Supplementary: Boundary Multiple Measurement Vectors for Multi-Coset Sampler

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This supplementary material is dedicated to the proofs for Theorems 1–3 in our main paper.

Before proceeding to the proofs, we review some useful notations. For a complex matrix $\mathbf{X} \in \mathbb{C}^{n\times L}$ and a set $S \subseteq \{1,\cdots,n\}$, $\mathbf{X}_S$ (or $\mathbf{X}^S$) denotes the submatrix of $\mathbf{X}$ with columns (or rows) indexed by S; $\mathbf{X}_{i,j}$, $\mathbf{X}_{i,:}$ and $\mathbf{X}_{:,i}$ are the (i,j)th entry, ith row and ith column of $\mathbf{X}$, respectively; $\mathbf{X}^\dagger$, $\mathbf{X}^H$ and $\mathbf{X}^\top$ mean the Moore-Penrose pseudo-inverse, conjugate transpose and transpose of $\mathbf{X}$, respectively; $\mathrm{supp}(\mathbf{X})$ is the non-zero row indices (i.e., joint sparsity) of $\mathbf{X}$; $\|\mathbf{X}\|_F$ and $\|\mathbf{X}\|_2$ signify the Frobenius and Euclidean norm of $\mathbf{X}$, respectively. Moreover, S^c is the complement of set S; $\mathbf{I}_L$ is an $L\times L$ identity matrix.

I. Proof of Theorem 1

Theorem 1. *The actual sampling rate of (4) is* $\min\left(pf_s, f_{nyq}\right)$, *which attains the theoretical lower bound of sampling rate in MCS when* $|supp(\mathbf{X})| \le \frac{N_{sig}B}{f_s}$.

Proof. Assume that a multi-coset sampler has p channels. In the ith channel, The sampling sequence is given by

$$x_{c_i}[n] = x(LTn+\tau_i), \quad n = 0,1,\cdots. \tag{S.1}$$

The sampling rate of each channel is determined by the sampled signal sequence. Because the sampling time interval is LT, the sampling rate of each channel is 1 in L of the Nyquist sampling rate (i.e. $\frac{f_{\text{nyq}}}{L}$). The average sampling rate of p channels is p times that of one channel (i.e. $\frac{pf_{\text{nyq}}}{L}$). Noting that when the sampling rate of p channels is greater than the Nyquist rate, the advantage of sub-Nyquist sampling structure no longer exists. Thus, we only need to sample at Nyquist sampling rate f_{nyq}, the actual sampling rate can be represented as

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right). \tag{S.2}$$

The theoretical lower bound of the sampling rate is given in [17], which is directly determined by the true bandwidth of the signal:

$$\min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \tag{S.3}$$

In most cases, the subband bandwidth $\lambda(\mathcal{T})$ and the actual sampling rate does not exceed f_{nyq} (when not satisfied, the sampling rate is f_{sig}). To ensure reconstruction performance, the parameter p is often not set too low (for instance, it's often chosen to be at least twice the sparsity level $\mathrm{supp}(\mathbf{X})$). It can be seen from above that the theoretical lower bound on the sampling rate is achieved only when $p/LT = pf_s \le 2\lambda(\mathcal{T}) = 2N_{sig}B$. In other words, when $p = 2\mathrm{supp}(\mathbf{X})$ for the worst case of p, the condition for the actual sampling rate to meet the theoretical lower bound is $|\mathrm{supp}(\mathbf{X})| \le \frac{N_{\text{sig}}B}{f_s}$. □

II. Proof of Theorem 2

Theorem 2. *When* $r \in [\lceil \frac{f_s}{(M-1)f_s-B}\rceil, \infty)$ *and* $B > f_s$, *we have* $\max_{i\in\{1,\cdots,r\}} \left|supp(\bar{\mathbf{X}}_{S_i})\right| \le \frac{N_{\text{sig}}B}{f_s}$.

Proof. Consider all blocks $\left\{\mathbf{X}^{U_1},\cdots,\mathbf{X}^{U_M}\right\}$ in $\mathbf{X}$, where there are consecutive corresponding frequency points of length B. For the case $B > f_s$ and each block occupies M rows in $\mathbf{X}$, the length of the frequency point in each $\bar{\mathbf{X}}^{U_i}$ meets

$$l = B-(M-2)f_s < f_s. \tag{S.4}$$

Because $l < f_s$, we know that the non-zero elements of an any sub-block (i.e. $\bar{\mathbf{X}}^{U_i}$) are distributed on both sides and do not intersect on the columns. Conside one block $\bar{\mathbf{X}}^{U_i}$, We let $r\to\infty$ and observe the change in $\bar{\mathbf{X}}^{U_i}$, $\bar{\mathbf{X}}^{U_i}$ gradually changes from an MMV form to an SMV form. In SMV, the sparsity of the signal is determined by the column in $\bar{\mathbf{X}}^{U_i}$ where it is located. It is observed that

$$\lim_{r\to\infty,i,j} \mathrm{supp}(\bar{\mathbf{X}}^{U_i}_{:,j}) \le 1. \tag{S.5}$$

Thus, the sparsity of each column in $\bar{\mathbf{X}}^{U_i}$ is less than N_{sig}.

