Reachability analysis

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Next few lectures

Focus on specific classes of hybrid automata for which safety properties (invariants) can be verified completely automatically

- Finite state machines
- Alur-Dill's Timed Automata[1] (Today)
- Rectangular initializaed hybrid automata
- Linear hybrid automata
- …

We will introduce ***abstractions:*** Simplifying or approximating one automaton A with another automaton B

[1] Rajeev Alur et al. The Algorithmic Analysis ofHybrid Systems. Theoretical Computer Science, volume 138, pages 3-34, 1995.

Today

- Finite state machines
- Algorithmic analysis of (Alur-Dill's) Timed Automata[1]
  - A restricted class of what we call hybrid automata in this course with only clock variables

[1] Rajeev Alur and David L. Dill. A theory of timed automata. Theoretical Computer Science, 126:183-235, 1994.

Reachability of Finite Automata

An **finite automaton** is a tuple $\mathcal{A} = \langle Q, \mathrm{Q}_0, \mathcal{D} \rangle$ where

- Q is a finite set of states
- $Q_0 \subseteq Q$ is the set of **initial** or **start states**
- $\mathcal{D} \subseteq Q \times Q$ is the set of **transitions**

An *execution* of $\mathcal{A}$ is an alternating sequence of states and actions $\alpha = q_0 q_1 q_2 \dots q_k$ such that:

1. $q_0 \in \mathrm{Q}_0$ 2. $\forall\, i$ in the sequence, $(q_i, q_{i+1}) \in \mathcal{D}$

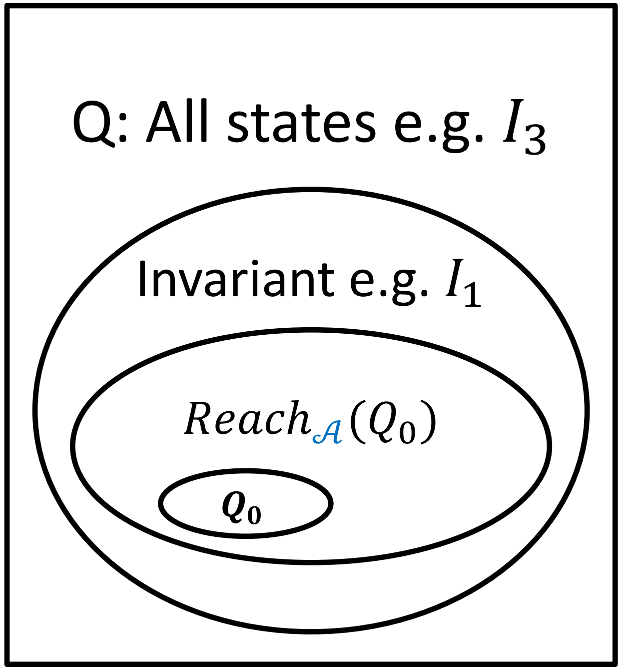
A state $\boldsymbol{u}$ is *reachable* if there exists an execution α such that $\alpha.lstate = q_k = \boldsymbol{u}$

Reachability in finite state machines

$Reach_{\mathcal{A}}(\Theta)$: set of states reachable from Θ by automaton $\mathcal{A}$

An *invariant* is a set of states I such that $Reach_{\mathcal{A}} \subseteq I$

How to check whether $\boldsymbol{u}$ is *reachable* ?



Reachability as graph search

Q1. Given $\mathcal{A}$, is a state $\boldsymbol{u} \in Q$ reachable?
Define a graph $G_{\mathcal{A}} = \langle V, E \rangle$ where

$V = Q$

$E = \{(q, q') | q \rightarrow q'\}$

Q2. Does there exist a path in $G_{\mathcal{A}}$ from any state in Θ to u ?

Perform Depth First or Breadth First Search on $G_{\mathcal{A}}$ from Q_0

Time complexity of BFS $O(| Q | + |D|)$
Space complexity is $O(| Q |)$

Nondeterministic reachability

Input: G = (V, E), $Q_0, U \subseteq V$

$n := |V|$

vcurrent := **choose** Q_0

If vcurrent ∈ T return ‘‘yes”

Else For i = 1 to n:

  vnext := **choose** V

  If (vcurrent, vnext) ∉E break

  If vnext ∈ T **return** ‘‘yes”

  vcurrent := vnext

Return ‘‘no”

Requires only O(log |Q|) bits of memory

Using Savitch’s construction we get a deterministic algorithm that uses O($\log^2$|Q|) bits

Adding Clocks and Clock Constraints

- A clock variable x is a continuous (analog) variable of type real such that along any trajectory τ of x, for all t ∈ $\tau.dom, (\tau \downarrow x)(t) = t$.

