

Supplementary: Boundary Multiple Measurement Vectors for Multi-Coset Sampler

Dong Xiao, Jian Wang and Yun Lin

This supplementary material is dedicated to the proofs for Theorems 1–3 in our main paper.

Before proceeding to the proofs, we review some useful notations. For a complex matrix $\mathbf{X} \in \mathbb{C}^{n \times L}$ and a set $S \subseteq \{1, \dots, n\}$, \mathbf{X}_S (or \mathbf{X}^S) denotes the submatrix of \mathbf{X} with columns (or rows) indexed by S ; $\mathbf{X}_{i,j}$, $\mathbf{X}_{i,:}$ and $\mathbf{X}_{:,i}$ are the (i, j) th entry, i th row and i th column of \mathbf{X} , respectively; \mathbf{X}^\dagger , \mathbf{X}^H and \mathbf{X}^\top mean the Moore-Penrose pseudo-inverse, conjugate transpose and transpose of \mathbf{X} , respectively; $\text{supp}(\mathbf{X})$ is the non-zero row indices (i.e., joint sparsity) of \mathbf{X} ; $\|\mathbf{X}\|_F$ and $\|\mathbf{X}\|_2$ signify the Frobenius and Euclidean norm of \mathbf{X} , respectively. Moreover, S^c is the complement of set S ; \mathbf{I}_L is an $L \times L$ identity matrix.

I. PROOF OF THEOREM 1

Theorem 1. *The actual sampling rate of (4) is $\min(p f_s, f_{\text{nyq}})$, which attains the theoretical lower bound of sampling rate in MCS when $|\text{supp}(\mathbf{X})| \leq \frac{N_{\text{sig}} B}{f_s}$.*

Proof. In the i th channel of a multi-coset sampler, the sampling sequence is given by

$$x_{c_i}[n] = x(LTn + \tau_i), \quad n = 0, 1, \dots \quad (\text{S.1})$$

The sampling rate of each channel is determined by the sampled signal sequence. To be specific, since the sampling time interval is LT , the sampling rate of each channel is

$$f_s = \frac{1}{LT} = \frac{f_{\text{nyq}}}{L}, \quad (\text{S.2})$$

i.e., one- L th of the Nyquist sampling rate.

Moreover, as the multi-coset sampler is assumed to have p channels, the overall sampling rate of p channels is p times that of each channel (i.e. $\frac{p f_{\text{nyq}}}{L}$). If this sampling rate is greater than the Nyquist rate f_{nyq} , then the advantage of sub-Nyquist sampling structure no longer exists. In this case, we only need to sample at Nyquist sampling rate f_{nyq} . Thus, the actual sampling rate can be given by

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right). \quad (\text{S.3})$$

The theoretical lower bound of the sampling rate is given in [17], which is determined directly by the true bandwidth of the signal:

$$\min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \quad (\text{S.4})$$

Thus, the theoretical lower bound on the sampling rate is achieved when

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right) \leq \min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \quad (\text{S.5})$$

In most cases, $2\lambda(\mathcal{T})$ and $\frac{p}{LT}$ do not exceed f_{nyq} . (If violated, the sampling rate would just be f_{nyq} .) Therefore, the condition (S.5) holds whenever

$$\frac{p}{LT} \leq 2\lambda(\mathcal{T}). \quad (\text{S.6})$$

Furthermore, to ensure a unique-solution reconstruction, the number p of channels should not be too small. In particular, its lower bound is twice the signal sparsity without the priori information about the signal \mathbf{X} [17],

$$p \geq 2|\text{supp}(\mathbf{X})|. \quad (\text{S.7})$$

For the worst case where $p = 2|\text{supp}(\mathbf{X})|$, (S.6) can be rewritten as

$$|\text{supp}(\mathbf{X})| \leq \lambda(\mathcal{T})LT = \frac{N_{\text{sig}} B}{f_s}, \quad (\text{S.8})$$

which completes the proof. \square

II. PROOF OF THEOREM 2

Theorem 2. *When $r \in [\lceil \frac{f_s}{(M-1)f_s - B} \rceil, \infty)$ and $B > f_s$, we have $\max_{i \in \{1, \dots, r\}} |\text{supp}(\bar{\mathbf{X}}_{S_i})| \leq \frac{N_{\text{sig}} B}{f_s}$.*

Proof. Consider all row blocks $\{\mathbf{X}^{U_1}, \dots, \mathbf{X}^{U_M}\}$ in \mathbf{X} , where there are consecutive corresponding frequency points of length B in occupied blocks. For the case $B > f_s$ and each block occupies M rows in \mathbf{X} , we assume that $(M-2)f_s < B \leq (M-1)f_s$ for $M \geq 3$. The frequency points of the sub-band signal may occupy $M-1$ or M rows actually.