A more complex situation is when r is a finite value, assuming $r = r^\star$ is a finite value. In this case, the length of frequency point in each sub-matrix is $\frac{f_s}{r^\star}$. We use reduction to absurdity to prove the condition that the sparsity of each sub-matrix is less than N_{sig}. Assuming that there exists a sub-matrix $\bar{\mathbf{X}}_{S_i}$ with $\mathrm{supp}(\bar{\mathbf{X}}_{S_i}) > N_{\text{sig}}$. Also, the non-zero elements of an any sub-block of $\bar{\mathbf{X}}_{S_i}$ do not intersect on the columns. $\bar{\mathbf{X}}_{S_i}$ must contain both non-zero elements on both sides of one block. From (S.4), we know that the length of any block of $\bar{\mathbf{X}}$ is less than f_s. We can draw a conclusion that

$$\frac{f_s}{r^\star} > f_s - l = (M-1)f_s - B. \tag{S.6}$$

As can be seen, there exist a contradiction between (S.6) and Theorem 2 , so the length of any column-partition sub-matrix non-zero elements in $\bar{\mathbf{X}}$ must be less or equal than $f_s - l$, which is equivalent to r satisfying

$$r \ge \left\lceil \frac{f_s}{(M-1)f_s-B} \right\rceil. \tag{S.7}$$

□

III. Proof of Theorem 3

Theorem 3. *Consider the column-partitioned MMV model (5) with* $\min_{i,j} \|(\mathbf{X}_{S_i})_{j,:}\|_2 / \|\mathbf{X}_{S_i}\|_F = \eta$ *and* $|supp(\mathbf{X}_{S_i})| \leq s$. *Let* $s_1 := \min_{i,k} |\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i})|$, $s_2 := \min_{i,k} |\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i}) \cap \tilde{S}^k_{S_i}|$ *and* $s_3 := \min_{i,k} |\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i}) \cap \tilde{S}^k_{S_i} \setminus S^k_{S_i}|$. *Then, if the sensing matrix* $\mathbf{A}$ *obeys the RIP with*

$$\delta_{3s} \leq \sqrt{\frac{\nu_1 \sqrt{\nu_1^2 + 4\nu_2^2} - \nu_1^2 - 1}{4\nu_1^2 \nu_2^2 - 2\nu_1^2 - 1}} \tag{S.8}$$

where $\nu_1 := \frac{1+\omega}{1+\eta\omega\sqrt{s_2}}$ *and* $\nu_2 := \frac{1+\omega}{1+\eta\omega\sqrt{s_3}}$, *SI-SSP produces an signal estimate* $\mathbf{X}^k = [\mathbf{X}^k_{S_1}, \cdots, \mathbf{X}^k_{S_r}]$ *satisfying*

$$\left\|\mathbf{X} - \mathbf{X}^k\right\|_F \leq \rho^k \left\|\mathbf{X}\right\|_F + \tau \left\|\mathbf{E}\right\|_F, \tag{S.9}$$

where $\rho \in (0,1)$ *and* τ *are constants depending on* δ_{3s}, ν_1 *and* ν_2. *Furthermore, after at most* $k^* = \lceil \log_\rho \frac{\|\mathbf{X}\|_F}{\tau\|\mathbf{E}\|_F} \rceil$ *iterations, SI-SSP estimates* $\mathbf{X}$ *with*

$$\|\mathbf{X} - \mathbf{X}^{k^*}\|_F \leq (\tau + 1)\|\mathbf{E}\|_F. \tag{S.10}$$

To prove Theorem 3, we first introduce six useful lemmas, whose proofs are left to appendices.

Lemma 1. *(Lemma 1 in [25]): For nonnegative numbers a, b, c, d, x, y,*

$$(ax + by)^2 + (cx + dy)^2 \leq (\sqrt{a^2 + c^2} x + (b + d) y)^2. \tag{S.11}$$

Lemma 2. *Consider the system model* $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$, $supp(\mathbf{X}) = T$ *and* $|T| = s$. *Suppose* S, $T_0 \subseteq \{1, 2, ..., n\}$, $|S| = t$. $\mathbf{W}_{T_0}$ *is constructed with diagonal entries indexed by* T_0 *being* $\omega \geq 0$. *Let* $\tilde{\mathbf{X}}$ *be the solution of the least squares problem* $\arg\min_{\mathbf{Z}} \{\|\mathbf{Y} - \mathbf{A}\mathbf{Z}\|_2, supp(\mathbf{Z}) \subseteq S\}$, *if* $\delta_{3s} < 1$, *then*

$$\|\mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}})_S\|_F \leq \omega\delta_{s+t}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \omega\sqrt{1 + \delta_t}\|\mathbf{E}\|_F \tag{S.12}$$

and

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{s+t}^2}}\|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1 + \delta_t}}{1 - \delta_{s+t}}\|\mathbf{E}\|_F. \tag{S.13}$$

