Supplementary: Boundary Multiple Measurement Vectors for Multi-Coset Sampler

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This supplementary material is dedicated to the proofs for Theorems 1–3 in our main paper.

Before proceeding to the proofs, we review some useful notations. For a complex matrix $\mathbf{X} \in \mathbb{C}^{n\times L}$ and a set $S \subseteq \{1, \cdots, n\}$, $\mathbf{X}_S$ (or $\mathbf{X}^S$) denotes the submatrix of $\mathbf{X}$ with columns (or rows) indexed by S; $\mathbf{X}_{i,j}$, $\mathbf{X}_{i,:}$ and $\mathbf{X}_{:,i}$ are the (i,j)th entry, ith row and ith column of $\mathbf{X}$, respectively; $\mathbf{X}^{\dagger}$, $\mathbf{X}^{H}$ and $\mathbf{X}^{\top}$ mean the Moore-Penrose pseudo-inverse, conjugate transpose and transpose of $\mathbf{X}$, respectively; $\text{supp}(\mathbf{X})$ is the non-zero row indices (i.e., joint sparsity) of $\mathbf{X}$; $\|\mathbf{X}\|_F$ and $\|\mathbf{X}\|_2$ signify the Frobenius and Euclidean norm of $\mathbf{X}$, respectively. Moreover, S^c is the complement of set S; $\mathbf{I}_L$ is an $L \times L$ identity matrix.

I. Proof of Theorem 1

Theorem 1. *The actual sampling rate of (4) is* $\min\left(pf_s, f_{nyq}\right)$, *which attains the theoretical lower bound of sampling rate in MCS when* $|supp(\mathbf{X})| \leq \frac{N_{sig}B}{f_s}$.

Proof. Assume that a multi-coset sampler has p channels. In the ith channel, The sampling sequence is given by

$$x_{c_i}[n] = x(LTn + \tau_i), \quad n = 0, 1, \cdots . \tag{S.1}$$

The sampling rate of each channel is determined by the sampled signal sequence. Because the sampling time interval is LT, the sampling rate of each channel is 1 in L of the Nyquist sampling rate (i.e. $\frac{f_{\text{nyq}}}{L}$). The average sampling rate of p channels is p times that of one channel (i.e. $\frac{pf_{\text{nyq}}}{L}$). Noting that when the sampling rate of p channels is greater than the Nyquist rate, the advantage of sub-Nyquist sampling structure no longer exists. Thus, we only need to sample at Nyquist sampling rate f_{nyq}, the actual sampling rate can be represented as

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right). \tag{S.2}$$

The theoretical lower bound of the sampling rate is given in [17], which is directly determined by the true bandwidth of the signal:

$$\min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \tag{S.3}$$

In most cases, the subband bandwidth $\lambda(\mathcal{T})$ and the actual sampling rate does not exceed f_{nyq} (when not satisfied, the sampling rate is f_{sig}). To ensure reconstruction performance, the parameter p is often not set too low (for instance, it's often chosen to be at least twice the sparsity level $\text{supp}(\mathbf{X})$). It can be seen from above that the theoretical lower bound on the sampling rate is achieved only when $p/LT = pf_s \leq 2\lambda(\mathcal{T}) = 2N_{sig}B$. In other words, when $p = 2\text{supp}(\mathbf{X})$ for the worst case of p, the condition for the actual sampling rate to meet the theoretical lower bound is $|\text{supp}(\mathbf{X})| \leq \frac{N_{\text{sig}}B}{f_s}$. □

II. Proof of Theorem 2

Theorem 2. *When* $r \in [\lceil\frac{f_s}{(M-1)f_s - B}\rceil, \infty)$ *and* $B > f_s$, *we have* $\max_{i\in\{1,\cdots,r\}} \left|supp(\bar{\mathbf{X}}_{S_i})\right| \leq \frac{N_{\text{sig}}B}{f_s}$.

Proof. Consider all blocks $\left\{\mathbf{X}^{U_1}, \cdots, \mathbf{X}^{U_M}\right\}$ in $\mathbf{X}$, where there are consecutive corresponding frequency points of length B. For the case $B > f_s$ and each block occupies M rows in $\mathbf{X}$, the length of the frequency point in each $\bar{\mathbf{X}}^{U_i}$ meets

$$l = B - (M-2)f_s < f_s. \tag{S.4}$$

Because $l < f_s$, we know that the non-zero elements of an any sub-block (i.e. $\bar{\mathbf{X}}^{U_i}$) are distributed on both sides and do not intersect on the columns. Conside one block $\bar{\mathbf{X}}^{U_i}$, We let $r \to \infty$ and observe the change in $\bar{\mathbf{X}}^{U_i}$, $\bar{\mathbf{X}}^{U_i}$ gradually changes from an MMV form to an SMV form. In SMV, the sparsity of the signal is determined by the column in $\bar{\mathbf{X}}^{U_i}$ where it is located. It is observed that

$$\lim_{r\to\infty,i,j} \text{supp}(\bar{\mathbf{X}}^{U_i}_{:,j}) \leq 1. \tag{S.5}$$

Thus, the sparsity of each column in $\bar{\mathbf{X}}^{U_i}$ is less than N_{sig}.

