Supplementary: Boundary Multiple Measurement Vectors for Multi-Coset Sampler

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This supplementary material is dedicated to the proofs for Theorems 1–3 in our main paper.

Before proceeding to the proofs, we review some useful notations. For a complex matrix $\mathbf{X} \in \mathbb{C}^{n\times L}$ and a set $S \subseteq \{1,\cdots,n\}$, $\mathbf{X}_S$ (or $\mathbf{X}^S$) denotes the submatrix of $\mathbf{X}$ with columns (or rows) indexed by S; $\mathbf{X}_{i,j}$, $\mathbf{X}_{i,:}$ and $\mathbf{X}_{:,i}$ are the (i,j)th entry, ith row and ith column of $\mathbf{X}$, respectively; $\mathbf{X}^{\dagger}$, $\mathbf{X}^H$ and $\mathbf{X}^{\top}$ mean the Moore-Penrose pseudo-inverse, conjugate transpose and transpose of $\mathbf{X}$, respectively; $\mathrm{supp}(\mathbf{X})$ is the non-zero row indices (i.e., joint sparsity) of $\mathbf{X}$; $\|\mathbf{X}\|_F$ and $\|\mathbf{X}\|_2$ signify the Frobenius and Euclidean norm of $\mathbf{X}$, respectively. Moreover, S^c is the complement of set S; $\mathbf{I}_L$ is an $L\times L$ identity matrix.

I. Proof of Theorem 1

Theorem 1. *The actual sampling rate of (4) is $\min\left(pf_s, f_{nyq}\right)$, which attains the theoretical lower bound of sampling rate in MCS when $|supp(\mathbf{X})| \le \frac{N_{sig}B}{f_s}$.*

Proof. In the ith channel of a multi-coset sampler, the sampling sequence is given by

$$x_{c_i}[n] = x(LTn+\tau_i), \quad n = 0,1,\cdots. \tag{S.1}$$

The sampling rate of each channel is determined by the sampled signal sequence. To be specific, since the sampling time interval is LT, the sampling rate of each channel is

$$f_s = \frac{1}{LT} = \frac{f_{\text{nyq}}}{L}, \tag{S.2}$$

i.e., one-Lth of the Nyquist sampling rate.

Moreover, as the multi-coset sampler is assumed to have p channels, the overall sampling rate of p channels is p times that of each channel (i.e. $\frac{pf_{\text{nyq}}}{L}$). If this sampling rate is greater than the Nyquist rate f_{nyq}, then the advantage of sub-Nyquist sampling structure no longer exists. In this case, we only need to sample at Nyquist sampling rate f_{nyq}. Thus, the actual sampling rate can be given by

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right). \tag{S.3}$$

The theoretical lower bound of the sampling rate is given in [17], which is determined directly by the true bandwidth of the signal:

$$\min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \tag{S.4}$$

Thus, the theoretical lower bound on the sampling rate is achieved when

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right) \le \min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \tag{S.5}$$

In most cases, $2\lambda(\mathcal{T})$ and $\frac{p}{LT}$ do not exceed f_{nyq}. (If violated, the sampling rate would just be f_{nyq}.) Therefore, the condition (S.5) holds whenever

$$\frac{p}{LT} \le 2\lambda(\mathcal{T}). \tag{S.6}$$

Furthermore, to ensure an unique-solution reconstruction, the number p of channels should not be too small. In particular, it's lower bound is twice the signal sparsity without the priori information about the signal $\mathbf{X}$ [17],

$$p \ge 2|\mathrm{supp}(\mathbf{X})|. \tag{S.7}$$

For the worst case where $p = 2|\mathrm{supp}(\mathbf{X})|$, (S.6) can be rewritten as

$$|\mathrm{supp}(\mathbf{X})| \le \lambda(\mathcal{T})LT = \frac{N_{\text{sig}}B}{f_s}, \tag{S.8}$$

which completes the proof. □

II. Proof of Theorem 2

Theorem 2. *When $r \in [\lceil\frac{f_s}{(M-1)f_s-B}\rceil, \infty)$ and $B > f_s$, we have $\max_{i\in\{1,\cdots,r\}}\left|supp(\bar{\mathbf{X}}_{S_i})\right| \le \frac{N_{sig}B}{f_s}$.*

Proof. Consider all row blocks $\left\{\mathbf{X}^{U_1},\cdots,\mathbf{X}^{U_M}\right\}$ in $\mathbf{X}$, where there are consecutive corresponding frequency points of length B in occupied blocks. For the case $B > f_s$ and each block occupies M rows in $\mathbf{X}$, we assume that $(M-2)f_s < B \le (M-1)f_s$ for $M \ge 3$. The frequency points of the sub-band signal may occupy $M-1$ or M rows actually.

