

Supplementary: Boundary Multiple Measurement Vectors for Multi-Coset Sampler

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This supplementary material is dedicated to the proofs for Theorems 1–3 in our main paper.

Before proceeding to the proofs, we review some useful notations. For a complex matrix $\mathbf{X} \in \mathbb{C}^{n \times L}$ and a set $S \subseteq \{1, \dots, n\}$, \mathbf{X}_S (or \mathbf{X}^S) denotes the submatrix of \mathbf{X} with columns (or rows) indexed by S ; $\mathbf{X}_{i,j}$, $\mathbf{X}_{i,:}$ and $\mathbf{X}_{:,i}$ are the (i, j) th entry, i th row and i th column of \mathbf{X} , respectively; \mathbf{X}^\dagger , \mathbf{X}^H and \mathbf{X}^\top mean the Moore-Penrose pseudo-inverse, conjugate transpose and transpose of \mathbf{X} , respectively; $\text{supp}(\mathbf{X})$ is the non-zero row indices (i.e., joint sparsity) of \mathbf{X} ; $\|\mathbf{X}\|_F$ and $\|\mathbf{X}\|_2$ signify the Frobenius and Euclidean norm of \mathbf{X} , respectively. Moreover, S^c is the complement of set S ; \mathbf{I}_L is an $L \times L$ identity matrix.

I. PROOF OF THEOREM 1

Theorem 1. *The actual sampling rate of (4) is $\min(pf_s, f_{\text{nyq}})$, which attains the theoretical lower bound of sampling rate in MCS when $|\text{supp}(\mathbf{X})| \leq \frac{N_{\text{sig}}B}{f_s}$.*

Proof. Assume that a multi-coset sampler has p channels. In the i th channel, The sampling sequence is given by

$$x_{ci}[n] = x(LTn + \tau_i), \quad n = 0, 1, \dots \quad (\text{S.1})$$

The sampling rate of each channel is determined by the sampled signal sequence. Because the sampling time interval is LT , the sampling rate of each channel is $1/L$ of the Nyquist sampling rate (i.e. $\frac{f_{\text{nyq}}}{L}$). The average sampling rate of p channels is p times that of one channel (i.e. $\frac{pf_{\text{nyq}}}{L}$). Noting that when the sampling rate of p channels is greater than the Nyquist rate, the advantage of sub-Nyquist sampling structure no longer exists. Thus, we only need to sample at Nyquist sampling rate f_{nyq} , the actual sampling rate can be represented as

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right). \quad (\text{S.2})$$

The theoretical lower bound of the sampling rate is given in [17], which is directly determined by the true bandwidth of the signal:

$$\min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \quad (\text{S.3})$$

In most cases, the subband bandwidth $\lambda(\mathcal{T})$ and the actual sampling rate does not exceed f_{nyq} (when not satisfied, the sampling rate is f_{sig}). To ensure reconstruction performance, the parameter p is often not set too low (for instance, it's often chosen to be at least twice the sparsity level $\text{supp}(\mathbf{X})$). It can be seen from above that the theoretical lower bound on the sampling rate is achieved only when $p/LT = pf_s \leq 2\lambda(\mathcal{T}) = 2N_{\text{sig}}B$. In other words, when $p = 2\text{supp}(\mathbf{X})$ for

the worst case of p , the condition for the actual sampling rate to meet the theoretical lower bound is $|\text{supp}(\mathbf{X})| \leq \frac{N_{\text{sig}}B}{f_s}$. \square

II. PROOF OF THEOREM 2

Theorem 2. *When $r \in [\lceil \frac{f_s}{(M-1)f_s - B} \rceil, \infty)$ and $B > f_s$, we have $\max_{i \in \{1, \dots, r\}} |\text{supp}(\bar{\mathbf{X}}_{S_i})| \leq \frac{N_{\text{sig}}B}{f_s}$.*

Proof. Consider all blocks $\{\mathbf{X}^{U_1}, \dots, \mathbf{X}^{U_M}\}$ in \mathbf{X} , where there are consecutive corresponding frequency points of length B . For the case $B > f_s$ and each block occupies M rows in \mathbf{X} , the length of the frequency point in each $\bar{\mathbf{X}}^{U_i}$ meets

$$l = B - (M - 2)f_s < f_s. \quad (\text{S.4})$$

Because $l < f_s$, we know that the non-zero elements of an any sub-block (i.e. $\bar{\mathbf{X}}^{U_i}$) are distributed on both sides and do not intersect on the columns. Consider one block $\bar{\mathbf{X}}^{U_i}$, We let $r \rightarrow \infty$ and observe the change in $\bar{\mathbf{X}}^{U_i}$, $\bar{\mathbf{X}}^{U_i}$ gradually changes from an MMV form to an SMV form. In SMV, the sparsity of the signal is determined by the column in $\bar{\mathbf{X}}^{U_i}$ where it is located. It is observed that

$$\lim_{r \rightarrow \infty, i, j} \text{supp}(\bar{\mathbf{X}}_{:,j}^{U_i}) \leq 1. \quad (\text{S.5})$$

Thus, the sparsity of each column in $\bar{\mathbf{X}}^{U_i}$ is less than N_{sig} .