- For a set X of clock variables, the set Φ(X) of integral clock constraints are expressions defined by the syntax:

  g ::= x $\leq q \mid x \geq q \mid \neg\, g \;\mid g_1 \wedge \;\; g_2$

  where $x \in X \; and \; q \in \; \mathbb{Z}$

- Examples: x = 10; x ∈ [2, 5); true are valid clock constraints
- What do clock constraints look like?

- Semantics of clock constraints $[g]$

Integral Timed Automata

Definition. A integral timed automaton is a HIOA $\mathcal{A} = \langle V, \Theta, A, \mathcal{D}, \mathcal{T} \rangle$ where

- V = X ∪ $\{l\}$, X is a set of n clocks and l is a discrete state variable of finite type L; **stata space** $val(X) \times L$
- A is a finite set
- $\mathcal{D}$ is a set of transitions such that
  - The guards are described by clock constrains $\Phi(X)$
  - $\langle x, l \rangle - a \rightarrow \langle x', l' \rangle$ implies either $x' = x$ or $x = 0$
- $\mathcal{T}$ set of clock trajectories for the clock variables in X

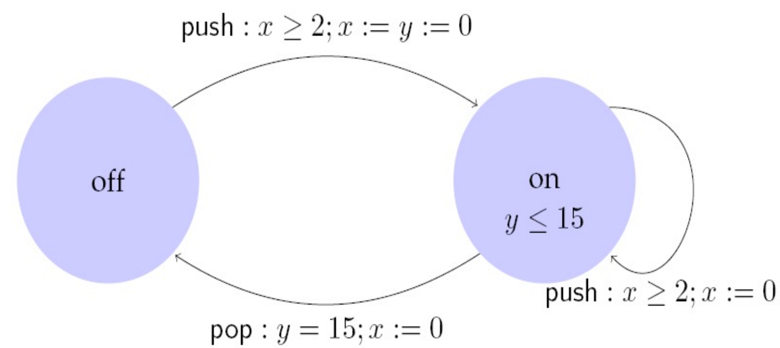
Example: Light switch

Math Formulation
automaton Switch
    variables
    internal x, y:Real := 0, loc: {on,off} := off

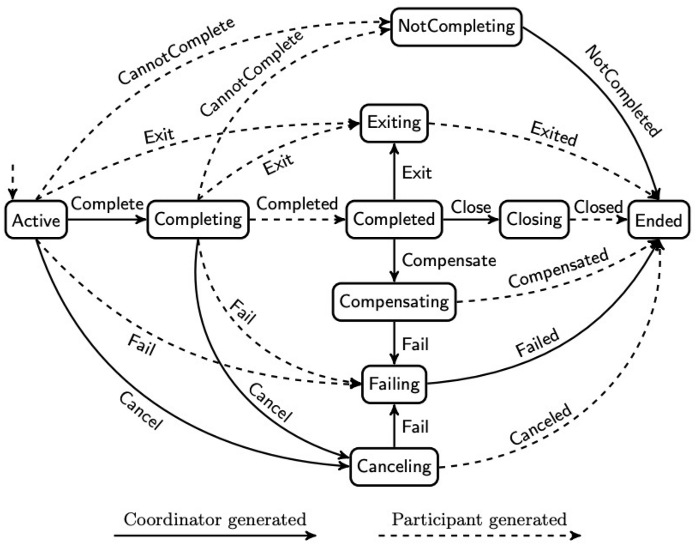
    transitions
    internal push
        pre x $\geq$ 2
        eff if loc = on **then** x := 0
          else x,y := 0; loc := off
    internal pop
        pre y = 15 $\wedge$ loc = off
        eff x := 0

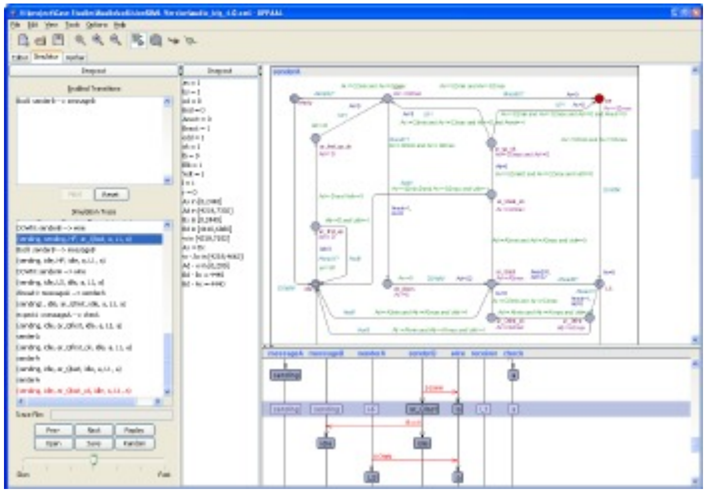
    trajectories
        invariant loc = off => y $\leq$ 15
        evolve d(x) = 1; d(y) = 1