For the case that $M-1$ rows in block (i.e. $\mathbf{X}^{U_i}$) are occupied actually, only one row of $\bar{\mathbf{X}}^{U_i}$ is occupied. Thus, we have

$$\left|\mathrm{supp}(\bar{\mathbf{X}}^{U_i}_{S_i})\right| \le \left|\mathrm{supp}(\bar{\mathbf{X}}^{U_i})\right| = 1. \tag{S.9}$$

For another case that M rows are occupied actually, the length of the frequency point in $\bar{\mathbf{X}}^{U_i}$ ($\mathbf{X}^{U_i}$ is occupied) meets

$$l = B-(M-2)f_s \le f_s. \tag{S.10}$$

Because $l \le f_s$, we know that the non-zero elements of any occupied partial-block (i.e. $\bar{\mathbf{X}}^{U_i}$) do not intersect on the column indices. Considering one occupied partial-block $\bar{\mathbf{X}}^{U_i}$, let $r \to \infty$ and observe the change in $\bar{\mathbf{X}}^{U_i}$, $\bar{\mathbf{X}}^{U_i}$ gradually changes from an MMV form to an SMV form. In the SMV form, the sparsity of $\bar{\mathbf{X}}^{U_i}$ is determined by the columns in $\bar{\mathbf{X}}^{U_i}$. It is observed that

$$\lim_{r\to\infty,i,j}\left|\mathrm{supp}(\bar{\mathbf{X}}^{U_i}_{S_j})\right| = \left|\mathrm{supp}(\bar{\mathbf{X}}^{U_i}_{:,j})\right| \le 1. \tag{S.11}$$

Thus, $|\text{supp}(\bar{\mathbf{X}}_{S_i})| = |\text{supp}(\bar{\mathbf{X}}_{:,i})|$ is less than N_{sig} (there exist N_{sig} subbands in $\mathbf{X}$). We proof the upper bound of the sub-MMV problems number r.

A more complex situation occurs when r is a finite value, assuming $r = r^\star$ is a finite value. In this case, the length of frequency points in each sub-matrix is less than the sub-matrix columns number $\left\lceil \frac{f_s}{r^\star} \right\rceil$. We use proof by contradiction to prove the condition that the sparsity of each sub-matrix is less than N_{sig}. Assuming that there exists a partial-block sub-matrix $\bar{\mathbf{X}}_{S_i}$ with $\text{supp}(\bar{\mathbf{X}}_{S_i}) > N_{\text{sig}}$ when $r^\star \in [\lceil \frac{f_s}{(M-1)f_s - B} \rceil, \infty)$. Also, the non-zero elements of any partial-block in $\bar{\mathbf{X}}$ do not intersect on the column indices. $\bar{\mathbf{X}}_{S_i}$ must contain both non-zero elements on both sides of one partial-block. From (S.10), we know that the length of any partial-block of $\mathbf{X}$ is less than f_s. We can draw a conclusion that

$$\left\lceil \frac{f_s}{r^\star} \right\rceil > f_s - l. \tag{S.12}$$

Combining (S.10) and (S.12), we can get

$$\left\lceil \frac{f_s}{r^\star} \right\rceil > (M-1)f_s - B. \tag{S.13}$$

As can be seen, there exist a contradiction between (S.13) and the assumption $r \in [\lceil \frac{f_s}{(M-1)f_s - B} \rceil, \infty)$, so the length of partial-block non-zero elements in any sub-matrix of $\bar{\mathbf{X}}$ must be less or equal than $(M-1)f_s - B$, which is equivalent to r satisfying

$$r \geq \left\lceil \frac{f_s}{(M-1)f_s - B} \right\rceil. \tag{S.14}$$

To sum up, when $r \in [\lceil \frac{f_s}{(M-1)f_s - B} \rceil, \infty)$, we have

$$\max_{i \in \{1, \cdots, r\}} \left|\text{supp}(\bar{\mathbf{X}}_{S_i})\right| \leq N_{\text{sig}} < \frac{N_{\text{sig}} B}{f_s}. \tag{S.15}$$

The proof is thus complete. □

III. Proof of Theorem 3

Theorem 3. *Consider the column-partitioned MMV model (5) with $\min_{i,j} \|(\mathbf{X}_{S_i})_{j,:}\|_2 / \|\mathbf{X}_{S_i}\|_F = \eta$ and $|supp(\mathbf{X}_{S_i})| \leq s$. Let $s_1 := \min_{i,k} |\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i})|$, $s_2 := \min_{i,k} |\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i}) \cap \tilde{S}^k_{S_i}|$ and $s_3 := \min_{i,k} |\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i}) \cap \tilde{S}^k_{S_i} \setminus S^k_{S_i}|$. Then, if the sensing matrix $\mathbf{A}$ obeys the RIP with*

$$\delta_{3s} \leq \sqrt{\frac{\nu_1 \sqrt{\nu_1^2 + 4\nu_2^2} - \nu_1^2 - 1}{4\nu_1^2 \nu_2^2 - 2\nu_1^2 - 1}} \tag{S.16}$$