A more complex situation is when r is a finite value, assuming $r = r^*$ is a finite value. In this case, the length of frequency point in each sub-matrix is $\frac{f_s}{r^*}$. We use reduction to absurdity to prove the condition that the sparsity of each sub-matrix is less than N_{sig} . Assuming that there exists a sub-matrix $\bar{\mathbf{X}}_{S_i}$ with $\text{supp}(\bar{\mathbf{X}}_{S_i}) > N_{\text{sig}}$. Also, the non-zero elements of an any sub-block of $\bar{\mathbf{X}}_{S_i}$ do not intersect on the columns. $\bar{\mathbf{X}}_{S_i}$ must contain both non-zero elements on both sides of one block. From (S.4), we know that the length of any block of $\bar{\mathbf{X}}$ is less than f_s . We can draw a conclusion that

$$\frac{f_s}{r^*} > f_s - l = (M - 1)f_s - B. \quad (\text{S.6})$$

As can be seen, there exist a contradiction between (S.6) and Theorem 2, so the length of any column-partition sub-matrix non-zero elements in $\bar{\mathbf{X}}$ must be less or equal than $f_s - l$, which is equivalent to r satisfying

$$r \geq \left\lceil \frac{f_s}{(M - 1)f_s - B} \right\rceil. \quad (\text{S.7})$$

\square

III. PROOF OF THEOREM 3

Theorem 3. Consider the column-partitioned MMV model (5) with $\min_{i,j} \|(\mathbf{X}_{S_i})_{j,:}\|_2 / \|\mathbf{X}_{S_i}\|_F = \eta$ and $|\text{supp}(\mathbf{X}_{S_i})| \leq s$. Let $s_1 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i})|$, $s_2 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i}) \cap \tilde{S}_{S_i}^k|$ and $s_3 := \min_{i,k} |\Lambda_{S_i}^k \cap \text{supp}(\mathbf{X}_{S_i}) \cap \tilde{S}_{S_i}^k \setminus S_{S_i}^k|$. Then, if the sensing matrix \mathbf{A} obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_1 \sqrt{\nu_1^2 + 4\nu_2^2} - \nu_1^2 - 1}{4\nu_1^2 \nu_2^2 - 2\nu_1^2 - 1}} \quad (\text{S.8})$$

where $\nu_1 := \frac{1+\omega}{1+\eta\omega\sqrt{s_2}}$ and $\nu_2 := \frac{1+\omega}{1+\eta\omega\sqrt{s_3}}$, SI-SSP produces an signal estimate $\mathbf{X}^k = [\mathbf{X}_{S_1}^k, \dots, \mathbf{X}_{S_r}^k]$ satisfying

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho^k \|\mathbf{X}\|_F + \tau \|\mathbf{E}\|_F, \quad (\text{S.9})$$

where $\rho \in (0, 1)$ and τ are constants depending on δ_{3s} , ν_1 and ν_2 . Furthermore, after at most $k^* = \lceil \log_{\rho} \frac{\|\mathbf{X}\|_F}{\tau \|\mathbf{E}\|_F} \rceil$ iterations, SI-SSP estimates \mathbf{X} with

$$\|\mathbf{X} - \mathbf{X}^{k^*}\|_F \leq (\tau + 1) \|\mathbf{E}\|_F. \quad (\text{S.10})$$

To prove Theorem 3, we first introduce six useful lemmas, whose proofs are left to appendices.

Lemma 1. (Lemma 1 in [25]): For nonnegative numbers a, b, c, d, x, y ,

$$(ax + by)^2 + (cx + dy)^2 \leq (\sqrt{a^2 + c^2}x + (b + d)y)^2. \quad (\text{S.11})$$

Lemma 2. Consider the system model $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$, $\text{supp}(\mathbf{X}) = T$ and $|T| = s$. Suppose $S, T_0 \subseteq \{1, 2, \dots, n\}$, $|S| = t$. \mathbf{W}_{T_0} is constructed with diagonal entries indexed by T_0 being $\omega \geq 0$. Let $\tilde{\mathbf{X}}$ be the solution of the least squares problem $\arg \min_{\mathbf{Z}} \{\|\mathbf{Y} - \mathbf{A}\mathbf{Z}\|_2, \text{supp}(\mathbf{Z}) \subseteq S\}$, if $\delta_{3s} < 1$, then

$$\|\mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}})_S\|_F \leq \omega \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \omega \sqrt{1 + \delta_t} \|\mathbf{E}\|_F \quad (\text{S.12})$$

and

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{s+t}^2}} \|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1 + \delta_t}}{1 - \delta_{s+t}} \|\mathbf{E}\|_F. \quad (\text{S.13})$$