Description
Switch can be turned on whenever at least 2 time units have elapsed since the last turn on. Switches off automatically 15 time units after the last on.



Timed Automaton application in Web Services (WS)





WS-Coordination describes a framework for coordinating transactional web services

Network protocol described in state tables

600+ lines of C-like code in the protocol model

Modeled and Verified using the UPPAAL tool

Analysis considers different channel models

The main safety property: protocol **does not enter invalid state**

Property violated in all but the FIFO channel model

Modelling and Verification of Web Services Business Activity Protocol Anders P. Ravn, Jiri Srba, and Saleem Vighio, RV 2010

Control State (Location) Reachability Problem

- Given an ITA $\mathcal{A}$, check if a particular (discrete) control state $l^* \in L$ is **reachable** from the initial states
- Why is control state reachability (CSR) good enough even if we are interested in checking reachability of $X^* \subseteq val(X)$?
- This problem is **decidable** [Alur Dill]
- That is, there is an algorithm that takes in $\mathcal{A}, l^*$ and terminates with the correct answer.
- Key idea:
  - Construct a finite automaton FA that is a ***time-abstract bisimilar*** to the ITA
  - That is, FA behaves identically to ITA w.r.t. control state reachability, but does not preserve timing information
  - Check reachability of FSM

An equivalence relation with a finite quotient

Under what conditions do two states x_1 and x_2 of the automaton $\mathcal{A}$ behave identically with respect to control state reachability (CSR)?

- When do they satisfy the same set of clock constraints?
- When would they continue to satisfy the same set of clock constraints?

$x_1.loc = x_2.loc$ and

x_1 and x_2 satisfy the same set of clock constraints

- For each clock y int($x_1.y$) = int($x_2.y$) or int($x_1.y) \geq c_{\mathcal{A}y}$ and int($x_2.y) \geq c_{\mathcal{A}y}$. ($c_{\mathcal{A}y}$ **is the maxium clock guard** of y)
- For each clock y with $x_1.y \leq c_{\mathcal{A}y}$, frac($x_1.y) = 0$ iff frac($x_2.y) = 0$
- For any two clocks y and z with $x_1.y \leq c_{\mathcal{A}y}$ and $x_1.z \leq c_{\mathcal{A}z}$, frac($x_1.y$) $\leq$ frac($x_1.z$) iff frac($x_2.y$) $\leq$ frac($x_2.z$)

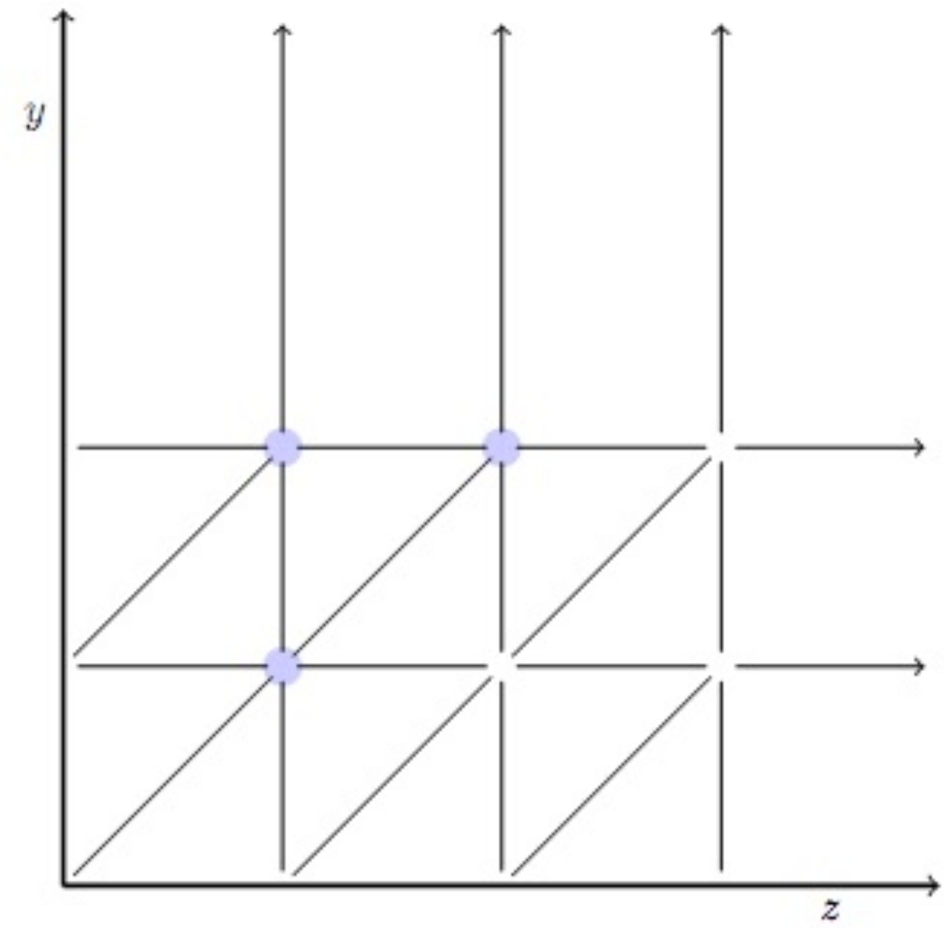
Lemma. This is a **equivalence relation** on *val(V)* the states of $\mathcal{A}$

