

Supplementary: Boundary Multiple Measurement Vectors for Multi-Coset Sampler

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This supplementary material is dedicated to the proofs for Theorems 1–3 in our main paper.

Before proceeding to the proofs, we review some useful notations. For a complex matrix $\mathbf{X} \in \mathbb{C}^{n \times L}$ and a set $S \subseteq \{1, \dots, n\}$, \mathbf{X}_S (or \mathbf{X}^S) denotes the submatrix of \mathbf{X} with columns (or rows) indexed by S ; $\mathbf{X}_{i,j}$, $\mathbf{X}_{i,:}$ and $\mathbf{X}_{:,i}$ are the (i, j) th entry, i th row and i th column of \mathbf{X} , respectively; \mathbf{X}^\dagger , \mathbf{X}^H and \mathbf{X}^\top mean the Moore-Penrose pseudo-inverse, conjugate transpose and transpose of \mathbf{X} , respectively; $\text{supp}(\mathbf{X})$ is the non-zero row indices (i.e., joint sparsity) of \mathbf{X} ; $\|\mathbf{X}\|_F$ and $\|\mathbf{X}\|_2$ signify the Frobenius and Euclidean norm of \mathbf{X} , respectively. Moreover, S^c is the complement of set S ; \mathbf{I}_L is an $L \times L$ identity matrix.

I. PROOF OF THEOREM 1

Theorem 1. *The actual sampling rate of (4) is $\min(p f_s, f_{\text{nyq}})$, which attains the theoretical lower bound of sampling rate in MCS when $|\text{supp}(\mathbf{X})| \leq \frac{N_{\text{sig}} B}{f_s}$.*

Proof. In the i th channel of a multi-coset sampler, the sampling sequence is given by

$$x_{c_i}[n] = x(LTn + \tau_i), \quad n = 0, 1, \dots \quad (\text{S.1})$$

The sampling rate of each channel is determined by the sampled signal sequence. To be specific, since the sampling time interval is LT , the sampling rate of each channel is

$$f_s = \frac{1}{LT} = \frac{f_{\text{nyq}}}{L}, \quad (\text{S.2})$$

i.e., one- L th of the Nyquist sampling rate.

Moreover, as the multi-coset sampler is assumed to have p channels, the overall sampling rate of p channels is p times that of each channel (i.e. $\frac{p f_{\text{nyq}}}{L}$). If this sampling rate is greater than the Nyquist rate f_{nyq} , then the advantage of sub-Nyquist sampling structure no longer exists. In this case, we only need to sample at Nyquist sampling rate f_{nyq} . Thus, the actual sampling rate can be given by

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right). \quad (\text{S.3})$$

The theoretical lower bound of the sampling rate is given in [17], which is determined directly by the true bandwidth of the signal:

$$\min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \quad (\text{S.4})$$

Thus, the theoretical lower bound on the sampling rate is achieved when

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right) \leq \min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \quad (\text{S.5})$$

In most cases, $2\lambda(\mathcal{T})$ and $\frac{p}{LT}$ do not exceed f_{nyq} . (If violated, the sampling rate would just be f_{nyq} .) Therefore, the condition (S.5) holds whenever

$$\frac{p}{LT} \leq 2\lambda(\mathcal{T}). \quad (\text{S.6})$$

Furthermore, to ensure a unique-solution reconstruction, the number p of channels should not be too small. In particular, its lower bound is twice the signal sparsity without the priori information about the signal \mathbf{X} [17],

$$p \geq 2|\text{supp}(\mathbf{X})|. \quad (\text{S.7})$$

For the worst case where $p = 2|\text{supp}(\mathbf{X})|$, (S.6) can be rewritten as

$$|\text{supp}(\mathbf{X})| \leq \lambda(\mathcal{T})LT = \frac{N_{\text{sig}} B}{f_s}, \quad (\text{S.8})$$

which completes the proof. \square

II. PROOF OF THEOREM 2

Theorem 2. *When $r \in [\lceil \frac{f_s}{(M-1)f_s - B} \rceil, \infty)$ and $B > f_s$, we have $\max_{i \in \{1, \dots, r\}} |\text{supp}(\bar{\mathbf{X}}_{S_i})| \leq \frac{N_{\text{sig}} B}{f_s}$.*

Proof. Consider all blocks $\{\mathbf{X}^{U_1}, \dots, \mathbf{X}^{U_M}\}$ in \mathbf{X} , where there are consecutive corresponding frequency points of length B in occupied blocks. For the case $B > f_s$ and each block occupies M rows in \mathbf{X} , we can assume that $(M-2)f_s < B \leq (M-1)f_s$ for $M \geq 3$. The frequency points of the subband signal may occupy $M-1$ or M rows actually. For the case that $M-1$ rows in any block (i.e. \mathbf{X}^{U_i}) are occupied actually, only N_{sig} rows of $\bar{\mathbf{X}}$ are occupied. Thus, we have

$$\max_{i \in \{1, \dots, r\}} |\text{supp}(\bar{\mathbf{X}}_{S_i})| = |\text{supp}(\bar{\mathbf{X}})| \leq N_{\text{sig}}. \quad (\text{S.9})$$