If $t > s$, define T_{∇} as the row-indices of the smallest $t - s$ row norm entries of $\tilde{\mathbf{X}}$ in S , we have

$$\|\mathbf{X}_{T_{\nabla}}\|_F \leq \sqrt{2\nu_2} \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F + \nu_2 \sqrt{2(1 + \delta_t)} \|\mathbf{E}\|_F. \quad (\text{S.14})$$

Remark 1. When we consider the atom selection strategy of $\|\tilde{\mathbf{X}}_{T_{\nabla}} + \mathbf{W}_{T_0} \tilde{\mathbf{X}}_{T_{\nabla}}\|_F \leq \|\tilde{\mathbf{X}}_{S'} + \mathbf{W}_{T_0} \tilde{\mathbf{X}}_{S'}\|_F$, we can also obtain another upper bound for $\|\mathbf{X}_{T_{\nabla}}\|_F$ in (S.14). In this case, we should allocate $2 \|\mathbf{X}_{T_{\nabla}}\|_F$ to the left hand side of (A.38), we have

$$\begin{aligned} \|\mathbf{X}_{T_{\nabla}}\|_F &\leq \sqrt{2\nu_3} \delta_{s+t} \|\mathbf{X} - \tilde{\mathbf{X}}\|_F \\ &\quad + \nu_4 \sqrt{2(1 + \delta_t)} \|\mathbf{E}\|_F. \end{aligned} \quad (\text{S.15})$$

where $\nu_3 = (1 - \omega + \omega \delta_{s+t} + \delta_{s+t}) / (2\delta_{s+t})$ and $\nu_4 = (1 + \omega) / (2\delta_{s+t})$.

Lemma 3. In steps 4 and 5 of SI-SSP, we have

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2\nu_1} \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \quad (\text{S.16})$$

Remark 2. When we consider the atom selection strategy in select step that

$$\begin{aligned} &\|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ &\leq \|((\mathbf{I}_L + \mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F. \end{aligned} \quad (\text{S.17})$$

We can also obtain another upper bound for $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ in (S.16). In this case, we should allocate $2 \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ to the left hand side of (A.51), we have

$$\begin{aligned} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F &\leq \sqrt{2\nu_4} \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F \\ &\quad + \nu_4 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \end{aligned} \quad (\text{S.18})$$

where $\nu_4 = (1 - \omega + \omega \delta_{3s} + \delta_{3s}) / (2\delta_{3s})$ and $\nu_4 = (1 + \omega) / (2\delta_{3s})$. Based on conclusions (S.15) and (S.18), we know that the sensing matrix \mathbf{A} obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_3 \sqrt{\nu_3^2 + 4\nu_4^2} - \nu_3^2 - 1}{4\nu_3^2 \nu_4^2 - 2\nu_3^2 - 1}}. \quad (\text{S.19})$$

Lemma 4. Let $T_0 \subseteq \{1, 2, \dots, n\}$, for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, if $|\text{supp}(\mathbf{u}) \cup \text{supp}(\mathbf{v})| \leq t$,

$$|\langle \mathbf{u}, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle| \leq \omega \delta_t \|\mathbf{u}\| \|\mathbf{v}\|; \quad (\text{S.20})$$

moreover, if $U \subseteq \{1, 2, \dots, n\}$ and $|U \cup \text{supp}(\mathbf{v})| \leq t$, then

$$|(\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v}| \leq \omega \delta_t \|\mathbf{v}\|. \quad (\text{S.21})$$

Lemma 5. For SMV model $\mathbf{y} = \Phi \mathbf{x} + \mathbf{e}$, let $T_0 \subseteq \{1, 2, \dots, n\}$, let $U \subseteq \{1, 2, \dots, n\}$ and $|U \cap T_0| \leq u$, we have

$$\|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U\|_2 \leq \omega \delta_u \|\mathbf{e}\|_2. \quad (\text{S.22})$$

Lemma 6. Consider MMV model $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$, let $\tilde{\mathbf{X}}$ be the solution of the least squares problem $\arg \min_{\mathbf{Z}} \{\|\mathbf{Y} - \mathbf{A}\mathbf{Z}\|_F, \text{supp}(\mathbf{Z}) \subseteq S\}$, then

$$\langle \mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}}, \mathbf{A}^H \mathbf{A} \mathbf{Z} \rangle + \omega \langle \mathbf{E}, \mathbf{A} \mathbf{Z} \rangle = 0. \quad (\text{S.23})$$

Proof of Theorem 3. Now we have all ingredients to prove Theorem 3. In step 3 and 4 of SI-SSP, by Lemma 3, in the k th iteration, we have

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2\nu_1} \delta_{3s} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1 \sqrt{2(1 + \delta_{3s})} \|\mathbf{E}\|_F. \quad (\text{S.24})$$