If $t > s$, *define* T_∇ *as the row-indices of the smallest* $t - s$ *row norm entries of* $\tilde{\mathbf{X}}$ *in* S, *we have*

$$\|\mathbf{X}_{T_\nabla}\|_F \leq \sqrt{2}\nu_2\delta_{s+t}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_2\sqrt{2(1 + \delta_t)}\|\mathbf{E}\|_F. \tag{S.14}$$

Remark 1. *When we consider the atom selection strategy of* $\|\tilde{\mathbf{X}}_{T_\nabla} + \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla}\|_F \leq \|\tilde{\mathbf{X}}_{S'} + \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{S'}\|_F$, *we can also obtain another upper bound for* $\|\mathbf{X}_{T_\nabla}\|_F$ *in* (S.14). *In this case, we should allocate* $2\|\mathbf{X}_{T_\nabla}\|_F$ *to the left hand side of* (A.38), *we have*

$$\|\mathbf{X}_{T_\nabla}\|_F \leq \sqrt{2}\nu_3\delta_{s+t}\left\|\mathbf{X} - \tilde{\mathbf{X}}\right\|_F + \nu_4\sqrt{2(1 + \delta_t)}\left\|\mathbf{E}\right\|_F. \tag{S.15}$$

where $\nu_3 = (1 - \omega + \omega\delta_{s+t} + \delta_{s+t})/(2\delta_{s+t})$ *and* $\nu_4 = (1 + \omega)/(2\delta_{s+t})$.

Lemma 3. *In steps 4 and 5 of SI-SSP, we have*

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2}\nu_1\delta_{3s}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F + \nu_1\sqrt{2(1 + \delta_{3s})}\left\|\mathbf{E}\right\|_F. \tag{S.16}$$

Remark 2. *When we consider the atom selection strategy in select step that*

$$\begin{aligned}&\left\|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\right\|_F \\ &\leq \left\|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\right\|_F.\end{aligned} \tag{S.17}$$

We can also obtain another upper bound for $\left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F$ *in* (S.16). *In this case, we should allocate* $2\left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F$ *to the left hand side of* (A.51), *we have*

$$\left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F \leq \sqrt{2}\nu_4\delta_{3s}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F + \nu_4\sqrt{2(1 + \delta_{3s})}\left\|\mathbf{E}\right\|_F. \tag{S.18}$$

where $\nu_4 = (1 - \omega + \omega\delta_{3s} + \delta_{3s})/(2\delta_{3s})$ *and* $\nu_4 = (1 + \omega)/(2\delta_{3s})$. *Based on conclusions* (S.15) *and* (S.18), *we know that the sensing matrix* $\mathbf{A}$ *obeys the RIP with*

$$\delta_{3s} \leq \sqrt{\frac{\nu_3\sqrt{\nu_3^2 + 4\nu_4^2} - \nu_3^2 - 1}{4\nu_3^2\nu_4^2 - 2\nu_3^2 - 1}}. \tag{S.19}$$

Lemma 4. *Let* $T_0 \subseteq \{1, 2, ..., n\}$, *for two vectors* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *if* $|supp(\mathbf{u}) \cup supp(\mathbf{v})| \leq t$,

$$\left|\left\langle \mathbf{u}, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\mathbf{v}\right\rangle\right| \leq \omega\delta_t\left\|\mathbf{u}\right\|\left\|\mathbf{v}\right\|; \tag{S.20}$$

moreover, if $U \subseteq \{1, 2, ..., n\}$ *and* $|U \cup supp(\mathbf{v})| \leq t$, *then*

$$\left|(\mathbf{W}_{T_0} - \mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\mathbf{v}\right| \leq \omega\delta_t\left\|\mathbf{v}\right\|. \tag{S.21}$$

Lemma 5. *For SMV model* $\mathbf{y} = \mathbf{\Phi}\mathbf{x} + \mathbf{e}$, *let* $T_0 \subseteq \{1, 2, ..., n\}$, *let* $U \subseteq \{1, 2, ..., n\}$ *and* $|U \cap T_0| \leq u$, *we have*

$$\left\|(\mathbf{W}_{T_0}\mathbf{A}^H e)_U\right\|_2 \leq \omega\delta_u\left\|e\right\|_2. \tag{S.22}$$

