

# Supplementary: Boundary Multiple Measurement Vectors for Multi-Coset Sampler

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This supplementary material is dedicated to the proofs for Theorems 1–3 in our main paper.

Before proceeding to the proofs, we review some useful notations. For a complex matrix  $\mathbf{X} \in \mathbb{C}^{n \times L}$  and a set  $S \subseteq \{1, \dots, n\}$ ,  $\mathbf{X}_S$  (or  $\mathbf{X}^S$ ) denotes the submatrix of  $\mathbf{X}$  with columns (or rows) indexed by  $S$ ;  $\mathbf{X}_{i,j}$ ,  $\mathbf{X}_{i,:}$  and  $\mathbf{X}_{:,i}$  are the  $(i, j)$ th entry,  $i$ th row and  $i$ th column of  $\mathbf{X}$ , respectively;  $\mathbf{X}^\dagger$ ,  $\mathbf{X}^H$  and  $\mathbf{X}^\top$  mean the Moore-Penrose pseudo-inverse, conjugate transpose and transpose of  $\mathbf{X}$ , respectively;  $\text{supp}(\mathbf{X})$  is the non-zero row indices (i.e., joint sparsity) of  $\mathbf{X}$ ;  $\|\mathbf{X}\|_F$  and  $\|\mathbf{X}\|_2$  signify the Frobenius and Euclidean norm of  $\mathbf{X}$ , respectively. Moreover,  $S^c$  is the complement of set  $S$ ;  $\mathbf{I}_L$  is an  $L \times L$  identity matrix.

## I. PROOF OF THEOREM 1

**Theorem 1.** *The actual sampling rate of (4) is  $\min(p f_s, f_{\text{nyq}})$ , which attains the theoretical lower bound of sampling rate in MCS when  $|\text{supp}(\mathbf{X})| \leq \frac{N_{\text{sig}} B}{f_s}$ .*

*Proof.* In the  $i$ th channel of a multi-coset sampler, the sampling sequence is given by

$$x_{ci}[n] = x(LTn + \tau_i), \quad n = 0, 1, \dots \quad (\text{S.1})$$

The sampling rate of each channel is determined by the sampled signal sequence. To be specific, since the sampling time interval is  $LT$ , the sampling rate of each channel is

$$f_s = \frac{1}{LT} = \frac{f_{\text{nyq}}}{L}, \quad (\text{S.2})$$

i.e., one- $L$ th of the Nyquist sampling rate.

Moreover, as the multi-coset sampler is assumed to have  $p$  channels, the overall sampling rate of  $p$  channels is  $p$  times that of each channel (i.e.  $\frac{p f_{\text{nyq}}}{L}$ ). If this sampling rate is greater than the Nyquist rate  $f_{\text{nyq}}$ , then the advantage of sub-Nyquist sampling structure no longer exists. In this case, we only need to sample at Nyquist sampling rate  $f_{\text{nyq}}$ . Thus, the actual sampling rate can be given by

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right). \quad (\text{S.3})$$

The theoretical lower bound of the sampling rate is given in [17], which is determined directly by the true bandwidth of the signal:

$$\min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \quad (\text{S.4})$$

Thus, the theoretical lower bound on the sampling rate is achieved when

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right) \leq \min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \quad (\text{S.5})$$

In most cases,  $2\lambda(\mathcal{T})$  and  $\frac{p}{LT}$  do not exceed  $f_{\text{nyq}}$ . (If violated, the sampling rate would just be  $f_{\text{nyq}}$ .) Therefore, the condition (S.5) holds whenever

$$\frac{p}{LT} \leq 2\lambda(\mathcal{T}). \quad (\text{S.6})$$

Furthermore, to ensure a unique-solution reconstruction, the number  $p$  of channels should not be too small. In particular, its lower bound is twice the signal sparsity without the priori information about the signal  $\mathbf{X}$  [17],

$$p \geq 2|\text{supp}(\mathbf{X})|. \quad (\text{S.7})$$

For the worst case where  $p = 2|\text{supp}(\mathbf{X})|$ , (S.6) can be rewritten as

$$|\text{supp}(\mathbf{X})| \leq \lambda(\mathcal{T})LT = \frac{N_{\text{sig}} B}{f_s}, \quad (\text{S.8})$$

which completes the proof.  $\square$

## II. PROOF OF THEOREM 2

**Theorem 2.** *When  $r \in [\lceil \frac{f_s}{(M-1)f_s - B} \rceil, \infty)$  and  $B > f_s$ , we have  $\max_{i \in \{1, \dots, r\}} |\text{supp}(\bar{\mathbf{X}}_{S_i})| \leq \frac{N_{\text{sig}} B}{f_s}$ .*