where $\nu_1 := \frac{1+\omega}{1+\eta\omega\sqrt{s_2}}$ and $\nu_2 := \frac{1+\omega}{1+\eta\omega\sqrt{s_3}}$, SI-SSP produces an signal estimate $\mathbf{X}^k = [\mathbf{X}^k_{S_1}, \cdots, \mathbf{X}^k_{S_r}]$ satisfying

$$\left\|\mathbf{X} - \mathbf{X}^k\right\|_F \leq \rho^k \|\mathbf{X}\|_F + \tau \|\mathbf{E}\|_F, \tag{S.17}$$

where $\rho \in (0,1)$ and τ are constants depending on δ_{3s}, ν_1 and ν_2. Furthermore, after at most $k^* = \lceil \log_\rho \frac{\|\mathbf{X}\|_F}{\tau \|\mathbf{E}\|_F} \rceil$ iterations, SI-SSP estimates $\mathbf{X}$ with

$$\|\mathbf{X} - \mathbf{X}^{k^*}\|_F \leq (\tau + 1)\|\mathbf{E}\|_F. \tag{S.18}$$

To prove Theorem 3, we first introduce six useful Lemmas, whose proofs are left to the appendices.

Lemma 1. *([25]): For nonnegative numbers a, b, c, d, x, y,*

$$(ax+by)^2 + (cx+dy)^2 \leq (\sqrt{a^2+c^2}x + (b+d)y)^2. \tag{S.19}$$

Lemma 2. *Consider the system model $\mathbf{Y} = \mathbf{AX} + \mathbf{E}$, where $supp(\mathbf{X}) = T$ and $|T| = s$. Let $S \subseteq \{1, 2, ..., n\}$ be an index set with $|S| = t$ and $\mathbf{W}_{T_0}$ be a side-information matrix with diagonal entries indexed by $T_0 \subseteq \{1, 2, ..., n\}$ being $\omega \geq 0$ and zero otherwise. Also, let $\tilde{\mathbf{X}} := \arg\min_{\mathbf{Z}: supp(\mathbf{Z}) \subseteq S} \|\mathbf{Y} - \mathbf{AZ}\|_2$. If $\delta_{3s} < 1$, then*

$$\|\mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}})_S\|_F \leq \omega\delta_{s+t}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \omega\sqrt{1+\delta_t}\|\mathbf{E}\|_F \tag{S.20}$$

and

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1-\delta^2_{s+t}}}\|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1+\delta_t}}{1-\delta_{s+t}}\|\mathbf{E}\|_F. \tag{S.21}$$

Furthermore, if $t > s$, define $T_\triangledown$ as the row-indices of the smallest $t-s$ row-norm entries of $\tilde{\mathbf{X}}$ in S, we have

$$\|\mathbf{X}_{T_\triangledown}\|_F \leq \sqrt{2}\nu_2\delta_{s+t}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_2\sqrt{2\,(1+\delta_t)}\|\mathbf{E}\|_F. \tag{S.22}$$

Remark 1. *When we consider the atom selection strategy of $\|\tilde{\mathbf{X}}_{T_\triangledown} + \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\triangledown}\|_F \leq \|\tilde{\mathbf{X}}_{S'} + \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{S'}\|_F$, we can also obtain another upper bound for $\|\mathbf{X}_{T_\triangledown}\|_F$ in (S.22). In this case, we should allocate $2\,\|\mathbf{X}_{T_\triangledown}\|_F$ to the left hand side of (A.46), we have*

$$\|\mathbf{X}_{T_\triangledown}\|_F \leq \sqrt{2}\nu_3\delta_{s+t}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_4\sqrt{2\,(1+\delta_t)}\,\|\mathbf{E}\|_F\,. \tag{S.23}$$

where $\nu_3 = (1 - \omega + \omega\delta_{s+t} + \delta_{s+t})/(2\delta_{s+t})$ and $\nu_4 = (1+\omega)/(2\delta_{s+t})$.

Lemma 3. *In steps 4 and 5 of SI-SSP, we have*

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2}\nu_1\delta_{3s}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F + \nu_1\sqrt{2(1+\delta_{3s})}\,\|\mathbf{E}\|_F\,. \tag{S.24}$$

Remark 2. *When we consider the atom selection strategy in select step that*

$$\begin{aligned}&\left\|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{AX}^{k-1}))_T\right\|_F \\ &\quad \leq \left\|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{AX}^{k-1}))_{\Delta S}\right\|_F.\end{aligned} \tag{S.25}$$

We can also obtain another upper bound for $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ in (S.24). In this case, we should allocate $2\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ to the left hand side of (A.59), we have

$$\begin{aligned}\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F &\leq \sqrt{2}\nu_4\delta_{3s}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F \\ &\quad + \nu_4\sqrt{2\,(1+\delta_{3s})}\,\|\mathbf{E}\|_F\,.\end{aligned} \tag{S.26}$$

where $\nu_4 = (1 - \omega + \omega\delta_{3s} + \delta_{3s})/(2\delta_{3s})$ and $\nu_4 = (1+\omega)/(2\delta_{3s})$. Based on conclusions (S.23) and (S.26), we know that the sensing matrix $\mathbf{A}$ obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_3\sqrt{\nu_3^2 + 4\nu_4^2} - \nu_3^2 - 1}{4\nu_3^2\nu_4^2 - 2\nu_3^2 - 1}}. \tag{S.27}$$