For another case that M rows are occupied actually, the length of the frequency point in each $\bar{\mathbf{X}}^{U_i}$ (\mathbf{X}^{U_i} is occupied) meets

$$l = B - (M-2)f_s \leq f_s. \quad (\text{S.10})$$

Because $l \leq f_s$, we know that the non-zero elements of any occupied partial-block (i.e. $\bar{\mathbf{X}}^{U_i}$) do not intersect on the column indices. Considering one occupied partial-block $\bar{\mathbf{X}}^{U_i}$, let $r \rightarrow \infty$ and observe the change in $\bar{\mathbf{X}}^{U_i}$, $\bar{\mathbf{X}}^{U_i}$ gradually changes from an MMV form to an SMV form. In the SMV form, the sparsity of $\bar{\mathbf{X}}^{U_i}$ is determined by the columns in $\bar{\mathbf{X}}^{U_i}$. It is observed that

$$\lim_{r \rightarrow \infty, i, j} \text{supp}(\bar{\mathbf{X}}_{:,j}^{U_i}) \leq 1. \quad (\text{S.11})$$

Thus, $|\text{supp}(\bar{\mathbf{X}}_{S_i})| = |\text{supp}(\bar{\mathbf{X}}_{:,i})|$ is less than N_{sig} (there exist N_{sig} subbands in \mathbf{X}). We proof the upper bound of the sub-MMV problems number r .

A more complex situation occurs when r is a finite value, assuming $r = r^*$ is a finite value. In this case, the length of frequency points in each sub-matrix is less than the sub-matrix columns number $\lceil \frac{f_s}{r^*} \rceil$. We use proof by contradiction to prove the condition that the sparsity of each sub-matrix is less than N_{sig} . Assuming that there exists a partial-block sub-matrix $\bar{\mathbf{X}}_{S_i}$ with $|\text{supp}(\bar{\mathbf{X}}_{S_i})| > N_{\text{sig}}$ when $r^* \in [\lceil \frac{f_s}{(M-1)f_s-B} \rceil, \infty)$. Also, the non-zero elements of any partial-block in $\bar{\mathbf{X}}$ do not intersect on the column indices. $\bar{\mathbf{X}}_{S_i}$ must contain both non-zero elements on both sides of one partial-block. From (S.10), we know that the length of any partial-block of \mathbf{X} is less than f_s . We can draw a conclusion that

$$\left\lceil \frac{f_s}{r^*} \right\rceil > f_s - l. \quad (\text{S.12})$$

Combining (S.10) and (S.12), we can get

$$\left\lceil \frac{f_s}{r^*} \right\rceil > (M-1)f_s - B. \quad (\text{S.13})$$

As can be seen, there exist a contradiction between (S.13) and the assumption $r \in [\lceil \frac{f_s}{(M-1)f_s-B} \rceil, \infty)$, so the length of partial-block non-zero elements in any sub-matrix of $\bar{\mathbf{X}}$ must be less or equal than $(M-1)f_s - B$, which is equivalent to r satisfying

$$r \geq \left\lceil \frac{f_s}{(M-1)f_s - B} \right\rceil. \quad (\text{S.14})$$

To sum up, when $r \in [\lceil \frac{f_s}{(M-1)f_s-B} \rceil, \infty)$, we have

$$\max_{i \in \{1, \dots, r\}} |\text{supp}(\bar{\mathbf{X}}_{S_i})| \leq N_{\text{sig}} < \frac{N_{\text{sig}} B}{f_s}. \quad (\text{S.15})$$

The proof is thus complete. \square

III. PROOF OF THEOREM 3

Theorem 3. Consider the column-partitioned MMV model (5) with $\min_{i,j} \|(\mathbf{X}_{S_i})_{j,:}\|_2 / \|\mathbf{X}_{S_i}\|_F = \eta$ and $|\text{supp}(\mathbf{X}_{S_i})| \leq s$. Let $s_1 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i})|$, $s_2 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i}) \cap \tilde{S}_{S_i}^k|$ and $s_3 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i}) \cap \tilde{S}_{S_i}^k \setminus S_{S_i}^k|$. Then, if the sensing matrix \mathbf{A} obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_1 \sqrt{\nu_1^2 + 4\nu_2^2} - \nu_1^2 - 1}{4\nu_1^2 \nu_2^2 - 2\nu_1^2 - 1}} \quad (\text{S.16})$$

where $\nu_1 := \frac{1+\omega}{1+\eta\omega\sqrt{s_2}}$ and $\nu_2 := \frac{1+\omega}{1+\eta\omega\sqrt{s_3}}$, SI-SSP produces an signal estimate $\mathbf{X}^k = [\mathbf{X}_{S_1}^k, \dots, \mathbf{X}_{S_r}^k]$ satisfying