For the case that $M-1$ rows in any block (i.e. \mathbf{X}^{U_i}) are occupied actually, only one row of $\bar{\mathbf{X}}^{U_i}$ is occupied. Thus, we have

$$|\text{supp}(\bar{\mathbf{X}}_{S_i}^{U_i})| \leq |\text{supp}(\bar{\mathbf{X}}^{U_i})| = 1. \quad (\text{S.9})$$

For another case that M rows are occupied actually, the length of the frequency point in each $\bar{\mathbf{X}}^{U_i}$ (\mathbf{X}^{U_i} is occupied) meets

$$l = B - (M-2)f_s \leq f_s. \quad (\text{S.10})$$

Because $l \leq f_s$, we know that the non-zero elements of any occupied partial-block (i.e. $\bar{\mathbf{X}}^{U_i}$) do not intersect on the column indices. Considering one occupied partial-block $\bar{\mathbf{X}}^{U_i}$, let $r \rightarrow \infty$ and observe the change in $\bar{\mathbf{X}}^{U_i}$, $\bar{\mathbf{X}}^{U_i}$ gradually changes from an MMV form to an SMV form. In the SMV form, the sparsity of $\bar{\mathbf{X}}^{U_i}$ is determined by the columns in $\bar{\mathbf{X}}^{U_i}$. It is observed that

$$\lim_{r \rightarrow \infty, i, j} \text{supp}(\bar{\mathbf{X}}_{:,j}^{U_i}) \leq 1. \quad (\text{S.11})$$

Thus, $|\text{supp}(\bar{\mathbf{X}}_{S_i})| = |\text{supp}(\bar{\mathbf{X}}_{:,i})|$ is less than N_{sig} (there exist N_{sig} subbands in \mathbf{X}). We proof the upper bound of the sub-MMV problems number r .

A more complex situation occurs when r is a finite value, assuming $r = r^*$ is a finite value. In this case, the length of frequency points in each sub-matrix is less than the sub-matrix columns number $\lceil \frac{f_s}{r^*} \rceil$. We use proof by contradiction to prove the condition that the sparsity of each sub-matrix is less than N_{sig} . Assuming that there exists a partial-block sub-matrix $\bar{\mathbf{X}}_{S_i}$ with $|\text{supp}(\bar{\mathbf{X}}_{S_i})| > N_{\text{sig}}$ when $r^* \in [\lceil \frac{f_s}{(M-1)f_s-B} \rceil, \infty)$. Also, the non-zero elements of any partial-block in $\bar{\mathbf{X}}$ do not intersect on the column indices. $\bar{\mathbf{X}}_{S_i}$ must contain both non-zero elements on both sides of one partial-block. From (S.10), we know that the length of any partial-block of \mathbf{X} is less than f_s . We can draw a conclusion that

$$\left\lceil \frac{f_s}{r^*} \right\rceil > f_s - l. \quad (\text{S.12})$$

Combining (S.10) and (S.12), we can get

$$\left\lceil \frac{f_s}{r^*} \right\rceil > (M-1)f_s - B. \quad (\text{S.13})$$

As can be seen, there exist a contradiction between (S.13) and the assumption $r \in [\lceil \frac{f_s}{(M-1)f_s-B} \rceil, \infty)$, so the length of partial-block non-zero elements in any sub-matrix of $\bar{\mathbf{X}}$ must be less or equal than $(M-1)f_s - B$, which is equivalent to r satisfying

$$r \geq \left\lceil \frac{f_s}{(M-1)f_s - B} \right\rceil. \quad (\text{S.14})$$

To sum up, when $r \in [\lceil \frac{f_s}{(M-1)f_s-B} \rceil, \infty)$, we have

$$\max_{i \in \{1, \dots, r\}} |\text{supp}(\bar{\mathbf{X}}_{S_i})| \leq N_{\text{sig}} < \frac{N_{\text{sig}} B}{f_s}. \quad (\text{S.15})$$

The proof is thus complete. \square

III. PROOF OF THEOREM 3

Theorem 3. Consider the column-partitioned MMV model (5) with $\min_{i,j} \|(\mathbf{X}_{S_i})_{j,:}\|_2 / \|\mathbf{X}_{S_i}\|_F = \eta$ and $|\text{supp}(\mathbf{X}_{S_i})| \leq s$. Let $s_1 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i})|$, $s_2 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i}) \cap \tilde{S}_{S_i}^k|$ and $s_3 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i}) \cap \tilde{S}_{S_i}^k \setminus S_{S_i}^k|$. Then, if the sensing matrix \mathbf{A} obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_1 \sqrt{\nu_1^2 + 4\nu_2^2} - \nu_1^2 - 1}{4\nu_1^2 \nu_2^2 - 2\nu_1^2 - 1}} \quad (\text{S.16})$$

where $\nu_1 := \frac{1+\omega}{1+\eta\omega\sqrt{s_2}}$ and $\nu_2 := \frac{1+\omega}{1+\eta\omega\sqrt{s_3}}$, SI-SSP produces an signal estimate $\mathbf{X}^k = [\mathbf{X}_{S_1}^k, \dots, \mathbf{X}_{S_r}^k]$ satisfying