Since the step 6 in SI-SSP is a process to solve a least squares problem. Let $S = \tilde{S}^k$ and $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $t = 2s$, by (S.13), we can know that

$$\|\mathbf{X} - \tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1 - \delta_{3s}^2}} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \frac{\sqrt{1 + \delta_{2s}}}{1 - \delta_{3s}} \|\mathbf{E}\|_F. \quad (\text{S.25})$$

Combining (S.24) and (S.25) and magnifying δ_{2s} to δ_{3s} , we have

$$\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F \leq \nu_1 \sqrt{\frac{2\delta_{3s}^2}{1 - \delta_{3s}^2}} \|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \tau_1 \|\mathbf{E}\|_F. \quad (\text{S.26})$$

Next, after step 7 of SI-SSP in k th iteration, let $T_{\nabla} = \tilde{S}^k \setminus S^k$ as the row-indices of the smallest $t - s$ row norm entries in

$\tilde{\mathbf{X}}^k$. Let $T = \tilde{S}^k$, $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $T_\nabla = S_\nabla$ and $t = 2s$, by (A.37), we have

$$\|\mathbf{X}_{S_\nabla}\|_F \leq \sqrt{2}\nu_2\delta_{3s}\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2\sqrt{2(1+\delta_{2s})}\|\mathbf{E}\|_F. \quad (\text{S.27})$$

Let $\tau_1 = (\nu_1\sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$ and $\tau_2 = \sqrt{1+\delta_{3s}}$. Dividing $(S^k)^c$ into 2 disjoint subsets: $(\tilde{S}^k)^c$ and S_∇ , we get

$$\begin{aligned} \|\mathbf{X}_{(S^k)^c}\|_F^2 &= \|\mathbf{X}_{S_\nabla}\|_F^2 + \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2 \\ &\stackrel{(\text{S.24}), (\text{S.27})}{\leq} 2\left(\nu_2\delta_{3s}\|\mathbf{X} - \tilde{\mathbf{X}}^k\|_F + \nu_2\tau_2\|\mathbf{E}\|_F\right)^2 \\ &\quad + 2\left(\nu_1\delta_{3s}\|\mathbf{X} - \mathbf{X}^{k-1}\|_2 + \nu_1\tau_2\|\mathbf{E}\|_F\right)^2 \\ &\stackrel{(\text{S.26})}{\leq} 2\left(\sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}}\|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_2(\tau_1\delta_{3s} + \tau_2)\right. \\ &\quad \times \|\mathbf{E}\|_F)^2 + 2\left(\nu_1\delta_{3s}\|\mathbf{X} - \mathbf{X}^{k-1}\|_F + \nu_1\tau_2\|\mathbf{E}\|_F\right)^2 \\ &\stackrel{(\text{S.11})}{\leq} 2\left(\sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}} + \nu_1^2\delta_{3s}^2\|\mathbf{X} - \mathbf{X}^{k-1}\|_F\right. \\ &\quad \left.+ ((\nu_1 + \nu_2)\tau_2 + \nu_2\delta_{3s}\tau_1)\|\mathbf{E}\|_F\right)^2. \end{aligned} \quad (\text{S.28})$$

Squaring both sides, we can get

$$\begin{aligned} \|\mathbf{X}_{(S^k)^c}\|_F &\leq \left(\sqrt{\frac{4\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}} + 2\nu_1^2\delta_{3s}^2\|\mathbf{X} - \mathbf{X}^{k-1}\|_F\right. \\ &\quad \left.+ \sqrt{2}((\nu_1 + \nu_2)\tau_2 + \nu_2\delta_{3s}\tau_1)\|\mathbf{E}\|_F\right)^2. \end{aligned} \quad (\text{S.29})$$

Step 9 of the k th iteration in SI-SSP also solves a least squares problem. Letting $T = S^k$, $\tilde{\mathbf{X}} = \mathbf{X}^k$ and $t = s$, by (S.13), we have

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \sqrt{\frac{1}{1-\delta_{2s}^2}}\|\mathbf{X}_{(S^k)^c}\|_F + \frac{\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\|\mathbf{E}\|_F. \quad (\text{S.30})$$

Combining (S.29) and (S.30) yields

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho\|\mathbf{X} - \mathbf{X}^{k-1}\|_F + (1-\rho)\tau\|\mathbf{E}\|_F \quad (\text{S.31})$$

where $\rho := \sqrt{2}\delta_{3s}\sqrt{2\nu_1^2\nu_2^2\delta_{3s}^2 + \nu_1^2 - \nu_1^2\delta_{3s}^2}(1-\delta_{3s}^2)^{-1}$ and $\tau := \sqrt{2}\delta_{3s}\nu_2(\nu_1\sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}})(1-\delta_{3s}^2)^{-1/2}(1-\delta_{3s})^{-1}(1-\rho)^{-1} + (\nu_1\nu_2\sqrt{2(1-\delta_{3s})} + \sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$.