A more complex situation is when r is a finite value, assuming $r = r^\star$ is a finite value. In this case, the length of frequency point in each sub-matrix is $\frac{f_s}{r^\star}$. We use reduction to absurdity to prove the condition that the sparsity of each sub-matrix is less than N_{sig}. Assuming that there exists a sub-matrix $\bar{\mathbf{X}}_{S_i}$ with $\text{supp}(\bar{\mathbf{X}}_{S_i}) > N_{\text{sig}}$. Also, the non-zero elements of an any sub-block of $\bar{\mathbf{X}}_{S_i}$ do not intersect on the columns. $\bar{\mathbf{X}}_{S_i}$ must contain both non-zero elements on both sides of one block. From (S.4), we know that the length of any block of $\bar{\mathbf{X}}$ is less than f_s. We can draw a conclusion that

$$\frac{f_s}{r^\star} > f_s - l = (M-1)f_s - B. \tag{S.6}$$

As can be seen, there exist a contradiction between (S.6) and Theorem 2 , so the length of any column-partition sub-matrix non-zero elements in $\bar{\mathbf{X}}$ must be less or equal than $f_s - l$, which is equivalent to r satisfying

$$r \geq \left\lceil \frac{f_s}{(M-1)f_s - B} \right\rceil. \tag{S.7}$$

□

III. Proof of Theorem 3

Theorem 3. *Consider the column-partitioned MMV model (5) with* $\min_{i,j}\|(\mathbf{X}_{S_i})_{j,:}\|_2/\|\mathbf{X}_{S_i}\|_F = \eta$ *and* $|supp(\mathbf{X}_{S_i})| \le s$. *Let* $s_1 := \min_{i,k}|\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i})|$, $s_2 := \min_{i,k}|\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i}) \cap \tilde{S}^k_{S_i}|$ *and* $s_3 := \min_{i,k}|\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i}) \cap \tilde{S}^k_{S_i} \setminus S^k_{S_i}|$. *Then, if the sensing matrix* $\mathbf{A}$ *obeys the RIP with*

$$\delta_{3s} \le \sqrt{\frac{\nu_1\sqrt{\nu_1^2+4\nu_2^2}-\nu_1^2-1}{4\nu_1^2\nu_2^2-2\nu_1^2-1}} \tag{S.8}$$

where $\nu_1 := \frac{1+\omega}{1+\eta\omega\sqrt{s_2}}$ *and* $\nu_2 := \frac{1+\omega}{1+\eta\omega\sqrt{s_3}}$, *SI-SSP produces an signal estimate* $\mathbf{X}^k = [\mathbf{X}^k_{S_1}, \cdots, \mathbf{X}^k_{S_r}]$ *satisfying*

$$\left\|\mathbf{X}-\mathbf{X}^k\right\|_F \le \rho^k \left\|\mathbf{X}\right\|_F + \tau \left\|\mathbf{E}\right\|_F, \tag{S.9}$$

where $\rho \in (0,1)$ *and* τ *are constants depending on* δ_{3s}, ν_1 *and* ν_2. *Furthermore, after at most* $k^* = \lceil \log_\rho \frac{\|\mathbf{X}\|_F}{\tau\|\mathbf{E}\|_F}\rceil$ *iterations, SI-SSP estimates* $\mathbf{X}$ *with*

$$\|\mathbf{X}-\mathbf{X}^{k^*}\|_F \le (\tau+1)\|\mathbf{E}\|_F. \tag{S.10}$$

To prove Theorem 3, we first introduce six useful lemmas, whose proofs are left to appendices.

Lemma 1. *(Lemma 1 in [25]): For nonnegative numbers a, b, c, d, x, y,*

$$(ax+by)^2+(cx+dy)^2 \le (\sqrt{a^2+c^2}x+(b+d)y)^2. \tag{S.11}$$

Lemma 2. *Consider the system model* $\mathbf{Y} = \mathbf{AX}+\mathbf{E}$, $supp(\mathbf{X}) = T$ *and* $|T| = s$. *Suppose* S, $T_0 \subseteq \{1,2,...,n\}$, $|S| = t$. $\mathbf{W}_{T_0}$ *is constructed with diagonal entries indexed by* T_0 *being* $\omega \ge 0$. *Let* $\tilde{\mathbf{X}}$ *be the solution of the least squares problem* $\arg\min_{\mathbf{Z}}\{\|\mathbf{Y}-\mathbf{AZ}\|_2, supp(\mathbf{Z}) \subseteq S\}$, *if* $\delta_{3s} < 1$, *then*

$$\|\mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}})_S\|_F \le \omega\delta_{s+t}\|\mathbf{X}-\tilde{\mathbf{X}}\|_F + \omega\sqrt{1+\delta_t}\|\mathbf{E}\|_F \tag{S.12}$$

and

$$\|\mathbf{X}-\tilde{\mathbf{X}}\|_F \le \sqrt{\frac{1}{1-\delta_{s+t}^2}}\|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1+\delta_t}}{1-\delta_{s+t}}\|\mathbf{E}\|_F. \tag{S.13}$$

