Supplementary: Boundary Multiple Measurement Vectors for Multi-Coset Sampler

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This supplementary material is dedicated to the proofs for Theorems 1–3 in our main paper.

Before proceeding to the proofs, we review some useful notations. For a complex matrix $\mathbf{X} \in \mathbb{C}^{n\times L}$ and a set $S \subseteq \{1, \cdots, n\}$, $\mathbf{X}_S$ (or $\mathbf{X}^S$) denotes the submatrix of $\mathbf{X}$ with columns (or rows) indexed by S; $\mathbf{X}_{i,j}$, $\mathbf{X}_{i,:}$ and $\mathbf{X}_{:,i}$ are the (i, j)th entry, ith row and ith column of $\mathbf{X}$, respectively; $\mathbf{X}^\dagger$, $\mathbf{X}^H$ and $\mathbf{X}^\top$ mean the Moore-Penrose pseudo-inverse, conjugate transpose and transpose of $\mathbf{X}$, respectively; $\text{supp}(\mathbf{X})$ is the non-zero row indices (i.e., joint sparsity) of $\mathbf{X}$; $\|\mathbf{X}\|_F$ and $\|\mathbf{X}\|_2$ signify the Frobenius and Euclidean norm of $\mathbf{X}$, respectively. Moreover, S^c is the complement of set S; $\mathbf{I}_L$ is an $L \times L$ identity matrix.

I. Proof of Theorem 1

Theorem 1. *The actual sampling rate of (4) is* $\min\left(pf_s, f_{nyq}\right)$, *which attains the theoretical lower bound of sampling rate in MCS when* $|supp(\mathbf{X})| \le \frac{N_{sig}B}{f_s}$.

Proof. In the ith channel of a multi-coset sampler, the sampling sequence is given by

$$x_{c_i}[n] = x(LTn + \tau_i), \quad n = 0, 1, \cdots . \tag{S.1}$$

The sampling rate of each channel is determined by the sampled signal sequence. To be specific, since the sampling time interval is LT, the sampling rate of each channel is

$$f_s = \frac{1}{LT} = \frac{f_{\text{nyq}}}{L}, \tag{S.2}$$

i.e., one-Lth of the Nyquist sampling rate.

Moreover, as the multi-coset sampler is assumed to have p channels, the overall sampling rate of p channels is p times that of each channel (i.e. $\frac{pf_{\text{nyq}}}{L}$). If this sampling rate is greater than the Nyquist rate f_{nyq}, then the advantage of sub-Nyquist sampling structure no longer exists. In this case, we only need to sample at Nyquist sampling rate f_{nyq}. Thus, the actual sampling rate can be given by

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right). \tag{S.3}$$

The theoretical lower bound of the sampling rate is given in [17], which is determined directly by the true bandwidth of the signal:

$$\min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \tag{S.4}$$

Thus, the theoretical lower bound on the sampling rate is achieved when

$$\min\left(\frac{p}{LT}, f_{\text{nyq}}\right) \le \min(2\lambda(\mathcal{T}), f_{\text{nyq}}). \tag{S.5}$$

In most cases, $2\lambda(\mathcal{T})$ and $\frac{p}{LT}$ do not exceed f_{nyq}. (If violated, the sampling rate would just be f_{nyq}.) Therefore, the condition (S.5) holds whenever

$$\frac{p}{LT} \le 2\lambda(\mathcal{T}). \tag{S.6}$$

Furthermore, to ensure an unique-solution reconstruction, the number p of channels should not be too small. In particular, it's lower bound is twice the signal sparsity without the priori information about the signal $\mathbf{X}$ [17],

$$p \ge 2|\text{supp}(\mathbf{X})|. \tag{S.7}$$

For the worst case where $p = 2|\text{supp}(\mathbf{X})|$, (S.6) can be rewritten as

$$|\text{supp}(\mathbf{X})| \le \lambda(\mathcal{T})LT = \frac{N_{\text{sig}}B}{f_s}, \tag{S.8}$$

which completes the proof. $\square$

II. Proof of Theorem 2

Theorem 2. *When* $r \in [\lceil\frac{f_s}{\lfloor(D-1)f_s-B\rfloor}\rceil, N]$ *and* $B > f_s$, *we have* $\max_{j\in\{1,\cdots,r\}}\left|supp(\bar{\mathbf{X}}_{S_j})\right| < \frac{N_{sig}B}{f_s}$.

Proof. Recall that the MMV model $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}$ is decomposed into r sub-MMV problems:

$$\mathbf{Y}_{S_j} = \mathbf{A}\mathbf{X}_{S_j} + \mathbf{E}_{S_j}, \quad j = 1, \cdots, r \tag{S.9}$$

and each problem is solved individually. Our goal is to determine the number r of sub-MMV problems that ensures the actual sampling rate to reach the lower bound of the theoretical sampling rate.

For all row blocks $\{\mathbf{X}^{U_1}, \cdots, \mathbf{X}^{U_M}\}$ in $\mathbf{X}$, assume that each block has consecutive frequency points of length B (i.e., the sub-band's width is B) if occupied and zero otherwise. In this work, we are primarily interested in the case where

$$B > f_s. \tag{S.10}$$

since the remaining case where $B \le f_s$ has been studied thoroughly in existing works (see. e.g., [16]).

Let D denote the height (i.e., number of rows) in each row block $\mathbf{X}^{U_i}$, which can be determined properly according to B and f_s. Specifically, we choose a $D \ge 3$ such that

$$(D-2)f_s < B \le (D-1)f_s. \tag{S.11}$$

In this case, the frequency points of each PU signal (i.e., occupied row block) occupy $D-1$ or D rows. An example of $D = 5$ is illustrated in Fig. 1, where PU signals 1 and 2 occupy 4 rows while PU signal 3 occupies 5 rows.

