# REAL ANALYSIS: ASSIGNMENT 2

DUE DATE: 2020/05/08

Unless otherwise specified, we assume that the space $\mathbb{R}^d$ has the measure space structure constructed in Theorem 2.20 (throught all the assignments). In addition, all the measurable subset $E \subset \mathbb{R}^d$ has the measure space structure constructed in Problem 3 of the recommending problems below.

In problems **1** and **2**, we prove that the Lebesgue integral is an extension of Riemann integral.

**1.** Let us consider the following function $f : [0,\ 1] \to \mathbb{R}$:

$$f(x) = \begin{cases} x^2 & \text{if } x \in [0,\ 1] \setminus \mathbb{Q} \\ 0 & \text{if } x \in [0,\ 1] \cap \mathbb{Q}. \end{cases}$$

Here, $\mathbb{Q}$ denotes the set of rational numbers.

**(1)** Prove that $f$ is **not** Riemann-integrable on $[0,\ 1]$. For the definition of Riemann integrability, I would like to mention the following link as a possible reference:
https://en.wikipedia.org/wiki/Riemann_integral
One can also check the undergraduate textbook by Kim, Kim and Kye.

**(2)** Prove that $f \in L^1([0,\ 1])$ (cf. Problem 3 of Recommending problems part) and compute $\int_0^1 f(x)dx$.

**2.** Let $f : [0,\ 1] \to \mathbb{R}$ be a Riemann integrable function. Prove that $f \in L^1([0,\ 1])$ and prove that the Riemann integral and Lebesgue integral of $f$ on $[0,\ 1]$ are same.

**3.** Let $X$ be a locally compact, Hausdorff, and $\sigma$-compact space, and denote by $(X,\ \mathfrak{M},\ \mu)$ the measure space constructed via Riesz representation theorem (for a certain positive functional $\Lambda$). Prove or disprove that arbitrary set $E \in \mathfrak{M}$ is both inner and outer regular. (We proved that all the Borel sets are both inner and outer regular.)

**4.** Review the definitions of lower- and upper-semicontinuity, and solve Exercise 1 of Chapter 2 of Textbook.

**5.** Exercise 3 of Chapter 2 of Textbook.

**6.** Exercise 5 of Chapter 2 (You can find the definition Cantor's set at: https://en.wikipedia.org/wiki/Cantor_set).

**7.** Exercise 17 of Chapter 2.

## Recommending Problems

**1.** Read and understand Section 2.23

**2.** Read and understand Section 2.25

**3.** (This problem is the one mentioned in the last lecture of chapter 2. This problem is trivial but enable us to use the notation $L^1(E)$ without any ambiguity.) Let $E \subset \mathbb{R}^d$ be a non-empty measurable function. Define $\mathfrak{M}_E$ as a collection of subsets of $E$ belonging to $\mathfrak{M}$. Define a measure $m_E$ on $\mathfrak{M}_E$ as $m_E(A) = m(A)$ for For $A \in \mathfrak{M}_E$. Answer the following questions.

**(1)** Prove that $(E, \mathfrak{M}_E)$ is a measurable space.

**(2)** Prove that $(E, \mathfrak{M}_E, m_E)$ is a measure space.

**(3)** We will say that $f : E \to \mathbb{R}$ (or $[-\infty, \infty]$) is measurable if $f^{-1}(A) \in \mathfrak{M}$. Check that $f$ is measurable if and only if it is measurable with respect to $\mathfrak{M}_E$.

**(4)** For measurable $f : E \to \mathbb{R}$, we write $f \in L^1(E)$ if $\int_E |f| dm < \infty$. Check that $f \in L^1(E)$ if and only if $f \in L^1(m_E)$

**4.** Solve Problem 2 of the required problem above for the following functions.

**(1)** Riemann integrable function $f : (0, 1) \to \mathbb{R}$. Here the Riemann integrability is defined by means of indefinite integral.

**(2)** $f \in C_c(\mathbb{R}^d)$

**(3)** Riemann integrable function $f : U \to \mathbb{R}$ for any domain $U \subset \mathbb{R}^d$.

**5.** Exercises 8, 9, 14, 15, 20, 21, 23 of Chapter 2.