If $t > s$, *define* T_∇ *as the row-indices of the smallest* $t-s$ *row norm entries of* $\tilde{\mathbf{X}}$ *in* S, *we have*

$$\|\mathbf{X}_{T_\nabla}\|_F \le \sqrt{2}\nu_2\delta_{s+t}\|\mathbf{X}-\tilde{\mathbf{X}}\|_F + \nu_2\sqrt{2\left(1+\delta_t\right)}\|\mathbf{E}\|_F. \tag{S.14}$$

Remark 1. *When we consider the atom selection strategy of* $\|\tilde{\mathbf{X}}_{T_\nabla} + \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla}\|_F \le \|\tilde{\mathbf{X}}_{S'} + \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{S'}\|_F$, *we can also obtain another upper bound for* $\|\mathbf{X}_{T_\nabla}\|_F$ *in (S.14). In this case, we should allocate* $2\|\mathbf{X}_{T_\nabla}\|_F$ *to the left hand side of (A.38), we have*

$$\begin{aligned}\|\mathbf{X}_{T_\nabla}\|_F \le \sqrt{2}\nu_3\delta_{s+t}\left\|\mathbf{X}-\tilde{\mathbf{X}}\right\|_F \\ + \nu_4\sqrt{2\left(1+\delta_t\right)}\left\|\mathbf{E}\right\|_F.\end{aligned} \tag{S.15}$$

where $\nu_3 = (1-\omega+\omega\delta_{s+t}+\delta_{s+t})/(2\delta_{s+t})$ *and* $\nu_4 = (1+\omega)/(2\delta_{s+t})$.

Lemma 3. *In steps 4 and 5 of SI-SSP, we have*

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \le \sqrt{2}\nu_1\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F + \nu_1\sqrt{2(1+\delta_{3s})}\left\|\mathbf{E}\right\|_F. \tag{S.16}$$

Remark 2. *When we consider the atom selection strategy in select step that*

$$\begin{aligned}&\left\|((\mathbf{I}_L+\mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y}-\mathbf{AX}^{k-1}))_T\right\|_F \\ &\le \left\|((\mathbf{I}_L+\mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y}-\mathbf{AX}^{k-1}))_{\Delta S}\right\|_F.\end{aligned} \tag{S.17}$$

We can also obtain another upper bound for $\left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F$ *in (S.16). In this case, we should allocate* $2\left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F$ *to the left hand side of (A.51), we have*

$$\begin{aligned}\left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F \le \sqrt{2}\nu_4\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F \\ + \nu_4\sqrt{2\left(1+\delta_{3s}\right)}\left\|\mathbf{E}\right\|_F.\end{aligned} \tag{S.18}$$

where $\nu_4 = (1-\omega+\omega\delta_{3s}+\delta_{3s})/(2\delta_{3s})$ *and* $\nu_4 = (1+\omega)/(2\delta_{3s})$. *Based on conclusions (S.15) and (S.18), we know that the sensing matrix* $\mathbf{A}$ *obeys the RIP with*

$$\delta_{3s} \le \sqrt{\frac{\nu_3\sqrt{\nu_3^2+4\nu_4^2}-\nu_3^2-1}{4\nu_3^2\nu_4^2-2\nu_3^2-1}}. \tag{S.19}$$

Lemma 4. *Let* $T_0 \subseteq \{1,2,...,n\}$, *for two vectors* $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$, *if* $|supp(\mathbf{u}) \cup supp(\mathbf{v})| \le t$,

$$\left|\left\langle \mathbf{u}, (\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\mathbf{v}\right\rangle\right| \le \omega\delta_t\left\|\mathbf{u}\right\|\left\|\mathbf{v}\right\|; \tag{S.20}$$

moreover, if $U \subseteq \{1,2,...,n\}$ *and* $|U \cup supp(\mathbf{v})| \le t$, *then*

$$\left|(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\mathbf{v}\right| \le \omega\delta_t\left\|\mathbf{v}\right\|. \tag{S.21}$$

Lemma 5. *For SMV model* $\mathbf{y} = \mathbf{\Phi x}+\mathbf{e}$, *let* $T_0 \subseteq \{1,2,...,n\}$, *let* $U \subseteq \{1,2,...,n\}$ *and* $|U \cap T_0| \le u$, *we have*

$$\left\|(\mathbf{W}_{T_0}\mathbf{A}^H e)_U\right\|_2 \le \omega\delta_u\left\|e\right\|_2. \tag{S.22}$$