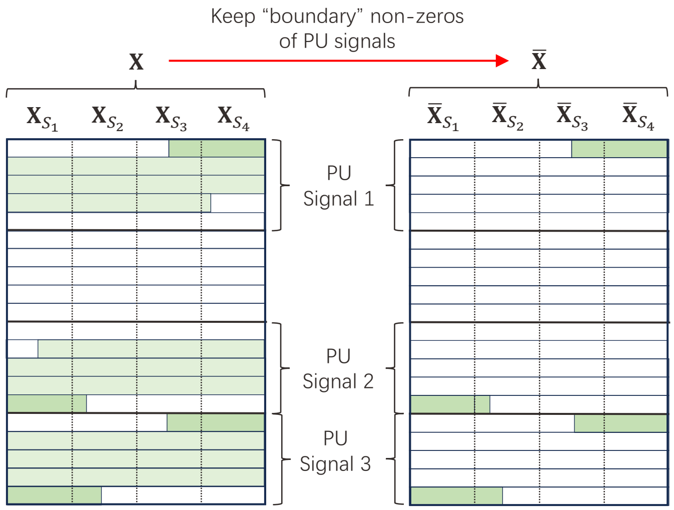

Fig. 1. An illustrative example of MCS signal $\mathbf{X}$ with 3 PU signals.

- When $D-1$ rows in PU signal $\mathbf{X}^{U_i}$ are occupied, only one row of $\bar{\mathbf{X}}^{U_i}$ is occupied, since $\bar{\mathbf{X}}^{U_i}$ only keeps the boundary non-zeros of $\mathbf{X}^{U_i}$. In this case, we can easily see that
$$\left|\text{supp}(\bar{\mathbf{X}}_{S_j}^{U_i})\right| \leq \left|\text{supp}(\bar{\mathbf{X}}^{U_i})\right| = 1. \tag{S.12}$$
Since there are N_{sig} sub-band signals in $\mathbf{X}$, and also noting that $B > f_s$, we have
$$\max_{j\in\{1,\cdots,r\}}\left|\text{supp}(\bar{\mathbf{X}}_{S_j})\right| \leq N_{\text{sig}} < \frac{N_{\text{sig}}B}{f_s}. \tag{S.13}$$
- When D rows in PU signal $\mathbf{X}^{U_i}$ are occupied, the length l of the frequency points in $\bar{\mathbf{X}}^{U_i}$ obeys
$$l = B - (D-2)f_s \overset{\text{(S.11)}}{\leq} f_s. \tag{S.14}$$
As a result, the column indices of non-zeros in any PU signal $\bar{\mathbf{X}}^{U_i}$ do not overlap. In other words, the total length of frequency points in each PU signal $\bar{\mathbf{X}}^{U_i}$ does not exceed f_s, as shown in Fig. 1.
Recall that $\bar{\mathbf{X}} \in \mathbb{R}^{L\times N}$ is column-partitioned into r sub-matrices (i.e., $\{\bar{\mathbf{X}}_{S_j}\}_{j=1,\cdots,r}$).
  - We first consider an extreme case where $r = N$. In this case, $\bar{\mathbf{X}}_{S_j}$ only have one column (i.e., $|S_j| = |\{j\}| = 1$), and so each column-partitioned PU signal $\bar{\mathbf{X}}_{S_j}^{U_i}$ satisfies
$$\left|\text{supp}(\bar{\mathbf{X}}_{S_j}^{U_i})\right| = \left|\text{supp}(\bar{\mathbf{X}}_{j,:}^{U_i})\right| \leq 1. \tag{S.15}$$
Similar to (S.13), we further have
$$\begin{aligned}\max_{j\in\{1,\cdots,r\}}\left|\text{supp}(\bar{\mathbf{X}}_{S_j})\right| &\leq N_{\text{sig}} \\ &\overset{\text{(S.10)}}{<} \frac{N_{\text{sig}}B}{f_s}.\end{aligned} \tag{S.16}$$
  - We then consider the general case where $r < N$. In this case, the length of frequency points in each sub-matrix $\bar{\mathbf{X}}_{S_j}^{U_i}$ does not exceed $\lceil \frac{f_s}{r} \rceil$. In the following, we shall prove that
$$\max_{j\in\{1,\cdots,r\}}|\text{supp}(\bar{\mathbf{X}}_{S_j})| \leq N_{\text{sig}} \tag{S.17}$$
when $r \in [\lceil \frac{f_s}{\lfloor (D-1)f_s - B\rfloor} \rceil, N)$.

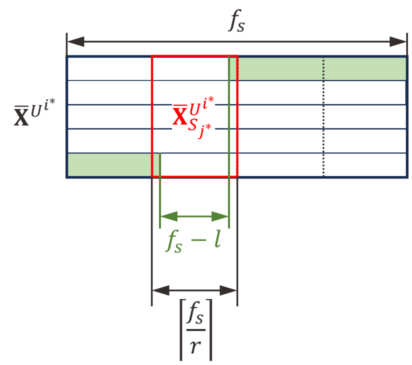

Fig. 2. An illustrative example of the PU signal $\bar{\mathbf{X}}^{U_{i^*}}$, which has a length l that is greater than $f_s - \lceil \frac{f_s}{r} \rceil$.