Lemma 6. *Consider MMV model* $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$, *let* $\tilde{\mathbf{X}}$ *be the solution of the least squares problem* $\arg\min_{\mathbf{Z}} \{\|\mathbf{Y} - \mathbf{A}\mathbf{Z}\|_F, supp(\mathbf{Z}) \subseteq S\}$, *then*

$$\left\langle \mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}}, \mathbf{A}^H\mathbf{A}\mathbf{Z}\right\rangle + \omega\left\langle \mathbf{E}, \mathbf{A}\mathbf{Z}\right\rangle = \mathbf{0}. \tag{S.23}$$

Proof of Theorem 3. Now we have all ingredients to prove Theorem 3. In step 3 and 4 of SI-SSP, by Lemma 3, in the kth iteration, we have

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2}\nu_1\delta_{3s}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F + \nu_1\sqrt{2(1 + \delta_{3s})}\left\|\mathbf{E}\right\|_F. \tag{S.24}$$

Since the step 6 in SI-SSP is a process to solve a least squares problem. Let $S = \tilde{S}^k$ and $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $t = 2s$, by (S.13), we can know that

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{3s}^2}}\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \frac{\sqrt{1 + \delta_{2s}}}{1 - \delta_{3s}}\|\mathbf{E}\|_F. \tag{S.25}$$

Combining (S.24) and (S.25) and magnifying δ_{2s} to δ_{3s}, we have

$$\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F \leq \nu_1\sqrt{\frac{2\delta_{3s}^2}{1 - \delta_{3s}^2}}\|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \tau_1\|\mathbf{E}\|_F. \tag{S.26}$$

Next, after step 7 of SI-SSP in kth iteration, let $S_\nabla = \tilde{S}^k \setminus S^k$ as the row-indices of the smallest $t - s$ row norm entries in $\tilde{\mathbf{X}}^k$. Let $T = \tilde{S}^k$, $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $T_\nabla = S_\nabla$ and $t = 2s$, by (A.37), we have

$$\|\mathbf{X}_{S_\nabla}\|_F \leq \sqrt{2}\nu_2\delta_{3s}\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2\sqrt{2\,(1+\delta_{2s})}\|\mathbf{E}\|_F. \tag{S.27}$$

Let $\tau_1 = (\nu_1\sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$ and $\tau_2 = \sqrt{1+\delta_{3s}}$. Dividing $(S^k)^c$ into 2 disjoint subsets: $(\tilde{S}^k)^c$ and S_∇, we get

$$\begin{aligned}
&\left\|\mathbf{X}_{(S^k)^c}\right\|_F^2 = \|\mathbf{X}_{S_\nabla}\|_F^2 + \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2 \\
&\overset{(S.24),\,(S.27)}{\leq} 2\left(\nu_2\delta_{3s}\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2\tau_2\|\mathbf{E}\|_F\right)^2 \\
&\quad + 2\left(\nu_1\delta_{3s}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_2 + \nu_1\tau_2\left\|\mathbf{E}\right\|_F\right)^2 \\
&\overset{(S.26)}{\leq} 2\left(\sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F + \nu_2(\tau_1\delta_{3s} + \tau_2)\right. \\
&\quad \left.\times\|\mathbf{E}\|_F\right)^2 + 2\left(\nu_1\delta_{3s}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F + \nu_1\tau_2\left\|\mathbf{E}\right\|_F\right)^2 \\
&\overset{(S.11)}{\leq} 2\left(\sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2} + \nu_1^2\delta_{3s}^2}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F\right. \\
&\quad \left. + \left((\nu_1+\nu_2)\tau_2 + \nu_2\delta_{3s}\tau_1\right)\left\|\mathbf{E}\right\|_F\right)^2.
\end{aligned} \tag{S.28}$$

Squaring both sides, we can get

$$\begin{aligned}
\left\|\mathbf{X}_{(S^k)^c}\right\|_F &\leq \sqrt{\tfrac{4\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2} + 2\nu_1^2\delta_{3s}^2}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F \\
&\quad + \sqrt{2}\left((\nu_1+\nu_2)\tau_2 + \nu_2\delta_{3s}\tau_1\right)\left\|\mathbf{E}\right\|_F.
\end{aligned} \tag{S.29}$$

Step 9 of the kth iteration in SI-SSP also solves a least squares problem. Letting $T = S^k$, $\tilde{\mathbf{X}} = \mathbf{X}^k$ and $t = s$, by (S.13), we have

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \sqrt{\frac{1}{1-\delta_{2s}^2}}\|\mathbf{X}_{(S^k)^c}\|_F + \frac{\sqrt{1+\delta_s}}{1-\delta_{2s}}\|\mathbf{E}\|_F. \tag{S.30}$$

Combining (S.29) and (S.30) yields

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho\|\mathbf{X} - \mathbf{X}^{k-1}\|_F + (1-\rho)\tau\|\mathbf{E}\|_F \tag{S.31}$$

where $\rho := \sqrt{2}\delta_{3s}\sqrt{2\nu_1^2\nu_2^2\delta_{3s}^2 + \nu_1^2 - \nu_1^2\delta_{3s}^2}(1-\delta_{3s}^2)^{-1}$ and $\tau := \sqrt{2}\delta_{3s}\nu_2(\nu_1\sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}})(1-\delta_{3s}^2)^{-1/2}(1-\delta_{3s})^{-1}(1-\rho)^{-1} + (\nu_1\nu_2\sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$.

