\frac{1}{3}$

$$\begin{cases} P(r=000)=\frac{2}{3}(1-f)^3+\frac{1}{3}f^3 \\ P(r=001)=\frac{2}{3}f(1-f)^2+\frac{1}{3}f^2(1-f) \\ P(r=011)=\frac{2}{3}f^2(1-f)+\frac{1}{3}f(1-f)^2 \\ P(r=111)=\frac{2}{3}f^3+\frac{1}{3}(1-f)^3 \end{cases} \rightarrow \begin{array}{l} P(s=1|r=000)=\frac{P(r=000|s=1)P(s=1)}{P(r=000)} \\ P(s=1|r=001)=\frac{P(r=001|s=1)P(s=1)}{P(r=001)} \\ P(s=0|r=011)=\frac{P(r=011|s=1)P(s=1)}{P(r=011)} \\ P(s=0|r=111)=\frac{P(r=111|s=1)P(s=1)}{P(r=111)} \end{array}$$

$$\begin{aligned} P_{err} &= \sum_r P(r)P(error|r) \\ &= P(r=000)P(s=1|r=000)+P(r=001)P(s=1|r=001)+P(r=010)P(s=1|r=010)+P(r=100)P(s=1|r=100) \\ &\quad +P(r=011)P(s=0|r=011)+P(r=101)P(s=0|r=101)+P(r=110)P(s=0|r=110)+P(r=111)P(s=0|r=111) \\ &= \{P(r=000|s=1)+P(r=001|s=1)+P(r=010|s=1)+P(r=100|s=1)\}P(s=1) \\ &\quad +\{P(r=011|s=0)+P(r=101|s=0)+P(r=110|s=0)+P(r=111|s=0)\}P(s=0) \\ &= \frac{1}{3}(f^3+3\cdot f^2(1-f))+\frac{2}{3}(3\cdot f^2(1-f)+f^3) \\ &= f^3+3f^2(1-f) \\ &= 3f^2-2f^3 \end{aligned}$$

$\rightarrow f=0.1,$

$P_{err}=3\cdot(0.1)^2-2\cdot(0.1)^3=0.028$

(c) **Repeat the process for when P(s=0) = 99/100 and P(s=1) = 1/100, and each bit is duplicated three times. What is the error rate? Can you suggest a better method?**

$P(s=0)=\frac{99}{100},\ P(s=1)=\frac{1}{100}$

$$\begin{cases} P(r=000)=\frac{99}{100}(1-f)^3+\frac{1}{100}f^3 \\ P(r=001)=\frac{99}{100}f(1-f)^2+\frac{1}{100}f^2(1-f) \\ P(r=011)=\frac{99}{100}f^2(1-f)+\frac{1}{100}f(1-f)^2 \\ P(r=111)=\frac{99}{100}f^3+\frac{1}{100}(1-f)^3 \end{cases} \rightarrow \begin{array}{l} P(s=1|r=000)=\frac{P(r=000|s=1)P(s=1)}{P(r=000)} \\ P(s=1|r=001)=\frac{P(r=001|s=1)P(s=1)}{P(r=001)} \\ P(s=0|r=011)=\frac{P(r=011|s=1)P(s=1)}{P(r=011)} \\ P(s=0|r=111)=\frac{P(r=111|s=1)P(s=1)}{P(r=111)} \end{array}$$

$$\begin{aligned} P_{err} &= \sum_r P(r)P(error|r) \\ &= P(r=000)P(s=1|r=000)+P(r=001)P(s=1|r=001)+P(r=010)P(s=1|r=010)+P(r=100)P(s=1|r=100) \\ &\quad +P(r=011)P(s=0|r=011)+P(r=101)P(s=0|r=101)+P(r=110)P(s=0|r=110)+P(r=111)P(s=0|r=111) \\ &= \{P(r=000|s=1)+P(r=001|s=1)+P(r=010|s=1)+P(r=100|s=1)\}P(s=1) \\ &\quad +\{P(r=011|s=0)+P(r=101|s=0)+P(r=110|s=0)+P(r=111|s=0)\}P(s=0) \\ &= \frac{99}{100}(f^3+3\cdot f^2(1-f))+\frac{1}{100}(3\cdot f^2(1-f)+f^3) \\ &= f^3+3f^2(1-f) \\ &= 3f^2-2f^3 \end{aligned}$$

$\rightarrow f=0.1,\ P_{err}=0.028$

$\Rightarrow$ Same as the (b)'s answer

Whatever the values of $P(s=0)$ and $P(s=1)$, the error rate is not affected by those values