*Proof.* Consider all blocks  $\{\mathbf{X}^{U_1}, \dots, \mathbf{X}^{U_M}\}$  in  $\mathbf{X}$ , where there are consecutive corresponding frequency points of length  $B$  in occupied blocks. For the case  $B > f_s$  and each block occupies  $M$  rows in  $\mathbf{X}$ , we can assume that  $(M-2)f_s < B \leq (M-1)f_s$  for  $M \geq 3$ . The length of the frequency point in each  $\bar{\mathbf{X}}^{U_i}$  ( $\mathbf{X}^{U_i}$  is occupied) meets

$$l = B - (M-2)f_s \leq f_s. \quad (\text{S.9})$$

Because  $l \leq f_s$ , we know that the non-zero elements of any occupied partial-block (i.e.  $\bar{\mathbf{X}}^{U_i}$ ) do not intersect on the column indices. Considering one occupied partial-block  $\bar{\mathbf{X}}^{U_i}$ , let  $r \rightarrow \infty$  and observe the change in  $\bar{\mathbf{X}}^{U_i}$ ,  $\bar{\mathbf{X}}^{U_i}$  gradually changes from an MMV form to an SMV form. In the SMV form, the sparsity of  $\bar{\mathbf{X}}^{U_i}$  is determined by the columns in  $\bar{\mathbf{X}}^{U_i}$ . It is observed that

$$\lim_{r \rightarrow \infty, i, j} \text{supp}(\bar{\mathbf{X}}_{:,j}^{U_i}) \leq 1. \quad (\text{S.10})$$

Thus,  $|\text{supp}(\bar{\mathbf{X}}_{S_i})| = |\text{supp}(\bar{\mathbf{X}}_{:,i})|$  is less than  $N_{\text{sig}}$  (there exist  $N_{\text{sig}}$  subbands in  $\mathbf{X}$ ). We proof the upper bound of the sub-MMV problems number  $r$ .

A more complex situation occurs when  $r$  is a finite value, assuming  $r = r^*$  is a finite value. In this case, the length of frequency points in each sub-matrix is less than the sub-matrix columns number  $\lceil \frac{f_s}{r^*} \rceil$ . We use proof by contradiction to prove the condition that the sparsity of each sub-matrix is less than

$N_{\text{sig}}$ . Assuming that there exists a partial-block sub-matrix  $\tilde{\mathbf{X}}_{S_i}$  with  $\text{supp}(\tilde{\mathbf{X}}_{S_i}) > N_{\text{sig}}$  when  $r^* \in [\lceil \frac{f_s}{(M-1)f_s-B} \rceil, \infty)$ . Also, the non-zero elements of any partial-block in  $\tilde{\mathbf{X}}$  do not intersect on the column indices.  $\tilde{\mathbf{X}}_{S_i}$  must contain both non-zero elements on both sides of one partial-block. From (S.9), we know that the length of any partial-block of  $\mathbf{X}$  is less than  $f_s$ . We can draw a conclusion that

$$\left\lceil \frac{f_s}{r^*} \right\rceil > f_s - l. \quad (\text{S.11})$$

Combining (S.9) and (S.11), we can get

$$\left\lceil \frac{f_s}{r^*} \right\rceil > (M-1)f_s - B. \quad (\text{S.12})$$

As can be seen, there exist a contradiction between (S.12) and the assumption  $r \in [\lceil \frac{f_s}{(M-1)f_s-B} \rceil, \infty)$ , so the length of partial-block non-zero elements in any sub-matrix of  $\tilde{\mathbf{X}}$  must be less or equal than  $(M-1)f_s - B$ , which is equivalent to  $r$  satisfying

$$r \geq \left\lceil \frac{f_s}{(M-1)f_s - B} \right\rceil. \quad (\text{S.13})$$

To sum up, when  $r \in [\lceil \frac{f_s}{(M-1)f_s-B} \rceil, \infty)$ , we have

$$\max_{i \in \{1, \dots, r\}} |\text{supp}(\tilde{\mathbf{X}}_{S_i})| \leq N_{\text{sig}} < \frac{N_{\text{sig}} B}{f_s}. \quad (\text{S.14})$$

The proof is thus complete.  $\square$

### III. PROOF OF THEOREM 3

**Theorem 3.** Consider the column-partitioned MMV model (5) with  $\min_{i,j} \|(\mathbf{X}_{S_i})_{j,:}\|_2 / \|\mathbf{X}_{S_i}\|_F = \eta$  and  $|\text{supp}(\mathbf{X}_{S_i})| \leq s$ . Let  $s_1 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i})|$ ,  $s_2 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i}) \cap \tilde{S}_{S_i}^k|$  and  $s_3 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i}) \cap \tilde{S}_{S_i}^k \setminus S_{S_i}^k|$ . Then, if the sensing matrix  $\mathbf{A}$  obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_1 \sqrt{\nu_1^2 + 4\nu_2^2} - \nu_1^2 - 1}{4\nu_1^2 \nu_2^2 - 2\nu_1^2 - 1}} \quad (\text{S.15})$$

where  $\nu_1 := \frac{1+\omega}{1+\eta\omega\sqrt{s_2}}$  and  $\nu_2 := \frac{1+\omega}{1+\eta\omega\sqrt{s_3}}$ , SI-SSP produces an signal estimate  $\mathbf{X}^k = [\mathbf{X}_{S_1}^k, \dots, \mathbf{X}_{S_r}^k]$  satisfying