The partition of *val(V)* induced by this relation is are called **clock regions**

What do the clock regions look like?

Example of
Two Clocks

X = {y,z}
$c_{\mathcal{A}y}$ = 2
$c_{\mathcal{A}z}$ = 3



Complexity

- Lemma. The number of clock regions is bounded by $|X|!\,2^{|X|}\prod_{z\in X}(2c_{\mathcal{A}z}+2)$.

Region automaton $\mathrm{R}(\mathcal{A})$

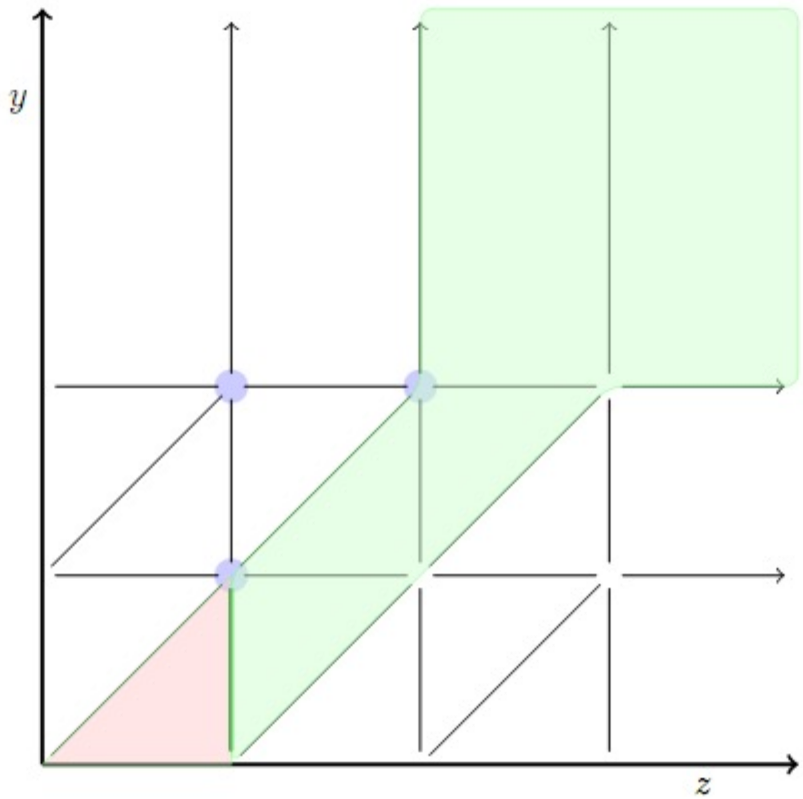
Given an ITA $\mathcal{A} = \langle V, \Theta, \mathcal{D}, \mathcal{T} \rangle$, we construct the corresponding **Region Automaton** $\mathrm{R}(\mathcal{A}) = \langle Q_R, \Theta_R, D_R \rangle$ such that (i) $\mathrm{R}(\mathcal{A})$ visits the same set of locations (but does not have timing information) and (ii) $\mathrm{R}(\mathcal{A})$ is finite state machine.

