GCT561 Scientific concept and thinking
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HW3: Majority Rule

1. Consider the symmetric channel.

(a) In class we saw that, when P(s=0) = P(s=1) = 1/2. The simple majority rule is the best option when the encoding process involved duplicating each bit three times.

- **Calculate the resulting error probability.**

$P(s=0)=P(s=1)=\frac{1}{2}$

f : flipping error rate

· $r=000$

$$P(s=0|r=000)=\frac{P(s=0,\ r=000)}{P(r=000)}$$
$$=\frac{P(r=000|s=0)P(s=0)}{P(r=000)} \quad (\because \text{Bayes theorem})$$

$$P(s=1|r=000)=\frac{P(r=000|s=1)P(s=1)}{P(r=000)}$$

$$\frac{P(s=0|r=000)}{P(s=1|r=000)}=\frac{\dfrac{P(r=000|s=0)\cancel{P(s=0)}^{\frac{1}{2}}}{\cancel{P(r=000)}}}{\dfrac{P(r=000|s=1)\cancel{P(s=1)}^{\frac{1}{2}}}{\cancel{P(r=000)}}}$$
$$=\frac{P(r=000|s=0)}{P(r=000|s=1)}$$
$$=\frac{(1-f)^3}{f^3}$$

· $r=100$

$$\frac{P(s=0|r=100)}{P(s=1|r=100)}=\frac{P(r=100|s=0)}{P(r=100|s=1)}$$
$$=\frac{f(1-f)^2}{(1-f)f^2}=\frac{1-f}{f}$$

· $r=101$

$$\frac{P(s=0|r=101)}{P(s=1|r=101)}=\frac{P(r=101|s=0)}{P(r=101|s=1)}$$
$$=\frac{f(1-f)f}{(1-f)f(1-f)}=\frac{f}{1-f}$$

· $r=001$

$$\frac{P(s=0|r=001)}{P(s=1|r=001)}=\frac{P(r=001|s=0)}{P(r=001|s=1)}$$
$$=\frac{(1-f)^2 f}{f^2(1-f)}=\frac{1-f}{f}$$

· $r=010$

$$\frac{P(s=0|r=010)}{P(s=1|r=010)}=\frac{P(r=010|s=0)}{P(r=010|s=1)}$$
$$=\frac{(1-f)f(1-f)}{f(1-f)f}=\frac{1-f}{f}$$

· $r=011$

$$\frac{P(s=0|r=011)}{P(s=1|r=011)}=\frac{P(r=011|s=0)}{P(r=011|s=1)}$$
$$=\frac{(1-f)f^2}{f(1-f)^2}=\frac{f}{1-f}$$

· $r=110$

$$\frac{P(s=0|r=110)}{P(s=1|r=110)}=\frac{P(r=110|s=0)}{P(r=110|s=1)}$$
$$=\frac{f^2(1-f)}{(1-f)^2 f}=\frac{f}{1-f}$$

· $r=111$

$$\frac{P(s=0|r=111)}{P(s=1|r=111)}=\frac{P(r=111|s=0)}{P(r=111|s=1)}$$
$$=\frac{f^3}{(1-f)^3}$$

$\Rightarrow$

Received code r	Likelihood ratio
000	$\frac{(1-f)^3}{f^3}$
001	$\frac{1-f}{f}$
010	$\frac{1-f}{f}$
100	$\frac{1-f}{f}$
011	$\frac{f}{1-f}$
101	$\frac{f}{1-f}$
110	$\frac{f}{1-f}$
111	$\frac{f^3}{(1-f)^3}$

$\Rightarrow$ Error rate

① 2 bits flipped : ${}_3C_2 f^2(1-f)=3f^2(1-f)$

$$\text{e.g.) } 000 \rightarrow \left.\begin{array}{ll} 011 & : (1-f)f^2 \\ 101 & : f(1-f)f \\ 110 & : f^2(1-f) \end{array}\right\} 3f^2(1-f)$$