Assume that there exists some a sub-matrix, say, $\bar{\mathbf{X}}_{S_{j^*}}$, such that
$$|\text{supp}(\bar{\mathbf{X}}_{S_{j^*}})| > N_{\text{sig}}. \tag{S.18}$$
Then, there exists at least a row-partitioned block of $\bar{\mathbf{X}}_{S_{j^*}}$, say, $\bar{\mathbf{X}}_{S_{j^*}}^{U_{i^*}}$, that satisfies
$$|\text{supp}(\bar{\mathbf{X}}_{S_{j^*}}^{U_{i^*}})| = 2. \tag{S.19}$$
This implies that the length l of the frequency points in PU signal $\bar{\mathbf{X}}^{U_{i^*}}$ must be greater than $f_s - \lceil \frac{f_s}{r} \rceil$, as illustrated in Fig. 2. That is,
$$\left\lceil \frac{f_s}{r} \right\rceil > f_s - l, \tag{S.20}$$
which, together with the fact that $l = B - (D-2)f_s$ (see (S.14)), leads to
$$\begin{aligned}\left\lceil \frac{f_s}{r} \right\rceil &> (D-1)f_s - B \\ &> \lfloor (D-1)f_s - B \rfloor.\end{aligned} \tag{S.21}$$
Since $\lfloor (D-1)f_s - B \rfloor$ is a integer, we have
$$r < \frac{f_s}{\lfloor (D-1)f_s - B \rfloor}. \tag{S.22}$$
Thus, if
$$r \geq \frac{f_s}{\lfloor (D-1)f_s - B \rfloor}, \tag{S.23}$$
the assumption of (S.18) must not be true, which implies (S.17).

To sum up, when $r \in [\lceil \frac{f_s}{\lfloor (D-1)f_s - B\rfloor} \rceil, N]$, we have
$$\max_{j\in\{1,\cdots,r\}} \left|\text{supp}(\bar{\mathbf{X}}_{S_j})\right| \leq N_{\text{sig}} < \frac{N_{\text{sig}}B}{f_s}. \tag{S.24}$$
The proof is thus complete. □

III. Proof of Theorem 3

Theorem 3. *Consider the column-partitioned MMV model (5) with* $\min_{i,j}\|(\mathbf{X}_{S_i})_{j,:}\|_2/\|\mathbf{X}_{S_i}\|_F = \eta$ *and* $|supp(\mathbf{X}_{S_i})| \leq s$. *Let* $s_1 := \min_{i,k}|\Lambda_{S_i}^k \cap supp(\mathbf{X}_{S_i})|$, $s_2 := \min_{i,k}|\Lambda_{S_i}^k \cap$

$supp(\mathbf{X}_{S_i}) \cap \tilde{S}^k_{S_i}|$ and $s_3 := \min_{i,k}|\Lambda^k_{S_i} \cap supp(\mathbf{X}_{S_i}) \cap \tilde{S}^k_{S_i} \setminus S^k_{S_i}|$. Then, if the sensing matrix $\mathbf{A}$ obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_1\sqrt{\nu_1^2+4\nu_2^2}-\nu_1^2-1}{4\nu_1^2\nu_2^2-2\nu_1^2-1}} \tag{S.25}$$

where $\nu_1 := \frac{1+\omega}{1+\eta\omega\sqrt{s_2}}$ and $\nu_2 := \frac{1+\omega}{1+\eta\omega\sqrt{s_3}}$, SI-SSP produces an signal estimate $\mathbf{X}^k = [\mathbf{X}^k_{S_1}, \cdots, \mathbf{X}^k_{S_r}]$ satisfying

$$\left\|\mathbf{X}-\mathbf{X}^k\right\|_F \leq \rho^k \left\|\mathbf{X}\right\|_F + \tau \left\|\mathbf{E}\right\|_F, \tag{S.26}$$

where $\rho \in (0,1)$ and τ are constants depending on δ_{3s}, ν_1 and ν_2. Furthermore, after at most $k^* = \lceil \log_\rho \frac{\|\mathbf{X}\|_F}{\tau\|\mathbf{E}\|_F} \rceil$ iterations, SI-SSP estimates $\mathbf{X}$ with

$$\|\mathbf{X}-\mathbf{X}^{k^*}\|_F \leq (\tau+1)\|\mathbf{E}\|_F. \tag{S.27}$$

To prove Theorem 3, we first introduce six useful Lemmas, whose proofs are left to the appendices.

Lemma 1. *([25]): For nonnegative numbers a, b, c, d, x, y,*

$$(ax+by)^2+(cx+dy)^2 \leq (\sqrt{a^2+c^2}x+(b+d)y)^2. \tag{S.28}$$

Lemma 2. *Consider the system model $\mathbf{Y} = \mathbf{AX}+\mathbf{E}$, where $supp(\mathbf{X}) = T$ and $|T| = s$. Let $S \subseteq \{1,2,...,n\}$ be an index set with $|S| = t$ and $\mathbf{W}_{T_0}$ be a side-information matrix with diagonal entries indexed by $T_0 \subseteq \{1,2,...,n\}$ being $\omega \geq 0$ and zero otherwise. Also, let $\tilde{\mathbf{X}} := \arg\min_{\mathbf{Z}:supp(\mathbf{Z})\subseteq S} \left\|\mathbf{Y}-\mathbf{AZ}\right\|_2$. If $\delta_{3s} < 1$, then*

$$\|\mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}})_S\|_F \leq \omega\delta_{s+t}\|\mathbf{X}-\tilde{\mathbf{X}}\|_F + \omega\sqrt{1+\delta_t}\|\mathbf{E}\|_F \tag{S.29}$$

and

$$\|\mathbf{X}-\tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1-\delta_{s+t}^2}}\|\mathbf{X}_{S^c}\|_F + \frac{\sqrt{1+\delta_t}}{1-\delta_{s+t}}\|\mathbf{E}\|_F. \tag{S.30}$$

Furthermore, if $t > s$, define T_∇ as the row-indices of the smallest $t-s$ row-norm entries of $\tilde{\mathbf{X}}$ in S, we have

$$\|\mathbf{X}_{T_\nabla}\|_F \leq \sqrt{2}\nu_2\delta_{s+t}\|\mathbf{X}-\tilde{\mathbf{X}}\|_F + \nu_2\sqrt{2\left(1+\delta_t\right)}\|\mathbf{E}\|_F. \tag{S.31}$$