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho^k \|\mathbf{X}\|_F + \tau \|\mathbf{E}\|_F, \quad (\text{S.17})$$

where $\rho \in (0, 1)$ and τ are constants depending on δ_{3s} , ν_1 and ν_2 . Furthermore, after at most $k^* = \lceil \log_{\rho} \frac{\|\mathbf{X}\|_F}{\tau \|\mathbf{E}\|_F} \rceil$ iterations, SI-SSP estimates \mathbf{X} with

$$\|\mathbf{X} - \mathbf{X}^{k^*}\|_F \leq (\tau + 1) \|\mathbf{E}\|_F. \quad (\text{S.18})$$

To prove Theorem 3, we first introduce six useful Lemmas, whose proofs are left to the appendices.

Lemma 1. ([25]): For nonnegative numbers a, b, c, d, x, y ,

$$(ax + by)^2 + (cx + dy)^2 \leq (\sqrt{a^2 + c^2}x + (b + d)y)^2. \quad (\text{S.19})$$

Lemma 2. Consider the system model $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$, where $\text{supp}(\mathbf{X}) = T$ and $|T| = s$. Let $S \subseteq \{1, 2, \dots, n\}$ be an index set with $|S| = t$ and \mathbf{W}_{T_0} be a side-information matrix with diagonal entries indexed by $T_0 \subseteq \{1, 2, \dots, n\}$ being $\omega \geq 0$ and zero otherwise. Also, let $\tilde{\mathbf{X}} := \arg \min_{\mathbf{Z}: \text{supp}(\mathbf{Z}) \subseteq S} \|\mathbf{Y} - \mathbf{A}\mathbf{Z}\|_2$. If $\delta_{3s} < 1$, then

$$\|\mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}})_S\|_F \leq \omega \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \omega \sqrt{1 + \delta_t} \|\mathbf{E}\|_F \quad (\text{S.20})$$

and

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{s+t}^2}} \|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1 + \delta_t}}{1 - \delta_{s+t}} \|\mathbf{E}\|_F. \quad (\text{S.21})$$

Furthermore, if $t > s$, define T_{∇} as the row-indices of the smallest $t - s$ row-norm entries of $\tilde{\mathbf{X}}$ in S , we have

$$\|\mathbf{X}_{T_{\nabla}}\|_F \leq \sqrt{2} \nu_2 \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_2 \sqrt{2(1 + \delta_t)} \|\mathbf{E}\|_F. \quad (\text{S.22})$$

Remark 1. When we consider the atom selection strategy of $\|\tilde{\mathbf{X}}_{T_{\nabla}} + \mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_{\nabla}}\|_F \leq \|\tilde{\mathbf{X}}_{S'} + \mathbf{W}_{T_0} \tilde{\mathbf{X}}_{S'}\|_F$, we can also obtain another upper bound for $\|\mathbf{X}_{T_{\nabla}}\|_F$ in (S.22). In this case, we should allocate $2\|\mathbf{X}_{T_{\nabla}}\|_F$ to the left hand side of (A.46), we have

$$\|\mathbf{X}_{T_{\nabla}}\|_F \leq \sqrt{2} \nu_3 \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_4 \sqrt{2(1 + \delta_t)} \|\mathbf{E}\|_F. \quad (\text{S.23})$$

where $\nu_3 = (1 - \omega + \omega \delta_{s+t} + \delta_{s+t}) / (2\delta_{s+t})$ and $\nu_4 = (1 + \omega) / (2\delta_{s+t})$.

Lemma 3. In steps 4 and 5 of SI-SSP, we have

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2} \nu_1 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \quad (\text{S.24})$$

Remark 2. When we consider the atom selection strategy in select step that

$$\begin{aligned} & \|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ & \leq \|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F. \end{aligned} \quad (\text{S.25})$$

We can also obtain another upper bound for $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ in (S.24). In this case, we should allocate $2\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ to the left hand side of (A.59), we have

$$\begin{aligned} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F & \leq \sqrt{2} \nu_4 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F \\ & + \nu_4 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \end{aligned} \quad (\text{S.26})$$

where $\nu_4 = (1 - \omega + \omega \delta_{3s} + \delta_{3s}) / (2\delta_{3s})$ and $\nu_4 = (1 + \omega) / (2\delta_{3s})$. Based on conclusions (S.23) and (S.26), we know that the sensing matrix \mathbf{A} obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_3 \sqrt{\nu_3^2 + 4\nu_4^2} - \nu_3^2 - 1}{4\nu_3^2 \nu_4^2 - 2\nu_3^2 - 1}}. \quad (\text{S.27})$$