② 3 bits flipped : ${}_3C_3 f^3(1-f)^0=f^3$

$\rightarrow$ ① + ② $=3f^2(1-f)+f^3$

$=3f^2-2f^3$

- **Calculate the resulting error when each bit is duplicated five times.**

· r = 00000

$$\frac{P(s=0|r=00000)}{P(s=1|r=00000)} = \frac{P(r=00000|s=0)}{P(r=00000|s=1)} = \frac{(1-f)^5}{f^5}$$

· r = 00001

$$\frac{P(s=0|r=00001)}{P(s=1|r=00001)} = \frac{P(r=00001|s=0)}{P(r=00001|s=1)} = \frac{(1-f)^4 f}{f^4(1-f)} = \frac{(1-f)^3}{f^3}$$

· r = 00011

$$\frac{P(s=0|r=00011)}{P(s=1|r=00011)} = \frac{P(r=00011|s=0)}{P(r=00011|s=1)} = \frac{(1-f)^3 f^2}{f^3(1-f)^2} = \frac{1-f}{f}$$

· r = 00111

$$\frac{P(s=0|r=00111)}{P(s=1|r=00111)} = \frac{P(r=00111|s=0)}{P(r=00111|s=1)} = \frac{(1-f)^2 f^3}{f^2(1-f)^3} = \frac{f}{1-f}$$

· r = 01111

$$\frac{P(s=0|r=01111)}{P(s=1|r=01111)} = \frac{P(r=01111|s=0)}{P(r=01111|s=1)} = \frac{(1-f)^4 f}{f^4(1-f)} = \frac{(1-f)^3}{f^3}$$

· r = 11111

$$\frac{P(s=0|r=11111)}{P(s=1|r=11111)} = \frac{P(r=11111|s=0)}{P(r=11111|s=1)} = \frac{f^5}{(1-f)^5}$$

⇒

Received r	Likelihood ratio	Received r	Likelihood ratio	Received r	Likelihood ratio	Received r	Likelihood ratio
00000	$\frac{(1-f)^5}{f^5}$	01001	$\frac{1-f}{f}$	00111	$\frac{f}{1-f}$	11010	$\frac{f}{1-f}$
00001	$\frac{(1-f)^3}{f^3}$	10001	$\frac{1-f}{f}$	01011	$\frac{f}{1-f}$	11100	$\frac{f}{1-f}$
00010	$\frac{(1-f)^3}{f^3}$	00110	$\frac{1-f}{f}$	10011	$\frac{f}{1-f}$	01111	$\frac{f^3}{(1-f)^3}$
00100	$\frac{(1-f)^3}{f^3}$	01010	$\frac{1-f}{f}$	01101	$\frac{f}{1-f}$	10111	$\frac{f^3}{(1-f)^3}$
01000	$\frac{(1-f)^3}{f^3}$	10010	$\frac{1-f}{f}$	10101	$\frac{f}{1-f}$	11011	$\frac{f^3}{(1-f)^3}$
10000	$\frac{(1-f)^3}{f^3}$	01100	$\frac{1-f}{f}$	11001	$\frac{f}{1-f}$	11101	$\frac{f^3}{(1-f)^3}$
00011	$\frac{1-f}{f}$	10100	$\frac{1-f}{f}$	01110	$\frac{f}{1-f}$	11110	$\frac{f^3}{(1-f)^3}$
00101	$\frac{1-f}{f}$	11000	$\frac{1-f}{f}$	10110	$\frac{f}{1-f}$	11111	$\frac{f^5}{(1-f)^5}$

⇒ Error rate

① 3 bits flipped : ${}_5C_3\, f^3(1-f)^2 = 10f^3(1-f)^2 = 10f^3 - 20f^4 + 10f^5$

② 4 bits flipped : ${}_5C_4\, f^4(1-f)^1 = 5f^4(1-f) = 5f^4 - 5f^5$

③ 5 bits flipped : ${}_5C_5\, f^5(1-f)^0 = f^5$

→ ① + ② + ③ = $10f^3 - 15f^4 + 6f^5$

- **Can you generalize it when each bit is duplicated 2n+1 times (n is a positive integer)?**

n : # of 1 in received code r

N : length of the code r

⇒ Likelihood ratio : $\left(\frac{1-f}{f}\right)^{N-2n}$

⇒ Error rate : $\sum_{n=\frac{N+1}{2}}^{N} \binom{N}{n} f^n (1-f)^{N-n}$

(b) Assume that P(s=0) = 2/3 and P(s=1) = 1/3. With each bit duplicated three times in the encoding phase, what is the error rate when one takes the majority rule?