Remark 1. *When we consider the atom selection strategy of $\|\tilde{\mathbf{X}}_{T_\nabla} + \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla}\|_F \leq \|\tilde{\mathbf{X}}_{S'} + \mathbf{W}_{T_0}\tilde{\mathbf{X}}_{S'}\|_F$, we can also obtain another upper bound for $\|\mathbf{X}_{T_\nabla}\|_F$ in (S.31). In this case, we should allocate $2\left\|\mathbf{X}_{T_\nabla}\right\|_F$ to the left hand side of (A.55), we have*

$$\|\mathbf{X}_{T_\nabla}\|_F \leq \sqrt{2}\nu_3\delta_{s+t}\|\mathbf{X}-\tilde{\mathbf{X}}\|_F + \nu_4\sqrt{2\left(1+\delta_t\right)}\left\|\mathbf{E}\right\|_F. \tag{S.32}$$

where $\nu_3 = (1-\omega+\omega\delta_{3s}+\delta_{3s})/(2\delta_{3s})$ and $\nu_4 = (1+\omega)/(2\delta_{3s})$.

Lemma 3. *In steps 4 and 5 of SI-SSP, we have*

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2}\nu_1\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F + \nu_1\sqrt{2(1+\delta_{3s})}\left\|\mathbf{E}\right\|_F. \tag{S.33}$$

Remark 2. *When we consider the atom selection strategy in select step that*

$$\begin{aligned}&\left\|((\mathbf{I}_L+\mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y}-\mathbf{AX}^{k-1}))_T\right\|_F\\ &\quad\leq \left\|((\mathbf{I}_L+\mathbf{W}_{T_0})\mathbf{A}^H(\mathbf{Y}-\mathbf{AX}^{k-1}))_{\Delta S}\right\|_F.\end{aligned} \tag{S.34}$$

We can also obtain another upper bound for $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ in (S.33). In this case, we should allocate $2\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ to the left hand side of (A.68), we have

$$\begin{aligned}\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq{}& \sqrt{2}\nu_3\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F\\ &+\nu_4\sqrt{2\left(1+\delta_{3s}\right)}\left\|\mathbf{E}\right\|_F.\end{aligned} \tag{S.35}$$

where $\nu_3 = (1-\omega+\omega\delta_{3s}+\delta_{3s})/(2\delta_{3s})$ and $\nu_4 = (1+\omega)/(2\delta_{3s})$. Based on conclusions (S.32) and (S.35), we know that the sensing matrix $\mathbf{A}$ obeys the RIP with

$$\delta_{3s} \leq \sqrt{\frac{\nu_3\sqrt{\nu_3^2+4\nu_4^2}-\nu_3^2-1}{4\nu_3^2\nu_4^2-2\nu_3^2-1}}. \tag{S.36}$$

Lemma 4. *Let $T_0 \subseteq \{1,2,...,n\}$, for two vectors $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$, if $|supp(\mathbf{u}) \cup supp(\mathbf{v})| \leq t$,*

$$\left|\left\langle \mathbf{u}, (\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\mathbf{v}\right\rangle\right| \leq \omega\delta_t\left\|\mathbf{u}\right\|\left\|\mathbf{v}\right\|; \tag{S.37}$$

Moreover, if $U \subseteq \{1,2,...,n\}$ and $|U \cup supp(\mathbf{v})| \leq t$, then

$$\left|(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A})\mathbf{v}\right| \leq \omega\delta_t\left\|\mathbf{v}\right\|. \tag{S.38}$$

Lemma 5. *For SMV model $\mathbf{y} = \mathbf{\Phi x}+\mathbf{e}$, let $T_0 \subseteq \{1,2,...,n\}$, let $U \subseteq \{1,2,...,n\}$ and $|U \cap T_0| \leq u$, we have*

$$\left\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{e})_U\right\|_2 \leq \omega\delta_u\left\|\mathbf{e}\right\|_2. \tag{S.39}$$

Lemma 6. *Consider the MMV model $\mathbf{Y} = \mathbf{AX}+\mathbf{E}$, let $\tilde{\mathbf{X}}$ be the solution of the least squares problem $\arg\min_{\mathbf{Z}}\{\left\|\mathbf{Y}-\mathbf{AZ}\right\|_F, supp(\mathbf{Z}) \subseteq S\}$, then*

$$\langle \mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}}, \mathbf{A}^H\mathbf{AZ}\rangle + \omega\left\langle \mathbf{E}, \mathbf{AZ}\right\rangle = \mathbf{0}. \tag{S.40}$$

Now we have all ingredients to prove Theorem 3.

Proof of Theorem 3. First, in Steps 4 and 5 of SI-SSP, Lemma 3 implies

$$\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \sqrt{2}\nu_1\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F + \nu_1\sqrt{2(1+\delta_{3s})}\left\|\mathbf{E}\right\|_F. \tag{S.41}$$

Note that Step 6 of SI-SSP solves a least squares problem. Let $S = \tilde{S}^k$ and $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $t = 2s$, by (S.30) we have

$$\|\mathbf{X}-\tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1-\delta_{3s}^2}}\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \frac{\sqrt{1+\delta_{2s}}}{1-\delta_{3s}}\|\mathbf{E}\|_F. \tag{S.42}$$

Combining (S.41) and (S.42) and also magnifying δ_{2s} to δ_{3s}, we further have

$$\|\mathbf{X}-\tilde{\mathbf{X}}^k\|_F \leq \nu_1\sqrt{\frac{2\delta_{3s}^2}{1-\delta_{3s}^2}}\|\mathbf{X}-\mathbf{X}^{k-1}\|_F + \tau_1\|\mathbf{E}\|_F. \tag{S.43}$$