Lemma 4. *Let $T_0 \subseteq \{1, 2, ..., n\}$, for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, if $|supp(\mathbf{u}) \cup supp(\mathbf{v})| \leq t$,*

$$\left|\left\langle \mathbf{u}, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\mathbf{v}\right\rangle\right| \leq \omega\delta_t\,\|\mathbf{u}\|\,\|\mathbf{v}\|\,; \tag{S.28}$$

Moreover, if $U \subseteq \{1, 2, ..., n\}$ *and* $|U \cup supp(\mathbf{v})| \le t$, *then*

$$\left|(\mathbf{W}_{T_0} - \mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\mathbf{v}\right| \le \omega\delta_t \|\mathbf{v}\|. \quad \text{(S.29)}$$

Lemma 5. *For SMV model* $\mathbf{y} = \mathbf{\Phi x}+\mathbf{e}$, *let* $T_0 \subseteq \{1, 2, ..., n\}$, *let* $U \subseteq \{1, 2, ..., n\}$ *and* $|U \cap T_0| \le u$, *we have*

$$\left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e})_U\right\|_2 \le \omega\delta_u \|\mathbf{e}\|_2. \quad \text{(S.30)}$$

Lemma 6. *Consider the MMV model* $\mathbf{Y} = \mathbf{AX} + \mathbf{E}$, *let* $\tilde{\mathbf{X}}$ *be the solution of the least squares problem* $\arg\min_{\mathbf{Z}} \{\|\mathbf{Y} - \mathbf{AZ}\|_F, supp(\mathbf{Z}) \subseteq S\}$, *then*

$$\langle \mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}}, \mathbf{A}^H\mathbf{AZ}\rangle + \omega\langle \mathbf{E}, \mathbf{AZ}\rangle = \mathbf{0}. \quad \text{(S.31)}$$

Now we have all ingredients to prove Theorem 3.

Proof of Theorem 3. First, in Steps 4 and 5 of SI-SSP, Lemma 3 implies

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \le \sqrt{2}\nu_1\delta_{3s}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F + \nu_1\sqrt{2(1+\delta_{3s})}\,\|\mathbf{E}\|_F. \quad \text{(S.32)}$$

Note that Step 6 of SI-SSP solves a least squares problem. Let $S = \tilde{S}^k$ and $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $t = 2s$, by (S.21) we have

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \le \sqrt{\frac{1}{1-\delta_{3s}^2}}\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \frac{\sqrt{1+\delta_{2s}}}{1-\delta_{3s}}\|\mathbf{E}\|_F. \quad \text{(S.33)}$$

Combining (S.32) and (S.33) and also magnifying δ_{2s} to δ_{3s}, we further have

$$\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F \le \nu_1\sqrt{\frac{2\delta_{3s}^2}{1-\delta_{3s}^2}}\|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \tau_1\|\mathbf{E}\|_F. \quad \text{(S.34)}$$

Next, after Step 7 of SI-SSP, let $S_\triangledown = \tilde{S}^k \setminus S^k$ be the row-indices of the smallest $t-s$ row norm entries in $\tilde{\mathbf{X}}^k$. Also, let $T = \tilde{S}^k$, $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $T_\triangledown = S_\triangledown$ and $t = 2s$. Then, by (A.45) we have

$$\|\mathbf{X}_{S_\triangledown}\|_F \le \sqrt{2}\nu_2\delta_{3s}\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2\sqrt{2\,(1+\delta_{2s})}\|\mathbf{E}\|_F. \quad \text{(S.35)}$$

Let $\tau_1 = (\nu_1\sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$ and $\tau_2 = \sqrt{1+\delta_{3s}}$. Dividing $(S^k)^c$ into two disjoint subsets: $(\tilde{S}^k)^c$ and $S_\triangledown$, we get

$$\begin{aligned}
\left\|\mathbf{X}_{(S^k)^c}\right\|_F^2 &= \|\mathbf{X}_{S_\triangledown}\|_F^2 + \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2 \\
&\overset{\text{(S.32), (S.35)}}{\le} 2\left(\nu_2\delta_{3s}\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2\tau_2\|\mathbf{E}\|_F\right)^2 \\
&\quad + 2\left(\nu_1\delta_{3s}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_2 + \nu_1\tau_2\|\mathbf{E}\|_F\right)^2 \\
&\overset{\text{(S.34)}}{\le} 2\left(\sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F + \nu_2(\tau_1\delta_{3s} + \tau_2)\right. \\
&\quad \left.\times\|\mathbf{E}\|_F\right)^2 + 2\left(\nu_1\delta_{3s}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F + \nu_1\tau_2\|\mathbf{E}\|_F\right)^2 \\
&\overset{\text{(S.19)}}{\le} 2\left(\sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2} + \nu_1^2\delta_{3s}^2}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F\right. \\
&\quad \left. + ((\nu_1+\nu_2)\tau_2 + \nu_2\delta_{3s}\tau_1)\|\mathbf{E}\|_F\right)^2. \quad \text{(S.36)}
\end{aligned}$$