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho^k \|\mathbf{X}\|_F + \tau \|\mathbf{E}\|_F, \quad (\text{S.17})$$

where $\rho \in (0, 1)$ and τ are constants depending on δ_{3s} , ν_1 and ν_2 . Furthermore, after at most $k^* = \lceil \log_{\rho} \frac{\|\mathbf{X}\|_F}{\tau \|\mathbf{E}\|_F} \rceil$ iterations, SI-SSP estimates \mathbf{X} with

$$\|\mathbf{X} - \mathbf{X}^{k^*}\|_F \leq (\tau + 1) \|\mathbf{E}\|_F. \quad (\text{S.18})$$

To prove Theorem 3, we first introduce six useful Lemmas, whose proofs are left to the appendices.

Lemma 1. ([25]): For nonnegative numbers a, b, c, d, x, y ,

$$(ax + by)^2 + (cx + dy)^2 \leq (\sqrt{a^2 + c^2}x + (b + d)y)^2. \quad (\text{S.19})$$

Lemma 2. Consider the system model $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$, where $\text{supp}(\mathbf{X}) = T$ and $|T| = s$. Let $S \subseteq \{1, 2, \dots, n\}$ be an index set with $|S| = t$ and \mathbf{W}_{T_0} be a side-information matrix with diagonal entries indexed by $T_0 \subseteq \{1, 2, \dots, n\}$ being $\omega \geq 0$ and zero otherwise. Also, let $\tilde{\mathbf{X}} := \arg \min_{\mathbf{Z}: \text{supp}(\mathbf{Z}) \subseteq S} \|\mathbf{Y} - \mathbf{A}\mathbf{Z}\|_2$. If $\delta_{3s} < 1$, then

$$\|\mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}})_S\|_F \leq \omega \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \omega \sqrt{1 + \delta_t} \|\mathbf{E}\|_F \quad (\text{S.20})$$

and

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{s+t}^2}} \|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1 + \delta_t}}{1 - \delta_{s+t}} \|\mathbf{E}\|_F. \quad (\text{S.21})$$

Furthermore, if $t > s$, define T_{∇} as the row-indices of the smallest $t - s$ row-norm entries of $\tilde{\mathbf{X}}$ in S , we have

$$\|\mathbf{X}_{T_{\nabla}}\|_F \leq \sqrt{2} \nu_2 \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_2 \sqrt{2(1 + \delta_t)} \|\mathbf{E}\|_F. \quad (\text{S.22})$$

Remark 1. When we consider the atom selection strategy of $\|\tilde{\mathbf{X}}_{T_{\nabla}} + \mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_{\nabla}}\|_F \leq \|\tilde{\mathbf{X}}_{S'} + \mathbf{W}_{T_0} \tilde{\mathbf{X}}_{S'}\|_F$, we can also obtain another upper bound for $\|\mathbf{X}_{T_{\nabla}}\|_F$ in (S.22). In this case, we should allocate $2\|\mathbf{X}_{T_{\nabla}}\|_F$ to the left hand side of (A.46), we have

$$\|\mathbf{X}_{T_{\nabla}}\|_F \leq \sqrt{2} \nu_3 \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_4 \sqrt{2(1 + \delta_t)} \|\mathbf{E}\|_F. \quad (\text{S.23})$$

where $\nu_3 = (1 - \omega + \omega \delta_{s+t} + \delta_{s+t}) / (2\delta_{s+t})$ and $\nu_4 = (1 + \omega) / (2\delta_{s+t})$.

Lemma 3. In steps 4 and 5 of SI-SSP, we have

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2} \nu_1 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \quad (\text{S.24})$$

Remark 2. When we consider the atom selection strategy in select step that

$$\begin{aligned} & \|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ & \leq \|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F. \end{aligned} \quad (\text{S.25})$$

We can also obtain another upper bound for $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ in (S.24). In this case, we should allocate $2\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ to the left hand side of (A.59), we have

$$\begin{aligned} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F & \leq \sqrt{2} \nu_4 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F \\ & + \nu_4 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \end{aligned} \quad (\text{S.26})$$

where $\nu_4 = (1 - \omega + \omega \delta_{3s} + \delta_{3s}) / (2\delta_{3s})$ and $\nu_4 = (1 + \omega) / (2\delta_{3s})$. Based on conclusions (S.23) and (S.26), we know that the sensing matrix \mathbf{A} obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_3 \sqrt{\nu_3^2 + 4\nu_4^2} - \nu_3^2 - 1}{4\nu_3^2 \nu_4^2 - 2\nu_3^2 - 1}}. \quad (\text{S.27})$$