Next, after Step 7 of SI-SSP, let $S_\nabla = \tilde{S}^k \setminus S^k$ be the row-indices of the smallest $t-s$ row norm entries in $\tilde{\mathbf{X}}^k$. Also, let $T = \tilde{S}^k$, $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^k$, $T_\nabla = S_\nabla$ and $t = 2s$. Then, by (A.54) we have

$$\|\mathbf{X}_{S_\nabla}\|_F \leq \sqrt{2}\nu_2\delta_{3s}\|\mathbf{X}-\tilde{\mathbf{X}}^k\|_F + \nu_2\sqrt{2\left(1+\delta_{2s}\right)}\|\mathbf{E}\|_F. \tag{S.44}$$

Let $\tau_1 = (\nu_1\sqrt{2(1-\delta_{3s})}+\sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$ and $\tau_2 = \sqrt{1+\delta_{3s}}$. Dividing $(S^k)^c$ into two disjoint subsets: $(\tilde{S}^k)^c$ and S_∇, we get

$$\left\|\mathbf{X}_{(S^k)^c}\right\|_F^2 = \|\mathbf{X}_{S_\nabla}\|_F^2 + \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F^2$$

$$\overset{(S.41),\,(S.44)}{\leq} 2\left(\nu_2\delta_{3s}\|\mathbf{X}-\tilde{\mathbf{X}}^k\|_F+\nu_2\tau_2\|\mathbf{E}\|_F\right)^2 + 2\left(\nu_1\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_2+\nu_1\tau_2\left\|\mathbf{E}\right\|_F\right)^2$$
$$\overset{(S.43)}{\leq} 2\left(\sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F+\nu_2(\tau_1\delta_{3s}+\tau_2)\times\|\mathbf{E}\|_F\right)^2+2\left(\nu_1\delta_{3s}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F+\nu_1\tau_2\left\|\mathbf{E}\right\|_F\right)^2$$
$$\overset{(S.28)}{\leq} 2\left(\sqrt{\frac{2\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}+\nu_1^2\delta_{3s}^2}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F + \left((\nu_1+\nu_2)\tau_2+\nu_2\delta_{3s}\tau_1\right)\left\|\mathbf{E}\right\|_F\right)^2. \tag{S.45}$$

Squaring both sides, we get

$$\left\|\mathbf{X}_{(S^k)^c}\right\|_F \leq \sqrt{\tfrac{4\nu_1^2\nu_2^2\delta_{3s}^4}{1-\delta_{3s}^2}+2\nu_1^2\delta_{3s}^2}\left\|\mathbf{X}-\mathbf{X}^{k-1}\right\|_F + \sqrt{2}\left((\nu_1+\nu_2)\tau_2+\nu_2\delta_{3s}\tau_1\right)\left\|\mathbf{E}\right\|_F. \tag{S.46}$$

Step 9 of SI-SSP also solves a least squares problem. Letting $T=S^k$, $\tilde{\mathbf{X}}=\mathbf{X}^k$ and $t=s$, by (S.30), we have

$$\|\mathbf{X}-\mathbf{X}^k\|_F \leq \sqrt{\frac{1}{1-\delta_{2s}^2}}\|\mathbf{X}_{(S^k)^c}\|_F+\frac{\sqrt{1+\delta_s}}{1-\delta_{2s}}\|\mathbf{E}\|_F. \tag{S.47}$$

Finally, combining (S.46) and (S.47) yields

$$\|\mathbf{X}-\mathbf{X}^k\|_F \leq \rho\|\mathbf{X}-\mathbf{X}^{k-1}\|_F+(1-\rho)\tau\|\mathbf{E}\|_F \tag{S.48}$$

where $\rho := \sqrt{2}\delta_{3s}\sqrt{2\nu_1^2\nu_2^2\delta_{3s}^2+\nu_1^2-\nu_1^2\delta_{3s}^2}(1-\delta_{3s}^2)^{-1}$ and $\tau := \sqrt{2}\delta_{3s}\nu_2(\nu_1\sqrt{2(1-\delta_{3s})}+\sqrt{1+\delta_{3s}})(1-\delta_{3s}^2)^{-1/2}(1-\delta_{3s})^{-1}(1-\rho)^{-1}+(\nu_1\nu_2\sqrt{2(1-\delta_{3s})}+\sqrt{1+\delta_{3s}})(1-\delta_{3s})^{-1}$.

We recursively apply (S.48) to obtain

$$\|\mathbf{X}-\mathbf{X}^k\|_F \leq \rho^k\|\mathbf{X}\|_F+\tau\|\mathbf{E}\|_F \tag{S.49}$$

where $\rho<1$ under (S.25). When $k^*=\lceil \log_\rho \frac{\|\mathbf{X}\|_F}{\tau\|\mathbf{E}\|_F}\rceil$, we have $\rho^k\|\mathbf{X}\|_F \leq \tau\|\mathbf{E}\|_F$, and thus the stability result (S.27). $\square$

Appendix A
Proof of Lemma 2

- First, we give a upper bound of $\|\mathbf{X}_{T_\nabla}\|_F$, by Lemma 6, let $\mathbf{Z}=(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S$, we have

$$\left\langle \mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}}), \mathbf{A}^H\mathbf{A}(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle + \left\langle \mathbf{W}_{T_0}\mathbf{E}, \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle = \mathbf{0}. \tag{A.50}$$

Noticing that $\text{supp}(\tilde{\mathbf{X}})\subseteq S$, we have

$$\begin{aligned}
&\|(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\|_F^2\\
=\;& \left\langle \mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}}), (\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle\\
\overset{(A.50)}{=}\;& \left\langle \mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}}), (\mathbf{I}_L-\mathbf{A}^H\mathbf{A})(\mathbf{X}-\tilde{\mathbf{X}})_S\right\rangle\\
-\;& \left\langle \mathbf{W}_{T_0}\mathbf{E}, \mathbf{A}(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\rangle\\
\overset{(7)}{\leq}\;& \omega\delta_{s+t}\left\|(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\right\|_F\left\|\mathbf{X}-\tilde{\mathbf{X}}\right\|_F\\
&+\omega\left\|\mathbf{E}\right\|_F\sqrt{1+\delta_t}\left\|\mathbf{W}_{T_0}(\mathbf{X}-\tilde{\mathbf{X}})_S\right\|_F.
\end{aligned} \tag{A.51}$$

Divide both sides by $\|(\mathbf{W}_{T_0}\mathbf{X}-\mathbf{W}_{T_0}\tilde{\mathbf{X}})_S\|_F$ to obtain (S.29).