- ITA (clock constants) defines a set of clock regions, say $\mathrm{C}_{\mathcal{A}}$. The set of states $Q_R = C_{\mathcal{A}} \times L$
- $Q_0 \subseteq Q$is the set of states contain initial set Θ of $\mathcal{A}$
- D: We add the transitions between Q (regions)
  - **Time successors**: Consider two clock regions γ and γ', we say that γ' is a time successor of γ if there exits a trajectory of ITA starting from γ that ends in γ'
  - **Discrete transitions**: Same as the ITA

Theorem. A location of ITA $\mathcal{A}$ is reachable iff it is also reachable in $\mathrm{R}(\mathcal{A})$.

(we say that $\mathrm{R}(\mathcal{A})$ is *time abstract bisimilar* to $\mathcal{A}$)

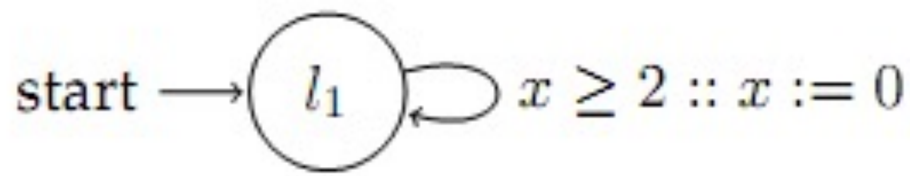
Time successors



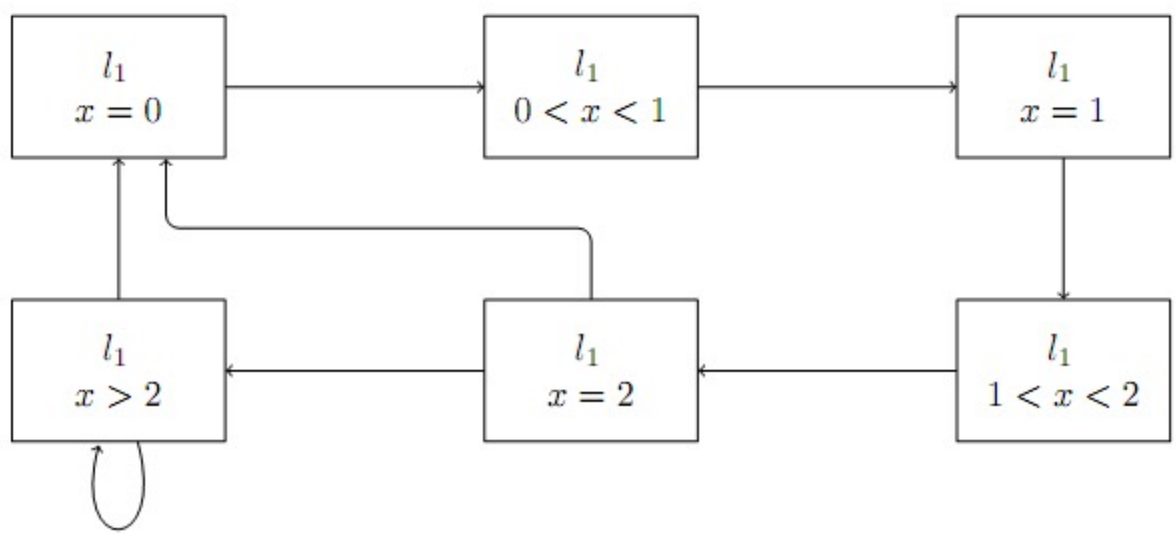
The clock regions in blue are time successors of the clock region in red.