Lemma 4. Let $T_0 \subseteq \{1, 2, \dots, n\}$, for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, if $|\text{supp}(\mathbf{u}) \cup \text{supp}(\mathbf{v})| \leq t$,

$$|\langle \mathbf{u}, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle| \leq \omega \delta_t \|\mathbf{u}\| \|\mathbf{v}\|; \quad (\text{S.28})$$

Moreover, if $U \subseteq \{1, 2, \dots, n\}$ and $|U \cup \text{supp}(\mathbf{v})| \leq t$, then

$$|(\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v}| \leq \omega \delta_t \|\mathbf{v}\|. \quad (\text{S.29})$$

Lemma 5. For SMV model $\mathbf{y} = \Phi \mathbf{x} + \mathbf{e}$, let $T_0 \subseteq \{1, 2, \dots, n\}$, let $U \subseteq \{1, 2, \dots, n\}$ and $|U \cap T_0| \leq u$, we have

$$\|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U\|_2 \leq \omega \delta_u \|\mathbf{e}\|_2. \quad (\text{S.30})$$

Lemma 6. Consider the MMV model $\mathbf{Y} = \mathbf{A} \mathbf{X} + \mathbf{E}$, let $\tilde{\mathbf{X}}$ be the solution of the least squares problem $\arg \min_{\mathbf{Z}} \{\|\mathbf{Y} - \mathbf{A} \mathbf{Z}\|_F, \text{supp}(\mathbf{Z}) \subseteq S\}$, then

$$\langle \mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}}, \mathbf{A}^H \mathbf{A} \mathbf{Z} \rangle + \omega \langle \mathbf{E}, \mathbf{A} \mathbf{Z} \rangle = 0. \quad (\text{S.31})$$

Now we have all ingredients to prove Theorem 3.

Proof of Theorem 3. First, in Steps 4 and 5 of SI-SSP, Lemma 3 implies

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2} \nu_1 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \quad (\text{S.32})$$

Note that Step 6 of SI-SSP solves a least squares problem. Let $S = \tilde{S}^k$ and $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $t = 2s$, by (S.21) we have

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{3s}^2}} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \frac{\sqrt{1 + \delta_{2s}}}{1 - \delta_{3s}} \|\mathbf{E}\|_F. \quad (\text{S.33})$$

Combining (S.32) and (S.33) and also magnifying δ_{2s} to δ_{3s} , we further have

$$\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F \leq \nu_1 \sqrt{\frac{2\delta_{3s}^2}{1 - \delta_{3s}^2}} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \tau_1 \|\mathbf{E}\|_F. \quad (\text{S.34})$$

Next, after Step 7 of SI-SSP, let $S_\nabla = \tilde{S}^k \setminus S^k$ be the row-indices of the smallest $t - s$ row norm entries in $\tilde{\mathbf{X}}^k$. Also, let $T = \tilde{S}^k$, $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $T_\nabla = S_\nabla$ and $t = 2s$. Then, by (A.45) we have

$$\|\mathbf{X}_{S_\nabla}\|_F \leq \sqrt{2} \nu_2 \delta_{3s} \|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2 \sqrt{2(1 + \delta_{2s})} \|\mathbf{E}\|_F. \quad (\text{S.35})$$

Let $\tau_1 = \frac{(\nu_1 \sqrt{2(1 - \delta_{3s})} + \sqrt{1 + \delta_{3s}})(1 - \delta_{3s})^{-1}}{\sqrt{1 + \delta_{3s}}}$. Dividing $(\tilde{S}^k)^c$ into two disjoint subsets: $(\tilde{S}^k)^c$ and S_∇ , we get

$$\begin{aligned} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2 &= \|\mathbf{X}_{S_\nabla}\|_F^2 + \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2 \\ &\stackrel{(\text{S.32}), (\text{S.35})}{\leq} 2 \left(\nu_2 \delta_{3s} \|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2 \tau_2 \|\mathbf{E}\|_F \right)^2 \\ &\quad + 2 \left(\nu_1 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \tau_2 \|\mathbf{E}\|_F \right)^2 \\ &\stackrel{(\text{S.34})}{\leq} 2 \left(\sqrt{\frac{2\nu_1^2 \nu_2^2 \delta_{3s}^4}{1 - \delta_{3s}^2}} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_2 (\tau_1 \delta_{3s} + \tau_2) \right. \\ &\quad \times \|\mathbf{E}\|_F^2 + 2 \left(\nu_1 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \tau_2 \|\mathbf{E}\|_F \right)^2 \\ &\stackrel{(\text{S.19})}{\leq} 2 \left(\sqrt{\frac{2\nu_1^2 \nu_2^2 \delta_{3s}^4}{1 - \delta_{3s}^2}} + \nu_1^2 \delta_{3s}^2 \|\mathbf{X} - \mathbf{X}^{k-1}\|_F \right. \\ &\quad \left. + ((\nu_1 + \nu_2) \tau_2 + \nu_2 \delta_{3s} \tau_1) \|\mathbf{E}\|_F \right)^2. \end{aligned} \quad (\text{S.36})$$

