Supplementary: Boundary Multiple Measurement Vectors for Multi-Coset Sampler

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This supplementary material is dedicated to the proofs for Theorems 1–3 in our main paper.

Before proceeding to the proofs, we review some useful notations. For a complex matrix $\mathbf{X} \in \mathbb{C}^{n\times L}$ and a set $S \subseteq \{1,\cdots,n\}$, $\mathbf{X}_S$ (or $\mathbf{X}^S$) denotes the submatrix of $\mathbf{X}$ with columns (or rows) indexed by S; $\mathbf{X}_{i,j}$, $\mathbf{X}_{i,:}$ and $\mathbf{X}_{:,i}$ are the (i,j)th entry, ith row and ith column of $\mathbf{X}$, respectively; $\mathbf{X}^\dagger$, $\mathbf{X}^H$ and $\mathbf{X}^\top$ mean the Moore-Penrose pseudo-inverse, conjugate transpose and transpose of $\mathbf{X}$, respectively; $\text{supp}(\mathbf{X})$ is the non-zero row indices (i.e., joint sparsity) of $\mathbf{X}$; $\|\mathbf{X}\|_F$ and $\|\mathbf{X}\|_2$ signify the Frobenius and Euclidean norm of $\mathbf{X}$, respectively. Moreover, S^c is the complement of set S; $\mathbf{I}_L$ is an $L\times L$ identity matrix.

I. Proof of Theorem 1

Theorem 1. *The actual sampling rate of (4) is* $\min\left(pf_s, f_{nyq}\right)$, *which attains the theoretical lower bound of sampling rate in MCS when* $|supp(\mathbf{X})| \leq \frac{N_{sig}B}{f_s}$.

Proof. In the ith channel of a multi-coset sampler, the sampling sequence is given by

$$x_{c_i}[n] = x(LTn+\tau_i), \quad n=0,1,\cdots. \tag{S.1}$$

The sampling rate of each channel is determined by the sampled signal sequence. To be specific, since the sampling time interval is LT, the sampling rate of each channel is

$$f_s = \frac{1}{LT} = \frac{f_{\text{nyq}}}{L}, \tag{S.2}$$

i.e., one-Lth of the Nyquist sampling rate.

Moreover, as the multi-coset sampler is assumed to have p channels, the overall sampling rate of p channels is p times that of each channel (i.e. $\frac{pf_{\text{nyq}}}{L}$). If this sampling rate is greater than the Nyquist rate f_{nyq}, then the advantage of sub-Nyquist sampling structure no longer exists. In this case, we only need to sample at Nyquist sampling rate f_{nyq}. Thus, the actual sampling rate can be given by

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right). \tag{S.3}$$

The theoretical lower bound of the sampling rate is given in [17], which is determined directly by the true bandwidth of the signal:

$$\min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \tag{S.4}$$

Thus, the theoretical lower bound on the sampling rate is achieved when

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right) \leq \min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \tag{S.5}$$

In most cases, $2\lambda(\mathcal{T})$ and $\frac{p}{LT}$ do not exceed f_{nyq}. (If violated, the sampling rate would just be f_{nyq}.) Therefore, the condition (S.5) holds whenever

$$\frac{p}{LT} \leq 2\lambda(\mathcal{T}). \tag{S.6}$$

Furthermore, to ensure an unique-solution reconstruction, the number p of channels should not be too small. In particular, it's lower bound is twice the signal sparsity without the priori information about the signal $\mathbf{X}$ [17],

$$p \geq 2|\text{supp}(\mathbf{X})|. \tag{S.7}$$

For the worst case where $p = 2|\text{supp}(\mathbf{X})|$, (S.6) can be rewritten as

$$|\text{supp}(\mathbf{X})| \leq \lambda(\mathcal{T})LT = \frac{N_{\text{sig}}B}{f_s}, \tag{S.8}$$

which completes the proof. □

II. Proof of Theorem 2

Theorem 2. *When* $r \in [\lceil \frac{f_s}{(M-1)f_s - B}\rceil, \infty)$ *and* $B > f_s$, *we have* $\max_{i\in\{1,\cdots,r\}} \left|supp(\bar{\mathbf{X}}_{S_i})\right| \leq \frac{N_{sig}B}{f_s}$.

Proof. Review that we decompose the MMV model $\mathbf{Y} = \mathbf{AX}+\mathbf{E}$ into r sub-MMV problems and solve each problem individually

$$\mathbf{Y}_{S_i} = \mathbf{A}\mathbf{X}_{S_i} + \mathbf{E}_{S_i}, \quad i=1,\cdots,r. \tag{S.9}$$

Theorem 2 indicates the number of sub-MMV problems that ensure reaching the lower bound of the theoretical sampling rate.

Consider all row blocks $\left\{\mathbf{X}^{U_1},\cdots,\mathbf{X}^{U_M}\right\}$ in $\mathbf{X}$, where there are consecutive corresponding frequency points of length B (the sub-band's width) in occupied blocks. For the case $B > f_s$, we assume that $(D-2)f_s < B \leq (D-1)f_s$ and $D \geq 3$ to represent any relationship between B and f_s. And we can select a block's height with D rows. As shown in Fig. 1, the frequency points of the sub-band signal may occupy $D-1$ (the PU signal 1 and 2) or D (the PU signal 3) rows actually.