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho^k \|\mathbf{X}\|_F + \tau \|\mathbf{E}\|_F, \quad (\text{S.16})$$

where  $\rho \in (0, 1)$  and  $\tau$  are constants depending on  $\delta_{3s}$ ,  $\nu_1$  and  $\nu_2$ . Furthermore, after at most  $k^* = \lceil \log_\rho \frac{\|\mathbf{X}\|_F}{\tau \|\mathbf{E}\|_F} \rceil$  iterations, SI-SSP estimates  $\mathbf{X}$  with

$$\|\mathbf{X} - \mathbf{X}^{k^*}\|_F \leq (\tau + 1) \|\mathbf{E}\|_F. \quad (\text{S.17})$$

To prove Theorem 3, we first introduce six useful Lemmas, whose proofs are left to the appendices.

**Lemma 1.** ([25]): For nonnegative numbers  $a, b, c, d, x, y$ ,

$$(ax + by)^2 + (cx + dy)^2 \leq (\sqrt{a^2 + c^2}x + (b + d)y)^2. \quad (\text{S.18})$$

**Lemma 2.** Consider the system model  $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$ , where  $\text{supp}(\mathbf{X}) = T$  and  $|T| = s$ . Let  $S \subseteq \{1, 2, \dots, n\}$  be an index set with  $|S| = t$  and  $\mathbf{W}_{T_0}$  be a side-information matrix with diagonal entries indexed by  $T_0 \subseteq$

$\{1, 2, \dots, n\}$  being  $\omega \geq 0$  and zero otherwise. Also, let  $\tilde{\mathbf{X}} := \arg \min_{\mathbf{Z}: \text{supp}(\mathbf{Z}) \subseteq S} \|\mathbf{Y} - \mathbf{A}\mathbf{Z}\|_2$ . If  $\delta_{3s} < 1$ , then

$$\|\mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}})_S\|_F \leq \omega \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \omega \sqrt{1 + \delta_t} \|\mathbf{E}\|_F \quad (\text{S.19})$$

and

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{s+t}^2}} \|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1 + \delta_t}}{1 - \delta_{s+t}} \|\mathbf{E}\|_F. \quad (\text{S.20})$$

Furthermore, if  $t > s$ , define  $T_\nabla$  as the row-indices of the smallest  $t - s$  row-norm entries of  $\tilde{\mathbf{X}}$  in  $S$ , we have

$$\|\mathbf{X}_{T_\nabla}\|_F \leq \sqrt{2\nu_2 \delta_{s+t}} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_2 \sqrt{2(1 + \delta_t)} \|\mathbf{E}\|_F. \quad (\text{S.21})$$

**Remark 1.** When we consider the atom selection strategy of  $\|\tilde{\mathbf{X}}_{T_\nabla} + \mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_\nabla}\|_F \leq \|\tilde{\mathbf{X}}_{S'} + \mathbf{W}_{T_0} \tilde{\mathbf{X}}_{S'}\|_F$ , we can also obtain another upper bound for  $\|\mathbf{X}_{T_\nabla}\|_F$  in (S.21). In this case, we should allocate  $2\|\mathbf{X}_{T_\nabla}\|_F$  to the left hand side of (A.45), we have

$$\|\mathbf{X}_{T_\nabla}\|_F \leq \sqrt{2\nu_3 \delta_{s+t}} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_4 \sqrt{2(1 + \delta_t)} \|\mathbf{E}\|_F. \quad (\text{S.22})$$

where  $\nu_3 = (1 - \omega + \omega \delta_{s+t} + \delta_{s+t}) / (2\delta_{s+t})$  and  $\nu_4 = (1 + \omega) / (2\delta_{s+t})$ .

**Lemma 3.** In steps 4 and 5 of SI-SSP, we have

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2\nu_1 \delta_{3s}} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \quad (\text{S.23})$$

**Remark 2.** When we consider the atom selection strategy in select step that

$$\begin{aligned} & \|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ & \leq \|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F. \end{aligned} \quad (\text{S.24})$$

We can also obtain another upper bound for  $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$  in (S.23). In this case, we should allocate  $2\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$  to the left hand side of (A.58), we have

$$\begin{aligned} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F & \leq \sqrt{2\nu_4 \delta_{3s}} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F \\ & + \nu_4 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \end{aligned} \quad (\text{S.25})$$

where  $\nu_4 = (1 - \omega + \omega \delta_{3s} + \delta_{3s}) / (2\delta_{3s})$  and  $\nu_4 = (1 + \omega) / (2\delta_{3s})$ . Based on conclusions (S.22) and (S.25), we know that the sensing matrix  $\mathbf{A}$  obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_3 \sqrt{\nu_3^2 + 4\nu_4^2} - \nu_3^2 - 1}{4\nu_3^2 \nu_4^2 - 2\nu_3^2 - 1}}. \quad (\text{S.26})$$