Squaring both sides, we get

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{\frac{4\nu_1^2 \nu_2^2 \delta_{3s}^4}{1 - \delta_{3s}^2} + 2\nu_1^2 \delta_{3s}^2} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F$$

$$+ \sqrt{2} ((\nu_1 + \nu_2) \tau_2 + \nu_2 \delta_{3s} \tau_1) \|\mathbf{E}\|_F. \quad (\text{S.37})$$

Step 9 of SI-SSP also solves a least squares problem. Letting $T = S^k$, $\tilde{\mathbf{X}} = \mathbf{X}^k$ and $t = s$, by (S.21), we have

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \sqrt{\frac{1}{1 - \delta_{2s}^2}} \|\mathbf{X}_{(S^k)^c}\|_F + \frac{\sqrt{1 + \delta_s}}{1 - \delta_{2s}} \|\mathbf{E}\|_F. \quad (\text{S.38})$$

Finally, combining (S.37) and (S.38) yields

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + (1 - \rho) \tau \|\mathbf{E}\|_F \quad (\text{S.39})$$

where $\rho := \sqrt{2} \delta_{3s} \sqrt{2\nu_1^2 \nu_2^2 \delta_{3s}^2 + \nu_1^2 - \nu_1^2 \delta_{3s}^2} (1 - \delta_{3s}^2)^{-1}$ and $\tau := \sqrt{2} \delta_{3s} \nu_2 (\nu_1 \sqrt{2(1 - \delta_{3s})} + \sqrt{1 + \delta_{3s}}) (1 - \delta_{3s}^2)^{-1/2} (1 - \delta_{3s})^{-1} (1 - \rho)^{-1} + (\nu_1 \nu_2 \sqrt{2(1 - \delta_{3s})} + \sqrt{1 + \delta_{3s}}) (1 - \delta_{3s})^{-1}$.

We recursively apply (S.39) to obtain

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho^k \|\mathbf{X}\|_F + \tau \|\mathbf{E}\|_F \quad (\text{S.40})$$

where $\rho < 1$ under (S.16). When $k^* = \lceil \log_\rho \frac{\|\mathbf{X}\|_F}{\tau \|\mathbf{E}\|_F} \rceil$, we have $\rho^{k^*} \|\mathbf{X}\|_F \leq \tau \|\mathbf{E}\|_F$, and thus the stability result (S.18). \square

APPENDIX A PROOF OF LEMMA 2

- First, we give an upper bound of $\|\mathbf{X}_{T_\nabla}\|_F$, by Lemma 6, let $\mathbf{Z} = (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S$, we have

$$\begin{aligned} &\langle \mathbf{W}_{T_0} (\mathbf{X} - \tilde{\mathbf{X}}), \mathbf{A}^H \mathbf{A} (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \rangle \\ &\quad + \langle \mathbf{W}_{T_0} \mathbf{E}, \mathbf{A} (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \rangle = 0. \end{aligned} \quad (\text{A.41})$$

Noticing that $\text{supp}(\tilde{\mathbf{X}}) \subseteq S$, we have

$$\begin{aligned} &\|(\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S\|_F^2 \\ &= \langle \mathbf{W}_{T_0} (\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \rangle \\ &\stackrel{(\text{A.41})}{=} \langle \mathbf{W}_{T_0} (\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{I}_L - \mathbf{A}^H \mathbf{A}) (\mathbf{X} - \tilde{\mathbf{X}})_S \rangle \\ &\quad - \langle \mathbf{W}_{T_0} \mathbf{E}, \mathbf{A} (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \rangle \\ &\stackrel{(7)}{\leq} \omega \delta_{s+t} \left\| (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \right\|_F \left\| \mathbf{X} - \tilde{\mathbf{X}} \right\|_F \\ &\quad + \omega \|\mathbf{E}\|_F \sqrt{1 + \delta_t} \left\| \mathbf{W}_{T_0} (\mathbf{X} - \tilde{\mathbf{X}})_S \right\|_F. \end{aligned} \quad (\text{A.42})$$

Divide both sides by $\|(\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S\|_F$ to obtain (S.20).