For the case that $D-1$ rows in block (i.e. $\mathbf{X}^{U_i}$) are occupied actually, only one row of $\bar{\mathbf{X}}^{U_i}$ is occupied. Thus, we have

$$\left|\text{supp}(\bar{\mathbf{X}}^{U_i}_{S_i})\right| \leq \left|\text{supp}(\bar{\mathbf{X}}^{U_i})\right| = 1. \tag{S.10}$$

For another case that D rows are occupied actually, the length of the frequency point in $\bar{\mathbf{X}}^{U_i}$ ($\mathbf{X}^{U_i}$ is occupied) meets

$$l = B-(M-2)f_s \leq f_s. \tag{S.11}$$

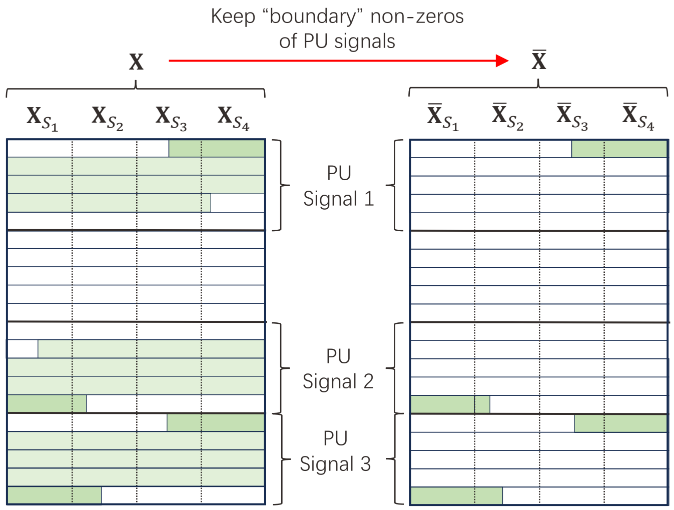

Fig. 1. An illustrative example of MCS signal $\mathbf{X}$ with 3 PU signals.

Because $l \leq f_s$, we know that the non-zero elements of any occupied partial-block (i.e. $\bar{\mathbf{X}}^{U_i}$) do not intersect on the column indices. Considering one occupied partial-block $\bar{\mathbf{X}}^{U_i}$, let $r \to \infty$ and observe the change in $\bar{\mathbf{X}}^{U_i}$, $\bar{\mathbf{X}}^{U_i}$ gradually changes from an MMV form to an SMV form. In the SMV form, the sparsity of $\bar{\mathbf{X}}^{U_i}$ is determined by the columns in $\bar{\mathbf{X}}^{U_i}$. It is observed that

$$\lim_{r\to\infty,i,j} \left|\text{supp}(\bar{\mathbf{X}}^{U_i}_{S_j})\right| = \left|\text{supp}(\bar{\mathbf{X}}^{U_i}_{:,j})\right| \leq 1. \tag{S.12}$$

Thus, $|\text{supp}(\bar{\mathbf{X}}_{S_i})| = |\text{supp}(\bar{\mathbf{X}}_{:,i})|$ is less than N_{sig} (there exist N_{sig} subbands in $\mathbf{X}$). We proof the upper bound of the sub-MMV problems number r.

A more complex situation occurs when r is a finite value, assuming $r = r^\star$ is a finite value. In this case, the length of frequency points in each sub-matrix is less than the sub-matrix columns number $\left\lceil \frac{f_s}{r^\star} \right\rceil$. We use proof by contradiction to prove the condition that the sparsity of each sub-matrix is less than N_{sig}. Assuming that there exists a partial-block sub-matrix $\bar{\mathbf{X}}_{S_i}$ with $\text{supp}(\bar{\mathbf{X}}_{S_i}) > N_{\text{sig}}$ when $r^\star \in [\lceil \frac{f_s}{(M-1)f_s - B} \rceil, \infty)$. Also, the non-zero elements of any partial-block in $\bar{\mathbf{X}}$ do not intersect on the column indices. $\bar{\mathbf{X}}_{S_i}$ must contain both non-zero elements on both sides of one partial-block. From (S.11), we know that the length of any partial-block of $\mathbf{X}$ is less than f_s. We can draw a conclusion that

$$\left\lceil \frac{f_s}{r^\star} \right\rceil > f_s - l. \tag{S.13}$$

Combining (S.11) and (S.13), we can get

$$\left\lceil \frac{f_s}{r^\star} \right\rceil > (M-1)f_s - B. \tag{S.14}$$

As can be seen, there exist a contradiction between (S.14) and the assumption $r \in [\lceil \frac{f_s}{(M-1)f_s - B} \rceil, \infty)$, so the length of partial-block non-zero elements in any sub-matrix of $\bar{\mathbf{X}}$ must be less or equal than $(M-1)f_s - B$, which is equivalent to r satisfying

$$r \geq \left\lceil \frac{f_s}{(M-1)f_s - B} \right\rceil. \tag{S.15}$$

To sum up, when $r \in [\lceil \frac{f_s}{(M-1)f_s - B} \rceil, \infty)$, we have

$$\max_{i\in\{1,\cdots,r\}} \left|\text{supp}(\bar{\mathbf{X}}_{S_i})\right| \leq N_{\text{sig}} < \frac{N_{\text{sig}} B}{f_s}. \tag{S.16}$$