Squaring both sides, we get

$$\begin{aligned}
\left\|\mathbf{X}_{(S^k)^c}\right\|_F &\le \sqrt{\tfrac{4\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2} + 2\nu_1^2\delta_{3s}^2}\left\|\mathbf{X} - \mathbf{X}^{k-1}\right\|_F \\
&\quad + \sqrt{2}\left((\nu_1+\nu_2)\tau_2 + \nu_2\delta_{3s}\tau_1\right)\|\mathbf{E}\|_F. \quad \text{(S.37)}
\end{aligned}$$

Step 9 of SI-SSP also solves a least squares problem. Letting $T = S^k$, $\tilde{\mathbf{X}} = \mathbf{X}^k$ and $t = s$, by (S.21), we have

$$\|\mathbf{X} - \mathbf{X}^k\|_F \le \sqrt{\frac{1}{1-\delta_{2s}^2}}\|\mathbf{X}_{(S^k)^c}\|_F + \frac{\sqrt{1+\delta_s}}{1-\delta_{2s}}\|\mathbf{E}\|_F. \quad \text{(S.38)}$$

Finally, combining (S.37) and (S.38) yields

$$\|\mathbf{X} - \mathbf{X}^k\|_F \le \rho\|\mathbf{X} - \mathbf{X}^{k-1}\|_F + (1-\rho)\tau\|\mathbf{E}\|_F \quad \text{(S.39)}$$

where $\rho := \sqrt{2}\delta_{3s}\sqrt{2\nu_1^2\nu_2^2\delta_{3s}^2 + \nu_1^2 - \nu_1^2\delta_{3s}^2}(1-\delta_{3s}^2)^{-1}$ and $\tau := \sqrt{2}\delta_{3s}\nu_2(\nu_1\sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}})(1-\delta_{3s}^2)^{-1/2}(1-\delta_{3s})^{-1}(1-\rho)^{-1} + (\nu_1\nu_2\sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$.

We recursively apply (S.39) to obtain

$$\|\mathbf{X} - \mathbf{X}^k\|_F \le \rho^k\|\mathbf{X}\|_F + \tau\|\mathbf{E}\|_F \quad \text{(S.40)}$$

where $\rho < 1$ under (S.16). When $k^* = \lceil \log_\rho \frac{\|\mathbf{X}\|_F}{\tau\|\mathbf{E}\|_F}\rceil$, we have $\rho^k\|\mathbf{X}\|_F \le \tau\|\mathbf{E}\|_F$, and thus the stability result (S.18). ☐

Appendix A
Proof of Lemma 2

- First, we give a upper bound of $\|\mathbf{X}_{T_\triangledown}\|_F$, by Lemma 6, let $\mathbf{Z} = (\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S$, we have

$$\begin{aligned}
&\left\langle \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}}), \mathbf{A}^H\mathbf{A}(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle \\
&\quad + \left\langle \mathbf{W}_{T_0}\mathbf{E}, \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle = \mathbf{0}. \quad \text{(A.41)}
\end{aligned}$$

Noticing that $\text{supp}(\tilde{\mathbf{X}}) \subseteq S$, we have

$$\begin{aligned}
&\|(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\|_F^2 \\
&= \left\langle \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle \\
&\overset{\text{(A.41)}}{=} \left\langle \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{I}_L - \mathbf{A}^H\mathbf{A})(\mathbf{X} - \tilde{\mathbf{X}})_S\right\rangle \\
&- \left\langle \mathbf{W}_{T_0}\mathbf{E}, \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle \\
&\overset{(7)}{\le} \omega\delta_{s+t}\left\|(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\|_F\left\|\mathbf{X} - \tilde{\mathbf{X}}\right\|_F \\
&\quad + \omega\|\mathbf{E}\|_F\sqrt{1+\delta_t}\left\|\mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}})_S\right\|_F. \quad \text{(A.42)}
\end{aligned}$$

Divide both sides by $\|(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\|_F$ to obtain (S.20).
- Next, by expanding [Lemma 2, 25] to the MMV model, we could get a relationship between $\|\mathbf{X} - \tilde{\mathbf{X}}\|_F$ and $\|\mathbf{X}_{S^c}\|_F$. We have

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \le \sqrt{\frac{1}{1-\delta_{s+t}^2}}\|\mathbf{X}_{S^c}\|_F^2 + \frac{\sqrt{1+\delta_t}}{1-\delta_{s+t}}\|\mathbf{E}\|_F. \quad \text{(A.43)}$$

