Modeling Cyberphysical Systems

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Verifying cyberphysical systems

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Map of CPS models

Discrete transition systems, automata

Dynamical systems
Differential inclusions

Hybrid systems

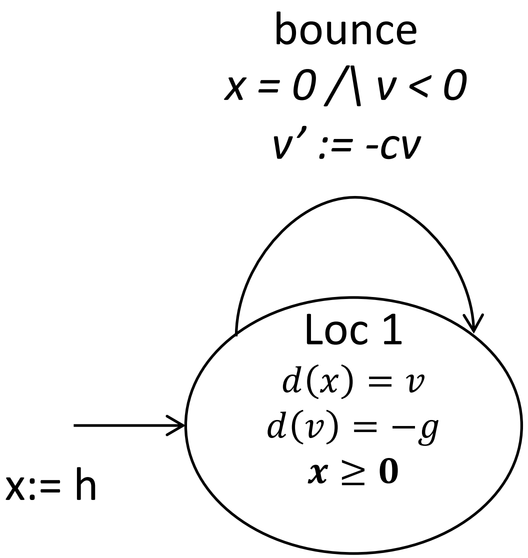
Markov chains

Probabilistic automata, Markov decision processes (MDP)

Continuous time, continuous state MDPs

Stochastic Hybrid systems

Bouncing Ball: Hello world of CPS



Graphical Representation used in many articles

automaton Bouncingball(c,h,g)

variables: x: Reals := h, v: Reals := 0

actions: bounce

transitions:

bounce

pre *x = 0 /\ v < 0*

eff v := -cv

trajectories:

Loc1

evolve d(x) = v; d(v) = -g

invariant $\boldsymbol{x} \geq \mathbf{0}$

mode invariant, not to be confused with invariants of the automaton

Recall from Lecture 1. language defines an automaton

An **automaton** is a tuple $\mathcal{A} = \langle X, \Theta, A, \mathcal{D} \rangle$ where

- X is a set of names of variables; each variable $x \in X$ is associated with a type, $type(x)$
  - A **valuation** for X maps each variable in X to its type
  - Set of all valuations:$val(X)$ this is sometimes identified as the **state space** of the automaton
- $\Theta \subseteq val(X)$ is the set of **initial** or **start states**
- A is a set of names of **actions** or **labels**
- $\mathcal{D} \subseteq val(X) \times A \times val(X)$ is the set of **transitions**
  - a transition is a triple (u, a, u')
  - We write it as $u \rightarrow_a u'$

```
automaton DijkstraTR(N:Nat, K:Nat), where K > N
 type ID: enumeration [0,...,N-1]
 type Val: enumeration [0,...,K]
 actions
    update(i:ID)
 variables
    x:[ID -> Val]
  transitions
    update(i:ID)
      pre i = 0 /\ x[i] = x[N-1]
      eff x[i] := (x[i] + 1) %  K

     update(i:ID)
       pre i >0  /\ x[i] ~= x[i-1]
       eff x[i] := x[i-1]
```

Trajectories

Given a set of variables X and a time interval J which can be of the form $[0, T], [0, T) or\ [0, \infty)$, a ***trajectory*** for X is a function $\tau\colon\ J \rightarrow val(X)$

We will specify τ as solutions of differential equations

The **first state** of a trajectory $\tau.fstate := \tau(0)$

If τ is right closed then the **limit state** of a trajectory $\tau.lstate = \tau(T)$