- Next, by expanding [Lemma 2, 25] to the MMV model, we could get a relationship between $\|\mathbf{X} - \tilde{\mathbf{X}}\|_F$ and $\|\mathbf{X}_{S^c}\|_F$. We have

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{s+t}^2}} \|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1 + \delta_t}}{1 - \delta_{s+t}} \|\mathbf{E}\|_F. \quad (\text{A.43})$$

- Then, we established the relationship between \mathbf{X}_{T_∇} and $\mathbf{X} - \tilde{\mathbf{X}}$. There exist a subset $S' \subseteq S$ and $S' \cap T = \emptyset$. Since T_∇ is defined by the set of indices of the $t - s$ smallest row entries of $\tilde{\mathbf{X}}$, we can conclude that

$$\begin{aligned} &\|\tilde{\mathbf{X}}_{T_\nabla}\|_F + \|\mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_\nabla}\|_F \\ &\leq \|\tilde{\mathbf{X}}_{S'}\|_F + \|\mathbf{W}_{T_0} \tilde{\mathbf{X}}_{S'}\|_F. \end{aligned} \quad (\text{A.44})$$

By eliminating the contribution from $T_{\nabla} \cap S'$ and noticing that $S' \cap T = \emptyset$, we have

$$\begin{aligned} \|\tilde{\mathbf{X}}_{T_{\nabla} \setminus S'}\|_F &+ \|\mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_{\nabla} \setminus S'}\|_F \\ &\leq \|(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_{\nabla}}\|_F \\ &+ \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_{\nabla}}\|_F. \end{aligned} \quad (\text{A.45})$$

For the left-hand side of (A.45), we have

$$\begin{aligned} \|\tilde{\mathbf{X}}_{T_{\nabla} \setminus S'}\|_F &+ \|\mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_{\nabla} \setminus S'}\|_F \\ &= \|(\tilde{\mathbf{X}} - \mathbf{X} + \mathbf{X})_{T_{\nabla} \setminus S'}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}) + \mathbf{W}_{T_0} \mathbf{X})_{T_{\nabla} \setminus S'}\|_F \\ &\geq \|\mathbf{X}_{T_{\nabla}}\|_F + \|\mathbf{W}_{T_0} \mathbf{X}_{T_{\nabla}}\|_F \\ &- \|(\tilde{\mathbf{X}} - \mathbf{X})_{T_{\nabla} \setminus S'}\|_F \\ &- \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{T_{\nabla} \setminus S'}\|_F. \end{aligned} \quad (\text{A.46})$$

Finally, combining (A.47) and (A.45), and noticing that

$$(T_{\nabla} \setminus S') \cap (S' \setminus T_{\nabla}) = \emptyset \quad (\text{A.48})$$

$$(T_{\nabla} \setminus S') \cup (S' \setminus T_{\nabla}) \subseteq T, \quad (\text{A.49})$$

we have

$$\begin{aligned} \|\mathbf{X}_{T_{\nabla}}\|_F &+ \|\mathbf{W}_{T_0} \mathbf{X}_{T_{\nabla}}\|_F \\ &\leq \|(\tilde{\mathbf{X}} - \mathbf{X})_{T_{\nabla} \setminus S'}\|_F + \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{T_{\nabla} \setminus S'}\|_F \\ &+ \|(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_{\nabla}}\|_F + \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{S' \setminus T_{\nabla}}\|_F \\ &\leq \sqrt{2}\|(\tilde{\mathbf{X}} - \mathbf{X})_{(S' \setminus T_{\nabla}) \cup (T_{\nabla} \setminus S')}\|_F \\ &+ \sqrt{2}\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{(S' \setminus T_{\nabla}) \cup (T_{\nabla} \setminus S')}\|_F \\ &\leq \sqrt{2}\|(\tilde{\mathbf{X}} - \mathbf{X})_S\|_F + \sqrt{2}\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_S\|_F \\ &\stackrel{(\text{S.20})}{\leq} \sqrt{2}(1 + \omega)\delta_{s+t}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\ &+ (1 + \omega)\sqrt{2(1 + \delta_t)}\|\mathbf{E}\|_F. \end{aligned} \quad (\text{A.50})$$

Also, we can obtain the relationship between $\|\mathbf{W}_{T_0} \mathbf{X}_{T_{\nabla}}\|_F$ and $\|\mathbf{X}_{T_{\nabla}}\|_F$:

$$\eta\omega\sqrt{s_3}\|\mathbf{X}_{T_{\nabla}}\|_F \leq \|\mathbf{W}_{T_0} \mathbf{X}_{T_{\nabla}}\|_F. \quad (\text{A.51})$$

Combining (A.50) and (A.51), we have

$$\begin{aligned} \|\mathbf{X}_{T_{\nabla}}\|_F &\leq \frac{\sqrt{2}(1 + \omega)\delta_{s+t}}{1 + \eta\omega\sqrt{s_3}}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\ &+ \frac{(1 + \omega)\sqrt{2(1 + \delta_t)}}{1 + \eta\omega\sqrt{s_3}}\|\mathbf{E}\|_F. \end{aligned} \quad (\text{A.52})$$

Noting the definition of ν_2 , we complete the proof of Lemma 2.