The proof is thus complete. □

III. Proof of Theorem 3

Theorem 3. *Consider the column-partitioned MMV model (5) with* $\min_{i,j} \|(\mathbf{X}_{S_i})_{j,:}\|_2 / \|\mathbf{X}_{S_i}\|_F = \eta$ *and* $|supp(\mathbf{X}_{S_i})| \leq s$. *Let* $s_1 := \min_{i,k} |\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i})|$, $s_2 := \min_{i,k} |\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i}) \cap \tilde{S}^k_{S_i}|$ *and* $s_3 := \min_{i,k} |\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i}) \cap \tilde{S}^k_{S_i} \setminus S^k_{S_i}|$. *Then, if the sensing matrix* $\mathbf{A}$ *obeys the RIP with*

$$\delta_{3s} \leq \sqrt{\frac{\nu_1 \sqrt{\nu_1^2 + 4\nu_2^2} - \nu_1^2 - 1}{4\nu_1^2 \nu_2^2 - 2\nu_1^2 - 1}} \tag{S.17}$$

where $\nu_1 := \frac{1+\omega}{1+\eta\omega\sqrt{s_2}}$ *and* $\nu_2 := \frac{1+\omega}{1+\eta\omega\sqrt{s_3}}$, *SI-SSP produces an signal estimate* $\mathbf{X}^k = [\mathbf{X}^k_{S_1}, \cdots, \mathbf{X}^k_{S_r}]$ *satisfying*

$$\left\|\mathbf{X} - \mathbf{X}^k\right\|_F \leq \rho^k \left\|\mathbf{X}\right\|_F + \tau \left\|\mathbf{E}\right\|_F, \tag{S.18}$$

where $\rho \in (0,1)$ *and* τ *are constants depending on* δ_{3s}, ν_1 *and* ν_2. *Furthermore, after at most* $k^* = \lceil \log_\rho \frac{\|\mathbf{X}\|_F}{\tau\|\mathbf{E}\|_F} \rceil$ *iterations, SI-SSP estimates* $\mathbf{X}$ *with*

$$\|\mathbf{X} - \mathbf{X}^{k^*}\|_F \leq (\tau + 1)\|\mathbf{E}\|_F. \tag{S.19}$$

To prove Theorem 3, we first introduce six useful Lemmas, whose proofs are left to the appendices.

Lemma 1. *([25]): For nonnegative numbers a, b, c, d, x, y,*

$$(ax+by)^2 + (cx+dy)^2 \leq (\sqrt{a^2+c^2}x + (b+d)y)^2. \tag{S.20}$$

Lemma 2. *Consider the system model* $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$, *where* $supp(\mathbf{X}) = T$ *and* $|T| = s$. *Let* $S \subseteq \{1, 2, ..., n\}$ *be an index set with* $|S| = t$ *and* $\mathbf{W}_{T_0}$ *be a side-information matrix with diagonal entries indexed by* $T_0 \subseteq \{1, 2, ..., n\}$ *being* $\omega \geq 0$ *and zero otherwise. Also, let* $\tilde{\mathbf{X}} := \arg\min_{\mathbf{Z}: supp(\mathbf{Z}) \subseteq S} \|\mathbf{Y} - \mathbf{A}\mathbf{Z}\|_2$. *If* $\delta_{3s} < 1$, *then*

$$\|\mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}})_S\|_F \leq \omega\delta_{s+t}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \omega\sqrt{1+\delta_t}\|\mathbf{E}\|_F \tag{S.21}$$

and

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1-\delta^2_{s+t}}}\|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1+\delta_t}}{1-\delta_{s+t}}\|\mathbf{E}\|_F. \tag{S.22}$$

Furthermore, if $t > s$, *define* $T_\triangledown$ *as the row-indices of the smallest* $t - s$ *row-norm entries of* $\tilde{\mathbf{X}}$ *in* S, *we have*

$$\|\mathbf{X}_{T_\triangledown}\|_F \leq \sqrt{2}\nu_2\delta_{s+t}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_2\sqrt{2\left(1+\delta_t\right)}\|\mathbf{E}\|_F. \tag{S.23}$$

Remark 1. *When we consider the atom selection strategy of* $\|\tilde{\mathbf{X}}_{T_\triangledown} + \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\triangledown}\|_F \leq \|\tilde{\mathbf{X}}_{S'} + \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{S'}\|_F$, *we can also obtain another upper bound for* $\|\mathbf{X}_{T_\triangledown}\|_F$ *in (S.23). In this case, we should allocate* $2\left\|\mathbf{X}_{T_\triangledown}\right\|_F$ *to the left hand side of (A.47), we have*

$$\|\mathbf{X}_{T_\triangledown}\|_F \leq \sqrt{2}\nu_3\delta_{s+t}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_4\sqrt{2\left(1+\delta_t\right)}\left\|\mathbf{E}\right\|_F. \tag{S.24}$$

where $\nu_3 = (1-\omega+\omega\delta_{s+t}+\delta_{s+t})/(2\delta_{s+t})$ *and* $\nu_4 = (1+\omega)/(2\delta_{s+t})$.