If τ is finite then **duration** of τ is $\tau.dur = T$

The domain of $\tau.dom = J$

A **point trajectory** is a trajectory with $\tau.dom = [0,0]$

Operations on trajectories: prefix, suffix, concatenation

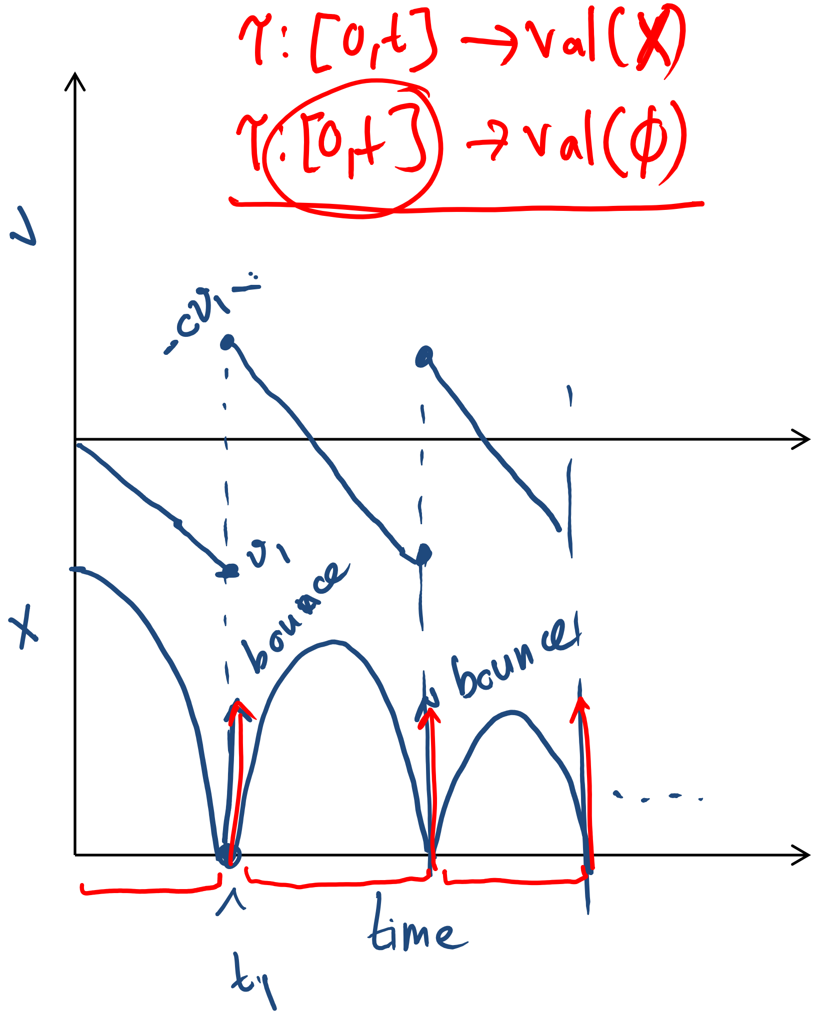
Hybrid Automaton

$\boldsymbol{\mathcal{A}} = (X, \Theta, A, \mathcal{D}, \mathcal{T})$

- X: set of **state variables**
  - $Q \subseteq val(X)$ set of **states**
- $\Theta \subseteq Q$ set of **start states**
- set of **actions**, A= E ∪ H
- $\mathcal{D} \subseteq Q \times A \times Q$
- $\mathcal{T}$: set of **trajectories** for X which is closed under **prefix, suffix, and concatenation**

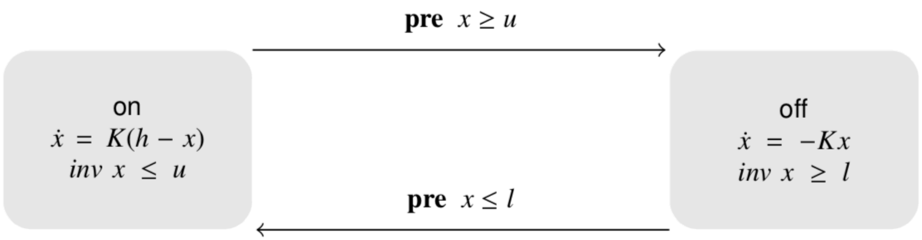
Semantics: Executions and Traces

- An *execution fragment* of $\mathcal{A}$ is an (possibly infinite) alternating (A, X)-sequence $\alpha = \tau_0\ a_1\ \tau_1 a_2 \tau_2\ \ldots$ where
  - $\forall$ i, $\tau_i.lstate \overset{a_{i+1}}{\longrightarrow} \tau_{i+1}.fstate$



- If τ_0.fstate ∈ Θ then α is an **execution**
- **Execs**$_{\mathcal{A}}$ set of all executions
- The first state of an execution α is $\alpha.fstate = \tau_0.fstate$
- If the execution α is **finite and closed** $\tau_0\ a_1\ \tau_1 a_2 \tau_2\ \ldots \tau_k$ then $\alpha.lstate = \tau_k.lstate$
- A state $\boldsymbol{x}$ is reachable if there exists an execution α with $\alpha.lstate = \boldsymbol{x}$