**Lemma 4.** Let  $T_0 \subseteq \{1, 2, \dots, n\}$ , for two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ , if  $|\text{supp}(\mathbf{u}) \cup \text{supp}(\mathbf{v})| \leq t$ ,

$$|\langle \mathbf{u}, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle| \leq \omega \delta_t \|\mathbf{u}\| \|\mathbf{v}\|; \quad (\text{S.27})$$

Moreover, if  $U \subseteq \{1, 2, \dots, n\}$  and  $|U \cup \text{supp}(\mathbf{v})| \leq t$ , then

$$|(\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v}| \leq \omega \delta_t \|\mathbf{v}\|. \quad (\text{S.28})$$

**Lemma 5.** For SMV model  $\mathbf{y} = \Phi \mathbf{x} + \mathbf{e}$ , let  $T_0 \subseteq \{1, 2, \dots, n\}$ , let  $U \subseteq \{1, 2, \dots, n\}$  and  $|U \cap T_0| \leq u$ , we have

$$\|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U\|_2 \leq \omega \delta_u \|\mathbf{e}\|_2. \quad (\text{S.29})$$

**Lemma 6.** Consider the MMV model  $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$ , let  $\tilde{\mathbf{X}}$  be the solution of the least squares problem  $\arg \min_{\mathbf{Z}} \{\|\mathbf{Y} - \mathbf{A}\mathbf{Z}\|_F, \text{supp}(\mathbf{Z}) \subseteq S\}$ , then

$$\langle \mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}}, \mathbf{A}^H \mathbf{A} \mathbf{Z} \rangle + \omega \langle \mathbf{E}, \mathbf{A} \mathbf{Z} \rangle = \mathbf{0}. \quad (\text{S.30})$$

Now we have all ingredients to prove Theorem 3.

*Proof of Theorem 3.* First, in Steps 4 and 5 of SI-SSP, Lemma 3 implies

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2}\nu_1\delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1\sqrt{2(1+\delta_{3s})} \|\mathbf{E}\|_F. \quad (\text{S.31})$$

Note that Step 6 of SI-SSP solves a least squares problem. Let  $S = \tilde{S}^k$  and  $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$ ,  $t = 2s$ , by (S.20) we have

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1-\delta_{3s}^2}} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \frac{\sqrt{1+\delta_{2s}}}{1-\delta_{3s}} \|\mathbf{E}\|_F. \quad (\text{S.32})$$

Combining (S.31) and (S.32) and also magnifying  $\delta_{2s}$  to  $\delta_{3s}$ , we further have

$$\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F \leq \nu_1 \sqrt{\frac{2\delta_{3s}^2}{1-\delta_{3s}^2}} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \tau_1 \|\mathbf{E}\|_F. \quad (\text{S.33})$$

Next, after Step 7 of SI-SSP, let  $S_\nabla = \tilde{S}^k \setminus S^k$  be the row-indices of the smallest  $t-s$  row norm entries in  $\tilde{\mathbf{X}}^k$ . Also, let  $T = \tilde{S}^k$ ,  $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$ ,  $T_\nabla = S_\nabla$  and  $t = 2s$ . Then, by (A.44) we have

$$\|\mathbf{X}_{S_\nabla}\|_F \leq \sqrt{2}\nu_2\delta_{3s} \|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2\sqrt{2(1+\delta_{2s})} \|\mathbf{E}\|_F. \quad (\text{S.34})$$

Let  $\tau_1 = (\nu_1\sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$  and  $\tau_2 = \sqrt{1+\delta_{3s}}$ . Dividing  $(S^k)^c$  into two disjoint subsets:  $(\tilde{S}^k)^c$  and  $S_\nabla$ , we get

$$\begin{aligned} \|\mathbf{X}_{(S^k)^c}\|_F^2 &= \|\mathbf{X}_{S_\nabla}\|_F^2 + \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2 \\ &\stackrel{(\text{S.31}), (\text{S.34})}{\leq} 2 \left( \nu_2\delta_{3s} \|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2\tau_2 \|\mathbf{E}\|_F \right)^2 \\ &\quad + 2 \left( \nu_1\delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1\tau_2 \|\mathbf{E}\|_F \right)^2 \\ &\stackrel{(\text{S.33})}{\leq} 2 \left( \sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_2(\tau_1\delta_{3s} + \tau_2) \right. \\ &\quad \times \|\mathbf{E}\|_F^2 + 2 \left( \nu_1\delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1\tau_2 \|\mathbf{E}\|_F \right)^2 \\ &\stackrel{(\text{S.18})}{\leq} 2 \left( \sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}} + \nu_1^2\delta_{3s}^2 \|\mathbf{X} - \mathbf{X}^{k-1}\|_F \right. \\ &\quad \left. + ((\nu_1 + \nu_2)\tau_2 + \nu_2\delta_{3s}\tau_1) \|\mathbf{E}\|_F \right)^2. \end{aligned} \quad (\text{S.35})$$