Lemma 3. *In steps 4 and 5 of SI-SSP, we have*

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \le \sqrt{2}\nu_1\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F + \nu_1\sqrt{2(1+\delta_{3s})}\,\|\mathbf{E}\|_F. \tag{S.25}$$

Remark 2. *When we consider the atom selection strategy in select step that*

$$\begin{aligned}&\left\|((\mathbf{I}_L+\mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_T\right\|_F\\ &\quad\le \left\|((\mathbf{I}_L+\mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\right\|_F.\end{aligned} \tag{S.26}$$

We can also obtain another upper bound for $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ *in* (S.25)*. In this case, we should allocate* $2\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ *to the left hand side of* (A.60)*, we have*

$$\begin{aligned}\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \le{}& \sqrt{2}\nu_4\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F\\ &+\nu_4\sqrt{2\,(1+\delta_{3s})}\,\|\mathbf{E}\|_F.\end{aligned} \tag{S.27}$$

where $\nu_4 = (1-\omega+\omega\delta_{3s}+\delta_{3s})/(2\delta_{3s})$ *and* $\nu_4 = (1+\omega)/(2\delta_{3s})$*. Based on conclusions* (S.24) *and* (S.27)*, we know that the sensing matrix* $\mathbf{A}$ *obeys the RIP with*

$$\delta_{3s} \le \sqrt{\frac{\nu_3\sqrt{\nu_3^2+4\nu_4^2}-\nu_3^2-1}{4\nu_3^2\nu_4^2-2\nu_3^2-1}}. \tag{S.28}$$

Lemma 4. *Let* $T_0 \subseteq \{1,2,...,n\}$*, for two vectors* $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$*, if* $|supp(\mathbf{u})\cup supp(\mathbf{v})| \le t$*,*

$$\left|\left\langle \mathbf{u},(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\mathbf{v}\right\rangle\right| \le \omega\delta_t\,\|\mathbf{u}\|\,\|\mathbf{v}\|; \tag{S.29}$$

Moreover, if $U\subseteq\{1,2,...,n\}$ *and* $|U\cup supp(\mathbf{v})|\le t$*, then*

$$\left|(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\mathbf{v}\right| \le \omega\delta_t\,\|\mathbf{v}\|. \tag{S.30}$$

Lemma 5. *For SMV model* $\mathbf{y}=\mathbf{\Phi}\mathbf{x}+\mathbf{e}$*, let* $T_0\subseteq\{1,2,...,n\}$*, let* $U\subseteq\{1,2,...,n\}$ *and* $|U\cap T_0|\le u$*, we have*

$$\left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e})_U\right\|_2 \le \omega\delta_u\,\|\mathbf{e}\|_2. \tag{S.31}$$

Lemma 6. *Consider the MMV model* $\mathbf{Y}=\mathbf{A}\mathbf{X}+\mathbf{E}$*, let* $\tilde{\mathbf{X}}$ *be the solution of the least squares problem* $\arg\min_{\mathbf{Z}}\{\|\mathbf{Y}-\mathbf{A}\mathbf{Z}\|_F, supp(\mathbf{Z})\subseteq S\}$*, then*

$$\langle\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}},\mathbf{A}^H\mathbf{A}\mathbf{Z}\rangle+\omega\langle\mathbf{E},\mathbf{A}\mathbf{Z}\rangle=\mathbf{0}. \tag{S.32}$$

Now we have all ingredients to prove Theorem 3.

Proof of Theorem 3. First, in Steps 4 and 5 of SI-SSP, Lemma 3 implies

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \le \sqrt{2}\nu_1\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F + \nu_1\sqrt{2(1+\delta_{3s})}\,\|\mathbf{E}\|_F. \tag{S.33}$$

Note that Step 6 of SI-SSP solves a least squares problem. Let $S=\tilde{S}^k$ and $\tilde{\mathbf{X}}=\tilde{\mathbf{X}}^k$, $t=2s$, by (S.22) we have

$$\|\mathbf{X}-\tilde{\mathbf{X}}\|_F \le \sqrt{\frac{1}{1-\delta_{3s}^2}}\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F+\frac{\sqrt{1+\delta_{2s}}}{1-\delta_{3s}}\|\mathbf{E}\|_F. \tag{S.34}$$

Combining (S.33) and (S.34) and also magnifying δ_{2s} to δ_{3s}, we further have

$$\|\mathbf{X}-\tilde{\mathbf{X}}^k\|_F \le \nu_1\sqrt{\frac{2\delta_{3s}^2}{1-\delta_{3s}^2}}\|\mathbf{X}-\mathbf{X}^{k-1}\|_F+\tau_1\|\mathbf{E}\|_F. \tag{S.35}$$

Next, after Step 7 of SI-SSP, let $S_\nabla=\tilde{S}^k\setminus S^k$ be the row-indices of the smallest $t-s$ row norm entries in $\tilde{\mathbf{X}}^k$. Also, let $T=\tilde{S}^k$, $\tilde{\mathbf{X}}=\tilde{\mathbf{X}}^k$, $T_\nabla=S_\nabla$ and $t=2s$. Then, by (A.46) we have

$$\|\mathbf{X}_{S_\nabla}\|_F \le \sqrt{2}\nu_2\delta_{3s}\|\mathbf{X}-\tilde{\mathbf{X}}^k\|_F+\nu_2\sqrt{2\,(1+\delta_{2s})}\|\mathbf{E}\|_F. \tag{S.36}$$