Lemma 4. Let $T_0 \subseteq \{1, 2, \dots, n\}$, for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, if $|\text{supp}(\mathbf{u}) \cup \text{supp}(\mathbf{v})| \leq t$,

$$|\langle \mathbf{u}, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle| \leq \omega \delta_t \|\mathbf{u}\| \|\mathbf{v}\|; \quad (\text{S.28})$$

Moreover, if $U \subseteq \{1, 2, \dots, n\}$ and $|U \cup \text{supp}(\mathbf{v})| \leq t$, then

$$|(\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v}| \leq \omega \delta_t \|\mathbf{v}\|. \quad (\text{S.29})$$

Lemma 5. For SMV model $\mathbf{y} = \Phi \mathbf{x} + \mathbf{e}$, let $T_0 \subseteq \{1, 2, \dots, n\}$, let $U \subseteq \{1, 2, \dots, n\}$ and $|U \cap T_0| \leq u$, we have

$$\|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U\|_2 \leq \omega \delta_u \|\mathbf{e}\|_2. \quad (\text{S.30})$$

Lemma 6. Consider the MMV model $\mathbf{Y} = \mathbf{A} \mathbf{X} + \mathbf{E}$, let $\tilde{\mathbf{X}}$ be the solution of the least squares problem $\arg \min_{\mathbf{Z}} \{\|\mathbf{Y} - \mathbf{A} \mathbf{Z}\|_F, \text{supp}(\mathbf{Z}) \subseteq S\}$, then

$$\langle \mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}}, \mathbf{A}^H \mathbf{A} \mathbf{Z} \rangle + \omega \langle \mathbf{E}, \mathbf{A} \mathbf{Z} \rangle = 0. \quad (\text{S.31})$$

Now we have all ingredients to prove Theorem 3.

Proof of Theorem 3. First, in Steps 4 and 5 of SI-SSP, Lemma 3 implies

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2} \nu_1 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \quad (\text{S.32})$$

Note that Step 6 of SI-SSP solves a least squares problem. Let $S = \tilde{S}^k$ and $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $t = 2s$, by (S.21) we have

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{3s}^2}} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \frac{\sqrt{1 + \delta_{2s}}}{1 - \delta_{3s}} \|\mathbf{E}\|_F. \quad (\text{S.33})$$

Combining (S.32) and (S.33) and also magnifying δ_{2s} to δ_{3s} , we further have

$$\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F \leq \nu_1 \sqrt{\frac{2\delta_{3s}^2}{1 - \delta_{3s}^2}} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \tau_1 \|\mathbf{E}\|_F. \quad (\text{S.34})$$

Next, after Step 7 of SI-SSP, let $S_\nabla = \tilde{S}^k \setminus S^k$ be the row-indices of the smallest $t - s$ row norm entries in $\tilde{\mathbf{X}}^k$. Also, let $T = \tilde{S}^k$, $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $T_\nabla = S_\nabla$ and $t = 2s$. Then, by (A.45) we have

$$\|\mathbf{X}_{S_\nabla}\|_F \leq \sqrt{2} \nu_2 \delta_{3s} \|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2 \sqrt{2(1 + \delta_{2s})} \|\mathbf{E}\|_F. \quad (\text{S.35})$$

Let $\tau_1 = \frac{\nu_1 \sqrt{2(1 - \delta_{3s})} + \sqrt{1 + \delta_{3s}}}{1 - \delta_{3s}} (1 - \delta_{3s})^{-1}$ and $\tau_2 = \sqrt{1 + \delta_{3s}}$. Dividing $(\tilde{S}^k)^c$ into two disjoint subsets: $(\tilde{S}^k)^c$ and S_∇ , we get

$$\begin{aligned} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2 &= \|\mathbf{X}_{S_\nabla}\|_F^2 + \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2 \\ &\stackrel{(\text{S.32}), (\text{S.35})}{\leq} 2 \left(\nu_2 \delta_{3s} \|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2 \tau_2 \|\mathbf{E}\|_F \right)^2 \\ &\quad + 2 \left(\nu_1 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \tau_2 \|\mathbf{E}\|_F \right)^2 \\ &\stackrel{(\text{S.34})}{\leq} 2 \left(\sqrt{\frac{2\nu_1^2 \nu_2^2 \delta_{3s}^4}{1 - \delta_{3s}^2}} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_2 (\tau_1 \delta_{3s} + \tau_2) \right. \\ &\quad \times \|\mathbf{E}\|_F^2 + 2 \left(\nu_1 \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \tau_2 \|\mathbf{E}\|_F \right)^2 \\ &\stackrel{(\text{S.19})}{\leq} 2 \left(\sqrt{\frac{2\nu_1^2 \nu_2^2 \delta_{3s}^4}{1 - \delta_{3s}^2}} + \nu_1^2 \delta_{3s}^2 \|\mathbf{X} - \mathbf{X}^{k-1}\|_F \right. \\ &\quad \left. + ((\nu_1 + \nu_2) \tau_2 + \nu_2 \delta_{3s} \tau_1) \|\mathbf{E}\|_F \right)^2. \end{aligned} \quad (\text{S.36})$$