Squaring both sides, we get

$$\begin{aligned} \|\mathbf{X}_{(S^k)^c}\|_F &\leq \sqrt{\frac{4\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2} + 2\nu_1^2\delta_{3s}^2} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F \\ &\quad + \sqrt{2}((\nu_1 + \nu_2)\tau_2 + \nu_2\delta_{3s}\tau_1) \|\mathbf{E}\|_F. \end{aligned} \quad (\text{S.36})$$

Step 9 of SI-SSP also solves a least squares problem. Letting  $T = S^k$ ,  $\tilde{\mathbf{X}} = \mathbf{X}^k$  and  $t = s$ , by (S.20), we have

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \sqrt{\frac{1}{1-\delta_{2s}^2}} \|\mathbf{X}_{(S^k)^c}\|_F + \frac{\sqrt{1+\delta_{2s}}}{1-\delta_{2s}} \|\mathbf{E}\|_F. \quad (\text{S.37})$$

Finally, combining (S.36) and (S.37) yields

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + (1-\rho)\tau \|\mathbf{E}\|_F \quad (\text{S.38})$$

where  $\rho := \sqrt{2}\delta_{3s}\sqrt{2\nu_1^2\nu_2^2\delta_{3s}^2 + \nu_1^2 - \nu_1^2\delta_{3s}^2}(1-\delta_{3s}^2)^{-1}$  and  $\tau := \sqrt{2}\delta_{3s}\nu_2(\nu_1\sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}})(1-\delta_{3s}^2)^{-1/2}(1-\delta_{3s}^2)^{-1}(1-\rho)^{-1} + (\nu_1\nu_2\sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}})(1-\delta_{3s}^2)^{-1}$ .

We recursively apply (S.38) to obtain

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho^k \|\mathbf{X}\|_F + \tau \|\mathbf{E}\|_F \quad (\text{S.39})$$

where  $\rho < 1$  under (S.15). When  $k^* = \lceil \log_\rho \frac{\|\mathbf{X}\|_F}{\tau \|\mathbf{E}\|_F} \rceil$ , we have  $\rho^{k^*} \|\mathbf{X}\|_F \leq \tau \|\mathbf{E}\|_F$ , and thus the stability result (S.17).  $\square$

## APPENDIX A PROOF OF LEMMA 2

- First, we give an upper bound of  $\|\mathbf{X}_{T_\nabla}\|_F$ , by Lemma 6, let  $\mathbf{Z} = (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S$ , we have

$$\begin{aligned} &\langle \mathbf{W}_{T_0} (\mathbf{X} - \tilde{\mathbf{X}}), \mathbf{A}^H \mathbf{A} (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \rangle \\ &\quad + \langle \mathbf{W}_{T_0} \mathbf{E}, \mathbf{A} (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \rangle = \mathbf{0}. \end{aligned} \quad (\text{A.40})$$

Noticing that  $\text{supp}(\tilde{\mathbf{X}}) \subseteq S$ , we have

$$\begin{aligned} &\|(\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S\|_F^2 \\ &= \langle \mathbf{W}_{T_0} (\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \rangle \\ &\stackrel{(\text{A.40})}{=} \langle \mathbf{W}_{T_0} (\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{I}_L - \mathbf{A}^H \mathbf{A}) (\mathbf{X} - \tilde{\mathbf{X}})_S \rangle \\ &\quad - \langle \mathbf{W}_{T_0} \mathbf{E}, \mathbf{A} (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \rangle \\ &\stackrel{(7)}{\leq} \omega \delta_{s+t} \left\| (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \right\|_F \left\| \mathbf{X} - \tilde{\mathbf{X}} \right\|_F \\ &\quad + \omega \|\mathbf{E}\|_F \sqrt{1+\delta_t} \left\| \mathbf{W}_{T_0} (\mathbf{X} - \tilde{\mathbf{X}})_S \right\|_F. \end{aligned} \quad (\text{A.41})$$

Divide both sides by  $\|(\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S\|_F$  to obtain (S.19).

