

# Supplementary: Boundary Multiple Measurement Vectors for Multi-Coset Sampler

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This supplementary material is dedicated to the proofs for Theorems 1–3 in our main paper.

Before proceeding to the proofs, we review some useful notations. For a complex matrix  $\mathbf{X} \in \mathbb{C}^{n \times L}$  and a set  $S \subseteq \{1, \dots, n\}$ ,  $\mathbf{X}_S$  (or  $\mathbf{X}^S$ ) denotes the submatrix of  $\mathbf{X}$  with columns (or rows) indexed by  $S$ ;  $\mathbf{X}_{i,j}$ ,  $\mathbf{X}_{i,:}$  and  $\mathbf{X}_{:,i}$  are the  $(i, j)$ th entry,  $i$ th row and  $i$ th column of  $\mathbf{X}$ , respectively;  $\mathbf{X}^\dagger$ ,  $\mathbf{X}^H$  and  $\mathbf{X}^\top$  mean the Moore-Penrose pseudo-inverse, conjugate transpose and transpose of  $\mathbf{X}$ , respectively;  $\text{supp}(\mathbf{X})$  is the non-zero row indices (i.e., joint sparsity) of  $\mathbf{X}$ ;  $\|\mathbf{X}\|_F$  and  $\|\mathbf{X}\|_2$  signify the Frobenius and Euclidean norm of  $\mathbf{X}$ , respectively. Moreover,  $S^c$  is the complement of set  $S$ ;  $\mathbf{I}_L$  is an  $L \times L$  identity matrix.

## I. PROOF OF THEOREM 1

**Theorem 1.** *The actual sampling rate of (4) is  $\min(p f_s, f_{\text{nyq}})$ , which attains the theoretical lower bound of sampling rate in MCS when  $|\text{supp}(\mathbf{X})| \leq \frac{N_{\text{sig}} B}{f_s}$ .*

*Proof.* In the  $i$ th channel of a multi-coset sampler, the sampling sequence is given by

$$x_{ci}[n] = x(LTn + \tau_i), \quad n = 0, 1, \dots \quad (\text{S.1})$$

The sampling rate of each channel is determined by the sampled signal sequence. To be specific, since the sampling time interval is  $LT$ , the sampling rate of each channel is

$$f_s = \frac{1}{LT} = \frac{f_{\text{nyq}}}{L}, \quad (\text{S.2})$$

i.e., one- $L$ th of the Nyquist sampling rate.

Moreover, as the multi-coset sampler is assumed to have  $p$  channels, the overall sampling rate of  $p$  channels is  $p$  times that of each channel (i.e.  $\frac{p f_{\text{nyq}}}{L}$ ). If this sampling rate is greater than the Nyquist rate  $f_{\text{nyq}}$ , then the advantage of sub-Nyquist sampling structure no longer exists. In this case, we only need to sample at Nyquist sampling rate  $f_{\text{nyq}}$ . Thus, the actual sampling rate can be given by

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right). \quad (\text{S.3})$$

The theoretical lower bound of the sampling rate is given in [17], which is determined directly by the true bandwidth of the signal:

$$\min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \quad (\text{S.4})$$

Thus, the theoretical lower bound on the sampling rate is achieved when

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right) \leq \min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \quad (\text{S.5})$$

In most cases,  $2\lambda(\mathcal{T})$  and  $\frac{p}{LT}$  do not exceed  $f_{\text{nyq}}$ . (If violated, the sampling rate would just be  $f_{\text{nyq}}$ .) Therefore, the condition (S.5) holds whenever

$$\frac{p}{LT} \leq 2\lambda(\mathcal{T}). \quad (\text{S.6})$$

Furthermore, to ensure a unique-solution reconstruction, the number  $p$  of channels should not be too small. In particular, its lower bound is twice the signal sparsity without the priori information about the signal  $\mathbf{X}$  [17],

$$p \geq 2|\text{supp}(\mathbf{X})|. \quad (\text{S.7})$$

For the worst case where  $p = 2|\text{supp}(\mathbf{X})|$ , (S.6) can be rewritten as

$$|\text{supp}(\mathbf{X})| \leq \lambda(\mathcal{T})LT = \frac{N_{\text{sig}} B}{f_s}, \quad (\text{S.8})$$

which completes the proof.  $\square$

## II. PROOF OF THEOREM 2

**Theorem 2.** *When  $r \in [\lceil \frac{f_s}{(D-1)f_s - B} \rceil, N]$  and  $B > f_s$ , we have  $\max_{j \in \{1, \dots, r\}} |\text{supp}(\mathbf{X}_{S_j})| < \frac{N_{\text{sig}} B}{f_s}$ .*

*Proof.* Recall that the MMV model  $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$  is decomposed into  $r$  sub-MMV problems:

$$\mathbf{Y}_{S_j} = \mathbf{A}\mathbf{X}_{S_j} + \mathbf{E}_{S_j}, \quad j = 1, \dots, r \quad (\text{S.9})$$

and each problem is solved individually. Our goal is to determine the number  $r$  of sub-MMV problems that ensures the actual sampling rate to reach the lower bound of the theoretical sampling rate.