Squaring both sides, we get

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{\frac{4\nu_1^2 \nu_2^2 \delta_{3s}^4}{1 - \delta_{3s}^2} + 2\nu_1^2 \delta_{3s}^2} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F$$

$$+ \sqrt{2} ((\nu_1 + \nu_2) \tau_2 + \nu_2 \delta_{3s} \tau_1) \|\mathbf{E}\|_F. \quad (\text{S.37})$$

Step 9 of SI-SSP also solves a least squares problem. Letting $T = S^k$, $\tilde{\mathbf{X}} = \mathbf{X}^k$ and $t = s$, by (S.21), we have

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \sqrt{\frac{1}{1 - \delta_{2s}^2}} \|\mathbf{X}_{(S^k)^c}\|_F + \frac{\sqrt{1 + \delta_s}}{1 - \delta_{2s}} \|\mathbf{E}\|_F. \quad (\text{S.38})$$

Finally, combining (S.37) and (S.38) yields

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + (1 - \rho) \tau \|\mathbf{E}\|_F \quad (\text{S.39})$$

where $\rho := \sqrt{2} \delta_{3s} \sqrt{2\nu_1^2 \nu_2^2 \delta_{3s}^2 + \nu_1^2 - \nu_1^2 \delta_{3s}^2} (1 - \delta_{3s}^2)^{-1}$ and $\tau := \sqrt{2} \delta_{3s} \nu_2 (\nu_1 \sqrt{2(1 - \delta_{3s})} + \sqrt{1 + \delta_{3s}}) (1 - \delta_{3s}^2)^{-1/2} (1 - \delta_{3s})^{-1} (1 - \rho)^{-1} + (\nu_1 \nu_2 \sqrt{2(1 - \delta_{3s})} + \sqrt{1 + \delta_{3s}}) (1 - \delta_{3s})^{-1}$.

We recursively apply (S.39) to obtain

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho^k \|\mathbf{X}\|_F + \tau \|\mathbf{E}\|_F \quad (\text{S.40})$$

where $\rho < 1$ under (S.16). When $k^* = \lceil \log_\rho \frac{\|\mathbf{X}\|_F}{\tau \|\mathbf{E}\|_F} \rceil$, we have $\rho^{k^*} \|\mathbf{X}\|_F \leq \tau \|\mathbf{E}\|_F$, and thus the stability result (S.18). \square

APPENDIX A PROOF OF LEMMA 2

- First, we give an upper bound of $\|\mathbf{X}_{T_\nabla}\|_F$, by Lemma 6, let $\mathbf{Z} = (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S$, we have

$$\begin{aligned} &\langle \mathbf{W}_{T_0} (\mathbf{X} - \tilde{\mathbf{X}}), \mathbf{A}^H \mathbf{A} (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \rangle \\ &\quad + \langle \mathbf{W}_{T_0} \mathbf{E}, \mathbf{A} (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \rangle = 0. \end{aligned} \quad (\text{A.41})$$

Noticing that $\text{supp}(\tilde{\mathbf{X}}) \subseteq S$, we have

$$\begin{aligned} &\|(\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S\|_F^2 \\ &= \langle \mathbf{W}_{T_0} (\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \rangle \\ &\stackrel{(\text{A.41})}{=} \langle \mathbf{W}_{T_0} (\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{I}_L - \mathbf{A}^H \mathbf{A}) (\mathbf{X} - \tilde{\mathbf{X}})_S \rangle \\ &\quad - \langle \mathbf{W}_{T_0} \mathbf{E}, \mathbf{A} (\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S \rangle \\ &\stackrel{(7)}{\leq} \omega \delta_{s+t} \|(\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S\|_F \|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\ &\quad + \omega \|\mathbf{E}\|_F \sqrt{1 + \delta_t} \|\mathbf{W}_{T_0} (\mathbf{X} - \tilde{\mathbf{X}})_S\|_F. \end{aligned} \quad (\text{A.42})$$

Divide both sides by $\|(\mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}})_S\|_F$ to obtain (S.20).