- Next, by expanding [Lemma 2, 25] to the MMV model, we could get a relationship between $\|\mathbf{X}-\tilde{\mathbf{X}}\|_F$ and $\|\mathbf{X}_{S^c}\|_F$. We have

$$\|\mathbf{X}-\tilde{\mathbf{X}}\|_F \leq \sqrt{\frac{1}{1-\delta_{s+t}^2}}\|\mathbf{X}_{S^c}\|_F^2+\frac{\sqrt{1+\delta_t}}{1-\delta_{s+t}}\left\|\mathbf{E}\right\|_F. \tag{A.52}$$

- Then, we established the relationship between $\mathbf{X}_{T_\nabla}$ and $\mathbf{X}-\tilde{\mathbf{X}}$. There exist a subset $S'\subseteq S$ and $S'\cap T=\emptyset$. Since T_∇ is defined by the set of indices of the $t-s$ smallest row entries of $\tilde{\mathbf{X}}$, we can conclude that

$$\begin{aligned}
&\|\tilde{\mathbf{X}}_{T_\nabla}\|_F+\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla}\|_F\\
\leq\;& \|\tilde{\mathbf{X}}_{S'}\|_F+\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{S'}\|_F.
\end{aligned} \tag{A.53}$$

By eliminating the contribution from $T_\nabla\cap S'$ and noticing that $S'\cap T=\emptyset$, we have

$$\begin{aligned}
\|\tilde{\mathbf{X}}_{T_\nabla\setminus S'}\|_F \;+\;& \|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla\setminus S'}\|_F\\
\leq\;& \|(\tilde{\mathbf{X}}-\mathbf{X})_{S'\setminus T_\nabla}\|_F\\
+\;& \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X})_{S'\setminus T_\nabla}\|_F.
\end{aligned} \tag{A.54}$$

For the left-hand side of (A.54), we have

$$\begin{aligned}
&\|\tilde{\mathbf{X}}_{T_\nabla\setminus S'}\|_F+\|\mathbf{W}_{T_0}\tilde{\mathbf{X}}_{T_\nabla\setminus S'}\|_F\\
=\;& \|(\tilde{\mathbf{X}}-\mathbf{X}+\mathbf{X})_{T_\nabla\setminus S'}\|_F\\
+\;& \|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X})+\mathbf{W}_{T_0}\mathbf{X})_{T_\nabla\setminus S'}\|_F\\
\geq\;& \|\mathbf{X}_{T_\nabla}\|_F+\|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F \qquad (A.55)\\
-\;& \|(\tilde{\mathbf{X}}-\mathbf{X})_{T_\nabla\setminus S'}\|_F\\
-\;& \|\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X})_{T_\nabla\setminus S'}\|_F.
\end{aligned} \tag{A.56}$$

Finally, combining (A.56) and (A.54), and noticing that

$$(T_\nabla\setminus S')\cap(S'\setminus T_\nabla) = \emptyset \tag{A.57}$$
$$(T_\nabla\setminus S')\cup(S'\setminus T_\nabla) \subseteq T, \tag{A.58}$$

we have

$$\begin{aligned}
&\|\mathbf{X}_{T_\nabla}\|_F+\|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F\\
\leq\;& \|(\tilde{\mathbf{X}}-\mathbf{X})_{T_\nabla\setminus S'}\|_F+\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_{T_\nabla\setminus S'}\|_F\\
+\;& \|(\tilde{\mathbf{X}}-\mathbf{X})_{S'\setminus T_\nabla}\|_F+\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_{S'\setminus T_\nabla}\|_F\\
\leq\;& \sqrt{2}\|(\tilde{\mathbf{X}}-\mathbf{X})_{(S'\setminus T_\nabla)\cup(T_\nabla\setminus S')}\|_F\\
+\;& \sqrt{2}\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_{(S'\setminus T_\nabla)\cup(T_\nabla\setminus S')}\|_F\\
\leq\;& \sqrt{2}\|(\tilde{\mathbf{X}}-\mathbf{X})_S\|_F+\sqrt{2}\|(\mathbf{W}_{T_0}(\tilde{\mathbf{X}}-\mathbf{X}))_S\|_F\\
\overset{(S.29)}{\leq}\;& \sqrt{2}(1+\omega)\delta_{s+t}\|\mathbf{X}-\tilde{\mathbf{X}}\|_F\\
+\;& (1+\omega)\sqrt{2(1+\delta_t)}\|\mathbf{E}\|_F.
\end{aligned} \tag{A.59}$$

Also, we can obtain the relationship between $\|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\|_F$ and $\|\mathbf{X}_{T_\nabla}\|_F$:

$$\eta\omega\sqrt{s_3}\left\|\mathbf{X}_{T_\nabla}\right\|_F \leq \left\|\mathbf{W}_{T_0}\mathbf{X}_{T_\nabla}\right\|_F. \tag{A.60}$$

Combining (A.59) and (A.60), we have

$$\begin{aligned}
\|\mathbf{X}_{T_\nabla}\|_F \;\leq\;& \frac{\sqrt{2}(1+\omega)\delta_{s+t}}{1+\eta\omega\sqrt{s_3}}\|\mathbf{X}-\tilde{\mathbf{X}}\|_F\\
+\;& \frac{(1+\omega)\sqrt{2(1+\delta_t)}}{1+\eta\omega\sqrt{s_3}}\left\|\mathbf{E}\right\|_F.
\end{aligned} \tag{A.61}$$

Noting the definition of ν_2, we complete the proof of Lemma 2.