For all row blocks  $\{\mathbf{X}^{U_1}, \dots, \mathbf{X}^{U_M}\}$  in  $\mathbf{X}$ , assume that each block has consecutive frequency points of length  $B$  (i.e., the sub-band's width is  $B$ ) if occupied and zero otherwise. In this work, we are primarily interested in the case where

$$B > f_s. \quad (\text{S.10})$$

since the remaining case where  $B \leq f_s$  has been studied thoroughly in existing works (see. e.g., [16]).

Let  $D$  denote the height (i.e., number of rows) in each row block  $\mathbf{X}^{U_i}$ , which can be determined properly according to  $B$  and  $f_s$ . Specifically, we choose a  $D \geq 3$  such that

$$(D-2)f_s < B \leq (D-1)f_s. \quad (\text{S.11})$$

In this case, the frequency points of each PU signal (i.e., occupied row block) occupy  $D-1$  or  $D$  rows. An example of  $D = 5$  is illustrated in Fig. 1, where PU signals 1 and 2 occupy 4 rows while PU signal 3 occupies 5 rows.

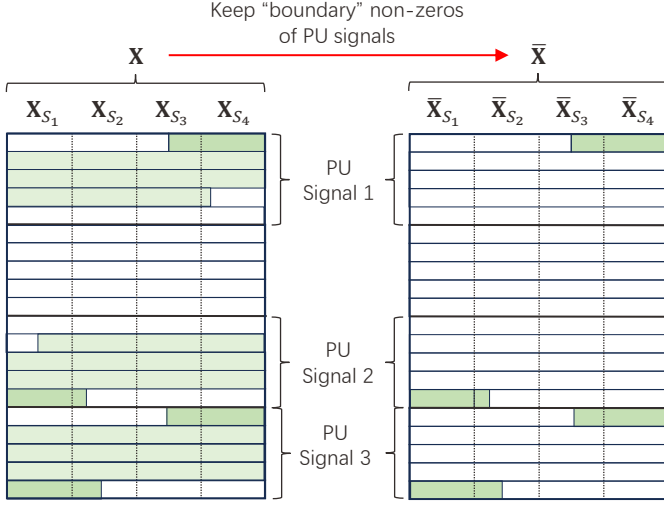


Fig. 1. An illustrative example of MCS signal  $\mathbf{X}$  with 3 PU signals.

- When  $D - 1$  rows in PU signal  $\mathbf{X}^{U_i}$  are occupied, only one row of  $\bar{\mathbf{X}}^{U_i}$  is occupied, since  $\bar{\mathbf{X}}^{U_i}$  only keeps the boundary non-zeros of  $\mathbf{X}^{U_i}$ . In this case, we can easily see that

$$|\text{supp}(\bar{\mathbf{X}}_{S_j}^{U_i})| \leq |\text{supp}(\bar{\mathbf{X}}^{U_i})| = 1. \quad (\text{S.12})$$

Since there are  $N_{\text{sig}}$  sub-band signals in  $\mathbf{X}$ , and also noting that  $B > f_s$ , we have

$$\max_{j \in \{1, \dots, r\}} |\text{supp}(\bar{\mathbf{X}}_{S_j})| \leq N_{\text{sig}} < \frac{N_{\text{sig}} B}{f_s}. \quad (\text{S.13})$$

- When  $D$  rows in PU signal  $\mathbf{X}^{U_i}$  are occupied, the length  $l$  of the frequency points in  $\bar{\mathbf{X}}^{U_i}$  obeys

$$l = B - (D - 2)f_s \stackrel{(\text{S.11})}{\leq} f_s. \quad (\text{S.14})$$

As a result, the column indices of non-zeros in any PU signal  $\bar{\mathbf{X}}^{U_i}$  do not overlap. In other words, the total length of frequency points in each PU signal  $\bar{\mathbf{X}}^{U_i}$  does not exceed  $f_s$ , as shown in Fig. 1.

Recall that  $\bar{\mathbf{X}} \in \mathbb{R}^{L \times N}$  is column-partitioned into  $r$  sub-matrices (i.e.,  $\{\bar{\mathbf{X}}_{S_j}\}_{j=1, \dots, r}$ ).

- We first consider an extreme case where  $r = N$ . In this case,  $\bar{\mathbf{X}}_{S_j}$  only have one column (i.e.,  $|S_j| = |\{j\}| = 1$ ), and so each column-partitioned PU signal  $\bar{\mathbf{X}}_{S_j}^{U_i}$  satisfies

$$|\text{supp}(\bar{\mathbf{X}}_{S_j}^{U_i})| = |\text{supp}(\bar{\mathbf{X}}_{j,:}^{U_i})| \leq 1. \quad (\text{S.15})$$

Similar to (S.13), we further have

$$\max_{j \in \{1, \dots, r\}} |\text{supp}(\bar{\mathbf{X}}_{S_j})| \leq N_{\text{sig}} \stackrel{(\text{S.10})}{<} \frac{N_{\text{sig}} B}{f_s}. \quad (\text{S.16})$$

- We then consider the general case where  $r < N$ . In this case, the length of frequency points in each sub-matrix  $\bar{\mathbf{X}}_{S_j}^{U_i}$  does not exceed  $\lceil \frac{f_s}{r} \rceil$ . In the following, we shall prove that

$$\max_{j \in \{1, \dots, r\}} |\text{supp}(\bar{\mathbf{X}}_{S_j})| \leq N_{\text{sig}} \quad (\text{S.17})$$

when  $r \in [\lceil \frac{f_s}{(D-1)f_s - B} \rceil, N)$ .