Second, we recursively apply (S.31) to obtain

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho^k\|\mathbf{X}\|_F + \tau\|\mathbf{E}\|_F \tag{S.32}$$

where $\rho < 1$ under (S.8). When $k^* = \lceil \log_\rho \frac{\|\mathbf{X}\|_F}{\tau\|\mathbf{E}\|_F} \rceil$, we have $\rho^k\|\mathbf{X}\|_F \leq \tau\|\mathbf{E}\|_F$, and thus the stability result (S.10). □

Appendix A
Proof of Lemma 2

- First, we give a upper bound of $\|\mathbf{X}_{T_\nabla}\|_F$, by Lemma 6, let $\mathbf{Z} = (\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S$, we have

$$\begin{aligned}
&\left\langle \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}}), \mathbf{A}^H\mathbf{A}(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle \\
&\quad + \left\langle \mathbf{W}_{T_0}\mathbf{E}, \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle = \mathbf{0}.
\end{aligned} \tag{A.33}$$

Noticing that $\mathrm{supp}(\tilde{\mathbf{X}}) \subseteq S$, we have

$$\begin{aligned}
&\left\|(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\|_F^2 \\
= &\left\langle \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle \\
\overset{(A.33)}{=} &\left\langle \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{I}_L - \mathbf{A}^H\mathbf{A})(\mathbf{X} - \tilde{\mathbf{X}})_S \right\rangle \\
- &\left\langle \mathbf{W}_{T_0}\mathbf{E}, \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle \\
\overset{(7)}{\leq} &\ \omega\delta_{s+t}\left\|(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\|_F\left\|\mathbf{X} - \tilde{\mathbf{X}}\right\|_F \\
&+ \omega\left\|\mathbf{E}\right\|_F\sqrt{1+\delta_t}\left\|\mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}})_S\right\|_F
\end{aligned} \tag{A.34}$$

Divide both sides by $\left\|(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\|_F$ to obtain (S.12).

- By expanding Lemma 2 in [25] to MMV model, we could get a relationship between $\left\|\mathbf{X} - \tilde{\mathbf{X}}\right\|_F$ and $\|\mathbf{X}_{S^c}\|_F$. We have

$$\left\|\mathbf{X} - \tilde{\mathbf{X}}\right\|_F \leq \sqrt{\frac{1}{1-\delta_{s+t}^2}}\|\mathbf{X}_{S^c}\|_F^2 + \frac{\sqrt{1+\delta_t}}{1-\delta_{s+t}}\|\mathbf{E}\|_F \tag{A.35}$$

- Finally, we established the relationship between $\mathbf{X}_{T_\nabla}$ and $\mathbf{X} - \tilde{\mathbf{X}}$. There exist a subset $S' \subseteq S$ and $S' \cap T = \emptyset$. Since T_∇ is defined by the set of indices of the $t-s$ smallest row entries of $\tilde{\mathbf{X}}$, we can conclude that

$$\begin{aligned}
&\left\|\tilde{\mathbf{X}}_{T_\nabla}\right\|_F + \left\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla}\right\|_F \\
&\quad \leq \left\|\tilde{\mathbf{X}}_{S'}\right\|_F + \left\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{S'}\right\|_F.
\end{aligned} \tag{A.36}$$

By eliminating the contribution from $T_\nabla \cap S'$ and noticing that $S' \cap T = \emptyset$, we have

$$\begin{aligned}
&\left\|\tilde{\mathbf{X}}_{T_\nabla\setminus S'}\right\|_F + \left\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla\setminus S'}\right\|_F \\
\leq &\left\|(\tilde{\mathbf{X}} - \mathbf{X})_{S'\setminus T_\nabla}\right\|_F \\
+ &\left\|\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{S'\setminus T_\nabla}\right\|_F.
\end{aligned} \tag{A.37}$$

For the left-hand side of (A.37), we have

$$\begin{aligned}
&\left\|\tilde{\mathbf{X}}_{T_\nabla\setminus S'}\right\|_F + \left\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla\setminus S'}\right\|_F \\
= &\left\|(\tilde{\mathbf{X}} - \mathbf{X} + \mathbf{X})_{T_\nabla\setminus S'}\right\|_F \\
+ &\left\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}) + \mathbf{W}_{T_0}\mathbf{X})_{T_\nabla\setminus S'}\right\|_F \\
\geq &\ \|\mathbf{X}_{T_\nabla}\|_F + \|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F \qquad\qquad (A.38) \\
- &\left\|(\tilde{\mathbf{X}} - \mathbf{X})_{T_\nabla\setminus S'}\right\|_F \\
- &\left\|\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{T_\nabla\setminus S'}\right\|_F.
\end{aligned} \tag{A.39}$$