- Next, by expanding [Lemma 2, 25] to the MMV model, we could get a relationship between $\|\mathbf{X} - \tilde{\mathbf{X}}\|_F$ and $\|\mathbf{X}_{S^c}\|_F$. We have

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{s+t}^2}} \|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1 + \delta_t}}{1 - \delta_{s+t}} \|\mathbf{E}\|_F. \quad (\text{A.43})$$

- Then, we established the relationship between \mathbf{X}_{T_∇} and $\mathbf{X} - \tilde{\mathbf{X}}$. There exist a subset $S' \subseteq S$ and $S' \cap T = \emptyset$. Since T_∇ is defined by the set of indices of the $t - s$ smallest row entries of $\tilde{\mathbf{X}}$, we can conclude that

$$\begin{aligned} &\|\tilde{\mathbf{X}}_{T_\nabla}\|_F + \|\mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_\nabla}\|_F \\ &\leq \|\tilde{\mathbf{X}}_{S'}\|_F + \|\mathbf{W}_{T_0} \tilde{\mathbf{X}}_{S'}\|_F. \end{aligned} \quad (\text{A.44})$$

By eliminating the contribution from $T_{\nabla} \cap S'$ and noticing that $S' \cap T = \emptyset$, we have

$$\begin{aligned} \|\tilde{\mathbf{X}}_{T_{\nabla} \setminus S'}\|_F &+ \|\mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_{\nabla} \setminus S'}\|_F \\ &\leq \|(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_{\nabla}}\|_F \\ &+ \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_{\nabla}}\|_F. \end{aligned} \quad (\text{A.45})$$

For the left-hand side of (A.45), we have

$$\begin{aligned} \|\tilde{\mathbf{X}}_{T_{\nabla} \setminus S'}\|_F &+ \|\mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_{\nabla} \setminus S'}\|_F \\ &= \|(\tilde{\mathbf{X}} - \mathbf{X} + \mathbf{X})_{T_{\nabla} \setminus S'}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}) + \mathbf{W}_{T_0} \mathbf{X})_{T_{\nabla} \setminus S'}\|_F \\ &\geq \|\mathbf{X}_{T_{\nabla}}\|_F + \|\mathbf{W}_{T_0} \mathbf{X}_{T_{\nabla}}\|_F \\ &- \|(\tilde{\mathbf{X}} - \mathbf{X})_{T_{\nabla} \setminus S'}\|_F \\ &- \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{T_{\nabla} \setminus S'}\|_F. \end{aligned} \quad (\text{A.46})$$

Finally, combining (A.47) and (A.45), and noticing that

$$(T_{\nabla} \setminus S') \cap (S' \setminus T_{\nabla}) = \emptyset \quad (\text{A.48})$$

$$(T_{\nabla} \setminus S') \cup (S' \setminus T_{\nabla}) \subseteq T, \quad (\text{A.49})$$

we have

$$\begin{aligned} \|\mathbf{X}_{T_{\nabla}}\|_F &+ \|\mathbf{W}_{T_0} \mathbf{X}_{T_{\nabla}}\|_F \\ &\leq \|(\tilde{\mathbf{X}} - \mathbf{X})_{T_{\nabla} \setminus S'}\|_F + \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{T_{\nabla} \setminus S'}\|_F \\ &+ \|(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_{\nabla}}\|_F + \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{S' \setminus T_{\nabla}}\|_F \\ &\leq \sqrt{2}\|(\tilde{\mathbf{X}} - \mathbf{X})_{(S' \setminus T_{\nabla}) \cup (T_{\nabla} \setminus S')}\|_F \\ &+ \sqrt{2}\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{(S' \setminus T_{\nabla}) \cup (T_{\nabla} \setminus S')}\|_F \\ &\leq \sqrt{2}\|(\tilde{\mathbf{X}} - \mathbf{X})_S\|_F + \sqrt{2}\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_S\|_F \\ &\stackrel{(\text{S.20})}{\leq} \sqrt{2}(1 + \omega)\delta_{s+t}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\ &+ (1 + \omega)\sqrt{2(1 + \delta_t)}\|\mathbf{E}\|_F. \end{aligned} \quad (\text{A.50})$$

Also, we can obtain the relationship between $\|\mathbf{W}_{T_0} \mathbf{X}_{T_{\nabla}}\|_F$ and $\|\mathbf{X}_{T_{\nabla}}\|_F$:

$$\eta\omega\sqrt{s_3}\|\mathbf{X}_{T_{\nabla}}\|_F \leq \|\mathbf{W}_{T_0} \mathbf{X}_{T_{\nabla}}\|_F. \quad (\text{A.51})$$

Combining (A.50) and (A.51), we have

$$\begin{aligned} \|\mathbf{X}_{T_{\nabla}}\|_F &\leq \frac{\sqrt{2}(1 + \omega)\delta_{s+t}}{1 + \eta\omega\sqrt{s_3}}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\ &+ \frac{(1 + \omega)\sqrt{2(1 + \delta_t)}}{1 + \eta\omega\sqrt{s_3}}\|\mathbf{E}\|_F. \end{aligned} \quad (\text{A.52})$$

Noting the definition of ν_2 , we complete the proof of Lemma 2.