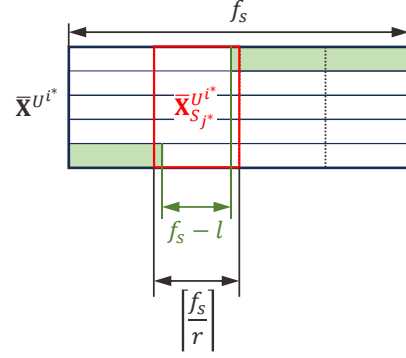


Fig. 2. An illustrative example of the PU signal  $\bar{\mathbf{X}}^{U_i*}$ , which has a length  $l$  that is greater than  $f_s - \lceil \frac{f_s}{r} \rceil$ .

Assume that there exists some a sub-matrix, say,  $\bar{\mathbf{X}}_{S_j*}$ , such that

$$|\text{supp}(\bar{\mathbf{X}}_{S_j*})| > N_{\text{sig}}. \quad (\text{S.18})$$

Then, there exists at least a row-partitioned block of  $\bar{\mathbf{X}}_{S_j*}$ , say,  $\bar{\mathbf{X}}_{S_j*}^{U_i*}$ , that satisfies

$$|\text{supp}(\bar{\mathbf{X}}_{S_j*}^{U_i*})| = 2. \quad (\text{S.19})$$

This implies that the length  $l$  of the frequency points in PU signal  $\bar{\mathbf{X}}^{U_i*}$  must be greater than  $f_s - \lceil \frac{f_s}{r} \rceil$ , as illustrated in Fig. 2. That is,

$$\left\lceil \frac{f_s}{r} \right\rceil > f_s - l, \quad (\text{S.20})$$

which, together with the fact that  $l = B - (D - 2)f_s$  (see (S.14)), leads to

$$\left\lceil \frac{f_s}{r} \right\rceil > (D - 1)f_s - B \quad (\text{S.21})$$

or

$$r < \frac{f_s}{[(D - 1)f_s - B]}. \quad (\text{S.22})$$

Thus, if

$$r \geq \frac{f_s}{[(D - 1)f_s - B]}, \quad (\text{S.23})$$

the assumption of (S.18) must not be true, which implies (S.17).

To sum up, when  $r \in [\lceil \frac{f_s}{(D-1)f_s - B} \rceil, N]$ , we have

$$\max_{j \in \{1, \dots, r\}} |\text{supp}(\bar{\mathbf{X}}_{S_j})| \leq N_{\text{sig}} < \frac{N_{\text{sig}} B}{f_s}. \quad (\text{S.24})$$

The proof is thus complete.  $\square$

### III. PROOF OF THEOREM 3

**Theorem 3.** Consider the column-partitioned MMV model (5) with  $\min_{i,j} \|(\mathbf{X}_{S_i})_{j,:}\|_2 / \|\mathbf{X}_{S_i}\|_F = \eta$  and  $|\text{supp}(\mathbf{X}_{S_i})| \leq s$ . Let  $s_1 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i})|$ ,  $s_2 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i}) \cap \tilde{S}_{S_i}^k|$  and  $s_3 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i}) \cap \tilde{S}_{S_i}^k \setminus S_{S_i}^k|$ . Then, if the sensing matrix  $\mathbf{A}$  obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_1 \sqrt{\nu_1^2 + 4\nu_2^2} - \nu_1^2 - 1}{4\nu_1^2 \nu_2^2 - 2\nu_1^2 - 1}} \quad (\text{S.25})$$

where  $\nu_1 := \frac{1+\omega}{1+\eta\omega\sqrt{s_2}}$  and  $\nu_2 := \frac{1+\omega}{1+\eta\omega\sqrt{s_3}}$ , SI-SSP produces an signal estimate  $\mathbf{X}^k = [\mathbf{X}_{S_1}^k, \dots, \mathbf{X}_{S_r}^k]$  satisfying

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho^k \|\mathbf{X}\|_F + \tau \|\mathbf{E}\|_F, \quad (\text{S.26})$$

where  $\rho \in (0, 1)$  and  $\tau$  are constants depending on  $\delta_{3s}$ ,  $\nu_1$  and  $\nu_2$ . Furthermore, after at most  $k^* = \lceil \log_{\rho} \frac{\|\mathbf{X}\|_F}{\tau \|\mathbf{E}\|_F} \rceil$  iterations, SI-SSP estimates  $\mathbf{X}$  with

$$\|\mathbf{X} - \mathbf{X}^{k^*}\|_F \leq (\tau + 1) \|\mathbf{E}\|_F. \quad (\text{S.27})$$

To prove Theorem 3, we first introduce six useful Lemmas, whose proofs are left to the appendices.