Example 1: Region Automata

ITA

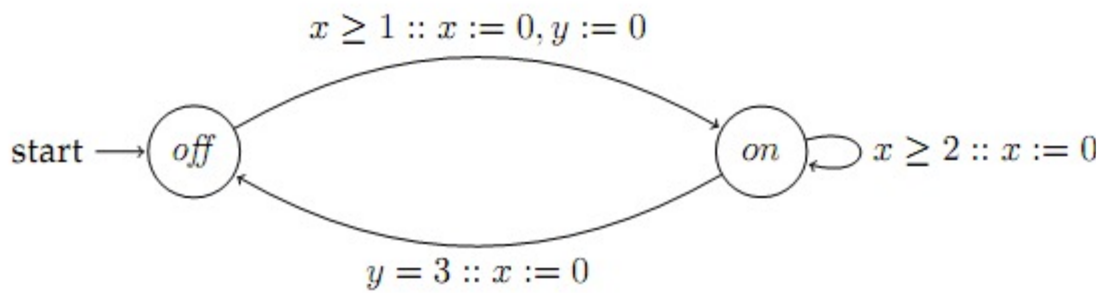


Corresponding FA

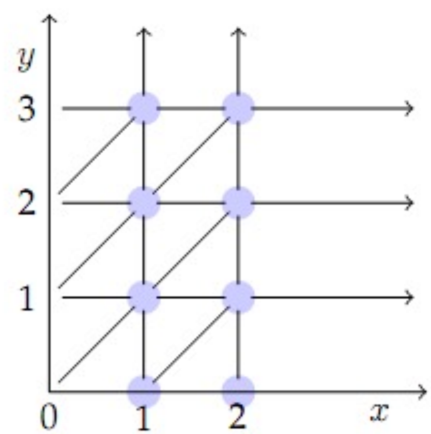


Example 2

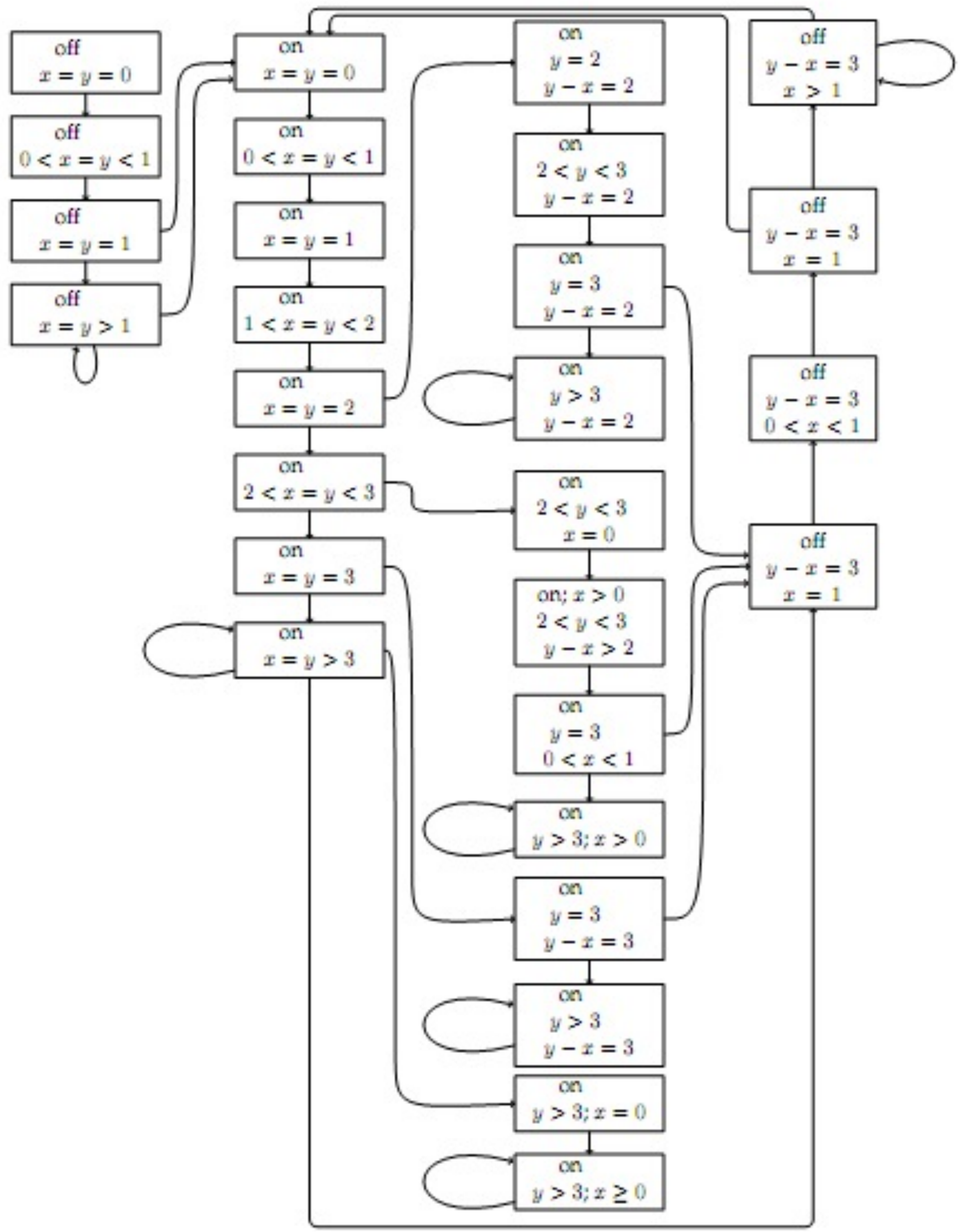
ITA



Clock Regions



Corresponding FA



$|X|!\,2^{|X|}\prod_{z\in X}(2c_{\mathcal{A}z} + 2)$

Drastically increasing with the number of clocks

Special Classes of Hybrid Automata

- Timed Automata ←
- Rational time automata
- Multirate automata
- Rectangular Initialized HA
- Rectangular HA
- Linear HA
- Nonlinear HA