APPENDIX B PROOF OF LEMMA 3

Proof: From Step 5 of SI-SSP, we have

$$\mathbf{X}_{S_i}^k = \arg \min_{\Theta: \text{supp}(\Theta) = S_{S_i}^k} \|\mathbf{Y}_{S_i} - \mathbf{A}\Theta\|_F. \quad (\text{A.53})$$

From Step 4 of SI-SSP, let $\mathbf{X}^k = [\mathbf{X}_{S_1}^k, \dots, \mathbf{X}_{S_r}^k]$. We have the following conclusion

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \end{aligned}$$

$$\begin{aligned} &\leq \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F. \end{aligned} \quad (\text{A.54})$$

By removing the same coordinates $T \cap \Delta S$, we get

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ &\leq \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F. \end{aligned} \quad (\text{A.55})$$

Because $\text{supp}(\mathbf{X}) = T$ and $\text{supp}(\mathbf{X}^{k-1}) = S^{k-1}$,

$$(\mathbf{X} - \mathbf{X}^{k-1})_{\Delta S \setminus (T \cup S^{k-1})} = \mathbf{0}. \quad (\text{A.56})$$

For the right-hand side of (A.55), we have

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &\leq \|((\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{A}^H \mathbf{E})_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{E})_{\Delta S \setminus T}\|_F. \end{aligned} \quad (\text{A.57})$$

Note that $\tilde{S}^k = S^{k-1} \cup \Delta S$, we have

$$(\mathbf{X} - \mathbf{X}^{k-1})_{T \setminus \tilde{S}^k} = \mathbf{X}_{(\tilde{S}^k)^c}. \quad (\text{A.58})$$

For the left-hand side of (A.55), we have

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0} \mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ &= \|((\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}) \\ &+ \mathbf{A}^H \mathbf{E} + \mathbf{X})_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &+ \|\mathbf{W}_{T_0}(\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}) \\ &+ \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{E} + \mathbf{W}_{T_0} \mathbf{X})_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &\geq \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \|(\mathbf{W}_{T_0} \mathbf{X})_{(\tilde{S}^k)^c}\|_F \\ &- \|((\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \\ &- \|(\mathbf{W}_{T_0}(\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \\ &- \|(\mathbf{A}^H \mathbf{E})_{(\tilde{S}^k)^c}\|_F - \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{E})_{(\tilde{S}^k)^c}\|_F. \end{aligned} \quad (\text{A.59})$$

Combining (A.61) and (A.60), we have

$$\begin{aligned} &\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \|(\mathbf{W}_{T_0} \mathbf{X})_{(\tilde{S}^k)^c}\|_F \\ &\leq \|((\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \setminus \tilde{S}^k}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \setminus \tilde{S}^k}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \end{aligned}$$

$$\begin{aligned}
& + \|((\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\
& + \|(\mathbf{A}^H \mathbf{E})_{T \setminus \tilde{S}^k}\|_F + \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{E})_{T \setminus \tilde{S}^k}\|_F \\
& + \|(\mathbf{A}^H \mathbf{E})_{\Delta S \setminus T}\|_F + \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{E})_{\Delta S \setminus T}\|_F \\
& \leq \sqrt{2} \|((\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \cup \Delta S}\|_F \\
& + \sqrt{2} \|(\mathbf{W}_{T_0} (\mathbf{A}^H \mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \cup \Delta S}\|_F \\
& + \sqrt{2} \|(\mathbf{A}^H \mathbf{E})_{T \cup \Delta S}\|_F + \sqrt{2} \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{E})_{T \cup \Delta S}\|_F \\
& \leq \sqrt{2}(1 + \omega) \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F \\
& + (1 + \omega) \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \tag{A.61}
\end{aligned}$$

We can obtain the relationship between $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ and $\|(\mathbf{W}_{T_0} \mathbf{X})_{(\tilde{S}^k)^c}\|_F$:

$$\eta \omega \sqrt{s_2} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \|(\mathbf{W}_{T_0} \mathbf{X})_{(\tilde{S}^k)^c}\|_F. \tag{A.62}$$

Combining (A.61) and (A.62), we have

$$\begin{aligned}
\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F & \leq \frac{\sqrt{2}(1 + \omega) \delta_{3s}}{1 + \eta \omega \sqrt{s_2}} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\
& + \frac{(1 + \omega) \sqrt{2(1 + \delta_{2s})}}{1 + \eta \omega \sqrt{s_2}} \|\mathbf{E}\|_F. \tag{A.63}
\end{aligned}$$

Noting the definition of ν_1 , we complete the proof of Lemma 3.