APPENDIX B
PROOF OF LEMMA 3

Proof: From Step 5 of SI-SSP, we have

$$\mathbf{X}^k_{S_i} = \underset{\boldsymbol{\Theta}:\text{supp}(\boldsymbol{\Theta})=S^k_{S_i}}{\arg\min} \|\mathbf{Y}_{S_i} - \mathbf{A}\boldsymbol{\Theta}\|_F\,. \tag{A.62}$$

From Step 4 of SI-SSP, let $\mathbf{X}^k = [\mathbf{X}^k_{S_1}, \cdots, \mathbf{X}^k_{S_r}]$. We have the following conclusion

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_T\|_F \\ &\leq \|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S}\|_F. \end{aligned} \tag{A.63}$$

By removing the same coordinates $T\cap\Delta S$, we get

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\|_F \\ &\leq \|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F. \end{aligned} \tag{A.64}$$

Because $\text{supp}(\mathbf{X}) = T$ and $\text{supp}(\mathbf{X}^{k-1}) = S^{k-1}$,

$$(\mathbf{X}-\mathbf{X}^{k-1})_{\Delta S\setminus(T\cup S^{k-1})} = 0. \tag{A.65}$$

For the right-hand side of (A.64), we have

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{\Delta S\setminus(T\cup S^{k-1})}\|_F \\ &\leq \|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\ &+ \|(\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\|_F. \end{aligned} \tag{A.66}$$

Note that $\tilde{S}^k = S^{k-1}\cup\Delta S$, we have

$$(\mathbf{X}-\mathbf{X}^{k-1})_{T\setminus\tilde{S}^k} = \mathbf{X}_{(\tilde{S}^k)^c}. \tag{A.67}$$

For the left-hand side of (A.64), we have

$$\begin{aligned} &\|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus\Delta S}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{Y}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\|_F \\ &= \|(\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\|_F \\ &+ \|(\mathbf{W}_{T_0}\mathbf{A}^H(\mathbf{A}\mathbf{X}+\mathbf{E}-\mathbf{A}\mathbf{X}^{k-1}))_{T\setminus(\Delta S\cup S^{k-1})}\|_F \\ &= \|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}) \\ &+ \mathbf{A}^H\mathbf{E}+\mathbf{X})_{\Delta S\setminus(T\cup S^{k-1})}\| \\ &+ \|\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}) \\ &+ \mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E}+\mathbf{W}_{T_0}\mathbf{X})_{\Delta S\setminus(T\cup S^{k-1})}\|_F \\ &\geq \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F \\ &- \|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \\ &- \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{(\tilde{S}^k)^c}\|_F \\ &- \|(\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\|_F - \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{(\tilde{S}^k)^c}\|_F. \end{aligned} \tag{A.68–A.69}$$

Combining (A.70) and (A.69), we have

$$\begin{aligned} &\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F + \|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F \\ &\leq \|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\setminus\tilde{S}^k}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\setminus\tilde{S}^k}\|_F \\ &+ \|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\ &+ \|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{\Delta S\setminus T}\|_F \\ &+ \|(\mathbf{A}^H\mathbf{E})_{T\setminus\tilde{S}^k}\|_F + \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T\setminus\tilde{S}^k}\|_F \\ &+ \|(\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\|_F + \|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{\Delta S\setminus T}\|_F \\ &\leq \sqrt{2}\|((\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\cup\Delta S}\|_F \\ &+ \sqrt{2}\|(\mathbf{W}_{T_0}(\mathbf{A}^H\mathbf{A}-\mathbf{I}_L)(\mathbf{X}-\mathbf{X}^{k-1}))_{T\cup\Delta S}\|_F \\ &+ \sqrt{2}\|(\mathbf{A}^H\mathbf{E})_{T\cup\Delta S}\|_F + \sqrt{2}\|(\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{E})_{T\cup\Delta S}\|_F \\ &\leq \sqrt{2}(1+\omega)\delta_{3s}\|\mathbf{X}-\mathbf{X}^{k-1}\|_F \\ &+ (1+\omega)\sqrt{2(1+\delta_{3s})}\|\mathbf{E}\|_F. \end{aligned} \tag{A.70}$$

We can obtain the relationship between $\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F$ $\|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F$:

$$\eta\omega\sqrt{s_2}\|\mathbf{X}_{(\tilde{S}^k)^c}\|_F \leq \|(\mathbf{W}_{T_0}\mathbf{X})_{(\tilde{S}^k)^c}\|_F. \tag{A.71}$$

Combining (A.70) and (A.71), we have

$$\begin{aligned} \|\mathbf{X}_{(\tilde{S}^k)^c}\|_F &\leq \frac{\sqrt{2}(1+\omega)\delta_{3s}}{1+\eta\omega\sqrt{s_2}}\|\mathbf{X}-\tilde{\mathbf{X}}\|_F \\ &+\frac{(1+\omega)\sqrt{2(1+\delta_{2s})}}{1+\eta\omega\sqrt{s_2}}\|\mathbf{E}\|_F. \end{aligned} \tag{A.72}$$

Noting the definition of ν_1, we complete the proof of Lemma 3.