Let $\tau_1 = (\nu_1\sqrt{2(1-\delta_{3s})}+\sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$ and $\tau_2=\sqrt{1+\delta_{3s}}$. Dividing $(S^k)^c$ into two disjoint subsets: $(\tilde{S}^k)^c$ and S_∇, we get

$$\begin{aligned}\left\|\mathbf{X}_{(S^k)^c}\right\|_F^2 ={}& \|\mathbf{X}_{S_\nabla}\|_F^2+\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2\\
\overset{(S.33),\,(S.36)}{\le}{}& 2\left(\nu_2\delta_{3s}\|\mathbf{X}-\tilde{\mathbf{X}}^k\|_F+\nu_2\tau_2\|\mathbf{E}\|_F\right)^2\\
&+2\left(\nu_1\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_2+\nu_1\tau_2\|\mathbf{E}\|_F\right)^2\\
\overset{(S.35)}{\le}{}& 2\left(\sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F+\nu_2(\tau_1\delta_{3s}+\tau_2)\right.\\
&\left.\times\|\mathbf{E}\|_F\right)^2+2\left(\nu_1\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F+\nu_1\tau_2\|\mathbf{E}\|_F\right)^2\\
\overset{(S.20)}{\le}{}& 2\left(\sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}+\nu_1^2\delta_{3s}^2}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F\right.\\
&\left.+((\nu_1+\nu_2)\tau_2+\nu_2\delta_{3s}\tau_1)\|\mathbf{E}\|_F\right)^2.\end{aligned} \tag{S.37}$$

Squaring both sides, we get

$$\begin{aligned}\left\|\mathbf{X}_{(S^k)^c}\right\|_F \le{}& \sqrt{\tfrac{4\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}+2\nu_1^2\delta_{3s}^2}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F\\ &+\sqrt{2}\,((\nu_1+\nu_2)\tau_2+\nu_2\delta_{3s}\tau_1)\|\mathbf{E}\|_F.\end{aligned} \tag{S.38}$$

Step 9 of SI-SSP also solves a least squares problem. Letting $T=S^k$, $\tilde{\mathbf{X}}=\mathbf{X}^k$ and $t=s$, by (S.22), we have

$$\|\mathbf{X}-\mathbf{X}^k\|_F \le \sqrt{\frac{1}{1-\delta_{2s}^2}}\|\mathbf{X}_{(S^k)^c}\|_F+\frac{\sqrt{1+\delta_s}}{1-\delta_{2s}}\|\mathbf{E}\|_F. \tag{S.39}$$

Finally, combining (S.38) and (S.39) yields

$$\|\mathbf{X}-\mathbf{X}^k\|_F \le \rho\|\mathbf{X}-\mathbf{X}^{k-1}\|_F+(1-\rho)\tau\|\mathbf{E}\|_F \tag{S.40}$$

where $\rho := \sqrt{2}\delta_{3s}\sqrt{2\nu_1^2\nu_2^2\delta_{3s}^2+\nu_1^2-\nu_1^2\delta_{3s}^2}(1-\delta_{3s}^2)^{-1}$ and $\tau := \sqrt{2}\delta_{3s}\nu_2(\nu_1\sqrt{2(1-\delta_{3s})}+\sqrt{1+\delta_{3s}})(1-\delta_{3s}^2)^{-1/2}(1-\delta_{3s})^{-1}(1-\rho)^{-1}+(\nu_1\nu_2\sqrt{2(1-\delta_{3s})}+\sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$.
We recursively apply (S.40) to obtain

$$\|\mathbf{X}-\mathbf{X}^k\|_F \le \rho^k\|\mathbf{X}\|_F+\tau\|\mathbf{E}\|_F \tag{S.41}$$

where $\rho<1$ under (S.17). When $k^*=\lceil\log_\rho\frac{\|\mathbf{X}\|_F}{\tau\|\mathbf{E}\|_F}\rceil$, we have $\rho^k\|\mathbf{X}\|_F\le\tau\|\mathbf{E}\|_F$, and thus the stability result (S.19). ☐

Appendix A
Proof of Lemma 2

- First, we give a upper bound of $\|\mathbf{X}_{T_\nabla}\|_F$, by Lemma 6, let $\mathbf{Z}=(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S$, we have

$$\begin{aligned}&\left\langle\mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}}),\mathbf{A}^H\mathbf{A}(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle\\ &\quad+\left\langle\mathbf{W}_{T_0}\mathbf{E},\mathbf{A}(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle=\mathbf{0}.\end{aligned} \tag{A.42}$$

  Noticing that $\text{supp}(\tilde{\mathbf{X}})\subseteq S$, we have

$$\|(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\|_F^2$$

$$
\begin{aligned}
&= \left\langle \mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}}), (\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle \\
&\overset{(A.42)}{=} \left\langle \mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}}), (\mathbf{I}_L-\mathbf{A}^H\mathbf{A})(\mathbf{X}-\tilde{\mathbf{X}})_S \right\rangle \\
&- \left\langle \mathbf{W}_{T_0}\mathbf{E}, \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle \\
&\overset{(7)}{\leq} \omega\delta_{s+t}\left\|(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\|_F\left\|\mathbf{X}-\tilde{\mathbf{X}}\right\|_F \\
&+\omega\left\|\mathbf{E}\right\|_F\sqrt{1+\delta_t}\left\|\mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}})_S\right\|_F. \quad (A.43)
\end{aligned}
$$

Divide both sides by $\|(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\|_F$ to obtain (S.21).