**Lemma 1.** ([25]): For nonnegative numbers  $a, b, c, d, x, y$ ,

$$(ax + by)^2 + (cx + dy)^2 \leq (\sqrt{a^2 + c^2}x + (b + d)y)^2. \quad (\text{S.28})$$

**Lemma 2.** Consider the system model  $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$ , where  $\text{supp}(\mathbf{X}) = T$  and  $|T| = s$ . Let  $S \subseteq \{1, 2, \dots, n\}$  be an index set with  $|S| = t$  and  $\mathbf{W}_{T_0}$  be a side-information matrix with diagonal entries indexed by  $T_0 \subseteq \{1, 2, \dots, n\}$  being  $\omega \geq 0$  and zero otherwise. Also, let  $\tilde{\mathbf{X}} := \arg \min_{\mathbf{Z}: \text{supp}(\mathbf{Z}) \subseteq S} \|\mathbf{Y} - \mathbf{A}\mathbf{Z}\|_2$ . If  $\delta_{3s} < 1$ , then

$$\|\mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}})_S\|_F \leq \omega \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \omega \sqrt{1 + \delta_t} \|\mathbf{E}\|_F \quad (\text{S.29})$$

and

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{s+t}^2}} \|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1 + \delta_t}}{1 - \delta_{s+t}} \|\mathbf{E}\|_F. \quad (\text{S.30})$$

Furthermore, if  $t > s$ , define  $T_{\nabla}$  as the row-indices of the smallest  $t - s$  row-norm entries of  $\tilde{\mathbf{X}}$  in  $S$ , we have

$$\|\mathbf{X}_{T_{\nabla}}\|_F \leq \sqrt{2} \nu_2 \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_2 \sqrt{2(1 + \delta_t)} \|\mathbf{E}\|_F. \quad (\text{S.31})$$

**Remark 1.** When we consider the atom selection strategy of  $\|\tilde{\mathbf{X}}_{T_{\nabla}} + \mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_{\nabla}}\|_F \leq \|\tilde{\mathbf{X}}_{S'} + \mathbf{W}_{T_0} \tilde{\mathbf{X}}_{S'}\|_F$ , we can also obtain another upper bound for  $\|\mathbf{X}_{T_{\nabla}}\|_F$  in (S.31). In this case, we should allocate  $2 \|\mathbf{X}_{T_{\nabla}}\|_F$  to the left hand side of (A.55), we have

$$\|\mathbf{X}_{T_{\nabla}}\|_F \leq \sqrt{2} \nu_3 \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_4 \sqrt{2(1 + \delta_t)} \|\mathbf{E}\|_F. \quad (\text{S.32})$$

where  $\nu_3 = (1 - \omega + \omega \delta_{s+t} + \delta_{s+t}) / (2\delta_{s+t})$  and  $\nu_4 = (1 + \omega) / (2\delta_{s+t})$ .

**Lemma 3.** In steps 4 and 5 of SI-SSP, we have

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2} \nu_1 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \quad (\text{S.33})$$

**Remark 2.** When we consider the atom selection strategy in select step that

$$\begin{aligned} & \|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ & \leq \|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F. \end{aligned} \quad (\text{S.34})$$

We can also obtain another upper bound for  $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$  in (S.33). In this case, we should allocate  $2 \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$  to the left hand side of (A.68), we have

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2} \nu_4 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F$$

$$+ \nu_4 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \quad (\text{S.35})$$

where  $\nu_4 = (1 - \omega + \omega \delta_{3s} + \delta_{3s}) / (2\delta_{3s})$  and  $\nu_4 = (1 + \omega) / (2\delta_{3s})$ . Based on conclusions (S.32) and (S.35), we know that the sensing matrix  $\mathbf{A}$  obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_3 \sqrt{\nu_3^2 + 4\nu_4^2} - \nu_3^2 - 1}{4\nu_3^2 \nu_4^2 - 2\nu_3^2 - 1}}. \quad (\text{S.36})$$

**Lemma 4.** Let  $T_0 \subseteq \{1, 2, \dots, n\}$ , for two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ , if  $|\text{supp}(\mathbf{u}) \cup \text{supp}(\mathbf{v})| \leq t$ ,

$$|\langle \mathbf{u}, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle| \leq \omega \delta_t \|\mathbf{u}\| \|\mathbf{v}\|; \quad (\text{S.37})$$

Moreover, if  $U \subseteq \{1, 2, \dots, n\}$  and  $|U \cup \text{supp}(\mathbf{v})| \leq t$ , then

$$|(\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v}| \leq \omega \delta_t \|\mathbf{v}\|. \quad (\text{S.38})$$

**Lemma 5.** For SMV model  $\mathbf{y} = \Phi \mathbf{x} + \mathbf{e}$ , let  $T_0 \subseteq \{1, 2, \dots, n\}$ , let  $U \subseteq \{1, 2, \dots, n\}$  and  $|U \cap T_0| \leq u$ , we have

$$\|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U\|_2 \leq \omega \delta_u \|\mathbf{e}\|_2. \quad (\text{S.39})$$

**Lemma 6.** Consider the MMV model  $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$ , let  $\tilde{\mathbf{X}}$  be the solution of the least squares problem  $\arg \min_{\mathbf{Z}} \{\|\mathbf{Y} - \mathbf{A}\mathbf{Z}\|_F, \text{supp}(\mathbf{Z}) \subseteq S\}$ , then

$$\langle \mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}}, \mathbf{A}^H \mathbf{A} \mathbf{Z} \rangle + \omega \langle \mathbf{E}, \mathbf{A} \mathbf{Z} \rangle = 0. \quad (\text{S.40})$$

Now we have all ingredients to prove Theorem 3.