- Then, we established the relationship between $\mathbf{X}_{T_\triangledown}$ and $\mathbf{X} - \tilde{\mathbf{X}}$. There exist a subset $S' \subseteq S$ and $S' \cap T = \emptyset$. Since $T_\triangledown$ is defined by the set of indices of the $t-s$ smallest row entries of $\tilde{\mathbf{X}}$, we can conclude that

$$\begin{aligned}
&\|\tilde{\mathbf{X}}_{T_\triangledown}\|_F + \|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\triangledown}\|_F \\
&\le \|\tilde{\mathbf{X}}_{S'}\|_F + \|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{S'}\|_F. \quad \text{(A.44)}
\end{aligned}$$

By eliminating the contribution from $T_\triangledown \cap S'$ and noticing that $S' \cap T = \emptyset$, we have

$$\begin{aligned} \|\tilde{\mathbf{X}}_{T_\triangledown \setminus S'}\|_F &+ \|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\triangledown \setminus S'}\|_F \\ &\leq \|(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_\triangledown}\|_F \\ &+ \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_\triangledown}\|_F. \quad \text{(A.45)} \end{aligned}$$

For the left-hand side of (A.45), we have

$$\begin{aligned} &\|\tilde{\mathbf{X}}_{T_\triangledown \setminus S'}\|_F + \|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\triangledown \setminus S'}\|_F \\ &= \|(\tilde{\mathbf{X}} - \mathbf{X} + \mathbf{X})_{T_\triangledown \setminus S'}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}) + \mathbf{W}_{T_0}\mathbf{X})_{T_\triangledown \setminus S'}\|_F \\ &\geq \|\mathbf{X}_{T_\triangledown}\|_F + \|\mathbf{W}_{T_0}\mathbf{X}_{T_\triangledown}\|_F \quad \text{(A.46)} \\ &- \|(\tilde{\mathbf{X}} - \mathbf{X})_{T_\triangledown \setminus S'}\|_F \\ &- \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{T_\triangledown \setminus S'}\|_F. \quad \text{(A.47)} \end{aligned}$$

Finally, combining (A.47) and (A.45), and noticing that

$$(T_\triangledown \setminus S') \cap (S' \setminus T_\triangledown) = \emptyset \quad \text{(A.48)}$$
$$(T_\triangledown \setminus S') \cup (S' \setminus T_\triangledown) \subseteq T, \quad \text{(A.49)}$$

we have

$$\begin{aligned} &\|\mathbf{X}_{T_\triangledown}\|_F + \|\mathbf{W}_{T_0}\mathbf{X}_{T_\triangledown}\|_F \\ &\leq \|(\tilde{\mathbf{X}} - \mathbf{X})_{T_\triangledown \setminus S'}\|_F + \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{T_\triangledown \setminus S'}\|_F \\ &+ \|(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_\triangledown}\|_F + \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{S' \setminus T_\triangledown}\|_F \\ &\leq \sqrt{2}\|(\tilde{\mathbf{X}} - \mathbf{X})_{(S' \setminus T_\triangledown) \cup (T_\triangledown \setminus S')}\|_F \\ &+ \sqrt{2}\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{(S' \setminus T_\triangledown) \cup (T_\triangledown \setminus S')}\|_F \\ &\leq \sqrt{2}\|(\tilde{\mathbf{X}} - \mathbf{X})_S\|_F + \sqrt{2}\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_S\|_F \\ &\overset{\text{(S.20)}}{\leq} \sqrt{2}(1+\omega)\delta_{s+t}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\ &+ (1+\omega)\sqrt{2(1+\delta_t)}\|\mathbf{E}\|_F. \quad \text{(A.50)} \end{aligned}$$

Also, we can obtain the relationship between $\|\mathbf{W}_{T_0}\mathbf{X}_{T_\triangledown}\|_F$ and $\|\mathbf{X}_{T_\triangledown}\|_F$:

$$\eta\omega\sqrt{s_3}\,\|\mathbf{X}_{T_\triangledown}\|_F \leq \|\mathbf{W}_{T_0}\mathbf{X}_{T_\triangledown}\|_F\,. \quad \text{(A.51)}$$

Combining (A.50) and (A.51), we have

$$\begin{aligned} \|\mathbf{X}_{T_\triangledown}\|_F &\leq \frac{\sqrt{2}(1+\omega)\delta_{s+t}}{1+\eta\omega\sqrt{s_3}}\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\ &+ \frac{(1+\omega)\sqrt{2(1+\delta_t)}}{1+\eta\omega\sqrt{s_3}}\,\|\mathbf{E}\|_F\,. \quad \text{(A.52)} \end{aligned}$$

Noting the definition of ν_2, we complete the proof of Lemma 2.