Clocks and **Rational** Clock Constraints

- A **clock variable** x is a continuous (analog) variable of type real such that along any trajectory τ of x, for all t ∈ $\tau.dom, (\tau \downarrow x)(t) = t$.

- For a set X of clock variables, the set Φ(X) of ***rational*** **clock constraints** are expressions defined by the syntax:

  g ::= x $\leq q \mid x \geq q \mid \neg\, g \mid g_1 \wedge g_2$

  where $x \in X$ and $q \in \mathbb{Q}$

- Examples: x = 10.125; x ∈ [2.99, 5); true are valid rational clock constraints

- Semantics of clock constraints $[g]$

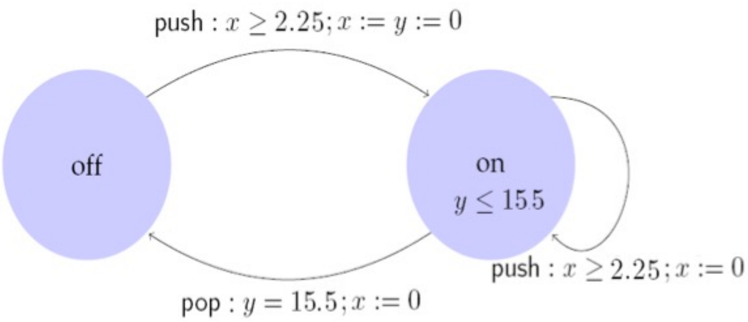
Step 1. Rational Timed Automata

- **Definition.** A *rational timed automaton* is a HA $\boldsymbol{\mathcal{A}} = \langle V, \Theta, A, \mathcal{D}, \mathcal{T} \rangle$ where
  - V = X ∪ $\{loc\}$, where X is a set of n clocks and l is a discrete state variable of finite type Ł
  - A is a finite set
  - $\mathcal{D}$ is a set of transitions such that
    - The guards are described by rational clock constraings $\Phi(X)$
    - $\langle x, l \rangle - a \rightarrow \langle x', l' \rangle$ implies either $x' = x$ or $x = 0$
  - $\mathcal{T}$ set of clock trajectories for the clock variables in X

Example: Rational Light switch

Switch can be turned on whenever at least 2.25 time units have elapsed since the last turn off or on. Switches off automatically 15.5 time units after the last on.

automaton Switch
  internal push; pop
    variables
      internal x, y:Real := 0, loc:{on,off} := off
    transitions
      push
        pre x >=2.25
        eff if loc = on **then** y := 0 **fi;** x := 0; loc := off
      pop
        pre y = 15.5 ∧ loc = off
        eff x := 0
    trajectories
      invariant loc = on ∨ loc = off
      stop when y = 15.5 ∧ loc = off
      evolve d(x) = 1; d(y) = 1



Control State (Location) Reachability Problem

- Given an RTA, check if a particular location is reachable from the initial states
- Is problem decidable?
- Yes
- Key idea:
  - Construct a ITA that is time-abstract bisimilar to the given RTA
  - Check CSR for ITA

Construction of ITA from RTA

- Multiply all rational constants by a factor q that make them integral
- Make d(x) = q for all the clocks

- RTA Switch is bisimilar to ITA Iswitch

- Simulation relation R is given by
- (**u**,**s**) $\in R$ iff **u**.x = 4 **s**.x and **u**.y = 4 **s**.y

automaton ISwitch
internal push; pop
variables
  internal x, y:Real := 0, loc:{on,off} := off
transitions
  push
    pre x >= 9
    eff if loc = on **then** y := 0 **fi;** x := 0; loc := off
  pop
    pre y = 62 ∧ loc = off
    eff x := 0
trajectories
  invariant loc = on ∨ loc = off
  stop when y = 62 ∧ loc = off
  evolve d(x) = 4; d(y) = 4