*Proof of Theorem 3.* First, in Steps 4 and 5 of SI-SSP, Lemma 3 implies

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2} \nu_1 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \quad (\text{S.41})$$

Note that Step 6 of SI-SSP solves a least squares problem. Let  $S = \tilde{S}^k$  and  $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$ ,  $t = 2s$ , by (S.30) we have

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{3s}^2}} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \frac{\sqrt{1 + \delta_{3s}}}{1 - \delta_{3s}} \|\mathbf{E}\|_F. \quad (\text{S.42})$$

Combining (S.41) and (S.42) and also magnifying  $\delta_{2s}$  to  $\delta_{3s}$ , we further have

$$\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F \leq \nu_1 \sqrt{\frac{2\delta_{3s}^2}{1 - \delta_{3s}^2}} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \tau_1 \|\mathbf{E}\|_F. \quad (\text{S.43})$$

Next, after Step 7 of SI-SSP, let  $S_{\nabla} = \tilde{S}^k \setminus S^k$  be the row-indices of the smallest  $t - s$  row norm entries in  $\tilde{\mathbf{X}}^k$ . Also, let  $T = \tilde{S}^k$ ,  $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$ ,  $T_{\nabla} = S_{\nabla}$  and  $t = 2s$ . Then, by (A.54) we have

$$\|\mathbf{X}_{S_{\nabla}}\|_F \leq \sqrt{2} \nu_2 \delta_{3s} \|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2 \sqrt{2(1 + \delta_{2s})} \|\mathbf{E}\|_F. \quad (\text{S.44})$$

Let  $\tau_1 = (\nu_1 \sqrt{2(1 - \delta_{3s})} + \sqrt{1 + \delta_{3s}})(1 - \delta_{3s})^{-1}$  and  $\tau_2 = \sqrt{1 + \delta_{3s}}$ . Dividing  $(\tilde{S}^k)^c$  into two disjoint subsets:  $(\tilde{S}^k)^c$  and  $S_{\nabla}$ , we get

$$\begin{aligned} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2 &= \|\mathbf{X}_{S_{\nabla}}\|_F^2 + \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2 \\ &\stackrel{(\text{S.41}), (\text{S.44})}{\leq} 2 \left( \nu_2 \delta_{3s} \|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2 \tau_2 \|\mathbf{E}\|_F \right)^2 \\ &\quad + 2 \left( \nu_1 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \tau_1 \|\mathbf{E}\|_F \right)^2 \end{aligned}$$

$$\begin{aligned}
&\stackrel{(S.43)}{\leq} 2 \left( \sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_2(\tau_1\delta_{3s} + \tau_2) \right. \\
&\quad \times \|\mathbf{E}\|_F^2 + 2(\nu_1\delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1\tau_2 \|\mathbf{E}\|_F)^2 \\
&\stackrel{(S.28)}{\leq} 2 \left( \sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}} + \nu_1^2\delta_{3s}^2 \|\mathbf{X} - \mathbf{X}^{k-1}\|_F \right. \\
&\quad \left. + ((\nu_1 + \nu_2)\tau_2 + \nu_2\delta_{3s}\tau_1) \|\mathbf{E}\|_F \right)^2. \quad (S.45)
\end{aligned}$$

Squaring both sides, we get

$$\begin{aligned}
\|\mathbf{X}_{(S^k)^c}\|_F &\leq \sqrt{\frac{4\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2} + 2\nu_1^2\delta_{3s}^2} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F \\
&\quad + \sqrt{2}((\nu_1 + \nu_2)\tau_2 + \nu_2\delta_{3s}\tau_1) \|\mathbf{E}\|_F. \quad (S.46)
\end{aligned}$$

Step 9 of SI-SSP also solves a least squares problem. Letting  $T = S^k$ ,  $\tilde{\mathbf{X}} = \mathbf{X}^k$  and  $t = s$ , by (S.30), we have

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \sqrt{\frac{1}{1-\delta_{2s}^2}} \|\mathbf{X}_{(S^k)^c}\|_F + \frac{\sqrt{1+\delta_s}}{1-\delta_{2s}^2} \|\mathbf{E}\|_F. \quad (S.47)$$

Finally, combining (S.46) and (S.47) yields

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + (1-\rho)\tau \|\mathbf{E}\|_F \quad (S.48)$$

where  $\rho := \sqrt{2\delta_{3s}} \sqrt{2\nu_1^2\nu_2^2\delta_{3s}^2 + \nu_1^2 - \nu_1^2\delta_{3s}^2} (1-\delta_{3s}^2)^{-1}$  and  $\tau := \sqrt{2\delta_{3s}} \nu_2 (\nu_1 \sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}}) (1-\delta_{3s}^2)^{-1/2} (1-\delta_{3s})^{-1} (1-\rho)^{-1} + (\nu_1\nu_2 \sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}}) (1-\delta_{3s})^{-1}$ .