Combining (A.39) and (A.37), and noticing that

$$(T_\nabla \setminus S') \cap (S' \setminus T_\nabla) = \emptyset \tag{A.40}$$

$$(T_\nabla \setminus S') \cup (S' \setminus T_\nabla) \subseteq T, \tag{A.41}$$

we have

$$\begin{aligned}
&\|\mathbf{X}_{T_\nabla}\|_F + \|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F \\
\leq &\left\|(\tilde{\mathbf{X}} - \mathbf{X})_{T_\nabla\setminus S'}\right\|_F + \left\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{T_\nabla\setminus S'}\right\|_F \\
+ &\left\|(\tilde{\mathbf{X}} - \mathbf{X})_{S'\setminus T_\nabla}\right\|_F + \left\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{S'\setminus T_\nabla}\right\|_F \\
\leq &\ \sqrt{2}\left\|(\tilde{\mathbf{X}} - \mathbf{X})_{(S'\setminus T_\nabla)\cup(T_\nabla\setminus S')}\right\|_F
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \sqrt{2}\left\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_{(S'\setminus T_\triangledown)\cup(T_\triangledown\setminus S')}\right\|_F \\
&\leq \quad \sqrt{2}\left\|(\tilde{\mathbf{X}}-\mathbf{X})_S\right\|_F + \sqrt{2}\left\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_S\right\|_F \\
&\overset{(S.12)}{\leq} \quad \sqrt{2}(1+\omega)\delta_{s+t}\left\|\mathbf{X}-\tilde{\mathbf{X}}\right\|_F \\
&+ \quad (1+\omega)\sqrt{2(1+\delta_t)}\left\|\mathbf{E}\right\|_F. \qquad \text{(A.42)}
\end{aligned}
$$

Also, we can obtain the relationship between $\|\mathbf{W}_{T_0}\mathbf{X}_{T_\triangledown}\|_F$ and $\|\mathbf{X}_{T_\triangledown}\|_F$:

$$
\eta\omega\sqrt{s_3}\left\|\mathbf{X}_{T_\triangledown}\right\|_F \leq \left\|\mathbf{W}_{T_0}\mathbf{X}_{T_\triangledown}\right\|_F. \qquad \text{(A.43)}
$$

Combining (A.42) and (A.43), we have

$$
\begin{aligned}
\left\|\mathbf{X}_{T_\triangledown}\right\|_F \leq & \frac{\sqrt{2}(1+\omega)\delta_{s+t}}{1+\eta\omega\sqrt{s_3}}\left\|\mathbf{X}-\tilde{\mathbf{X}}\right\|_F \\
& + \frac{(1+\omega)\sqrt{2(1+\delta_t)}}{1+\eta\omega\sqrt{s_3}}\left\|\mathbf{E}\right\|_F. \qquad \text{(A.44)}
\end{aligned}
$$

Noting the definition of ν_2, we complete the proof of Lemma 2.

Appendix B
Proof of Lemma 3

Proof: From step 5 of SI-SSP, we have

$$
\mathbf{X}^k_{S_i} = \underset{\boldsymbol{\Theta}:\text{supp}(\boldsymbol{\Theta})=S^k_{S_i}}{\arg\min}\left\|\mathbf{Y}_{S_i}-\mathbf{A}\boldsymbol{\Theta}\right\|_F. \qquad \text{(A.45)}
$$

From the step 4 of SI-SSP, let $\mathbf{X}^k = [\mathbf{X}^k_{S_1},\cdots,\mathbf{X}^k_{S_r}]$, we have the following conclusion

$$
\begin{aligned}
& \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_T\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_T\right\|_F \\
\leq \quad & \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\right\|_F. \qquad \text{(A.46)}
\end{aligned}
$$

By removing the same coordinates $T\cap\Delta S$, we can get

$$
\begin{aligned}
& \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\right\|_F \\
\leq \quad & \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F. \qquad \text{(A.47)}
\end{aligned}
$$

Because $\text{supp}(\mathbf{X}) = T$ and $\text{supp}(\mathbf{X}^{k-1}) = S^{k-1}$,

$$
(\mathbf{X}-\mathbf{X}^{k-1})_{\Delta S\setminus(T\cup S^{k-1})} = 0. \qquad \text{(A.48)}
$$

For the right-hand of (A.47), we have

$$
\begin{aligned}
& \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
= \quad & \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\right\|_F \\
= \quad & \left\|(\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\right\|_F \\
\leq \quad & \left\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
+ \quad & \left\|(\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\right\|_F. \qquad \text{(A.49)}
\end{aligned}
$$