Lemma 6. *Consider MMV model* $\mathbf{Y} = \mathbf{AX}+\mathbf{E}$, *let* $\tilde{\mathbf{X}}$ *be the solution of the least squares problem* $\arg\min_{\mathbf{Z}}\{\|\mathbf{Y}-\mathbf{AZ}\|_F, supp(\mathbf{Z}) \subseteq S\}$, *then*

$$\left\langle \mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}}, \mathbf{A}^H\mathbf{AZ}\right\rangle + \omega\left\langle \mathbf{E}, \mathbf{AZ}\right\rangle = \mathbf{0}. \tag{S.23}$$

Proof of Theerem 3. Now we have all ingredients to prove Theorem 3. In step 3 and 4 of SI-SSP, by Lemma 3, in the kth iteration, we have

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \le \sqrt{2}\nu_1\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F + \nu_1\sqrt{2(1+\delta_{3s})}\left\|\mathbf{E}\right\|_F. \tag{S.24}$$

Since the step 6 in SI-SSP is a process to solve a least squares problem. Let $S = \tilde{S}^k$ and $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $t = 2s$, by (S.13), we can know that

$$\|\mathbf{X}-\tilde{\mathbf{X}}\|_F \le \sqrt{\frac{1}{1-\delta_{3s}^2}}\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \frac{\sqrt{1+\delta_{2s}}}{1-\delta_{3s}}\|\mathbf{E}\|_F. \tag{S.25}$$

Combining (S.24) and (S.25) and magnifying δ_{2s} to δ_{3s}, we have

$$\|\mathbf{X}-\tilde{\mathbf{X}}^k\|_F \le \nu_1\sqrt{\frac{2\delta_{3s}^2}{1-\delta_{3s}^2}}\|\mathbf{X}-\mathbf{X}^{k-1}\|_F + \tau_1\|\mathbf{E}\|_F. \tag{S.26}$$

Next, after step 7 of SI-SSP in kth iteration, let $S_\nabla = \tilde{S}^k \setminus S^k$ as the row-indices of the smallest $t-s$ row norm entries in $\tilde{\mathbf{X}}^k$. Let $T=\tilde{S}^k$, $\tilde{\mathbf{X}}=\tilde{\mathbf{X}}^k$, $T_\nabla=S_\nabla$ and $t=2s$, by (A.37), we have

$$\|\mathbf{X}_{S_\nabla}\|_F \le \sqrt{2}\nu_2\delta_{3s}\|\mathbf{X}-\tilde{\mathbf{X}}^k\|_F+\nu_2\sqrt{2\left(1+\delta_{2s}\right)}\|\mathbf{E}\|_F. \tag{S.27}$$

Let $\tau_1=(\nu_1\sqrt{2(1-\delta_{3s})}+\sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$ and $\tau_2=\sqrt{1+\delta_{3s}}$. Dividing $(S^k)^c$ into 2 disjoint subsets: $(\tilde{S}^k)^c$ and S_∇, we get

$$\begin{aligned}
&\left\|\mathbf{X}_{(S^k)^c}\right\|_F^2=\|\mathbf{X}_{S_\nabla}\|_F^2+\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2\\
&\overset{(S.24),(S.27)}{\le} 2\left(\nu_2\delta_{3s}\|\mathbf{X}-\tilde{\mathbf{X}}^k\|_F+\nu_2\tau_2\|\mathbf{E}\|_F\right)^2\\
&\qquad+2\left(\nu_1\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_2+\nu_1\tau_2\left\|\mathbf{E}\right\|_F\right)^2\\
&\overset{(S.26)}{\le} 2\left(\sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F+\nu_2(\tau_1\delta_{3s}+\tau_2)\right.\\
&\qquad\left.\times\|\mathbf{E}\|_F\right)^2+2\left(\nu_1\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F+\nu_1\tau_2\left\|\mathbf{E}\right\|_F\right)^2\\
&\overset{(S.11)}{\le} 2\left(\sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}+\nu_1^2\delta_{3s}^2}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F\right.\\
&\qquad\left.+\left((\nu_1+\nu_2)\tau_2+\nu_2\delta_{3s}\tau_1\right)\|\mathbf{E}\|_F\right)^2.
\end{aligned}\tag{S.28}$$

Squaring both sides, we can get

$$\begin{aligned}
\left\|\mathbf{X}_{(S^k)^c}\right\|_F &\le \sqrt{\tfrac{4\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}+2\nu_1^2\delta_{3s}^2}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F\\
&\quad+\sqrt{2}\left((\nu_1+\nu_2)\tau_2+\nu_2\delta_{3s}\tau_1\right)\|\mathbf{E}\|_F.
\end{aligned}\tag{S.29}$$

Step 9 of the kth iteration in SI-SSP also solves a least squares problem. Letting $T=S^k$, $\tilde{\mathbf{X}}=\mathbf{X}^k$ and $t=s$, by (S.13), we have

$$\|\mathbf{X}-\mathbf{X}^k\|_F\le\sqrt{\frac{1}{1-\delta_{2s}^2}}\|\mathbf{X}_{(S^k)^c}\|_F+\frac{\sqrt{1+\delta_s}}{1-\delta_{2s}}\|\mathbf{E}\|_F. \tag{S.30}$$