We recursively apply (S.48) to obtain

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho^k \|\mathbf{X}\|_F + \tau \|\mathbf{E}\|_F \quad (S.49)$$

where  $\rho < 1$  under (S.25). When  $k^* = \lceil \log_{\rho} \frac{\|\mathbf{X}\|_F}{\tau \|\mathbf{E}\|_F} \rceil$ , we have  $\rho^k \|\mathbf{X}\|_F \leq \tau \|\mathbf{E}\|_F$ , and thus the stability result (S.27).  $\square$

## APPENDIX A PROOF OF LEMMA 2

- First, we give an upper bound of  $\|\mathbf{X}_{T_\nabla}\|_F$ , by Lemma 6, let  $\mathbf{Z} = (\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S$ , we have

$$\begin{aligned}
&\left\langle \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}}), \mathbf{A}^H \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle \\
&\quad + \left\langle \mathbf{W}_{T_0}\mathbf{E}, \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle = \mathbf{0}. \quad (A.50)
\end{aligned}$$

Noticing that  $\text{supp}(\tilde{\mathbf{X}}) \subseteq S$ , we have

$$\begin{aligned}
&\|(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\|_F^2 \\
&= \left\langle \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle \\
&\stackrel{(A.50)}{=} \left\langle \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{I}_L - \mathbf{A}^H \mathbf{A})(\mathbf{X} - \tilde{\mathbf{X}})_S \right\rangle \\
&\quad - \left\langle \mathbf{W}_{T_0}\mathbf{E}, \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle \\
&\stackrel{(7)}{\leq} \omega\delta_{s+t} \left\| (\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\|_F \left\| \mathbf{X} - \tilde{\mathbf{X}} \right\|_F \\
&\quad + \omega \|\mathbf{E}\|_F \sqrt{1+\delta_t} \left\| \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}})_S \right\|_F. \quad (A.51)
\end{aligned}$$

Divide both sides by  $\|(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\|_F$  to obtain (S.29).

- Next, by expanding [Lemma 2, 25] to the MMV model, we could get a relationship between  $\|\mathbf{X} - \tilde{\mathbf{X}}\|_F$  and  $\|\mathbf{X}_{S^c}\|_F$ . We have

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1-\delta_{s+t}^2}} \|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1+\delta_t}}{1-\delta_{s+t}} \|\mathbf{E}\|_F. \quad (A.52)$$

- Then, we established the relationship between  $\mathbf{X}_{T_\nabla}$  and  $\mathbf{X} - \tilde{\mathbf{X}}$ . There exist a subset  $S' \subseteq S$  and  $S' \cap T = \emptyset$ . Since  $T_\nabla$  is defined by the set of indices of the  $t-s$  smallest row entries of  $\tilde{\mathbf{X}}$ , we can conclude that

$$\begin{aligned}
&\|\tilde{\mathbf{X}}_{T_\nabla}\|_F + \|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla}\|_F \\
&\leq \|\tilde{\mathbf{X}}_{S'}\|_F + \|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{S'}\|_F. \quad (A.53)
\end{aligned}$$

By eliminating the contribution from  $T_\nabla \cap S'$  and noticing that  $S' \cap T = \emptyset$ , we have

$$\begin{aligned}
&\|\tilde{\mathbf{X}}_{T_\nabla \setminus S'}\|_F + \|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla \setminus S'}\|_F \\
&\leq \|(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_\nabla}\|_F \\
&\quad + \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_\nabla}\|_F. \quad (A.54)
\end{aligned}$$

For the left-hand side of (A.54), we have

$$\begin{aligned}
&\|\tilde{\mathbf{X}}_{T_\nabla \setminus S'}\|_F + \|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla \setminus S'}\|_F \\
&= \|(\tilde{\mathbf{X}} - \mathbf{X} + \mathbf{X})_{T_\nabla \setminus S'}\|_F \\
&\quad + \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}) + \mathbf{W}_{T_0}\mathbf{X})_{T_\nabla \setminus S'}\|_F \\
&\geq \|\mathbf{X}_{T_\nabla}\|_F + \|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F \quad (A.55)
\end{aligned}$$

$$\begin{aligned}
&- \|(\tilde{\mathbf{X}} - \mathbf{X})_{T_\nabla \setminus S'}\|_F \\
&- \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{T_\nabla \setminus S'}\|_F. \quad (A.56)
\end{aligned}$$

Finally, combining (A.56) and (A.54), and noticing that

$$(T_\nabla \setminus S') \cap (S' \setminus T_\nabla) = \emptyset \quad (A.57)$$

$$(T_\nabla \setminus S') \cup (S' \setminus T_\nabla) \subseteq T, \quad (A.58)$$

we have

$$\begin{aligned}
&\|\mathbf{X}_{T_\nabla}\|_F + \|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F \\
&\leq \|(\tilde{\mathbf{X}} - \mathbf{X})_{T_\nabla \setminus S'}\|_F + \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{T_\nabla \setminus S'}\|_F \\
&\quad + \|(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_\nabla}\|_F + \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{S' \setminus T_\nabla}\|_F \\
&\leq \sqrt{2} \|(\tilde{\mathbf{X}} - \mathbf{X})_{(S' \setminus T_\nabla) \cup (T_\nabla \setminus S')}\|_F \\
&\quad + \sqrt{2} \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{(S' \setminus T_\nabla) \cup (T_\nabla \setminus S')}\|_F \\
&\leq \sqrt{2} \|(\tilde{\mathbf{X}} - \mathbf{X})_S\|_F + \sqrt{2} \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_S\|_F \\
&\stackrel{(S.29)}{\leq} \sqrt{2}(1+\omega)\delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\
&\quad + (1+\omega)\sqrt{2(1+\delta_t)} \|\mathbf{E}\|_F. \quad (A.59)
\end{aligned}$$

Also, we can obtain the relationship between  $\|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F$  and  $\|\mathbf{X}_{T_\nabla}\|_F$ :

$$\eta\omega\sqrt{s_3} \|\mathbf{X}_{T_\nabla}\|_F \leq \|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F. \quad (A.60)$$

Combining (A.59) and (A.60), we have

$$\begin{aligned}
\|\mathbf{X}_{T_\nabla}\|_F &\leq \frac{\sqrt{2}(1+\omega)\delta_{s+t}}{1+\eta\omega\sqrt{s_3}} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\
&\quad + \frac{(1+\omega)\sqrt{2(1+\delta_t)}}{1+\eta\omega\sqrt{s_3}} \|\mathbf{E}\|_F. \quad (A.61)
\end{aligned}$$

Noting the definition of  $\nu_2$ , we complete the proof of Lemma 2.