APPENDIX C PROOF OF LEMMA 4

Proof: the RIC δ_t can be expressed as [25]

$$\delta_t = \max_{S \subseteq \{1, 2, \dots, N\}, |S| \leq t} \|\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}\|_{2 \rightarrow 2}, \tag{A.64}$$

where

$$\|\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}\|_{2 \rightarrow 2} = \sup_{\mathbf{a} \in \mathbb{R}^{|S|} \setminus \{0\}} \frac{\|(\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}) \mathbf{a}\|_2}{\|\mathbf{a}\|_2}. \tag{A.65}$$

Let $S = \text{supp}(\mathbf{u}) \cup \text{supp}(\mathbf{v})$, then $|S| \leq t$. Let $\mathbf{u}_{|S}$, $\mathbf{v}_{|S}$ denote respectively the S -dimensional sub-vectors of \mathbf{u} and \mathbf{v} obtained by only keeping the components indexed by S . We have

$$\begin{aligned}
& |\langle \mathbf{u}, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle| \\
& = |\langle \mathbf{W}_{T_0} \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{A} \mathbf{W}_{T_0} \mathbf{u}, \mathbf{A} \mathbf{v} \rangle| \\
& = |\langle \mathbf{W}_{T_0} \mathbf{u}_{|S}, (\mathbf{I}_L - \mathbf{A}_S^H \mathbf{A}_S) \mathbf{v}_{|S} \rangle| \\
& \leq \|\mathbf{W}_{T_0} \mathbf{u}_{|S}\|_2 \|(\mathbf{I}_L - \mathbf{A}_S^H \mathbf{A}_S) \mathbf{v}_{|S}\|_2 \\
& \stackrel{(A.65)}{\leq} \|\mathbf{W}_{T_0} \mathbf{u}_{|S}\|_2 \|\mathbf{I}_L - \mathbf{A}_S^H \mathbf{A}_S\|_{2 \rightarrow 2} \|\mathbf{v}_{|S}\|_2 \\
& \stackrel{(A.64)}{\leq} \omega \delta_t \|\mathbf{u}_{|T}\|_2 \|\mathbf{v}_{|S}\|_2 \\
& = \omega \delta_t \|\mathbf{u}\|_2 \|\mathbf{v}\|_2, \tag{A.66}
\end{aligned}$$

moreover, we have

$$\begin{aligned}
& \|((\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v})_U\|_2^2 \\
& = \langle ((\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v})_U, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle \\
& \stackrel{(S.28)}{\leq} \delta_t \|((\mathbf{W}_{T_0} - \mathbf{A}^H \mathbf{A}) \mathbf{v})_U\|_2 \|\mathbf{v}\|_2 \tag{A.67}
\end{aligned}$$

which completes the proof of Lemma 4.

APPENDIX D PROOF OF LEMMA 5

Proof: The lemma follows trivially from the fact that

$$\begin{aligned}
& \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U\|_2^2 \\
& = \langle \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e}, (\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U \rangle \\
& = \langle \mathbf{e}, \mathbf{W}_{T_0} \mathbf{A} ((\mathbf{A}^H \mathbf{e})_U) \rangle \\
& \leq \|\mathbf{e}\|_2 \|\mathbf{W}_{T_0} \mathbf{A} ((\mathbf{A}^H \mathbf{e})_U)\|_2 \\
& \stackrel{(7)}{\leq} \|\mathbf{e}'\|_2 \omega \sqrt{1 + \delta_u} \|(\mathbf{A}^H \mathbf{e})_U\|_2. \tag{A.68}
\end{aligned}$$

APPENDIX E PROOF OF LEMMA 6

Proof: Due to the orthogonality, the residue $\mathbf{Y} - \mathbf{A} \tilde{\mathbf{X}}$ is orthogonal to the space $\mathbf{A} \mathbf{Z}$. This means that for all $\mathbf{Z} \in \mathbb{C}^{L \times N}$ with $\text{supp}(\mathbf{Z}) \subseteq S$,

$$\langle \mathbf{A}(\mathbf{Y} - \mathbf{A} \tilde{\mathbf{X}}), \mathbf{Z} \rangle = 0. \tag{A.69}$$

Let $\tilde{\mathbf{X}}'$ be the solution of the least squares problem $\arg \min_{\mathbf{Z}} \{\|\mathbf{Y}' - \mathbf{A} \mathbf{Z}\|_F, \text{supp}(\mathbf{Z}) \subseteq S\}$, where $\mathbf{Y}' = \frac{\mathbf{A} \mathbf{W}_{T_0} \mathbf{X}_{T_0}}{\omega} + \mathbf{E}$. We have

$$\tilde{\mathbf{X}}' = \frac{\mathbf{W}_{T_0} \tilde{\mathbf{X}}}{\omega}. \tag{A.70}$$

Then, by (A.69), we have

$$\begin{aligned}
0 & = \left\langle \frac{\mathbf{A} \mathbf{W}_{T_0} \mathbf{X}_{T_0}}{\omega} + \mathbf{E} - \mathbf{A} \frac{\mathbf{W}_{T_0} \tilde{\mathbf{X}}}{\omega}, \mathbf{A} \mathbf{Z} \right\rangle \\
& = \left\langle \mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}}, \mathbf{A}^H \mathbf{A} \mathbf{Z} \right\rangle + \omega \langle \mathbf{E}, \mathbf{A} \mathbf{Z} \rangle. \tag{A.71}
\end{aligned}$$