· $r=000$

$$\frac{P(s=0|r=000)}{P(s=1|r=000)}=\frac{\dfrac{P(r=000|s=0)P(s=0)}{\cancel{P(r=000)}}}{\dfrac{P(r=000|s=1)P(s=1)}{\cancel{P(r=000)}}}=\frac{P(r=000|s=0)\cdot 2/3}{P(r=000|s=1)\cdot 1/3}=\frac{2(1-f)^3}{f^3}$$

· $r=001$

$$\frac{P(s=0|r=001)}{P(s=1|r=001)}=\frac{P(r=001|s=0)P(s=0)}{P(r=001|s=1)P(s=1)}=\frac{2(1-f)^2 f}{f^2(1-f)}=\frac{2(1-f)}{f}$$

· $r=011$

$$\frac{P(s=0|r=011)}{P(s=1|r=011)}=\frac{P(r=011|s=0)P(s=0)}{P(r=011|s=1)P(s=1)}=\frac{2(1-f)f^2}{f(1-f)^2}=\frac{2f}{1-f}$$

· $r=111$

$$\frac{P(s=0|r=111)}{P(s=1|r=111)}=\frac{P(r=111|s=0)P(s=0)}{P(r=111|s=1)P(s=1)}=\frac{2f^3}{(1-f)^3}$$

Received code r	Likelihood ratio
000	$\frac{2(1-f)^3}{f^3}$
001	$\frac{2(1-f)}{f}$
010	$\frac{2(1-f)}{f}$
100	$\frac{2(1-f)}{f}$
011	$\frac{2f}{1-f}$
101	$\frac{2f}{1-f}$
110	$\frac{2f}{1-f}$
111	$\frac{2f^3}{(1-f)^3}$

$P(s=0)=\frac{2}{3},\ P(s=1)=\frac{1}{3}$

$$\begin{cases} P(r=000)=\frac{2}{3}(1-f)^3+\frac{1}{3}f^3 \\ P(r=001)=\frac{2}{3}f(1-f)^2+\frac{1}{3}f^2(1-f) \\ P(r=011)=\frac{2}{3}f^2(1-f)+\frac{1}{3}f(1-f)^2 \\ P(r=111)=\frac{2}{3}f^3+\frac{1}{3}(1-f)^3 \end{cases}$$

$\longrightarrow$

$$P(s=1|r=000)=\frac{P(r=000|s=1)P(s=1)}{P(r=000)}$$

$$P(s=1|r=001)=\frac{P(r=001|s=1)P(s=1)}{P(r=001)}$$

$$P(s=0|r=011)=\frac{P(r=011|s=1)P(s=1)}{P(r=011)}$$

$$P(s=0|r=111)=\frac{P(r=111|s=1)P(s=1)}{P(r=111)}$$

$$\begin{aligned}
P_{err} &= \sum_r P(r)P(error|r)\\
&= P(r=000)P(s=1|r=000)+P(r=001)P(s=1|r=001)+P(r=010)P(s=1|r=010)+P(r=100)P(s=1|r=100)\\
&\quad +P(r=011)P(s=0|r=011)+P(r=101)P(s=0|r=101)+P(r=110)P(s=0|r=110)+P(r=111)P(s=0|r=111)\\
&= \{P(r=000|s=1)+P(r=001|s=1)+P(r=010|s=1)+P(r=100|s=1)\}P(s=1)\\
&\quad +\{P(r=011|s=0)+P(r=101|s=0)+P(r=110|s=0)+P(r=111|s=0)\}P(s=0)\\
&= \frac{1}{3}(f^3+3\cdot f^2(1-f))+\frac{2}{3}(3\cdot f^2(1-f)+f^3)\\
&= f^3+3f^2(1-f)\\
&= 3f^2-2f^3
\end{aligned}$$

(c) Repeat the process for when P(s=0) = 99/100 and P(s=1) = 1/100, and each bit is duplicated three times. What is the error rate? Can you suggest a better method?