Example 2: Thermostat



automaton Thermostat(*u*, *l*, *K*, *h* : Real) where *u* > *l*
type *Status* enumeration [*on*, *off*]
actions
turnOn; turnOff;

variables
x: Real := *l* *loc*: *Status* := *on*
transitions
turnOn
pre *x*≤*l* ∧ *loc*=*off*
eff *loc* := *on*

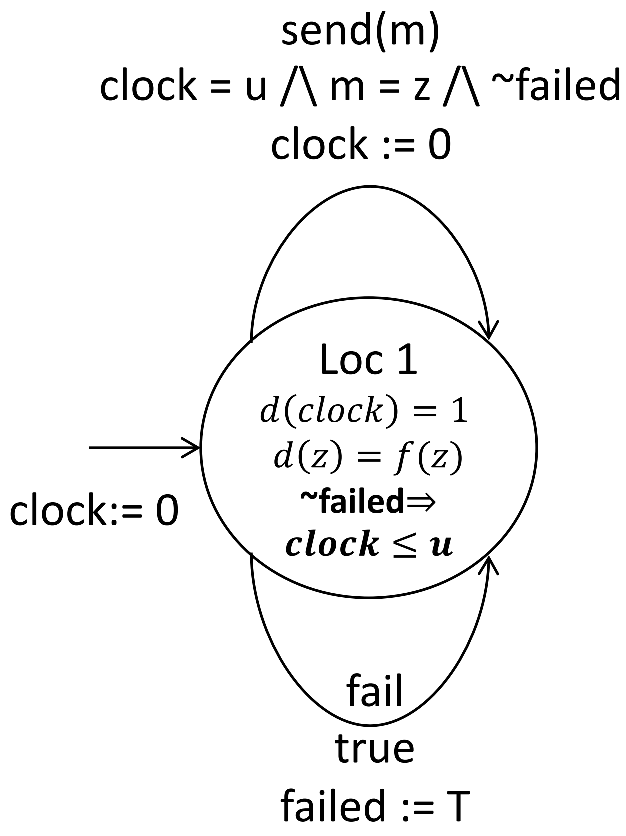
turnOff
pre *x*≥*u* ∧ *loc*=*on*
eff *loc* := *off*

trajectories
modeOn
evolve *d*(*x*) = *K*(*h* − *x*)
invariant *loc* = *on* ∧ *x* ≤ *u*

modeOff
evolve *d*(*x*) = −*Kx*
invariant *loc* = *off* ∧ *x* ≥ *l*

Determinism vs nondeterminism
mode invariants

Another Example: Periodically Sending Process



Automaton PeriodicSend(u)

  variables: analog

    clock: Reals := 0, z:Reals, failed:Boolean := F

  actions: send(m:Reals), fail

  transitions:

    send(m)

      pre clock = u /\ m = z /\ ~failed

      eff clock := 0

    fail

      pre true

      eff failed := T

  trajectories:

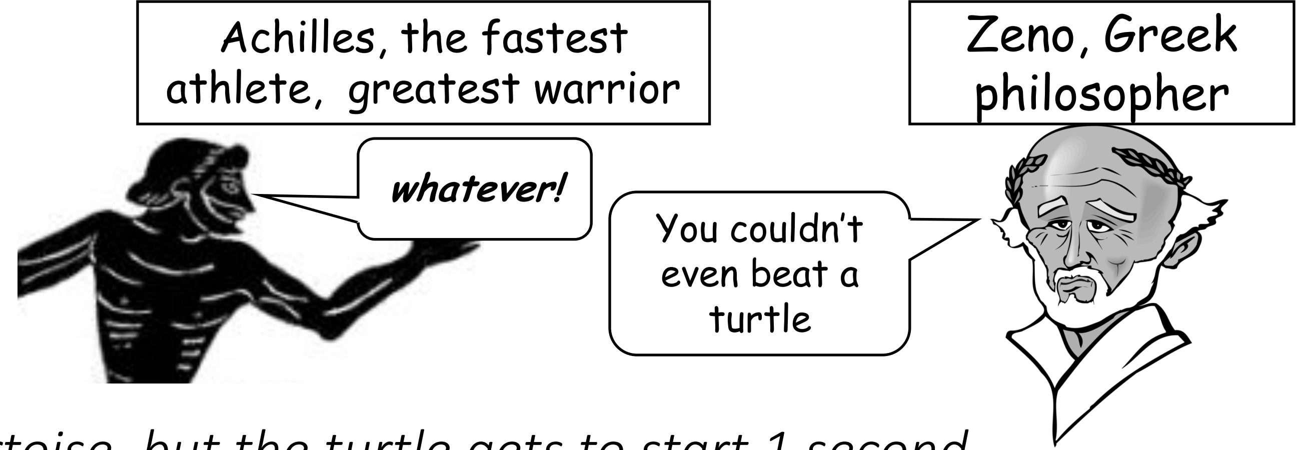
    evolve d(clock) = 1, d(z) = f(z)