- Next, by expanding [Lemma 2, 25] to the MMV model, we could get a relationship between  $\|\mathbf{X} - \tilde{\mathbf{X}}\|_F$  and  $\|\mathbf{X}_{S^c}\|_F$ . We have

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1-\delta_{s+t}^2}} \|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1+\delta_t}}{1-\delta_{s+t}} \|\mathbf{E}\|_F. \quad (\text{A.42})$$

- Then, we established the relationship between  $\mathbf{X}_{T_\nabla}$  and  $\mathbf{X} - \tilde{\mathbf{X}}$ . There exist a subset  $S' \subseteq S$  and  $S' \cap T = \emptyset$ . Since  $T_\nabla$  is defined by the set of indices of the  $t-s$  smallest row entries of  $\tilde{\mathbf{X}}$ , we can conclude that

$$\begin{aligned} &\|\tilde{\mathbf{X}}_{T_\nabla}\|_F + \|\mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_\nabla}\|_F \\ &\leq \|\tilde{\mathbf{X}}_{S'}\|_F + \|\mathbf{W}_{T_0} \tilde{\mathbf{X}}_{S'}\|_F. \end{aligned} \quad (\text{A.43})$$

By eliminating the contribution from  $T_\nabla \cap S'$  and noticing that  $S' \cap T = \emptyset$ , we have

$$\begin{aligned} &\|\tilde{\mathbf{X}}_{T_\nabla \setminus S'}\|_F + \|\mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_\nabla \setminus S'}\|_F \\ &\leq \|(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_\nabla}\|_F \\ &\quad + \|\mathbf{W}_{T_0} (\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_\nabla}\|_F. \end{aligned} \quad (\text{A.44})$$

For the left-hand side of (A.44), we have

$$\|\tilde{\mathbf{X}}_{T_\nabla \setminus S'}\|_F + \|\mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_\nabla \setminus S'}\|_F$$

$$\begin{aligned}
&= \|(\tilde{\mathbf{X}} - \mathbf{X} + \mathbf{X})_{T_{\nabla} \setminus S'}\|_F \\
&+ \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}) + \mathbf{W}_{T_0}\mathbf{X})_{T_{\nabla} \setminus S'}\|_F \\
&\geq \|\mathbf{X}_{T_{\nabla}}\|_F + \|\mathbf{W}_{T_0}\mathbf{X}_{T_{\nabla}}\|_F \quad (\text{A.45}) \\
&- \|(\tilde{\mathbf{X}} - \mathbf{X})_{T_{\nabla} \setminus S'}\|_F \\
&- \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{T_{\nabla} \setminus S'}\|_F. \quad (\text{A.46})
\end{aligned}$$

Finally, combining (A.46) and (A.44), and noticing that

$$(T_{\nabla} \setminus S') \cap (S' \setminus T_{\nabla}) = \emptyset \quad (\text{A.47})$$

$$(T_{\nabla} \setminus S') \cup (S' \setminus T_{\nabla}) \subseteq T, \quad (\text{A.48})$$

we have

$$\begin{aligned}
&\|\mathbf{X}_{T_{\nabla}}\|_F + \|\mathbf{W}_{T_0}\mathbf{X}_{T_{\nabla}}\|_F \\
&\leq \|(\tilde{\mathbf{X}} - \mathbf{X})_{T_{\nabla} \setminus S'}\|_F + \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{T_{\nabla} \setminus S'}\|_F \\
&+ \|(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_{\nabla}}\|_F + \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{S' \setminus T_{\nabla}}\|_F \\
&\leq \sqrt{2}\|(\tilde{\mathbf{X}} - \mathbf{X})_{(S' \setminus T_{\nabla}) \cup (T_{\nabla} \setminus S')}\|_F \\
&+ \sqrt{2}\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{(S' \setminus T_{\nabla}) \cup (T_{\nabla} \setminus S')}\|_F \\
&\leq \sqrt{2}\|(\tilde{\mathbf{X}} - \mathbf{X})_S\|_F + \sqrt{2}\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_S\|_F \\
&\stackrel{(\text{S.19})}{\leq} \sqrt{2}(1 + \omega)\delta_{s+t}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\
&+ (1 + \omega)\sqrt{2(1 + \delta_t)}\|\mathbf{E}\|_F. \quad (\text{A.49})
\end{aligned}$$

Also, we can obtain the relationship between  $\|\mathbf{W}_{T_0}\mathbf{X}_{T_{\nabla}}\|_F$  and  $\|\mathbf{X}_{T_{\nabla}}\|_F$ :

$$\eta\omega\sqrt{s_3}\|\mathbf{X}_{T_{\nabla}}\|_F \leq \|\mathbf{W}_{T_0}\mathbf{X}_{T_{\nabla}}\|_F. \quad (\text{A.50})$$

Combining (A.49) and (A.50), we have

$$\begin{aligned}
\|\mathbf{X}_{T_{\nabla}}\|_F &\leq \frac{\sqrt{2}(1 + \omega)\delta_{s+t}}{1 + \eta\omega\sqrt{s_3}}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\
&+ \frac{(1 + \omega)\sqrt{2(1 + \delta_t)}}{1 + \eta\omega\sqrt{s_3}}\|\mathbf{E}\|_F. \quad (\text{A.51})
\end{aligned}$$

Noting the definition of  $\nu_2$ , we complete the proof of Lemma 2.