Second, we recursively apply (S.31) to obtain

$$\|\mathbf{X} - \mathbf{X}^k\|_F \leq \rho^k\|\mathbf{X}\|_F + \tau\|\mathbf{E}\|_F \quad (\text{S.32})$$

where $\rho < 1$ under (S.8). When $k^* = \lceil \log_{\rho} \frac{\|\mathbf{X}\|_F}{\tau\|\mathbf{E}\|_F} \rceil$, we have $\rho^k\|\mathbf{X}\|_F \leq \tau\|\mathbf{E}\|_F$, and thus the stability result (S.10). \square

APPENDIX A PROOF OF LEMMA 2

- First, we give an upper bound of $\|\mathbf{X}_{T_\nabla}\|_F$, by Lemma 6, let $\mathbf{Z} = (\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S$, we have

$$\begin{aligned} &\left\langle \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}}), \mathbf{A}^H \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle \\ &\quad + \left\langle \mathbf{W}_{T_0}\mathbf{E}, \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle = \mathbf{0}. \end{aligned} \quad (\text{A.33})$$

Noticing that $\text{supp}(\tilde{\mathbf{X}}) \subseteq S$, we have

$$\left\| (\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\|_F^2$$

$$\begin{aligned} &= \left\langle \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle \\ &\stackrel{(\text{A.33})}{=} \left\langle \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}}), (\mathbf{I}_L - \mathbf{A}^H \mathbf{A})(\mathbf{X} - \tilde{\mathbf{X}})_S \right\rangle \\ &\quad - \left\langle \mathbf{W}_{T_0}\mathbf{E}, \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\rangle \\ &\stackrel{(7)}{\leq} \omega\delta_{s+t}\left\| (\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\|_F \left\| \mathbf{X} - \tilde{\mathbf{X}} \right\|_F \\ &\quad + \omega\|\mathbf{E}\|_F\sqrt{1+\delta_t}\left\| \mathbf{W}_{T_0}(\mathbf{X} - \tilde{\mathbf{X}})_S \right\|_F. \end{aligned} \quad (\text{A.34})$$

Divide both sides by $\left\| (\mathbf{W}_{T_0}\mathbf{X} - \mathbf{W}_{T_0}\tilde{\mathbf{X}})_S \right\|_F$ to obtain (S.12).

- By expanding Lemma 2 in [25] to MMV model, we could get a relationship between $\left\| \mathbf{X} - \tilde{\mathbf{X}} \right\|_F$ and $\|\mathbf{X}_{S^c}\|_F$. We have
- $$\left\| \mathbf{X} - \tilde{\mathbf{X}} \right\|_F \leq \sqrt{\frac{1}{1-\delta_{s+t}^2}}\|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1+\delta_t}}{1-\delta_{s+t}}\|\mathbf{E}\|_F. \quad (\text{A.35})$$
- Finally, we established the relationship between \mathbf{X}_{T_∇} and $\mathbf{X} - \tilde{\mathbf{X}}$. There exist a subset $S' \subseteq S$ and $S' \cap T = \emptyset$. Since T_∇ is defined by the set of indices of the $t-s$ smallest row entries of $\tilde{\mathbf{X}}$, we can conclude that

$$\begin{aligned} &\left\| \tilde{\mathbf{X}}_{T_\nabla} \right\|_F + \left\| \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla} \right\|_F \\ &\leq \left\| \tilde{\mathbf{X}}_{S'} \right\|_F + \left\| \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{S'} \right\|_F. \end{aligned} \quad (\text{A.36})$$

By eliminating the contribution from $T_\nabla \cap S'$ and noticing that $S' \cap T = \emptyset$, we have

$$\begin{aligned} &\left\| \tilde{\mathbf{X}}_{T_\nabla \setminus S'} \right\|_F + \left\| \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla \setminus S'} \right\|_F \\ &\leq \left\| (\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_\nabla} \right\|_F \\ &\quad + \left\| \mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_\nabla} \right\|_F. \end{aligned} \quad (\text{A.37})$$

For the left-hand side of (A.37), we have

$$\begin{aligned} &\left\| \tilde{\mathbf{X}}_{T_\nabla \setminus S'} \right\|_F + \left\| \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla \setminus S'} \right\|_F \\ &= \left\| (\tilde{\mathbf{X}} - \mathbf{X} + \mathbf{X})_{T_\nabla \setminus S'} \right\|_F \\ &\quad + \left\| (\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}) + \mathbf{W}_{T_0}\mathbf{X})_{T_\nabla \setminus S'} \right\|_F \\ &\geq \left\| \mathbf{X}_{T_\nabla} \right\|_F + \left\| \mathbf{W}_{T_0}\mathbf{X}_{T_\nabla} \right\|_F \end{aligned} \quad (\text{A.38})$$

$$\begin{aligned} &- \left\| (\tilde{\mathbf{X}} - \mathbf{X})_{T_\nabla \setminus S'} \right\|_F \\ &- \left\| \mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X})_{T_\nabla \setminus S'} \right\|_F. \end{aligned} \quad (\text{A.39})$$