Note that $\tilde{S}^k = S^{k-1}\cup\Delta S$, we have

$$
(\mathbf{X}-\mathbf{X}^{k-1})_{T\setminus\tilde{S}^k} = \mathbf{X}_{(\tilde{S}^k)^c}. \qquad \text{(A.50)}
$$

For the left-side of (A.47),we have

$$
\begin{aligned}
& \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\right\|_F \\
= \quad & \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\right\|_F \\
= \quad & \left\|(\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\right\|_F \\
= \quad & \|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}) \\
+ \quad & \mathbf{A}^H\mathbf{E}+\mathbf{X})_{\Delta S\setminus(T\cup S^{k-1})}\| \\
+ \quad & \|\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}) \\
+ \quad & \mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E}+\mathbf{W}_{T_0}\mathbf{X})_{\Delta S\setminus(T\cup S^{k-1})}\|_F \\
\geq \quad & \left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F + \left\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\right\|_F \\
- \quad & \left\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\right\|_F \qquad \text{(A.51)} \\
- \quad & \left\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\right\|_F \\
- \quad & \left\|(\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\right\|_F - \left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\right\|_F. \qquad \text{(A.52)}
\end{aligned}
$$

Combining (A.53) and (A.52), we have

$$
\begin{aligned}
& \left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F + \left\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\right\|_F \\
\leq \quad & \left\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\setminus\tilde{S}^k}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\setminus\tilde{S}^k}\right\|_F \\
+ \quad & \left\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
+ \quad & \left\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
+ \quad & \left\|(\mathbf{A}^H\mathbf{E})_{T\setminus\tilde{S}^k}\right\|_F + \left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T\setminus\tilde{S}^k}\right\|_F \\
+ \quad & \left\|(\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\right\|_F + \left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\right\|_F \\
\leq \quad & \sqrt{2}\left\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\cup\Delta S}\right\|_F \\
+ \quad & \sqrt{2}\left\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\cup\Delta S}\right\|_F \\
+ \quad & \sqrt{2}\left\|(\mathbf{A}^H\mathbf{E})_{T\cup\Delta S}\right\|_F + \sqrt{2}\left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T\cup\Delta S}\right\|_F \\
\leq \quad & \sqrt{2}(1+\omega)\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F \\
+ \quad & (1+\omega)\sqrt{2(1+\delta_{3s})}\left\|\mathbf{E}\right\|_F. \qquad \text{(A.53)}
\end{aligned}
$$

We can obtain the relationship between $\left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F$ $\left\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\right\|_F$:

$$
\eta\omega\sqrt{s_2}\left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F \leq \left\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\right\|_F. \qquad \text{(A.54)}
$$

Combining (A.53) and (A.54), we have

$$
\left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F \leq \frac{\sqrt{2}(1+\omega)\delta_{3s}}{1+\eta\omega\sqrt{s_2}}\left\|\mathbf{X}-\tilde{\mathbf{X}}\right\|_F
$$

$$+\frac{(1+\omega)\sqrt{2(1+\delta_{2s})}}{1+\eta\omega\sqrt{s_2}}\left\|\mathbf{E}\right\|_F. \quad \text{(A.55)}$$

Noting the definition of ν_1, we complete the proof of Lemma 3.

APPENDIX C
PROOF OF LEMMA 4

Proof: the RIC δ_t can be expressed as [25]

$$\delta_t = \max_{S\subseteq\{1,2,\ldots,N\},|S|\le t}\left\|\mathbf{A}_S^*\mathbf{A}_S-\mathbf{I}\right\|_{2\to 2}, \quad \text{(A.56)}$$

where

$$\left\|\mathbf{A}_S^*\mathbf{A}_S-\mathbf{I}\right\|_{2\to 2} = \sup_{\mathbf{a}\in\mathbb{R}^{|S|}\setminus\{\mathbf{0}\}}\frac{\left\|\left(\mathbf{A}_S^*\mathbf{A}_S-\mathbf{I}\right)\mathbf{a}\right\|_2}{\left\|\mathbf{a}\right\|_2}. \quad \text{(A.57)}$$