➔ Another method for data transmission

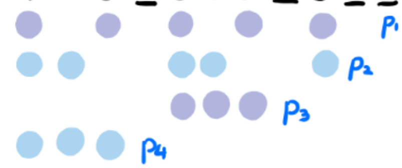
Using parity bit

- Number of parity bit
  $2^p + 1 \geq d + p$
  - p : # of parity bit
  - d : size of data
- Place of parity bit : 2^n
- Even parity bit
  : Making # of 1's in code even

e.g.) data = 1100100

→ $2^p + 1 \geq 7 + p$ ∴ p : 4 bit

11	10	9	8	7	6	5	4	3	2	1
1	1	0	_	0	1	0	_	0	_	_



p_1 : 10000 (1) → 1　　p_3 : 010 (1) → 1

p_2 : 11010 (3) → 1　　p_4 : 110 (2) → 0

∴ Hamming code ⇒ 11000101011

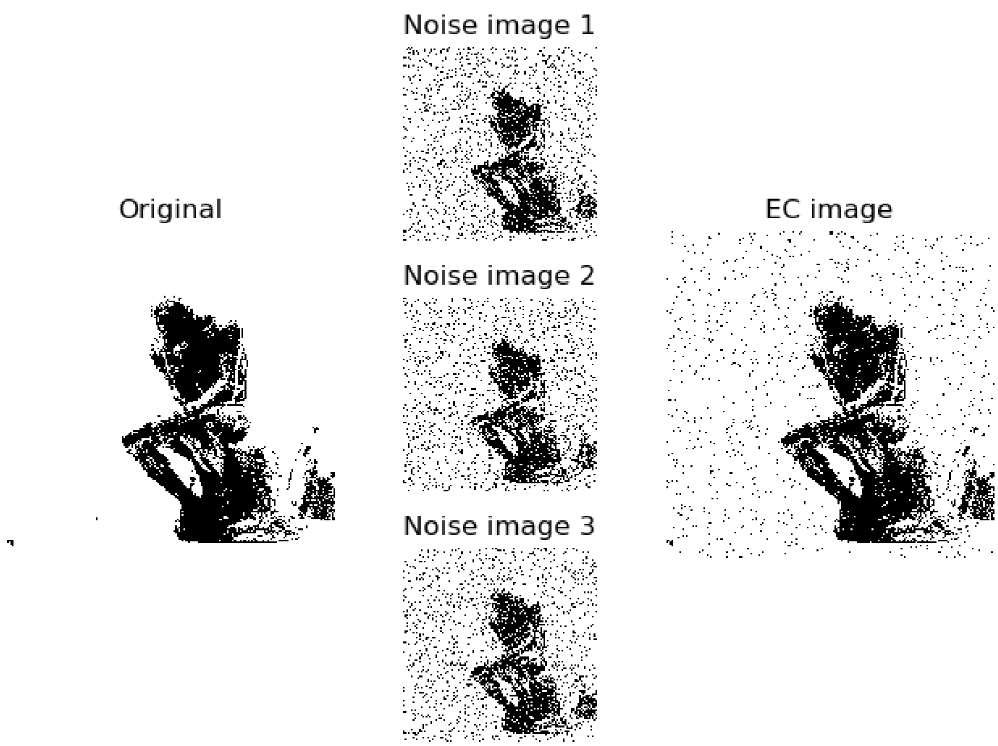
2. Attached is a b&w pixel art version of *Auguste Rodin's The Thinker*. The image is 200x200. We write black=0 and white=1.

(The code is attached – hw3_Q2_majorityrule.py)

(a) What are P(0) and P(1)?

```
### Original image ###
Counting pixels (black/white) : 6834 / 33166
P(0), P(1): [0.17085, 0.82915]
```

(b) Generate three images that have passed through a symmetric noisy channel with f=0.1.



(c) Using the simple majority rule, find the error-corrected image.

Refer to (b)'s result image (EC image)

(d) What is the error rate?

- In mathematics,

Error rate : $\sum_{n=\frac{3+1}{2}}^{3} \binom{3}{n} f^{n}(1-f)^{n} = 3f^{2}(1-f)+f^{3}$

$$= 3f^{2}-2f^{3}$$

$$= 3\cdot(0.1)^{2}-2\cdot(0.1)^{3}$$

$$= 0.028$$

- In example case

```
### Noise image 1 ###
Counting errors (correct/error) : 35973, 4027
P_correct, P_error: [0.899325, 0.100675]

### Noise image 2 ###
Counting errors (correct/error) : 36101, 3899
P_correct, P_error: [0.902525, 0.097475]

### Noise image 3 ###
Counting errors (correct/error) : 36035, 3965
P_correct, P_error: [0.900875, 0.099125]

### Error corrected image  ###
Counting errors (correct/error) : 38873, 1127
P_correct, P_error: [0.971825, 0.028175]
```