- Next, by expanding [Lemma 2, 25] to the MMV model, we could get a relationship between $\|\mathbf{X}-\tilde{\mathbf{X}}\|_F$ and $\|\mathbf{X}_{S^c}\|_F$. We have

$$
\|\mathbf{X}-\tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1-\delta_{s+t}^2}\|\mathbf{X}_{S^c}\|_F^2}+\frac{\sqrt{1+\delta_t}}{1-\delta_{s+t}}\|\mathbf{E}\|_F. \quad (A.44)
$$

- Then, we established the relationship between $\mathbf{X}_{T_\triangledown}$ and $\mathbf{X}-\tilde{\mathbf{X}}$. There exist a subset $S'\subseteq S$ and $S'\cap T=\emptyset$. Since $T_\triangledown$ is defined by the set of indices of the $t-s$ smallest row entries of $\tilde{\mathbf{X}}$, we can conclude that

$$
\begin{aligned}
&\|\tilde{\mathbf{X}}_{T_\triangledown}\|_F+\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\triangledown}\|_F \\
\leq\ &\|\tilde{\mathbf{X}}_{S'}\|_F+\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{S'}\|_F. \quad (A.45)
\end{aligned}
$$

By eliminating the contribution from $T_\triangledown\cap S'$ and noticing that $S'\cap T=\emptyset$, we have

$$
\begin{aligned}
\|\tilde{\mathbf{X}}_{T_\triangledown\setminus S'}\|_F &+ \|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\triangledown\setminus S'}\|_F \\
&\leq \|(\tilde{\mathbf{X}}-\mathbf{X})_{S'\setminus T_\triangledown}\|_F \\
&+ \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X})_{S'\setminus T_\triangledown}\|_F. \quad (A.46)
\end{aligned}
$$

For the left-hand side of (A.46), we have

$$
\begin{aligned}
&\|\tilde{\mathbf{X}}_{T_\triangledown\setminus S'}\|_F+\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\triangledown\setminus S'}\|_F \\
&= \|(\tilde{\mathbf{X}}-\mathbf{X}+\mathbf{X})_{T_\triangledown\setminus S'}\|_F \\
&+ \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X})+\mathbf{W}_{T_0}\mathbf{X})_{T_\triangledown\setminus S'}\|_F \\
&\geq \|\mathbf{X}_{T_\triangledown}\|_F+\|\mathbf{W}_{T_0}\mathbf{X}_{T_\triangledown}\|_F \quad (A.47) \\
&- \|(\tilde{\mathbf{X}}-\mathbf{X})_{T_\triangledown\setminus S'}\|_F \\
&- \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X})_{T_\triangledown\setminus S'}\|_F. \quad (A.48)
\end{aligned}
$$

Finally, combining (A.48) and (A.46), and noticing that

$$
\begin{aligned}
(T_\triangledown\setminus S')\cap(S'\setminus T_\triangledown) &= \emptyset \quad (A.49) \\
(T_\triangledown\setminus S')\cup(S'\setminus T_\triangledown) &\subseteq T, \quad (A.50)
\end{aligned}
$$

we have

$$
\begin{aligned}
&\|\mathbf{X}_{T_\triangledown}\|_F+\|\mathbf{W}_{T_0}\mathbf{X}_{T_\triangledown}\|_F \\
&\leq \|(\tilde{\mathbf{X}}-\mathbf{X})_{T_\triangledown\setminus S'}\|_F+\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_{T_\triangledown\setminus S'}\|_F \\
&+ \|(\tilde{\mathbf{X}}-\mathbf{X})_{S'\setminus T_\triangledown}\|_F+\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_{S'\setminus T_\triangledown}\|_F \\
&\leq \sqrt{2}\|(\tilde{\mathbf{X}}-\mathbf{X})_{(S'\setminus T_\triangledown)\cup(T_\triangledown\setminus S')}\|_F \\
&+ \sqrt{2}\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_{(S'\setminus T_\triangledown)\cup(T_\triangledown\setminus S')}\|_F \\
&\leq \sqrt{2}\|(\tilde{\mathbf{X}}-\mathbf{X})_S\|_F+\sqrt{2}\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_S\|_F \\
&\overset{(S.21)}{\leq} \sqrt{2}(1+\omega)\delta_{s+t}\|\mathbf{X}-\tilde{\mathbf{X}}\|_F \\
&+ (1+\omega)\sqrt{2(1+\delta_t)}\|\mathbf{E}\|_F. \quad (A.51)
\end{aligned}
$$

Also, we can obtain the relationship between $\|\mathbf{W}_{T_0}\mathbf{X}_{T_\triangledown}\|_F$ and $\|\mathbf{X}_{T_\triangledown}\|_F$:

$$
\eta\omega\sqrt{s_3}\,\|\mathbf{X}_{T_\triangledown}\|_F \leq \|\mathbf{W}_{T_0}\mathbf{X}_{T_\triangledown}\|_F. \quad (A.52)
$$

Combining (A.51) and (A.52), we have

$$
\begin{aligned}
\|\mathbf{X}_{T_\triangledown}\|_F &\leq \frac{\sqrt{2}(1+\omega)\delta_{s+t}}{1+\eta\omega\sqrt{s_3}}\|\mathbf{X}-\tilde{\mathbf{X}}\|_F \\
&+ \frac{(1+\omega)\sqrt{2(1+\delta_t)}}{1+\eta\omega\sqrt{s_3}}\|\mathbf{E}\|_F. \quad (A.53)
\end{aligned}
$$

Noting the definition of ν_2, we complete the proof of Lemma 2.