Let $S=\mathrm{supp}(\mathbf{u})\cup\mathrm{supp}(\mathbf{v})$, then $|S|\le t$. Let $\mathbf{u}_{|S},\mathbf{v}_{|S}$ denote respectively the S-dimensional sub-vectors of $\mathbf{u}$ and $\mathbf{v}$ obtained by only keeping the components indexed by S. We have

$$\begin{aligned}
&\left|\left\langle\mathbf{u},\left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\right\rangle\right| \\
=&\left|\left\langle\mathbf{W}_{T_0}\mathbf{u},\mathbf{v}\right\rangle-\left\langle\mathbf{A}\mathbf{W}_{T_0}\mathbf{u},\mathbf{A}\mathbf{v}\right\rangle\right| \\
=&\left|\left\langle\mathbf{W}_{T_0}\mathbf{u}_{|T},\left(\mathbf{I}_L-\mathbf{A}_T^H\mathbf{A}_T\right)\mathbf{v}_{|T}\right\rangle\right| \\
\le&\left\|\mathbf{W}_{T_0}\mathbf{u}_{|T}\right\|_2\left\|\left(\mathbf{I}_L-\mathbf{A}_T^H\mathbf{A}_T\right)\mathbf{v}_{|T}\right\|_2 \\
\overset{\text{(A.57)}}{\le}&\left\|\mathbf{W}_{T_0}\mathbf{u}_{|T}\right\|_2\left\|\mathbf{I}_L-\mathbf{A}_T^H\mathbf{A}_T\right\|_{2\to 2}\left\|\mathbf{v}_{|T}\right\|_2 \\
\overset{\text{(A.56)}}{\le}&\omega\delta_t\left\|\mathbf{u}_{|T}\right\|_2\left\|\mathbf{v}_{|T}\right\|_2 \\
=&\omega\delta_t\|\mathbf{u}\|_2\|\mathbf{v}\|_2,
\end{aligned} \quad \text{(A.58)}$$

moreover, we have

$$\begin{aligned}
&\left\|\left(\left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\right)_U\right\|_2^2 \\
=&\left\langle\left(\left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\right)_U,\right. \\
&\left.\left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\right\rangle \\
\overset{\text{(S.20)}}{\le}&\delta_t\left\|\left(\left(\mathbf{W}_{T_0}-\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\right)_U\right\|_2\|\mathbf{v}\|_2
\end{aligned} \quad \text{(A.59)}$$

which completes the proof of Lemma 4.

APPENDIX D
PROOF OF LEMMA 5

Proof: The lemma easily follows from the fact that

$$\begin{aligned}
&\left\|\left(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e}\right)_U\right\|_2^2 \\
=&\left\langle\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e},\left(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e}\right)_U\right\rangle \\
=&\left\langle\mathbf{e},\mathbf{W}_{T_0}\mathbf{A}\left(\left(\mathbf{A}^H\mathbf{e}\right)_U\right)\right\rangle \\
\le&\|\mathbf{e}\|_2\left\|\mathbf{W}_{T_0}\mathbf{A}\left(\left(\mathbf{A}^H\mathbf{e}\right)_U\right)\right\|_2 \\
\overset{(7)}{\le}&\|\mathbf{e}'\|_2\,\omega\sqrt{1+\delta_u}\left\|\left(\mathbf{A}^H\mathbf{e}\right)_U\right\|_2.
\end{aligned} \quad \text{(A.60)}$$

APPENDIX E
PROOF OF LEMMA 6

Proof: Due to the orthogonality, the residue $\mathbf{Y}-\mathbf{A}\tilde{\mathbf{X}}$ is orthogonal to the space $\mathbf{A}\mathbf{Z},\mathrm{supp}(\mathbf{Z})\subseteq S$. This means that for all $\mathbf{Z}\in\mathbb{R}$ with $\mathrm{supp}(\mathbf{Z})\subseteq S$,

$$\left\langle\mathbf{A}(\mathbf{Y}-\mathbf{A}\tilde{\mathbf{X}}),\mathbf{Z}\right\rangle=\mathbf{0} \quad \text{(A.61)}$$

then, let $\tilde{X}'$ be the solution of the least squares problem $\arg\min_{\mathbf{Z}}\left\{\left\|\mathbf{Y}'-\mathbf{A}\mathbf{Z}\right\|_F,\mathrm{supp}(\mathbf{Z})\subseteq S\right\}$, where $\mathbf{Y}'=\frac{\mathbf{A}\mathbf{W}_{T_0}\mathbf{X}_{T_0}}{\omega}+\mathbf{E}$. We have

$$\tilde{\mathbf{X}}'=\frac{\mathbf{W}_{T_0}\tilde{\mathbf{X}}}{\omega}. \quad \text{(A.62)}$$

Then, by (A.61), we have

$$\begin{aligned}
\mathbf{0} &= \left\langle\frac{\mathbf{A}\mathbf{W}_{T_0}\mathbf{X}_{T_0}}{\omega}+\mathbf{E}-\mathbf{A}\frac{\mathbf{W}_{T_0}\tilde{X}}{\omega},\mathbf{A}\mathbf{Z}\right\rangle \\
&= \left\langle\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}},\mathbf{A}^H\mathbf{A}\mathbf{Z}\right\rangle+\omega\left\langle\mathbf{E},\mathbf{A}\mathbf{Z}\right\rangle.
\end{aligned} \quad \text{(A.63)}$$