Combining (S.29) and (S.30) yields

$$\|\mathbf{X}-\mathbf{X}^k\|_F\le\rho\|\mathbf{X}-\mathbf{X}^{k-1}\|_F+(1-\rho)\tau\|\mathbf{E}\|_F \tag{S.31}$$

where $\rho:=\sqrt{2}\delta_{3s}\sqrt{2\nu_1^2\nu_2^2\delta_{3s}^2+\nu_1^2-\nu_1^2\delta_{3s}^2}(1-\delta_{3s}^2)^{-1}$ and $\tau:=\sqrt{2}\delta_{3s}\nu_2(\nu_1\sqrt{2(1-\delta_{3s})}+\sqrt{1+\delta_{3s}})(1-\delta_{3s}^2)^{-1/2}(1-\delta_{3s})^{-1}(1-\rho)^{-1}+(\nu_1\nu_2\sqrt{2(1-\delta_{3s})}+\sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$.

Second, we recursively apply (S.31) to obtain

$$\|\mathbf{X}-\mathbf{X}^k\|_F\le\rho^k\|\mathbf{X}\|_F+\tau\|\mathbf{E}\|_F \tag{S.32}$$

where $\rho<1$ under (S.8). When $k^*=\lceil\log_\rho\frac{\|\mathbf{X}\|_F}{\tau\|\mathbf{E}\|_F}\rceil$, we have $\rho^k\|\mathbf{X}\|_F\le\tau\|\mathbf{E}\|_F$, and thus the stability result (S.10). □

Appendix A
Proof of Lemma 2

- First, we give a upper bound of $\|\mathbf{X}_{T_\nabla}\|_F$, by Lemma 6, let $\mathbf{Z}=(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S$, we have

$$\begin{aligned}
&\left\langle\mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}}),\mathbf{A}^H\mathbf{A}(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle\\
&\qquad+\left\langle\mathbf{W}_{T_0}\mathbf{E},\mathbf{A}(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle=\mathbf{0}.
\end{aligned}\tag{A.33}$$

Noticing that $\mathrm{supp}(\tilde{\mathbf{X}})\subseteq S$, we have

$$\begin{aligned}
&\left\|(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\|_F^2\\
=\;&\left\langle\mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}}),(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle\\
\overset{(A.33)}{=}\;&\left\langle\mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}}),(\mathbf{I}_L-\mathbf{A}^H\mathbf{A})(\mathbf{X}-\tilde{\mathbf{X}})_S\right\rangle\\
-\;&\left\langle\mathbf{W}_{T_0}\mathbf{E},\mathbf{A}(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle\\
\overset{(7)}{\le}\;&\omega\delta_{s+t}\left\|(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\|_F\left\|\mathbf{X}-\tilde{\mathbf{X}}\right\|_F\\
&+\omega\left\|\mathbf{E}\right\|_F\sqrt{1+\delta_t}\left\|\mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}})_S\right\|_F
\end{aligned}\tag{A.34}$$

Divide both sides by $\left\|(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\|_F$ to obtain (S.12).

- By expanding Lemma 2 in [25] to MMV model, we could get a relationship between $\left\|\mathbf{X}-\tilde{\mathbf{X}}\right\|_F$ and $\|\mathbf{X}_{S^c}\|_F$. We have

$$\left\|\mathbf{X}-\tilde{\mathbf{X}}\right\|_F\le\sqrt{\frac{1}{1-\delta_{s+t}^2}}\left\|\mathbf{X}_{S^c}\right\|_F^2+\frac{\sqrt{1+\delta_t}}{1-\delta_{s+t}}\left\|\mathbf{E}\right\|_F \tag{A.35}$$

- Finally, we established the relationship between $\mathbf{X}_{T_\nabla}$ and $\mathbf{X}-\tilde{\mathbf{X}}$. There exist a subset $S'\subseteq S$ and $S'\cap T=\emptyset$. Since T_∇ is defined by the set of indices of the $t-s$ smallest row entries of $\tilde{\mathbf{X}}$, we can conclude that

$$\begin{aligned}
&\left\|\tilde{\mathbf{X}}_{T_\nabla}\right\|_F+\left\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla}\right\|_F\\
&\qquad\le\left\|\tilde{\mathbf{X}}_{S'}\right\|_F+\left\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{S'}\right\|_F.
\end{aligned}\tag{A.36}$$

By eliminating the contribution from $T_\nabla\cap S'$ and noticing that $S'\cap T=\emptyset$, we have

$$\begin{aligned}
&\left\|\tilde{\mathbf{X}}_{T_\nabla\setminus S'}\right\|_F+\left\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla\setminus S'}\right\|_F\\
\le\;&\left\|(\tilde{\mathbf{X}}-\mathbf{X})_{S'\setminus T_\nabla}\right\|_F\\
+\;&\left\|\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X})_{S'\setminus T_\nabla}\right\|_F.
\end{aligned}\tag{A.37}$$