APPENDIX B PROOF OF LEMMA 3

Proof: From Step 5 of SI-SSP, we have

$$\mathbf{X}_{S_i}^k = \arg \min_{\Theta: \text{supp}(\Theta) = S_{S_i}^k} \|\mathbf{Y}_{S_i} - \mathbf{A}\Theta\|_F. \quad (\text{A.53})$$

From Step 4 of SI-SSP, let $\mathbf{X}^k = [\mathbf{X}_{S_1}^k, \dots, \mathbf{X}_{S_r}^k]$. We have the following conclusion

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \end{aligned}$$

$$\begin{aligned} &\leq \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F. \end{aligned} \quad (\text{A.54})$$

By removing the same coordinates $T \cap \Delta S$, we get

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ &\leq \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F. \end{aligned} \quad (\text{A.55})$$

Because $\text{supp}(\mathbf{X}) = T$ and $\text{supp}(\mathbf{X}^{k-1}) = S^{k-1}$,

$$(\mathbf{X} - \mathbf{X}^{k-1})_{\Delta S \setminus (T \cup S^{k-1})} = \mathbf{0}. \quad (\text{A.56})$$

For the right-hand side of (A.55), we have

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &\leq \|((\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{A}^H \mathbf{E})_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{E})_{\Delta S \setminus T}\|_F. \end{aligned} \quad (\text{A.57})$$

Note that $\tilde{S}^k = S^{k-1} \cup \Delta S$, we have

$$(\mathbf{X} - \mathbf{X}^{k-1})_{T \setminus \tilde{S}^k} = \mathbf{X}_{(\tilde{S}^k)^c}. \quad (\text{A.58})$$

For the left-hand side of (A.55), we have

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ &= \|((\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}) \\ &+ \mathbf{A}^H \mathbf{E} + \mathbf{X})_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &+ \|\mathbf{W}_{T_0}(\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}) \\ &+ \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{E} + \mathbf{W}_{T_0} \mathbf{X})_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &\geq \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \|(\mathbf{W}_{T_0} \mathbf{X})_{(\tilde{S}^k)^c}\|_F \\ &- \|((\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \\ &- \|(\mathbf{W}_{T_0}(\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \\ &- \|(\mathbf{A}^H \mathbf{E})_{(\tilde{S}^k)^c}\|_F - \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{E})_{(\tilde{S}^k)^c}\|_F. \end{aligned} \quad (\text{A.59})$$

Combining (A.61) and (A.60), we have

$$\begin{aligned} &\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \|(\mathbf{W}_{T_0} \mathbf{X})_{(\tilde{S}^k)^c}\|_F \\ &\leq \|((\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \setminus \tilde{S}^k}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \setminus \tilde{S}^k}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \end{aligned}$$

$$\begin{aligned}
& + \|((\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\
& + \|(\mathbf{A}^H \mathbf{E})_{T \setminus \tilde{S}^k}\|_F + \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{E})_{T \setminus \tilde{S}^k}\|_F \\
& + \|(\mathbf{A}^H \mathbf{E})_{\Delta S \setminus T}\|_F + \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{E})_{\Delta S \setminus T}\|_F \\
& \leq \sqrt{2} \|((\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \cup \Delta S}\|_F \\
& + \sqrt{2} \|(\mathbf{W}_{T_0} (\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \cup \Delta S}\|_F \\
& + \sqrt{2} \|(\mathbf{A}^H \mathbf{E})_{T \cup \Delta S}\|_F + \sqrt{2} \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{E})_{T \cup \Delta S}\|_F \\
& \leq \sqrt{2}(1 + \omega) \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F \\
& + (1 + \omega) \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \tag{A.61}
\end{aligned}$$

We can obtain the relationship between $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ and $\|(\mathbf{W}_{T_0} \mathbf{X})_{(\tilde{S}^k)^c}\|_F$:

$$\eta \omega \sqrt{s_2} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \|(\mathbf{W}_{T_0} \mathbf{X})_{(\tilde{S}^k)^c}\|_F. \tag{A.62}$$

Combining (A.61) and (A.62), we have

$$\begin{aligned}
\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F & \leq \frac{\sqrt{2}(1 + \omega) \delta_{3s}}{1 + \eta \omega \sqrt{s_2}} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\
& + \frac{(1 + \omega) \sqrt{2(1 + \delta_{2s})}}{1 + \eta \omega \sqrt{s_2}} \|\mathbf{E}\|_F. \tag{A.63}
\end{aligned}$$

Noting the definition of ν_1 , we complete the proof of Lemma 3.