    invariant ~failed \/ clock<=u

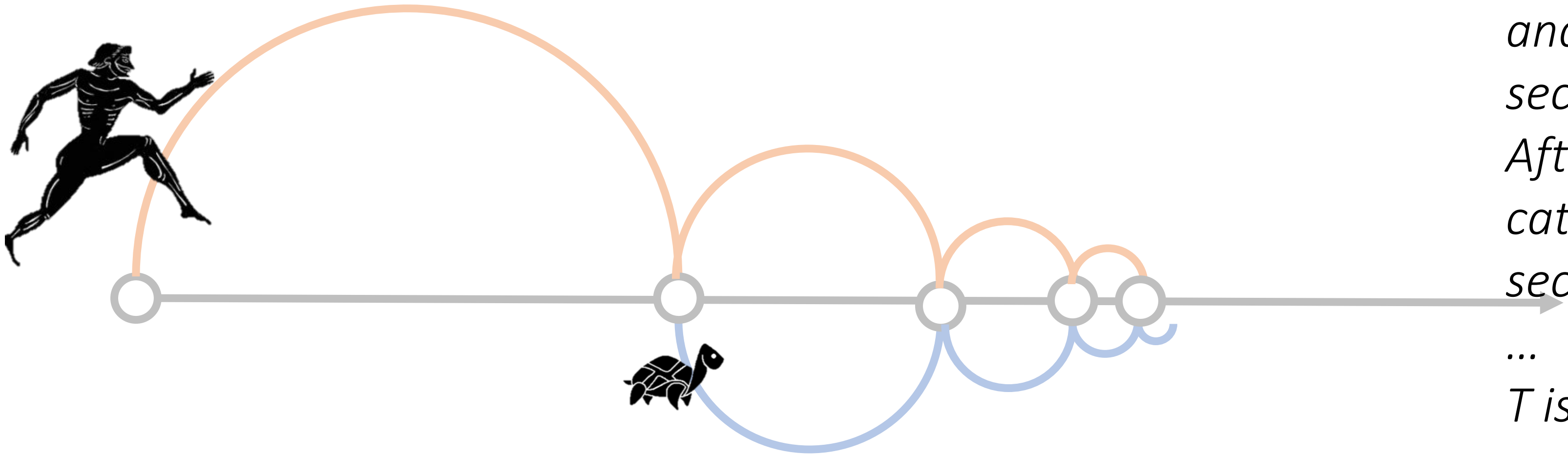
Special kinds of executions

- **Infinite**: Infinite sequence of transitions and trajectories $\tau_0\ a_1\ \tau_1 a_2 \tau_2\ \ldots$
- **Closed**: Finite with final trajectory with closed domain $\tau_0\ a_1\ \tau_1 a_2 \tau_2\ \ldots \tau_k$ and $\tau_k.dom = [0, T]$
- **Admissible**: Infinite duration
  - May or may not be infinite
  - $\tau_0\ a_1\ \tau_1 a_2 \tau_2\ \ldots$
  - $\tau_0\ a_1\ \tau_1 a_2 \tau_2\ \ldots \tau_k$ with $\tau_k.dom = [0, \infty)$
- **Zeno**: Infinite but not admissable
  - Infinite number of transitions in finite time

Zeno's Paradox



Achilles runs 10 times faster than than the tortoise, but the turtle gets to start 1 second earlier. Can Achilles ever catch Turtle?



After 1/10th of a second, Achilles reaches where the Turtle (T) started, and T has a head start of 1/10th second.

After another 1/100th of a second, A catches up to where T was at t=1/10 sec, but T has a head start of 1/100th

…

T is always ahead …

Lesson: Mixing discrete transitions with continuous motion can be tricky!