For the left-hand side of (A.37), we have

$$\begin{aligned}
&\left\|\tilde{\mathbf{X}}_{T_\nabla\setminus S'}\right\|_F+\left\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla\setminus S'}\right\|_F\\
=\;&\left\|(\tilde{\mathbf{X}}-\mathbf{X}+\mathbf{X})_{T_\nabla\setminus S'}\right\|_F\\
+\;&\left\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X})+\mathbf{W}_{T_0}\mathbf{X})_{T_\nabla\setminus S'}\right\|_F\\
\ge\;&\|\mathbf{X}_{T_\nabla}\|_F+\|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F \qquad (A.38)\\
-\;&\left\|(\tilde{\mathbf{X}}-\mathbf{X})_{T_\nabla\setminus S'}\right\|_F\\
-\;&\left\|\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X})_{T_\nabla\setminus S'}\right\|_F.
\end{aligned}\tag{A.39}$$

Combining (A.39) and (A.37), and noticing that

$$(T_\nabla\setminus S')\cap(S'\setminus T_\nabla)=\emptyset \tag{A.40}$$

$$(T_\nabla\setminus S')\cup(S'\setminus T_\nabla)\subseteq T, \tag{A.41}$$

we have

$$\begin{aligned}
&\|\mathbf{X}_{T_\nabla}\|_F+\|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F\\
\le\;&\left\|(\tilde{\mathbf{X}}-\mathbf{X})_{T_\nabla\setminus S'}\right\|_F+\left\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_{T_\nabla\setminus S'}\right\|_F\\
+\;&\left\|(\tilde{\mathbf{X}}-\mathbf{X})_{S'\setminus T_\nabla}\right\|_F+\left\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_{S'\setminus T_\nabla}\right\|_F\\
\le\;&\sqrt{2}\left\|(\tilde{\mathbf{X}}-\mathbf{X})_{(S'\setminus T_\nabla)\cup(T_\nabla\setminus S')}\right\|_F
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \sqrt{2}\left\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_{(S'\setminus T_\triangledown)\cup(T_\triangledown\setminus S')}\right\|_F \\
&\leq \quad \sqrt{2}\left\|(\tilde{\mathbf{X}}-\mathbf{X})_S\right\|_F+\sqrt{2}\left\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_S\right\|_F \\
&\overset{(S.12)}{\leq} \quad \sqrt{2}(1+\omega)\delta_{s+t}\left\|\mathbf{X}-\tilde{\mathbf{X}}\right\|_F \\
&+ \quad (1+\omega)\sqrt{2(1+\delta_t)}\left\|\mathbf{E}\right\|_F . \qquad (A.42)
\end{aligned}
$$

Also, we can obtain the relationship between $\|\mathbf{W}_{T_0}\mathbf{X}_{T_\triangledown}\|_F$ and $\|\mathbf{X}_{T_\triangledown}\|_F$:

$$
\eta\omega\sqrt{s_3}\left\|\mathbf{X}_{T_\triangledown}\right\|_F \leq \left\|\mathbf{W}_{T_0}\mathbf{X}_{T_\triangledown}\right\|_F . \qquad (A.43)
$$

Combining (A.42) and (A.43), we have

$$
\begin{aligned}
\left\|\mathbf{X}_{T_\triangledown}\right\|_F \leq & \frac{\sqrt{2}(1+\omega)\delta_{s+t}}{1+\eta\omega\sqrt{s_3}}\left\|\mathbf{X}-\tilde{\mathbf{X}}\right\|_F \\
&+\frac{(1+\omega)\sqrt{2(1+\delta_t)}}{1+\eta\omega\sqrt{s_3}}\left\|\mathbf{E}\right\|_F . \qquad (A.44)
\end{aligned}
$$

Noting the definition of ν_2, we complete the proof of Lemma 2.

Appendix B
Proof of Lemma 3

Proof: From step 5 of SI-SSP, we have

$$
\mathbf{X}^k_{S_i} = \underset{\boldsymbol{\Theta}:\text{supp}(\boldsymbol{\Theta})=S^k_{S_i}}{\arg\min}\left\|\mathbf{Y}_{S_i}-\mathbf{A}\boldsymbol{\Theta}\right\|_F . \qquad (A.45)
$$

From the step 4 of SI-SSP, let $\mathbf{X}^k = [\mathbf{X}^k_{S_1},\cdots,\mathbf{X}^k_{S_r}]$, we have the following conclusion

$$
\begin{aligned}
&\quad \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_T\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_T\right\|_F \\
&\leq \quad \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\right\|_F . \qquad (A.46)
\end{aligned}
$$

By removing the same coordinates $T\cap\Delta S$, we can get

$$
\begin{aligned}
&\quad \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\right\|_F \\
&\leq \quad \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F . \qquad (A.47)
\end{aligned}
$$