APPENDIX C PROOF OF LEMMA 4

Proof: the RIC δ_t can be expressed as [25]

$$\delta_t = \max_{S \subseteq \{1, 2, \dots, N\}, |S| \leq t} \|\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}\|_{2 \rightarrow 2}, \tag{A.64}$$

where

$$\|\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}\|_{2 \rightarrow 2} = \sup_{\mathbf{a} \in \mathbb{R}^{|S|} \setminus \{0\}} \frac{\|(\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}) \mathbf{a}\|_2}{\|\mathbf{a}\|_2}. \tag{A.65}$$

Let $S = \text{supp}(\mathbf{u}) \cup \text{supp}(\mathbf{v})$, then $|S| \leq t$. Let $\mathbf{u}_{|S}$, $\mathbf{v}_{|S}$ denote respectively the S -dimensional sub-vectors of \mathbf{u} and \mathbf{v} obtained by only keeping the components indexed by S . We have

$$\begin{aligned}
& |\langle \mathbf{u}, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle| \\
& = |\langle \mathbf{W}_{T_0} \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{A} \mathbf{W}_{T_0} \mathbf{u}, \mathbf{A} \mathbf{v} \rangle| \\
& = |\langle \mathbf{W}_{T_0} \mathbf{u}_{|S}, (\mathbf{I}_L - \mathbf{A}_S^H \mathbf{A}_S) \mathbf{v}_{|S} \rangle| \\
& \leq \|\mathbf{W}_{T_0} \mathbf{u}_{|S}\|_2 \|(\mathbf{I}_L - \mathbf{A}_S^H \mathbf{A}_S) \mathbf{v}_{|S}\|_2 \\
& \stackrel{(A.65)}{\leq} \|\mathbf{W}_{T_0} \mathbf{u}_{|S}\|_2 \|\mathbf{I}_L - \mathbf{A}_S^H \mathbf{A}_S\|_{2 \rightarrow 2} \|\mathbf{v}_{|S}\|_2 \\
& \stackrel{(A.64)}{\leq} \omega \delta_t \|\mathbf{u}_{|T}\|_2 \|\mathbf{v}_{|S}\|_2 \\
& = \omega \delta_t \|\mathbf{u}\|_2 \|\mathbf{v}\|_2, \tag{A.66}
\end{aligned}$$

moreover, we have

$$\begin{aligned}
& \|((\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v})_U\|_2^2 \\
& = \langle ((\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v})_U, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle \\
& \stackrel{(S.28)}{\leq} \delta_t \|((\mathbf{W}_{T_0} - \mathbf{A}^H \mathbf{A}) \mathbf{v})_U\|_2 \|\mathbf{v}\|_2 \tag{A.67}
\end{aligned}$$

which completes the proof of Lemma 4.

APPENDIX D PROOF OF LEMMA 5

Proof: The lemma follows trivially from the fact that

$$\begin{aligned}
& \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U\|_2^2 \\
& = \langle \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e}, (\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U \rangle \\
& = \langle \mathbf{e}, \mathbf{W}_{T_0} \mathbf{A} ((\mathbf{A}^H \mathbf{e})_U) \rangle \\
& \leq \|\mathbf{e}\|_2 \|\mathbf{W}_{T_0} \mathbf{A} ((\mathbf{A}^H \mathbf{e})_U)\|_2 \\
& \stackrel{(7)}{\leq} \|\mathbf{e}'\|_2 \omega \sqrt{1 + \delta_u} \|(\mathbf{A}^H \mathbf{e})_U\|_2. \tag{A.68}
\end{aligned}$$

APPENDIX E PROOF OF LEMMA 6

Proof: Due to the orthogonality, the residue $\mathbf{Y} - \mathbf{A} \tilde{\mathbf{X}}$ is orthogonal to the space $\mathbf{A} \mathbf{Z}$. This means that for all $\mathbf{Z} \in \mathbb{C}^{L \times N}$ with $\text{supp}(\mathbf{Z}) \subseteq S$,

$$\langle \mathbf{A}(\mathbf{Y} - \mathbf{A} \tilde{\mathbf{X}}), \mathbf{Z} \rangle = 0. \tag{A.69}$$

Let $\tilde{\mathbf{X}}'$ be the solution of the least squares problem $\arg \min_{\mathbf{Z}} \{\|\mathbf{Y}' - \mathbf{A} \mathbf{Z}\|_F, \text{supp}(\mathbf{Z}) \subseteq S\}$, where $\mathbf{Y}' = \frac{\mathbf{A} \mathbf{W}_{T_0} \mathbf{X}_{T_0}}{\omega} + \mathbf{E}$. We have

$$\tilde{\mathbf{X}}' = \frac{\mathbf{W}_{T_0} \tilde{\mathbf{X}}}{\omega}. \tag{A.70}$$

Then, by (A.69), we have

$$\begin{aligned}
0 & = \left\langle \frac{\mathbf{A} \mathbf{W}_{T_0} \mathbf{X}_{T_0}}{\omega} + \mathbf{E} - \mathbf{A} \frac{\mathbf{W}_{T_0} \tilde{\mathbf{X}}}{\omega}, \mathbf{A} \mathbf{Z} \right\rangle \\
& = \left\langle \mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}}, \mathbf{A}^H \mathbf{A} \mathbf{Z} \right\rangle + \omega \langle \mathbf{E}, \mathbf{A} \mathbf{Z} \rangle. \tag{A.71}
\end{aligned}$$