APPENDIX B  
PROOF OF LEMMA 3

*Proof:* From Step 5 of SI-SSP, we have

$$\mathbf{X}_{S_i}^k = \arg \min_{\Theta: \text{supp}(\Theta) = S_i^k} \|\mathbf{Y}_{S_i} - \mathbf{A}\Theta\|_F. \quad (\text{A.62})$$

From Step 4 of SI-SSP, let  $\mathbf{X}^k = [\mathbf{X}_{S_1}^k, \dots, \mathbf{X}_{S_r}^k]$ . We have the following conclusion

$$\begin{aligned} & \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ & + \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ & \leq \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F \\ & + \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F. \end{aligned} \quad (\text{A.63})$$

By removing the same coordinates  $T \cap \Delta S$ , we get

$$\begin{aligned} & \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ & + \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ & \leq \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ & + \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F. \end{aligned} \quad (\text{A.64})$$

Because  $\text{supp}(\mathbf{X}) = T$  and  $\text{supp}(\mathbf{X}^{k-1}) = S^{k-1}$ ,

$$(\mathbf{X} - \mathbf{X}^{k-1})_{\Delta S \setminus (T \cup S^{k-1})} = 0. \quad (\text{A.65})$$

For the right-hand side of (A.64), we have

$$\begin{aligned} & \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ & + \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ & = \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ & + \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ & = \|(\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ & + \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ & \leq \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ & + \|(\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T}\|_F \\ & + \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ & + \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T}\|_F. \end{aligned} \quad (\text{A.66})$$

Note that  $\tilde{S}^k = S^{k-1} \cup \Delta S$ , we have

$$(\mathbf{X} - \mathbf{X}^{k-1})_{T \setminus \tilde{S}^k} = \mathbf{X}_{(\tilde{S}^k)^c}. \quad (\text{A.67})$$

For the left-hand side of (A.64), we have

$$\begin{aligned} & \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ & + \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ & = \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ & + \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ & = \|(\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ & + \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ & = \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}) \\ & + \mathbf{A}^H\mathbf{E} + \mathbf{X})_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ & + \|\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}) \\ & + \mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E} + \mathbf{W}_{T_0}\mathbf{X})_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ & \geq \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F \end{aligned}$$

$$- \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \quad (\text{A.68})$$

$$- \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F$$

$$- \|(\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\|_F - \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\|_F. \quad (\text{A.69})$$

Combining (A.70) and (A.69), we have

$$\begin{aligned} & \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F \\ & \leq \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \setminus \tilde{S}^k}\|_F \\ & + \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \setminus \tilde{S}^k}\|_F \\ & + \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ & + \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ & + \|(\mathbf{A}^H\mathbf{E})_{T \setminus \tilde{S}^k}\|_F + \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T \setminus \tilde{S}^k}\|_F \\ & + \|(\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T}\|_F + \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T}\|_F \\ & \leq \sqrt{2}\|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \cup \Delta S}\|_F \\ & + \sqrt{2}\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \cup \Delta S}\|_F \\ & + \sqrt{2}\|(\mathbf{A}^H\mathbf{E})_{T \cup \Delta S}\|_F + \sqrt{2}\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T \cup \Delta S}\|_F \\ & \leq \sqrt{2}(1 + \omega)\delta_{3s}\|\mathbf{X} - \mathbf{X}^{k-1}\|_F \\ & + (1 + \omega)\sqrt{2(1 + \delta_{3s})}\|\mathbf{E}\|_F. \end{aligned} \quad (\text{A.70})$$

We can obtain the relationship between  $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$  and  $\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F$ :

$$\eta\omega\sqrt{s_2}\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F. \quad (\text{A.71})$$

Combining (A.70) and (A.71), we have

$$\begin{aligned} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F & \leq \frac{\sqrt{2}(1 + \omega)\delta_{3s}}{1 + \eta\omega\sqrt{s_2}}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\ & + \frac{(1 + \omega)\sqrt{2(1 + \delta_{2s})}}{1 + \eta\omega\sqrt{s_2}}\|\mathbf{E}\|_F. \end{aligned} \quad (\text{A.72})$$

Noting the definition of  $\nu_1$ , we complete the proof of Lemma 3.