Appendix B
Proof of Lemma 3

Proof: From Step 5 of SI-SSP, we have

$$
\mathbf{X}_{S_i}^k = \underset{\mathbf{\Theta}:\mathrm{supp}(\mathbf{\Theta})=S_{S_i}^k}{\arg\min}\|\mathbf{Y}_{S_i}-\mathbf{A}\mathbf{\Theta}\|_F. \quad (A.54)
$$

From Step 4 of SI-SSP, let $\mathbf{X}^k=[\mathbf{X}_{S_1}^k,\cdots,\mathbf{X}_{S_r}^k]$. We have the following conclusion

$$
\begin{aligned}
&\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\
&\leq \|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F. \quad (A.55)
\end{aligned}
$$

By removing the same coordinates $T\cap\Delta S$, we get

$$
\begin{aligned}
&\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\|_F \\
&\leq \|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F. \quad (A.56)
\end{aligned}
$$

Because $\mathrm{supp}(\mathbf{X})=T$ and $\mathrm{supp}(\mathbf{X}^{k-1})=S^{k-1}$,

$$
(\mathbf{X}-\mathbf{X}^{k-1})_{\Delta S\setminus(T\cup S^{k-1})}=0. \quad (A.57)
$$

For the right-hand side of (A.56), we have

$$
\begin{aligned}
&\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\
&= \|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\|_F \\
&= \|(\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\|_F \\
&\leq \|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\
&+ \|(\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\|_F \\
&+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\|_F. \quad (A.58)
\end{aligned}
$$

Note that $\tilde{S}^k=S^{k-1}\cup\Delta S$, we have

$$
(\mathbf{X}-\mathbf{X}^{k-1})_{T\setminus\tilde{S}^k}=\mathbf{X}_{(\tilde{S}^k)^c}. \quad (A.59)
$$

For the left-hand side of (A.56), we have

$$
\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\|_F
$$

$$
\begin{aligned}
&+ \quad \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\|_F \\
&= \quad \|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\|_F \\
&+ \quad \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\|_F \\
&= \quad \|(\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\|_F \\
&+ \quad \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\|_F \\
&= \quad \|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}) \\
&+ \quad \mathbf{A}^H\mathbf{E}+\mathbf{X})_{\Delta S\setminus(T\cup S^{k-1})}\| \\
&+ \quad \|\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}) \\
&+ \quad \mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E}+\mathbf{W}_{T_0}\mathbf{X})_{\Delta S\setminus(T\cup S^{k-1})}\|_F \\
&\geq \quad \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F+\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F \\
&- \quad \|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \qquad \text{(A.60)} \\
&- \quad \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \\
&- \quad \|(\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\|_F-\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\|_F. \qquad \text{(A.61)}
\end{aligned}
$$

Combining (A.62) and (A.61), we have

$$
\begin{aligned}
&\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F+\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F \\
&\leq \quad \|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\setminus\tilde{S}^k}\|_F \\
&+ \quad \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\setminus\tilde{S}^k}\|_F \\
&+ \quad \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\
&+ \quad \|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\
&+ \quad \|(\mathbf{A}^H\mathbf{E})_{T\setminus\tilde{S}^k}\|_F+\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T\setminus\tilde{S}^k}\|_F \\
&+ \quad \|(\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\|_F+\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\|_F \\
&\leq \quad \sqrt{2}\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\cup\Delta S}\|_F \\
&+ \quad \sqrt{2}\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\cup\Delta S}\|_F \\
&+ \quad \sqrt{2}\|(\mathbf{A}^H\mathbf{E})_{T\cup\Delta S}\|_F+\sqrt{2}\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T\cup\Delta S}\|_F \\
&\leq \quad \sqrt{2}(1+\omega)\delta_{3s}\|\mathbf{X}-\mathbf{X}^{k-1}\|_F \\
&+ \quad (1+\omega)\sqrt{2(1+\delta_{3s})}\|\mathbf{E}\|_F. \qquad \text{(A.62)}
\end{aligned}
$$

We can obtain the relationship between $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ $\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F$:

$$\eta\omega\sqrt{s_2}\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F. \qquad \text{(A.63)}$$

Combining (A.62) and (A.63), we have

$$
\begin{aligned}
\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq & \frac{\sqrt{2}(1+\omega)\delta_{3s}}{1+\eta\omega\sqrt{s_2}}\|\mathbf{X}-\tilde{\mathbf{X}}\|_F \\
& + \frac{(1+\omega)\sqrt{2(1+\delta_{2s})}}{1+\eta\omega\sqrt{s_2}}\|\mathbf{E}\|_F. \qquad \text{(A.64)}
\end{aligned}
$$

Noting the definition of ν_1, we complete the proof of Lemma 3.