Combining (A.39) and (A.37), and noticing that

$$(T_\nabla \setminus S') \cap (S' \setminus T_\nabla) = \emptyset \quad (\text{A.40})$$

$$(T_\nabla \setminus S') \cup (S' \setminus T_\nabla) \subseteq T, \quad (\text{A.41})$$

we have

$$\begin{aligned} &\left\| \mathbf{X}_{T_\nabla} \right\|_F + \left\| \mathbf{W}_{T_0}\mathbf{X}_{T_\nabla} \right\|_F \\ &\leq \left\| (\tilde{\mathbf{X}} - \mathbf{X})_{T_\nabla \setminus S'} \right\|_F + \left\| (\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{T_\nabla \setminus S'} \right\|_F \\ &\quad + \left\| (\tilde{\mathbf{X}} - \mathbf{X})_{S' \setminus T_\nabla} \right\|_F + \left\| (\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{S' \setminus T_\nabla} \right\|_F \\ &\leq \sqrt{2}\left\| (\tilde{\mathbf{X}} - \mathbf{X})_{(S' \setminus T_\nabla) \cup (T_\nabla \setminus S')} \right\|_F \end{aligned}$$

$$\begin{aligned}
& + \sqrt{2} \left\| (\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_{(S' \setminus T_\nabla) \cup (T_\nabla \setminus S')} \right\|_F \\
& \leq \sqrt{2} \left\| (\tilde{\mathbf{X}} - \mathbf{X})_S \right\|_F + \sqrt{2} \left\| (\mathbf{W}_{T_0}(\tilde{\mathbf{X}} - \mathbf{X}))_S \right\|_F \\
& \stackrel{(S.12)}{\leq} \sqrt{2}(1 + \omega)\delta_{s+t} \left\| \mathbf{X} - \tilde{\mathbf{X}} \right\|_F \\
& + (1 + \omega)\sqrt{2(1 + \delta_t)} \left\| \mathbf{E} \right\|_F. \tag{A.42}
\end{aligned}$$

Also, we can obtain the relationship between $\|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F$ and $\|\mathbf{X}_{T_\nabla}\|_F$:

$$\eta\omega\sqrt{s_3}\|\mathbf{X}_{T_\nabla}\|_F \leq \|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F. \tag{A.43}$$

Combining (A.42) and (A.43), we have

$$\begin{aligned}
\|\mathbf{X}_{T_\nabla}\|_F & \leq \frac{\sqrt{2}(1 + \omega)\delta_{s+t}}{1 + \eta\omega\sqrt{s_3}} \left\| \mathbf{X} - \tilde{\mathbf{X}} \right\|_F \\
& + \frac{(1 + \omega)\sqrt{2(1 + \delta_t)}}{1 + \eta\omega\sqrt{s_3}} \left\| \mathbf{E} \right\|_F. \tag{A.44}
\end{aligned}$$

Noting the definition of ν_2 , we complete the proof of Lemma 2.

APPENDIX B PROOF OF LEMMA 3

Proof: From step 5 of SI-SSP, we have

$$\mathbf{X}_{S_i}^k = \arg \min_{\Theta: \text{supp}(\Theta) = S_i^k} \|\mathbf{Y}_{S_i} - \mathbf{A}\Theta\|_F. \tag{A.45}$$

From the step 4 of SI-SSP, let $\mathbf{X}^k = [\mathbf{X}_{S_1}^k, \dots, \mathbf{X}_{S_r}^k]$, we have the following conclusion

$$\begin{aligned}
& \left\| (\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_T \right\|_F \\
& \leq \left\| (\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S} \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S} \right\|_F. \tag{A.46}
\end{aligned}$$

By removing the same coordinates $T \cap \Delta S$, we can get

$$\begin{aligned}
& \left\| (\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S} \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S} \right\|_F \\
& \leq \left\| (\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T} \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T} \right\|_F. \tag{A.47}
\end{aligned}$$

Because $\text{supp}(\mathbf{X}) = T$ and $\text{supp}(\mathbf{X}^{k-1}) = S^{k-1}$,

$$(\mathbf{X} - \mathbf{X}^{k-1})_{\Delta S \setminus (T \cup S^{k-1})} = 0. \tag{A.48}$$

For the right-hand of (A.47), we have

$$\begin{aligned}
& \left\| (\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T} \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus T} \right\|_F \\
& = \left\| (\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})} \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})} \right\|_F \\
& = \left\| (\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})} \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{\Delta S \setminus (T \cup S^{k-1})} \right\|_F \\
& \leq \left\| ((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T} \right\|_F \\
& + \left\| (\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T} \right\|_F
\end{aligned}$$

$$\begin{aligned}
& + \left\| (\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T} \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T} \right\|_F. \tag{A.49}
\end{aligned}$$