Because $\text{supp}(\mathbf{X}) = T$ and $\text{supp}(\mathbf{X}^{k-1}) = S^{k-1}$,

$$
(\mathbf{X}-\mathbf{X}^{k-1})_{\Delta S\setminus(T\cup S^{k-1})} = 0. \qquad (A.48)
$$

For the right-hand of (A.47), we have

$$
\begin{aligned}
&\quad \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
&= \quad \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\right\|_F \\
&= \quad \left\|(\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\right\|_F \\
&\leq \quad \left\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
&+ \quad \left\|(\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\right\|_F . \qquad (A.49)
\end{aligned}
$$

Note that $\tilde{S}^k = S^{k-1}\cup\Delta S$, we have

$$
(\mathbf{X}-\mathbf{X}^{k-1})_{T\setminus\tilde{S}^k} = \mathbf{X}_{(\tilde{S}^k)^c}. \qquad (A.50)
$$

For the left-side of (A.47),we have

$$
\begin{aligned}
&\quad \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\right\|_F \\
&= \quad \left\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\right\|_F \\
&= \quad \left\|(\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\right\|_F \\
&= \quad \left\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1})\right. \\
&+ \quad \left.\mathbf{A}^H\mathbf{E}+\mathbf{X})_{\Delta S\setminus(T\cup S^{k-1})}\right\| \\
&+ \quad \left\|\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1})\right. \\
&+ \quad \left.\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E}+\mathbf{W}_{T_0}\mathbf{X})_{\Delta S\setminus(T\cup S^{k-1})}\right\|_F \\
&\geq \quad \left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F+\left\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\right\|_F \\
&- \quad \left\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\right\|_F \qquad (A.51) \\
&- \quad \left\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\right\|_F \\
&- \quad \left\|(\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\right\|_F-\left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\right\|_F . \qquad (A.52)
\end{aligned}
$$

Combining (A.53) and (A.52), we have

$$
\begin{aligned}
&\quad \left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F+\left\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\right\|_F \\
&\leq \quad \left\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\setminus\tilde{S}^k}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\setminus\tilde{S}^k}\right\|_F \\
&+ \quad \left\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
&+ \quad \left\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\right\|_F \\
&+ \quad \left\|(\mathbf{A}^H\mathbf{E})_{T\setminus\tilde{S}^k}\right\|_F+\left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T\setminus\tilde{S}^k}\right\|_F \\
&+ \quad \left\|(\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\right\|_F+\left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\right\|_F \\
&\leq \quad \sqrt{2}\left\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\cup\Delta S}\right\|_F \\
&+ \quad \sqrt{2}\left\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\cup\Delta S}\right\|_F \\
&+ \quad \sqrt{2}\left\|(\mathbf{A}^H\mathbf{E})_{T\cup\Delta S}\right\|_F+\sqrt{2}\left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T\cup\Delta S}\right\|_F \\
&\leq \quad \sqrt{2}(1+\omega)\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F \\
&+ \quad (1+\omega)\sqrt{2(1+\delta_{3s})}\left\|\mathbf{E}\right\|_F . \qquad (A.53)
\end{aligned}
$$

We can obtain the relationship between $\left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F$ $\left\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\right\|_F$:

$$
\eta\omega\sqrt{s_2}\left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F \leq \left\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\right\|_F . \qquad (A.54)
$$

Combining (A.53) and (A.54), we have

$$
\left\|\mathbf{X}_{(\tilde{S}^k)^c}\right\|_F \leq \frac{\sqrt{2}(1+\omega)\delta_{3s}}{1+\eta\omega\sqrt{s_2}}\left\|\mathbf{X}-\tilde{\mathbf{X}}\right\|_F
$$

$$+\frac{(1+\omega)\sqrt{2(1+\delta_{2s})}}{1+\eta\omega\sqrt{s_2}}\|\mathbf{E}\|_F. \tag{A.55}$$

Noting the definition of ν_1, we complete the proof of Lemma 3.

Appendix C
Proof of Lemma 4

Proof: the RIC δ_t can be expressed as [25]

$$\delta_t = \max_{S\subseteq\{1,2,\ldots,N\},|S|\le t}\|\mathbf{A}_S^*\mathbf{A}_S-\mathbf{I}\|_{2\to 2}, \tag{A.56}$$

where

$$\|\mathbf{A}_S^*\mathbf{A}_S-\mathbf{I}\|_{2\to 2} = \sup_{\mathbf{a}\in\mathbb{R}^{|S|}\setminus\{\mathbf{0}\}}\frac{\|(\mathbf{A}_S^*\mathbf{A}_S-\mathbf{I})\,\mathbf{a}\|_2}{\|\mathbf{a}\|_2}. \tag{A.57}$$