· r = 000

$$\frac{P(s=0|r=000)}{P(s=1|r=000)}=\frac{P(r=000|s=0)P(s=0)}{P(r=000|s=1)P(s=1)}=\frac{P(r=000|s=0)\cdot 99/100}{P(r=000|s=1)\cdot 1/100}=\frac{99(1-f)^3}{f^3}$$

⇒

Received code r	Likelihood ratio
000	$\frac{99(1-f)^3}{f^3}$
001	$\frac{99(1-f)}{f}$
010	$\frac{99(1-f)}{f}$
100	$\frac{99(1-f)}{f}$
011	$\frac{99f}{1-f}$
101	$\frac{99f}{1-f}$
110	$\frac{99f}{1-f}$
111	$\frac{99f^3}{(1-f)^3}$

$P(s=0)=\frac{99}{100}$, $P(s=1)=\frac{1}{100}$

$$P(r=000)=\frac{99}{100}(1-f)^3+\frac{1}{100}f^3$$
$$P(r=001)=\frac{99}{100}f(1-f)^2+\frac{1}{100}f^2(1-f)$$
$$P(r=011)=\frac{99}{100}f^2(1-f)+\frac{1}{100}f(1-f)^2$$
$$P(r=111)=\frac{99}{100}f^3+\frac{1}{100}(1-f)^3$$

$$\longrightarrow P(s=1|r=000)=\frac{P(r=000|s=1)P(s=1)}{P(r=000)}$$
$$P(s=1|r=001)=\frac{P(r=001|s=1)P(s=1)}{P(r=001)}$$
$$P(s=0|r=011)=\frac{P(r=011|s=1)P(s=1)}{P(r=011)}$$
$$P(s=0|r=111)=\frac{P(r=111|s=1)P(s=1)}{P(r=111)}$$

$$P_{err}=\sum_r P(r)P(error|r)$$
$$=P(r=000)P(s=1|r=000)+P(r=001)P(s=1|r=001)+P(r=010)P(s=1|r=010)+P(r=100)P(s=1|r=100)$$
$$+P(r=011)P(s=0|r=011)+P(r=101)P(s=0|r=101)+P(r=110)P(s=0|r=110)+P(r=111)P(s=0|r=111)$$
$$=\{P(r=000|s=1)+P(r=001|s=1)+P(r=010|s=1)+P(r=100|s=1)\}P(s=1)$$
$$+\{P(r=011|s=0)+P(r=101|s=0)+P(r=110|s=0)+P(r=111|s=0)\}P(s=0)$$
$$=\frac{99}{100}(f^3+3\cdot f^2(1-f))+\frac{1}{100}(3\cdot f^2(1-f)+f^3)$$
$$=f^3+3f^2(1-f)$$
$$=3f^2-2f^3$$

↓

Same as the (b)'s answer

Whatever the values of P(s=0) and P(s=1), the error rate is not affected by those values

➔ Another method for data transmission

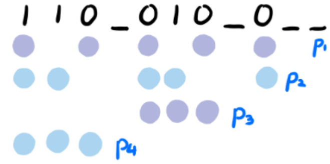
Using parity bit

- Number of parity bit
  - $2^p+1\geq d+p$
  - · p : # of parity bit
  - · d : size of data
- Place of parity bit : 2^n
- Even parity bit
  - : Making # of 1's in code even