Note that $\tilde{S}^k = S^{k-1} \cup \Delta S$, we have

$$(\mathbf{X} - \mathbf{X}^{k-1})_{T \setminus \tilde{S}^k} = \mathbf{X}_{(\tilde{S}^k)^c}. \tag{A.50}$$

For the left-side of (A.47), we have

$$\begin{aligned}
& \left\| (\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S} \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus \Delta S} \right\|_F \\
& = \left\| (\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})} \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})} \right\|_F \\
& = \left\| (\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})} \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X} + \mathbf{E} - \mathbf{A}\mathbf{X}^{k-1}))_{T \setminus (\Delta S \cup S^{k-1})} \right\|_F \\
& = \left\| ((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}) + \mathbf{A}^H\mathbf{E} + \mathbf{X})_{\Delta S \setminus (T \cup S^{k-1})} \right\|_F \\
& + \left\| \mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}) + \mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E} + \mathbf{W}_{T_0}\mathbf{X} \right\|_{\Delta S \setminus (T \cup S^{k-1})} \right\|_F \\
& \geq \left\| \mathbf{X}_{(\tilde{S}^k)^c} \right\|_F + \left\| (\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c} \right\|_F \\
& - \left\| ((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{(\tilde{S}^k)^c} \right\|_F \tag{A.51}
\end{aligned}$$

$$\begin{aligned}
& - \left\| (\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{(\tilde{S}^k)^c} \right\|_F \\
& - \left\| (\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c} \right\|_F - \left\| (\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c} \right\|_F. \tag{A.52}
\end{aligned}$$

Combining (A.53) and (A.52), we have

$$\begin{aligned}
& \left\| \mathbf{X}_{(\tilde{S}^k)^c} \right\|_F + \left\| (\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c} \right\|_F \\
& \leq \left\| ((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \setminus \tilde{S}^k} \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \setminus \tilde{S}^k} \right\|_F \\
& + \left\| (\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T} \right\|_F \\
& + \left\| ((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{\Delta S \setminus T} \right\|_F \\
& + \left\| (\mathbf{A}^H\mathbf{E})_{T \setminus \tilde{S}^k} \right\|_F + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T \setminus \tilde{S}^k} \right\|_F \\
& + \left\| (\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T} \right\|_F + \left\| (\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S \setminus T} \right\|_F \\
& \leq \sqrt{2} \left\| ((\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \cup \Delta S} \right\|_F \\
& + \sqrt{2} \left\| (\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A} - \mathbf{I}_L)(\mathbf{X} - \mathbf{X}^{k-1}))_{T \cup \Delta S} \right\|_F \\
& + \sqrt{2} \left\| (\mathbf{A}^H\mathbf{E})_{T \cup \Delta S} \right\|_F + \sqrt{2} \left\| (\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T \cup \Delta S} \right\|_F \\
& \leq \sqrt{2}(1 + \omega)\delta_{3s} \left\| \mathbf{X} - \mathbf{X}^{k-1} \right\|_F \\
& + (1 + \omega)\sqrt{2(1 + \delta_{3s})} \left\| \mathbf{E} \right\|_F. \tag{A.53}
\end{aligned}$$

We can obtain the relationship between $\left\| \mathbf{X}_{(\tilde{S}^k)^c} \right\|_F$ and $\left\| (\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c} \right\|_F$:

$$\eta\omega\sqrt{s_2} \left\| \mathbf{X}_{(\tilde{S}^k)^c} \right\|_F \leq \left\| (\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c} \right\|_F. \tag{A.54}$$

Combining (A.53) and (A.54), we have

$$\left\| \mathbf{X}_{(\tilde{S}^k)^c} \right\|_F \leq \frac{\sqrt{2}(1 + \omega)\delta_{3s}}{1 + \eta\omega\sqrt{s_2}} \left\| \mathbf{X} - \tilde{\mathbf{X}} \right\|_F$$

$$+ \frac{(1+\omega)\sqrt{2(1+\delta_{2s})}}{1+\eta\omega\sqrt{s_2}} \|\mathbf{E}\|_F. \quad (\text{A.55})$$

Noting the definition of ν_1 , we complete the proof of Lemma 3.