Let $S=\mathrm{supp}(\mathbf{u})\cup\mathrm{supp}(\mathbf{v})$, then $|S|\le t$. Let $\mathbf{u}_{|S},\mathbf{v}_{|S}$ denote respectively the S-dimensional sub-vectors of $\mathbf{u}$ and $\mathbf{v}$ obtained by only keeping the components indexed by S. We have

$$\begin{aligned}
&\left|\left\langle\mathbf{u},\left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\right\rangle\right|\\
=&|\langle\mathbf{W}_{T_0}\mathbf{u},\mathbf{v}\rangle-\langle\mathbf{A}\mathbf{W}_{T_0}\mathbf{u},\mathbf{A}\mathbf{v}\rangle|\\
=&\left|\left\langle\mathbf{W}_{T_0}\mathbf{u}_{|T},\left(\mathbf{I}_L-\mathbf{A}_T^H\mathbf{A}_T\right)\mathbf{v}_{|T}\right\rangle\right|\\
\le&\left\|\mathbf{W}_{T_0}\mathbf{u}_{|T}\right\|_2\left\|\left(\mathbf{I}_L-\mathbf{A}_T^H\mathbf{A}_T\right)\mathbf{v}_{|T}\right\|_2\\
\overset{(A.57)}{\le}&\left\|\mathbf{W}_{T_0}\mathbf{u}_{|T}\right\|_2\left\|\mathbf{I}_L-\mathbf{A}_T^H\mathbf{A}_T\right\|_{2\to 2}\left\|\mathbf{v}_{|T}\right\|_2\\
\overset{(A.56)}{\le}&\omega\delta_t\left\|\mathbf{u}_{|T}\right\|_2\left\|\mathbf{v}_{|T}\right\|_2\\
=&\omega\delta_t\|\mathbf{u}\|_2\|\mathbf{v}\|_2,
\end{aligned} \tag{A.58}$$

moreover, we have

$$\begin{aligned}
&\left\|\left(\left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\right)_U\right\|_2^2\\
=&\langle\left(\left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\right)_U,\\
&\left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\rangle\\
\overset{(S.20)}{\le}&\delta_t\left\|\left(\left(\mathbf{W}_{T_0}-\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\right)_U\right\|_2\|\mathbf{v}\|_2
\end{aligned} \tag{A.59}$$

which completes the proof of Lemma 4.

Appendix D
Proof of Lemma 5

Proof: The lemma easily follows from the fact that

$$\begin{aligned}
&\left\|\left(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e}\right)_U\right\|_2^2\\
=&\left\langle\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e},\left(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e}\right)_U\right\rangle\\
=&\left\langle\mathbf{e},\mathbf{W}_{T_0}\mathbf{A}\left(\left(\mathbf{A}^H\mathbf{e}\right)_U\right)\right\rangle\\
\le&\|\mathbf{e}\|_2\left\|\mathbf{W}_{T_0}\mathbf{A}\left(\left(\mathbf{A}^H\mathbf{e}\right)_U\right)\right\|_2\\
\overset{(7)}{\le}&\|\mathbf{e}'\|_2\,\omega\sqrt{1+\delta_u}\left\|\left(\mathbf{A}^H\mathbf{e}\right)_U\right\|_2.
\end{aligned} \tag{A.60}$$

Appendix E
Proof of Lemma 6

Proof: Due to the orthogonality, the residue $\mathbf{Y}-\mathbf{A}\tilde{\mathbf{X}}$ is orthogonal to the space $\mathbf{AZ},\mathrm{supp}(\mathbf{Z})\subseteq S$. This means that for all $\mathbf{Z}\in\mathbb{R}$ with $\mathrm{supp}(\mathbf{Z})\subseteq S$,

$$\left\langle\mathbf{A}(\mathbf{Y}-\mathbf{A}\tilde{\mathbf{X}}),\mathbf{Z}\right\rangle=\mathbf{0} \tag{A.61}$$

then, let $\tilde{X}'$ be the solution of the least squares problem $\arg\min_{\mathbf{Z}}\{\|\mathbf{Y}'-\mathbf{AZ}\|_F,\mathrm{supp}(\mathbf{Z})\subseteq S\}$, where $\mathbf{Y}'=\frac{\mathbf{A}\mathbf{W}_{T_0}\mathbf{X}_{T_0}}{\omega}+\mathbf{E}$. We have

$$\tilde{\mathbf{X}}'=\frac{\mathbf{W}_{T_0}\tilde{\mathbf{X}}}{\omega}. \tag{A.62}$$

Then, by (A.61), we have

$$\begin{aligned}
\mathbf{0} &= \left\langle\frac{\mathbf{A}\mathbf{W}_{T_0}\mathbf{X}_{T_0}}{\omega}+\mathbf{E}-\mathbf{A}\frac{\mathbf{W}_{T_0}\tilde{X}}{\omega},\mathbf{AZ}\right\rangle\\
&= \left\langle\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}},\mathbf{A}^H\mathbf{AZ}\right\rangle+\omega\langle\mathbf{E},\mathbf{AZ}\rangle.
\end{aligned} \tag{A.63}$$