## APPENDIX B PROOF OF LEMMA 3

*Proof:* From Step 5 of SI-SSP, we have

$$\mathbf{X}_{S_i}^k = \arg \min_{\Theta: \text{supp}(\Theta) = S_i^k} \|\mathbf{Y}_{S_i} - \mathbf{A}\Theta\|_F. \quad (\text{A.52})$$

From Step 4 of SI-SSP, let  $\mathbf{X}^k = [\mathbf{X}_{S_1}^k, \dots, \mathbf{X}_{S_r}^k]$ . We have the following conclusion

$$\begin{aligned}
&\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\
&\leq \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F. \quad (\text{A.53})
\end{aligned}$$

By removing the same coordinates  $T \cap \Delta S$ , we get

$$\begin{aligned}
&\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\
&\leq \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F. \quad (\text{A.54})
\end{aligned}$$

Because  $\text{supp}(\mathbf{X}) = T$  and  $\text{supp}(\mathbf{X}^{k-1}) = S^{k-1}$ ,

$$(\mathbf{X} - \mathbf{X}^{k-1})_{\Delta S \setminus (T \cup S^{k-1})} = 0. \quad (\text{A.55})$$

For the right-hand side of (A.54), we have

$$\begin{aligned}
&\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\
&= \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\
&= \|(\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\
&\leq \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\
&+ \|(\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T}\|_F \\
&+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T}\|_F. \quad (\text{A.56})
\end{aligned}$$

Note that  $\tilde{S}^k = S^{k-1} \cup \Delta S$ , we have

$$(\mathbf{X} - \mathbf{X}^{k-1})_{T \setminus \tilde{S}^k} = \mathbf{X}_{(\tilde{S}^k)^c}. \quad (\text{A.57})$$

For the left-hand side of (A.54), we have

$$\begin{aligned}
&\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\
&= \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\
&= \|(\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\
&+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\
&= \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}) + \mathbf{A}^H\mathbf{E} + \mathbf{X})_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\
&+ \|\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}) + \mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E} + \mathbf{W}_{T_0}\mathbf{X}\|_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\
&\geq \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F \\
&- \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \quad (\text{A.58}) \\
&- \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \\
&- \|(\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\|_F - \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\|_F. \quad (\text{A.59})
\end{aligned}$$

Combining (A.60) and (A.59), we have

$$\begin{aligned}
&\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F \\
&\leq \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \setminus \tilde{S}^k}\|_F \\
&+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \setminus \tilde{S}^k}\|_F \\
&+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\
&+ \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\
&+ \|(\mathbf{A}^H\mathbf{E})_{T \setminus \tilde{S}^k}\|_F + \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T \setminus \tilde{S}^k}\|_F \\
&+ \|(\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T}\|_F + \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T}\|_F \\
&\leq \sqrt{2}\|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \cup \Delta S}\|_F \\
&+ \sqrt{2}\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \cup \Delta S}\|_F \\
&+ \sqrt{2}\|(\mathbf{A}^H\mathbf{E})_{T \cup \Delta S}\|_F + \sqrt{2}\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T \cup \Delta S}\|_F \\
&\leq \sqrt{2}(1 + \omega)\delta_{3s}\|\mathbf{X} - \mathbf{X}^{k-1}\|_F
\end{aligned}$$

$$+ (1 + \omega) \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \quad (\text{A.60})$$

We can obtain the relationship between  $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$  and  $\|(\mathbf{W}_{T_0} \mathbf{X})_{(\tilde{S}^k)^c}\|_F$ :

$$\eta \omega \sqrt{s_2} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \|(\mathbf{W}_{T_0} \mathbf{X})_{(\tilde{S}^k)^c}\|_F. \quad (\text{A.61})$$

Combining (A.60) and (A.61), we have

$$\begin{aligned} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F &\leq \frac{\sqrt{2}(1 + \omega)\delta_{3s}}{1 + \eta \omega \sqrt{s_2}} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\ &\quad + \frac{(1 + \omega) \sqrt{2(1 + \delta_{2s})}}{1 + \eta \omega \sqrt{s_2}} \|\mathbf{E}\|_F. \end{aligned} \quad (\text{A.62})$$

Noting the definition of  $\nu_1$ , we complete the proof of Lemma 3.

#### APPENDIX C PROOF OF LEMMA 4

*Proof:* the RIC  $\delta_t$  can be expressed as [25]

$$\delta_t = \max_{S \subseteq \{1, 2, \dots, N\}, |S| \leq t} \|\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}\|_{2 \rightarrow 2}, \quad (\text{A.63})$$

where

$$\|\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}\|_{2 \rightarrow 2} = \sup_{\mathbf{a} \in \mathbb{R}^{|S|} \setminus \{0\}} \frac{\|(\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}) \mathbf{a}\|_2}{\|\mathbf{a}\|_2}. \quad (\text{A.64})$$