Appendix B
Proof of Lemma 3

Proof: From Step 5 of SI-SSP, we have

$$\mathbf{X}^k_{S_i} = \operatorname*{arg\,min}_{\mathbf{\Theta}:\text{supp}(\mathbf{\Theta}) = S^k_{S_i}} \|\mathbf{Y}_{S_i} - \mathbf{A}\mathbf{\Theta}\|_F\,. \quad \text{(A.53)}$$

From Step 4 of SI-SSP, let $\mathbf{X}^k = [\mathbf{X}^k_{S_1}, \cdots, \mathbf{X}^k_{S_r}]$. We have the following conclusion

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ &\leq \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F. \quad \text{(A.54)} \end{aligned}$$

By removing the same coordinates $T \cap \Delta S$, we get

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ &\leq \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F. \quad \text{(A.55)} \end{aligned}$$

Because $\text{supp}(\mathbf{X}) = T$ and $\text{supp}(\mathbf{X}^{k-1}) = S^{k-1}$,

$$(\mathbf{X} - \mathbf{X}^{k-1})_{\Delta S \setminus (T \cup S^{k-1})} = 0. \quad \text{(A.56)}$$

For the right-hand side of (A.55), we have

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &\leq \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T}\|_F. \quad \text{(A.57)} \end{aligned}$$

Note that $\tilde{S}^k = S^{k-1} \cup \Delta S$, we have

$$(\mathbf{X} - \mathbf{X}^{k-1})_{T \setminus \tilde{S}^k} = \mathbf{X}_{(\tilde{S}^k)^c}. \quad \text{(A.58)}$$

For the left-hand side of (A.55), we have

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})}\|_F \\ &= \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}) \\ &+ \mathbf{A}^H\mathbf{E} + \mathbf{X})_{\Delta S \setminus (T \cup S^{k-1})}\| \\ &+ \|\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}) \\ &+ \mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E} + \mathbf{W}_{T_0}\mathbf{X})_{\Delta S \setminus (T \cup S^{k-1})}\|_F \\ &\geq \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F \\ &- \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \quad \text{(A.59)} \\ &- \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \\ &- \|(\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\|_F - \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\|_F. \quad \text{(A.60)} \end{aligned}$$

Combining (A.61) and (A.60), we have

$$\begin{aligned} &\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F \\ &\leq \|((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \setminus \tilde{S}^k}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \setminus \tilde{S}^k}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T}\|_F \end{aligned}$$

$$
\begin{aligned}
&+ \quad \|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\
&+ \quad \|(\mathbf{A}^H\mathbf{E})_{T\setminus\tilde{S}^k}\|_F + \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T\setminus\tilde{S}^k}\|_F \\
&+ \quad \|(\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\|_F + \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\|_F \\
&\leq \quad \sqrt{2}\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\cup\Delta S}\|_F \\
&+ \quad \sqrt{2}\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\cup\Delta S}\|_F \\
&+ \quad \sqrt{2}\|(\mathbf{A}^H\mathbf{E})_{T\cup\Delta S}\|_F + \sqrt{2}\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T\cup\Delta S}\|_F \\
&\leq \quad \sqrt{2}(1+\omega)\delta_{3s}\|\mathbf{X}-\mathbf{X}^{k-1}\|_F \\
&+ \quad (1+\omega)\sqrt{2(1+\delta_{3s})}\|\mathbf{E}\|_F. \qquad \text{(A.61)}
\end{aligned}
$$

We can obtain the relationship between $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ $\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F$:

$$
\eta\omega\sqrt{s_2}\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F. \qquad \text{(A.62)}
$$

Combining (A.61) and (A.62), we have

$$
\begin{aligned}
\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq & \frac{\sqrt{2}(1+\omega)\delta_{3s}}{1+\eta\omega\sqrt{s_2}}\|\mathbf{X}-\tilde{\mathbf{X}}\|_F \\
& + \frac{(1+\omega)\sqrt{2(1+\delta_{2s})}}{1+\eta\omega\sqrt{s_2}}\|\mathbf{E}\|_F. \qquad \text{(A.63)}
\end{aligned}
$$

Noting the definition of ν_1, we complete the proof of Lemma 3.

Appendix C
Proof of Lemma 4

Proof: the RIC δ_t can be expressed as [25]

$$
\delta_t = \max_{S\subseteq\{1,2,\ldots,N\},|S|\leq t}\|\mathbf{A}_S^*\mathbf{A}_S-\mathbf{I}\|_{2\to 2}, \qquad \text{(A.64)}
$$

where

$$
\|\mathbf{A}_S^*\mathbf{A}_S-\mathbf{I}\|_{2\to 2} = \sup_{\mathbf{a}\in\mathbb{R}^{|S|}\setminus\{\mathbf{0}\}}\frac{\|(\mathbf{A}_S^*\mathbf{A}_S-\mathbf{I})\,\mathbf{a}\|_2}{\|\mathbf{a}\|_2}. \qquad \text{(A.65)}
$$