APPENDIX C
PROOF OF LEMMA 4

Proof: the RIC δ_t can be expressed as [25]

$$\delta_t = \max_{S\subseteq\{1,2,\ldots,N\},|S|\leq t}\|\mathbf{A}^*_S\mathbf{A}_S-\mathbf{I}\|_{2\to2}\,, \tag{A.73}$$

where

$$\|\mathbf{A}^*_S\mathbf{A}_S-\mathbf{I}\|_{2\to2} = \sup_{\mathbf{a}\in\mathbb{R}^{|S|}\setminus\{\mathbf{0}\}}\frac{\|(\mathbf{A}^*_S\mathbf{A}_S-\mathbf{I})\,\mathbf{a}\|_2}{\|\mathbf{a}\|_2}. \tag{A.74}$$

Let $S = \text{supp}(\mathbf{u})\cup\text{supp}(\mathbf{v})$, then $|S|\leq t$. Let $\mathbf{u}_{|S}, \mathbf{v}_{|S}$ denote respectively the S-dimensional sub-vectors of $\mathbf{u}$ and $\mathbf{v}$ obtained by only keeping the components indexed by S. We have

$$\begin{aligned} &|\langle\mathbf{u},\left(\mathbf{W}_{T_0}-\mathbf{W}_{T_0}\mathbf{A}^H\mathbf{A}\right)\mathbf{v}\rangle| \\ =\; &|\langle\mathbf{W}_{T_0}\mathbf{u},\mathbf{v}\rangle - \langle\mathbf{A}\mathbf{W}_{T_0}\mathbf{u},\mathbf{A}\mathbf{v}\rangle| \\ =\; &|\langle\mathbf{W}_{T_0}\mathbf{u}_{|S},\left(\mathbf{I}_L-\mathbf{A}^H_S\mathbf{A}_S\right)\mathbf{v}_{|S}\rangle| \\ \leq\; &\|\mathbf{W}_{T_0}\mathbf{u}_{|S}\|_2\left\|\left(\mathbf{I}_L-\mathbf{A}^H_S\mathbf{A}_S\right)\mathbf{v}_{|S}\right\|_2 \\ \overset{(A.74)}{\leq}\; &\|\mathbf{W}_{T_0}\mathbf{u}_{|S}\|_2\|\mathbf{I}_L-\mathbf{A}^H_S\mathbf{A}_S\|_{2\to2}\|\mathbf{v}_{|S}\|_2 \\ \overset{(A.73)}{\leq}\; &\omega\delta_t\|\mathbf{u}_{|T}\|_2\|\mathbf{v}_{|S}\|_2 \end{aligned}$$

$$= \quad \omega\delta_t \|\mathbf{u}\|_2 \|\mathbf{v}\|_2, \tag{A.75}$$

moreover, we have

$$\begin{aligned}
& \| \left(\left(\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A} \right) \mathbf{v} \right)_U \|_2^2 \\
= \quad & \langle \left(\left(\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A} \right) \mathbf{v} \right)_U , \\
& \left(\mathbf{W}_{T_0} - \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{A} \right) \mathbf{v} \rangle \\
\overset{(S.37)}{\leq} \quad & \delta_t \left\| \left(\left(\mathbf{W}_{T_0} - \mathbf{A}^H \mathbf{A} \right) \mathbf{v} \right)_U \right\|_2 \|\mathbf{v}\|_2
\end{aligned} \tag{A.76}$$

which completes the proof of Lemma 4.

Appendix D
Proof of Lemma 5

Proof: The lemma follows trivially from the fact that

$$\begin{aligned}
& \left\| \left(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e} \right)_U \right\|_2^2 \\
= & \left\langle \mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e}, \left(\mathbf{W}_{T_0} \mathbf{A}^H \mathbf{e} \right)_U \right\rangle \\
= & \left\langle \mathbf{e}, \mathbf{W}_{T_0} \mathbf{A} \left(\left(\mathbf{A}^H \mathbf{e} \right)_U \right) \right\rangle \\
\leq & \|\mathbf{e}\|_2 \left\| \mathbf{W}_{T_0} \mathbf{A} \left(\left(\mathbf{A}^H \mathbf{e} \right)_U \right) \right\|_2 \\
\overset{(7)}{\leq} & \|\mathbf{e}'\|_2 \, \omega \sqrt{1 + \delta_u} \left\| \left(\mathbf{A}^H \mathbf{e} \right)_U \right\|_2 .
\end{aligned} \tag{A.77}$$

Appendix E
Proof of Lemma 6

Proof: Due to the orthogonality, the residue $\mathbf{Y} - \mathbf{A}\tilde{\mathbf{X}}$ is orthogonal to the space $\mathbf{AZ}$. This means that for all $\mathbf{Z} \in \mathbb{C}^{L \times N}$ with $\mathrm{supp}(\mathbf{Z}) \subseteq S$,

$$\langle \mathbf{A}(\mathbf{Y} - \mathbf{A}\tilde{\mathbf{X}}), \mathbf{Z} \rangle = \mathbf{0}. \tag{A.78}$$

Let $\tilde{\mathbf{X}}'$ be the solution of the least squares problem $\arg\min_{\mathbf{Z}} \{ \|\mathbf{Y}' - \mathbf{AZ}\|_F , \mathrm{supp}(\mathbf{Z}) \subseteq S \}$, where $\mathbf{Y}' = \frac{\mathbf{A}\mathbf{W}_{T_0}\mathbf{X}_{T_0}}{\omega} + \mathbf{E}$. We have

$$\tilde{\mathbf{X}}' = \frac{\mathbf{W}_{T_0} \tilde{\mathbf{X}}}{\omega}. \tag{A.79}$$

Then, by (A.78), we have

$$\begin{aligned}
\mathbf{0} \quad = \quad & \left\langle \frac{\mathbf{A}\mathbf{W}_{T_0}\mathbf{X}_{T_0}}{\omega} + \mathbf{E} - \mathbf{A} \frac{\mathbf{W}_{T_0} \tilde{X}}{\omega}, \mathbf{AZ} \right\rangle \\
= \quad & \left\langle \mathbf{W}_{T_0} \mathbf{X} - \mathbf{W}_{T_0} \tilde{\mathbf{X}}, \mathbf{A}^H \mathbf{AZ} \right\rangle + \omega \left\langle \mathbf{E}, \mathbf{AZ} \right\rangle .
\end{aligned} \tag{A.80}$$