Let  $S = \text{supp}(\mathbf{u}) \cup \text{supp}(\mathbf{v})$ , then  $|S| \leq t$ . Let  $\mathbf{u}_{|S}, \mathbf{v}_{|S}$  denote respectively the  $S$ -dimensional sub-vectors of  $\mathbf{u}$  and  $\mathbf{v}$  obtained by only keeping the components indexed by  $S$ . We have

$$\begin{aligned} &|\langle \mathbf{u}, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle| \\ &= |\langle \mathbf{W}_{T_0} \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{A} \mathbf{W}_{T_0} \mathbf{u}, \mathbf{A} \mathbf{v} \rangle| \\ &= |\langle \mathbf{W}_{T_0} \mathbf{u}_{|S}, (\mathbf{I}_L - \mathbf{A}_S^H \mathbf{A}_S) \mathbf{v}_{|S} \rangle| \\ &\leq \|\mathbf{W}_{T_0} \mathbf{u}_{|S}\|_2 \|(\mathbf{I}_L - \mathbf{A}_S^H \mathbf{A}_S) \mathbf{v}_{|S}\|_2 \\ &\stackrel{(\text{A.64})}{\leq} \|\mathbf{W}_{T_0} \mathbf{u}_{|S}\|_2 \|\mathbf{I}_L - \mathbf{A}_S^H \mathbf{A}_S\|_{2 \rightarrow 2} \|\mathbf{v}_{|S}\|_2 \\ &\stackrel{(\text{A.63})}{\leq} \omega \delta_t \|\mathbf{u}_{|T}\|_2 \|\mathbf{v}_{|S}\|_2 \\ &= \omega \delta_t \|\mathbf{u}\|_2 \|\mathbf{v}\|_2, \end{aligned} \quad (\text{A.65})$$

moreover, we have

$$\begin{aligned} &\|((\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v})_U\|_2^2 \\ &= \langle ((\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v})_U, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle \\ &\stackrel{(\text{S.27})}{\leq} \delta_t \|((\mathbf{W}_{T_0} - \mathbf{A}^H \mathbf{A}) \mathbf{v})_U\|_2 \|\mathbf{v}\|_2 \end{aligned} \quad (\text{A.66})$$

which completes the proof of Lemma 4.

#### APPENDIX D PROOF OF LEMMA 5

*Proof:* The lemma follows trivially from the fact that

$$\begin{aligned} &\|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U\|_2^2 \\ &= \langle \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e}, (\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U \rangle \\ &= \langle \mathbf{e}, \mathbf{W}_{T_0} \mathbf{A} ((\mathbf{A}^H \mathbf{e})_U) \rangle \\ &\leq \|\mathbf{e}\|_2 \|\mathbf{W}_{T_0} \mathbf{A} ((\mathbf{A}^H \mathbf{e})_U)\|_2 \\ &\stackrel{(7)}{\leq} \|\mathbf{e}'\|_2 \omega \sqrt{1 + \delta_u} \|(\mathbf{A}^H \mathbf{e})_U\|_2. \end{aligned} \quad (\text{A.67})$$

#### APPENDIX E PROOF OF LEMMA 6

*Proof:* Due to the orthogonality, the residue  $\mathbf{Y} - \mathbf{A} \tilde{\mathbf{X}}$  is orthogonal to the space  $\mathbf{A} \mathbf{Z}$ . This means that for all  $\mathbf{Z} \in \mathbb{C}^{L \times N}$  with  $\text{supp}(\mathbf{Z}) \subseteq S$ ,

$$\langle \mathbf{A}(\mathbf{Y} - \mathbf{A} \tilde{\mathbf{X}}), \mathbf{Z} \rangle = 0. \quad (\text{A.68})$$

Let  $\tilde{\mathbf{X}}'$  be the solution of the least squares problem  $\arg \min_{\mathbf{Z}} \{\|\mathbf{Y}' - \mathbf{A} \mathbf{Z}\|_F, \text{supp}(\mathbf{Z}) \subseteq S\}$ , where  $\mathbf{Y}' = \frac{\mathbf{A} \mathbf{W}_{T_0} \mathbf{X}_{T_0}}{\omega} + \mathbf{E}$ . We have

$$\tilde{\mathbf{X}}' = \frac{\mathbf{W}_{T_0} \tilde{\mathbf{X}}}{\omega}. \quad (\text{A.69})$$

Then, by (A.68), we have

$$\begin{aligned} 0 &= \left\langle \frac{\mathbf{A} \mathbf{W}_{T_0} \mathbf{X}_{T_0}}{\omega} + \mathbf{E} - \mathbf{A} \frac{\mathbf{W}_{T_0} \tilde{\mathbf{X}}}{\omega}, \mathbf{A} \mathbf{Z} \right\rangle \\ &= \left\langle \mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}}, \mathbf{A}^H \mathbf{A} \mathbf{Z} \right\rangle + \omega \langle \mathbf{E}, \mathbf{A} \mathbf{Z} \rangle. \end{aligned} \quad (\text{A.70})$$