Let $S = \mathrm{supp}(\mathbf{u})\cup\mathrm{supp}(\mathbf{v})$, then $|S|\leq t$. Let $\mathbf{u}_{|S}, \mathbf{v}_{|S}$ denote respectively the S-dimensional sub-vectors of $\mathbf{u}$ and $\mathbf{v}$ obtained by only keeping the components indexed by S. We have

$$
\begin{aligned}
& |\langle\mathbf{u}, (\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\,\mathbf{v}\rangle| \\
= \quad & |\langle\mathbf{W}_{T_0}\mathbf{u},\mathbf{v}\rangle - \langle\mathbf{A}\mathbf{W}_{T_0}\mathbf{u},\mathbf{A}\mathbf{v}\rangle| \\
= \quad & |\langle\mathbf{W}_{T_0}\mathbf{u}_{|S}, (\mathbf{I}_L-\mathbf{A}_S^H\mathbf{A}_S)\,\mathbf{v}_{|S}\rangle| \\
\leq \quad & \|\mathbf{W}_{T_0}\mathbf{u}_{|S}\|_2\left\|(\mathbf{I}_L-\mathbf{A}_S^H\mathbf{A}_S)\,\mathbf{v}_{|S}\right\|_2 \\
\overset{\text{(A.65)}}{\leq} \quad & \|\mathbf{W}_{T_0}\mathbf{u}_{|S}\|_2\|\mathbf{I}_L-\mathbf{A}_S^H\mathbf{A}_S\|_{2\to 2}\|\mathbf{v}_{|S}\|_2 \\
\overset{\text{(A.64)}}{\leq} \quad & \omega\delta_t\|\mathbf{u}_{|T}\|_2\|\mathbf{v}_{|S}\|_2 \\
= \quad & \omega\delta_t\|\mathbf{u}\|_2\|\mathbf{v}\|_2, \qquad \text{(A.66)}
\end{aligned}
$$

moreover, we have

$$
\begin{aligned}
& \|\left((\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\,\mathbf{v}\right)_U\|_2^2 \\
= \quad & \langle\left((\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\,\mathbf{v}\right)_U, \\
& (\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\,\mathbf{v}\rangle \\
\overset{\text{(S.28)}}{\leq} \quad & \delta_t\left\|\left((\mathbf{W}_{T_0}-\mathbf{A}^H\mathbf{A})\,\mathbf{v}\right)_U\right\|_2\|\mathbf{v}\|_2 \qquad \text{(A.67)}
\end{aligned}
$$

which completes the proof of Lemma 4.

Appendix D
Proof of Lemma 5

Proof: The lemma follows trivially from the fact that

$$
\begin{aligned}
& \left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e})_U\right\|_2^2 \\
& = \left\langle\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e}, (\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e})_U\right\rangle \\
& = \left\langle\mathbf{e}, \mathbf{W}_{T_0}\mathbf{A}\left((\mathbf{A}^H\mathbf{e})_U\right)\right\rangle \qquad \text{(A.68)} \\
& \leq \|\mathbf{e}\|_2\left\|\mathbf{W}_{T_0}\mathbf{A}\left((\mathbf{A}^H\mathbf{e})_U\right)\right\|_2 \\
& \overset{(7)}{\leq} \|\mathbf{e}'\|_2\,\omega\sqrt{1+\delta_u}\left\|(\mathbf{A}^H\mathbf{e})_U\right\|_2.
\end{aligned}
$$

Appendix E
Proof of Lemma 6

Proof: Due to the orthogonality, the residue $\mathbf{Y}-\mathbf{A}\tilde{\mathbf{X}}$ is orthogonal to the space $\mathbf{A}\mathbf{Z}$. This means that for all $\mathbf{Z}\in\mathbb{C}^{L\times N}$ with $\mathrm{supp}(\mathbf{Z})\subseteq S$,

$$
\langle\mathbf{A}(\mathbf{Y}-\mathbf{A}\tilde{\mathbf{X}}),\mathbf{Z}\rangle = \mathbf{0}. \qquad \text{(A.69)}
$$

Let $\tilde{\mathbf{X}}'$ be the solution of the least squares problem $\arg\min_{\mathbf{Z}}\{\|\mathbf{Y}'-\mathbf{A}\mathbf{Z}\|_F, \mathrm{supp}(\mathbf{Z})\subseteq S\}$, where $\mathbf{Y}' = \frac{\mathbf{A}\mathbf{W}_{T_0}\mathbf{X}_{T_0}}{\omega}+\mathbf{E}$. We have

$$
\tilde{\mathbf{X}}' = \frac{\mathbf{W}_{T_0}\tilde{\mathbf{X}}}{\omega}. \qquad \text{(A.70)}
$$

Then, by (A.69), we have

$$
\begin{aligned}
\mathbf{0} \quad = & \quad \left\langle\frac{\mathbf{A}\mathbf{W}_{T_0}\mathbf{X}_{T_0}}{\omega}+\mathbf{E}-\mathbf{A}\frac{\mathbf{W}_{T_0}\tilde{X}}{\omega},\mathbf{A}\mathbf{Z}\right\rangle \\
= & \quad \left\langle\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}},\mathbf{A}^H\mathbf{A}\mathbf{Z}\right\rangle + \omega\langle\mathbf{E},\mathbf{A}\mathbf{Z}\rangle. \qquad \text{(A.71)}
\end{aligned}
$$