e.g.) data = 1100100

→ $2^p+1\geq 7+p$ ∴ p : 4 bit

11	10	9	8	7	6	5	4	3	2	1
1	1	0	_	0	1	0	_	0	_	_



p_1 : 10000 (1) → 1 p_3 : 010 (1) → 1

p_2 : 11010 (3) → 1 p_4 : 110 (2) → 0

∴ Hamming code ⇒ 11000101011

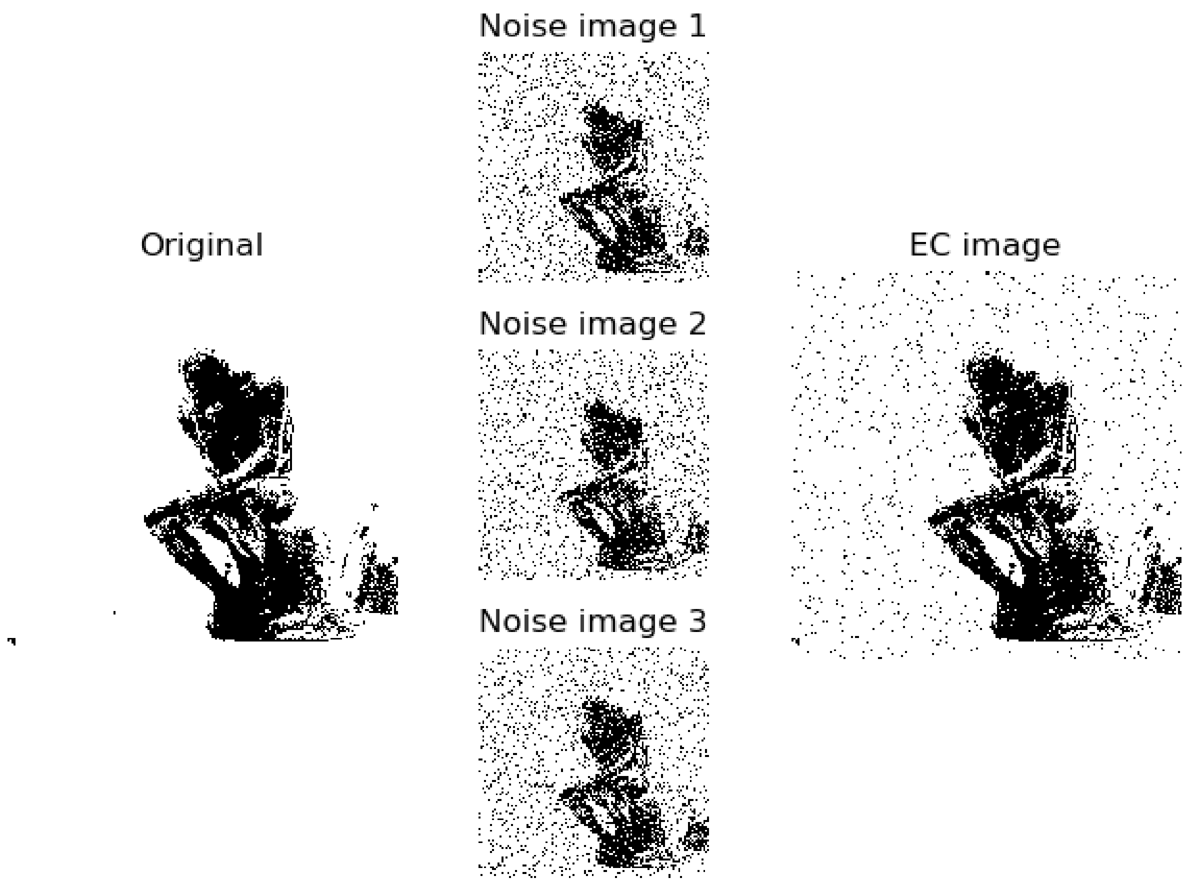
2. Attached is a b&w pixel art version of *Auguste Rodin's The Thinker*. The image is 200x200. We write black=0 and white=1.

(a) What are P(0) and P(1)?

P(0) = 1/2, P(1) = 1/2

In binary image, the pixel value can be 0 or 1.

(b) Generate three images that have passed through a symmetric noisy channel with f=0.1.



(The code is attached – hw3_Q2_majorityrule.py)

(c) Using the simple majority rule, find the error-corrected image.

Refer to (b)'s result image (EC image)

(d) What is the error rate?

Error rate : $\sum_{n=\frac{3+1}{2}}^{3} \binom{3}{n} f^n (1-f)^n = 3f^2(1-f)+f^3$

$= 3f^2 - 2f^3$

$= 3\cdot(0.1)^2 - 2\cdot(0.1)^3$

$= 0.028$