Appendix C
Proof of Lemma 4

Proof: the RIC δ_t can be expressed as [25]

$$\delta_t = \max_{S\subseteq\{1,2,\ldots,N\},|S|\leq t}\|\mathbf{A}_S^*\mathbf{A}_S-\mathbf{I}\|_{2\to 2}, \qquad \text{(A.65)}$$

where

$$\|\mathbf{A}_S^*\mathbf{A}_S-\mathbf{I}\|_{2\to 2} = \sup_{\mathbf{a}\in\mathbb{R}^{|S|}\setminus\{\mathbf{0}\}}\frac{\|(\mathbf{A}_S^*\mathbf{A}_S-\mathbf{I})\,\mathbf{a}\|_2}{\|\mathbf{a}\|_2}. \qquad \text{(A.66)}$$

Let $S = \operatorname{supp}(\mathbf{u})\cup\operatorname{supp}(\mathbf{v})$, then $|S|\leq t$. Let $\mathbf{u}_{|S}, \mathbf{v}_{|S}$ denote respectively the S-dimensional sub-vectors of $\mathbf{u}$ and $\mathbf{v}$ obtained by only keeping the components indexed by S. We have

$$
\begin{aligned}
& |\langle\mathbf{u},\left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\rangle| \\
= \quad & |\langle\mathbf{W}_{T_0}\mathbf{u},\mathbf{v}\rangle-\langle\mathbf{A}\mathbf{W}_{T_0}\mathbf{u},\mathbf{A}\mathbf{v}\rangle| \\
= \quad & |\langle\mathbf{W}_{T_0}\mathbf{u}_{|S},\left(\mathbf{I}_L-\mathbf{A}_S^H\mathbf{A}_S\right)\mathbf{v}_{|S}\rangle| \\
\leq \quad & \|\mathbf{W}_{T_0}\mathbf{u}_{|S}\|_2\left\|\left(\mathbf{I}_L-\mathbf{A}_S^H\mathbf{A}_S\right)\mathbf{v}_{|S}\right\|_2 \\
\overset{\text{(A.66)}}{\leq} \quad & \|\mathbf{W}_{T_0}\mathbf{u}_{|S}\|_2\|\mathbf{I}_L-\mathbf{A}_S^H\mathbf{A}_S\|_{2\to 2}\|\mathbf{v}_{|S}\|_2 \\
\overset{\text{(A.65)}}{\leq} \quad & \omega\delta_t\|\mathbf{u}_{|T}\|_2\|\mathbf{v}_{|S}\|_2 \\
= \quad & \omega\delta_t\|\mathbf{u}\|_2\|\mathbf{v}\|_2, \qquad \text{(A.67)}
\end{aligned}
$$

moreover, we have

$$
\begin{aligned}
& \|\left(\left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\right)_U\|_2^2 \\
= \quad & \langle\left(\left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\right)_U, \\
& \left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\rangle \\
\overset{\text{(S.29)}}{\leq} \quad & \delta_t\left\|\left(\left(\mathbf{W}_{T_0}-\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\right)_U\right\|_2\|\mathbf{v}\|_2 \qquad \text{(A.68)}
\end{aligned}
$$

which completes the proof of Lemma 4.

Appendix D
Proof of Lemma 5

Proof: The lemma follows trivially from the fact that

$$
\begin{aligned}
& \left\|\left(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e}\right)_U\right\|_2^2 \\
& = \left\langle\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e},\left(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e}\right)_U\right\rangle \\
& = \left\langle\mathbf{e},\mathbf{W}_{T_0}\mathbf{A}\left(\left(\mathbf{A}^H\mathbf{e}\right)_U\right)\right\rangle \qquad \text{(A.69)} \\
& \leq \|\mathbf{e}\|_2\left\|\mathbf{W}_{T_0}\mathbf{A}\left(\left(\mathbf{A}^H\mathbf{e}\right)_U\right)\right\|_2 \\
& \overset{(7)}{\leq} \|\mathbf{e}'\|_2\,\omega\sqrt{1+\delta_u}\left\|\left(\mathbf{A}^H\mathbf{e}\right)_U\right\|_2.
\end{aligned}
$$

Appendix E
Proof of Lemma 6

Proof: Due to the orthogonality, the residue $\mathbf{Y}-\mathbf{A}\tilde{\mathbf{X}}$ is orthogonal to the space $\mathbf{A}\mathbf{Z}$. This means that for all $\mathbf{Z}\in\mathbb{C}^{L\times N}$ with $\operatorname{supp}(\mathbf{Z})\subseteq S$,

$$\langle\mathbf{A}(\mathbf{Y}-\mathbf{A}\tilde{\mathbf{X}}),\mathbf{Z}\rangle = \mathbf{0}. \qquad \text{(A.70)}$$

Let $\tilde{\mathbf{X}}'$ be the solution of the least squares problem $\arg\min_{\mathbf{Z}}\left\{\|\mathbf{Y}'-\mathbf{A}\mathbf{Z}\|_F, \operatorname{supp}(\mathbf{Z})\subseteq S\right\}$, where $\mathbf{Y}' = \frac{\mathbf{A}\mathbf{W}_{T_0}\mathbf{X}_{T_0}}{\omega}+\mathbf{E}$. We have

$$\tilde{\mathbf{X}}' = \frac{\mathbf{W}_{T_0}\tilde{\mathbf{X}}}{\omega}. \qquad \text{(A.71)}$$

Then, by (A.70), we have

$$
\begin{aligned}
\mathbf{0} \quad = \quad & \left\langle\frac{\mathbf{A}\mathbf{W}_{T_0}\mathbf{X}_{T_0}}{\omega}+\mathbf{E}-\mathbf{A}\frac{\mathbf{W}_{T_0}\tilde{X}}{\omega},\mathbf{A}\mathbf{Z}\right\rangle \\
= \quad & \left\langle\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}},\mathbf{A}^H\mathbf{A}\mathbf{Z}\right\rangle+\omega\left\langle\mathbf{E},\mathbf{A}\mathbf{Z}\right\rangle. \qquad \text{(A.72)}
\end{aligned}
$$