APPENDIX C  
PROOF OF LEMMA 4

*Proof:* the RIC  $\delta_t$  can be expressed as [25]

$$\delta_t = \max_{S \subseteq \{1, 2, \dots, N\}, |S| \leq t} \|\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}\|_{2 \rightarrow 2}, \quad (\text{A.73})$$

where

$$\|\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}\|_{2 \rightarrow 2} = \sup_{\mathbf{a} \in \mathbb{R}^{|S|} \setminus \{0\}} \frac{\|(\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I})\mathbf{a}\|_2}{\|\mathbf{a}\|_2}. \quad (\text{A.74})$$

Let  $S = \text{supp}(\mathbf{u}) \cup \text{supp}(\mathbf{v})$ , then  $|S| \leq t$ . Let  $\mathbf{u}_{|S}, \mathbf{v}_{|S}$  denote respectively the  $S$ -dimensional sub-vectors of  $\mathbf{u}$  and  $\mathbf{v}$  obtained by only keeping the components indexed by  $S$ . We have

$$\begin{aligned} & |\langle \mathbf{u}, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\mathbf{v} \rangle| \\ & = |\langle \mathbf{W}_{T_0}\mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{A}\mathbf{W}_{T_0}\mathbf{u}, \mathbf{A}\mathbf{v} \rangle| \\ & = |\langle \mathbf{W}_{T_0}\mathbf{u}_{|S}, (\mathbf{I}_L - \mathbf{A}_S^H\mathbf{A}_S)\mathbf{v}_{|S} \rangle| \\ & \leq \|\mathbf{W}_{T_0}\mathbf{u}_{|S}\|_2 \|(\mathbf{I}_L - \mathbf{A}_S^H\mathbf{A}_S)\mathbf{v}_{|S}\|_2 \\ & \stackrel{(\text{A.74})}{\leq} \|\mathbf{W}_{T_0}\mathbf{u}_{|S}\|_2 \|\mathbf{I}_L - \mathbf{A}_S^H\mathbf{A}_S\|_{2 \rightarrow 2} \|\mathbf{v}_{|S}\|_2 \\ & \stackrel{(\text{A.73})}{\leq} \omega\delta_t \|\mathbf{u}_{|T}\|_2 \|\mathbf{v}_{|S}\|_2 \end{aligned}$$

$$= \omega \delta_t \|\mathbf{u}\|_2 \|\mathbf{v}\|_2, \quad (\text{A.75})$$

moreover, we have

$$\begin{aligned} & \left\| \left( (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \right)_U \right\|_2^2 \\ &= \left\langle \left( (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \right)_U, \right. \\ & \quad \left. (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \right\rangle \\ &\stackrel{(\text{S.37})}{\leq} \delta_t \left\| \left( (\mathbf{W}_{T_0} - \mathbf{A}^H \mathbf{A}) \mathbf{v} \right)_U \right\|_2 \|\mathbf{v}\|_2 \end{aligned} \quad (\text{A.76})$$

which completes the proof of Lemma 4.

#### APPENDIX D PROOF OF LEMMA 5

*Proof:* The lemma follows trivially from the fact that

$$\begin{aligned} & \left\| (\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U \right\|_2^2 \\ &= \left\langle \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e}, (\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U \right\rangle \\ &= \left\langle \mathbf{e}, \mathbf{W}_{T_0} \mathbf{A} ((\mathbf{A}^H \mathbf{e})_U) \right\rangle \\ &\leq \|\mathbf{e}\|_2 \left\| \mathbf{W}_{T_0} \mathbf{A} ((\mathbf{A}^H \mathbf{e})_U) \right\|_2 \\ &\stackrel{(7)}{\leq} \|\mathbf{e}'\|_2 \omega \sqrt{1 + \delta_u} \left\| (\mathbf{A}^H \mathbf{e})_U \right\|_2. \end{aligned} \quad (\text{A.77})$$

#### APPENDIX E PROOF OF LEMMA 6

*Proof:* Due to the orthogonality, the residue  $\mathbf{Y} - \mathbf{A}\tilde{\mathbf{X}}$  is orthogonal to the space  $\mathbf{AZ}$ . This means that for all  $\mathbf{Z} \in \mathbb{C}^{L \times N}$  with  $\text{supp}(\mathbf{Z}) \subseteq S$ ,

$$\langle \mathbf{A}(\mathbf{Y} - \mathbf{A}\tilde{\mathbf{X}}), \mathbf{Z} \rangle = 0. \quad (\text{A.78})$$

Let  $\tilde{\mathbf{X}}'$  be the solution of the least squares problem  $\arg \min_{\mathbf{Z}} \{ \|\mathbf{Y}' - \mathbf{AZ}\|_F, \text{supp}(\mathbf{Z}) \subseteq S \}$ , where  $\mathbf{Y}' = \frac{\mathbf{AW}_{T_0} \mathbf{X}_{T_0}}{\omega} + \mathbf{E}$ . We have

$$\tilde{\mathbf{X}}' = \frac{\mathbf{W}_{T_0} \tilde{\mathbf{X}}}{\omega}. \quad (\text{A.79})$$

Then, by (A.78), we have

$$\begin{aligned} 0 &= \left\langle \frac{\mathbf{AW}_{T_0} \mathbf{X}_{T_0}}{\omega} + \mathbf{E} - \mathbf{A} \frac{\mathbf{W}_{T_0} \tilde{\mathbf{X}}}{\omega}, \mathbf{AZ} \right\rangle \\ &= \left\langle \mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}}, \mathbf{A}^H \mathbf{AZ} \right\rangle + \omega \langle \mathbf{E}, \mathbf{AZ} \rangle. \end{aligned} \quad (\text{A.80})$$