APPENDIX C PROOF OF LEMMA 4

Proof: the RIC δ_t can be expressed as [25]

$$\delta_t = \max_{S \subseteq \{1,2,\dots,N\}, |S| \leq t} \|\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}\|_{2 \rightarrow 2}, \quad (\text{A.56})$$

where

$$\|\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}\|_{2 \rightarrow 2} = \sup_{\mathbf{a} \in \mathbb{R}^{|S|} \setminus \{\mathbf{0}\}} \frac{\|(\mathbf{A}_S^* \mathbf{A}_S - \mathbf{I}) \mathbf{a}\|_2}{\|\mathbf{a}\|_2}. \quad (\text{A.57})$$

Let $S = \text{supp}(\mathbf{u}) \cup \text{supp}(\mathbf{v})$, then $|S| \leq t$. Let $\mathbf{u}_{|S}, \mathbf{v}_{|S}$ denote respectively the S -dimensional sub-vectors of \mathbf{u} and \mathbf{v} obtained by only keeping the components indexed by S . We have

$$\begin{aligned} & |\langle \mathbf{u}, (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle| \\ &= |\langle \mathbf{W}_{T_0} \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{A} \mathbf{W}_{T_0} \mathbf{u}, \mathbf{A} \mathbf{v} \rangle| \\ &= |\langle \mathbf{W}_{T_0} \mathbf{u}_{|T}, (\mathbf{I}_L - \mathbf{A}_T^H \mathbf{A}_T) \mathbf{v}_{|T} \rangle| \\ &\leq \|\mathbf{W}_{T_0} \mathbf{u}_{|T}\|_2 \|(\mathbf{I}_L - \mathbf{A}_T^H \mathbf{A}_T) \mathbf{v}_{|T}\|_2 \\ &\stackrel{(\text{A.57})}{\leq} \|\mathbf{W}_{T_0} \mathbf{u}_{|T}\|_2 \|\mathbf{I}_L - \mathbf{A}_T^H \mathbf{A}_T\|_{2 \rightarrow 2} \|\mathbf{v}_{|T}\|_2 \\ &\stackrel{(\text{A.56})}{\leq} \omega \delta_t \|\mathbf{u}_{|T}\|_2 \|\mathbf{v}_{|T}\|_2 \\ &= \omega \delta_t \|\mathbf{u}\|_2 \|\mathbf{v}\|_2, \end{aligned} \quad (\text{A.58})$$

moreover, we have

$$\begin{aligned} & \|((\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v})_U\|_2^2 \\ &= \langle ((\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v})_U, \\ & \quad (\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A}) \mathbf{v} \rangle \\ &\stackrel{(\text{S.20})}{\leq} \delta_t \|((\mathbf{W}_{T_0} - \mathbf{A}^H \mathbf{A}) \mathbf{v})_U\|_2 \|\mathbf{v}\|_2 \end{aligned} \quad (\text{A.59})$$

which completes the proof of Lemma 4.

APPENDIX D PROOF OF LEMMA 5

Proof: The lemma easily follows from the fact that

$$\begin{aligned} & \|(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U\|_2^2 \\ &= \langle \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e}, (\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e})_U \rangle \\ &= \langle \mathbf{e}, \mathbf{W}_{T_0} \mathbf{A} ((\mathbf{A}^H \mathbf{e})_U) \rangle \\ &\leq \|\mathbf{e}\|_2 \|\mathbf{W}_{T_0} \mathbf{A} ((\mathbf{A}^H \mathbf{e})_U)\|_2 \\ &\stackrel{(7)}{\leq} \|\mathbf{e}'\|_2 \omega \sqrt{1+\delta_u} \|(\mathbf{A}^H \mathbf{e})_U\|_2. \end{aligned} \quad (\text{A.60})$$

APPENDIX E PROOF OF LEMMA 6

Proof: Due to the orthogonality, the residue $\mathbf{Y} - \mathbf{A}\tilde{\mathbf{X}}$ is orthogonal to the space $\mathbf{AZ}, \text{supp}(\mathbf{Z}) \subseteq S$. This means that for all $\mathbf{Z} \in \mathbb{R}$ with $\text{supp}(\mathbf{Z}) \subseteq S$,

$$\langle \mathbf{A}(\mathbf{Y} - \mathbf{A}\tilde{\mathbf{X}}), \mathbf{Z} \rangle = 0 \quad (\text{A.61})$$

then, let $\tilde{\mathbf{X}}'$ be the solution of the least squares problem $\arg \min_{\mathbf{Z}} \{\|\mathbf{Y}' - \mathbf{AZ}\|_F, \text{supp}(\mathbf{Z}) \subseteq S\}$, where $\mathbf{Y}' = \frac{\mathbf{AW}_{T_0} \mathbf{X}_{T_0}}{\omega} + \mathbf{E}$. We have

$$\tilde{\mathbf{X}}' = \frac{\mathbf{W}_{T_0} \tilde{\mathbf{X}}}{\omega}. \quad (\text{A.62})$$

Then, by (A.61), we have

$$\begin{aligned} \mathbf{0} &= \left\langle \frac{\mathbf{AW}_{T_0} \mathbf{X}_{T_0}}{\omega} + \mathbf{E} - \mathbf{A} \frac{\mathbf{W}_{T_0} \tilde{\mathbf{X}}}{\omega}, \mathbf{AZ} \right\rangle \\ &= \left\langle \mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}}, \mathbf{A}^H \mathbf{AZ} \right\rangle + \omega \langle \mathbf{E}, \mathbf{AZ} \rangle. \end{aligned} \quad (\text{A.63})$$