Step 2. Multi-Rate Automaton

- **Definition.** A multirate automaton is $\boldsymbol{\mathcal{A}} = \langle V, Q, \Theta, A, \mathcal{D}, \mathcal{T} \rangle$ where
  - V = X ∪ $\{loc\}$, where X is a set of n continuous variables and loc is a discrete state variable of finite type Ł
  - A is a finite set of actions
  - $\mathcal{D}$ is a set of transitions such that
    - The guards are described by rational clock constrings $\Phi(X)$
    - $\langle x, l \rangle - a \rightarrow \langle x', l' \rangle$ implies either $x' = c \text{ } or \text{ } x' = x$
  - $\mathcal{T}$ set of trajectories such that

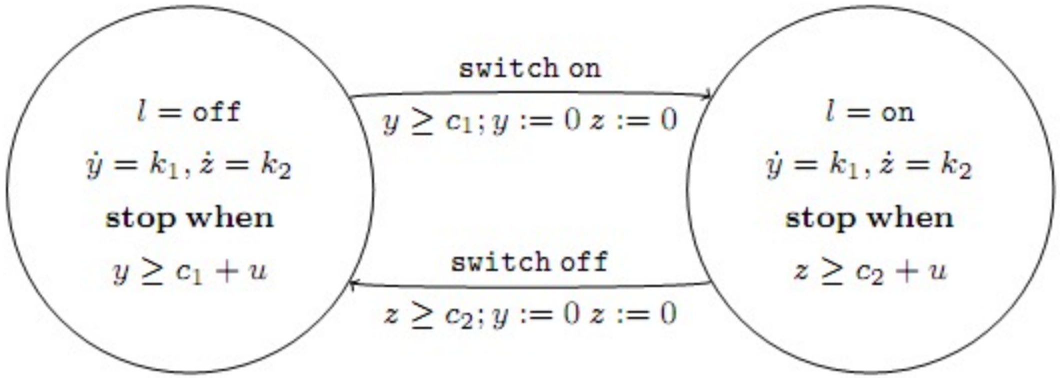
    for each variable $x \in X \; \exists k \; such \; that \; \tau \in \mathcal{T}, t \in \tau.dom$

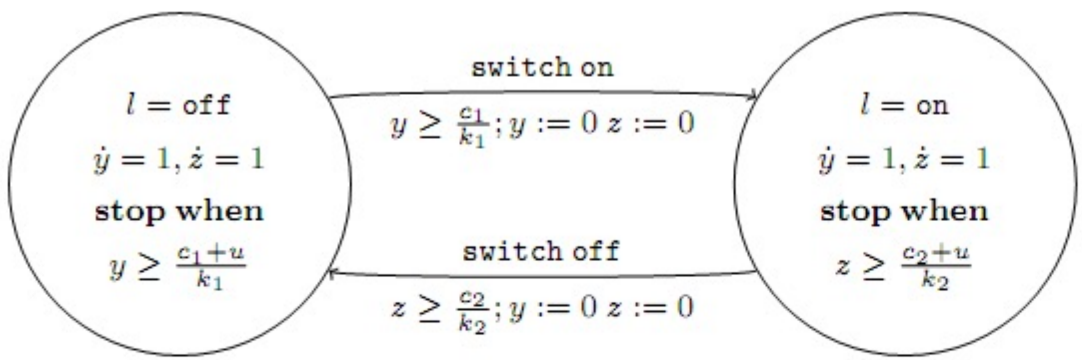
$$\tau(t).x = \tau(0).x + k\,t$$

Control State (Location) Reachability Problem

- Given an MRA, check if a particular location is reachable from the initial states
- Is problem is decidable? Yes
- Key idea:
  – Construct a RTA that is bisimilar to the given MRA

Example: Multi-rate to rational TA





Step 3. Rectangular HA

Definition. An rectangular hybrid automaton (RHA) is a HA $\boldsymbol{\mathcal{A}} = \langle V, A, \mathcal{T}, \mathcal{D} \rangle$ where

- V = X ∪ $\{loc\}$, where X is a set of n continuous variables and loc is a discrete state variable of finite type Ł
- A is a finite set
- $\mathcal{T}$ $= \cup_\ell \mathcal{T}_\ell$ set of trajectories for X
  - For each $\tau \in \mathcal{T}_\ell, x \in X$ either (i) $d(x) = k_\ell$ or (ii) $d(x) \in [k_{\ell 1}, k_{\ell 2}]$
  - Equivalently, (i) $\tau(t)\lceil x = \tau(0)\lceil x + k_\ell t$

    (ii) $\tau(0)\lceil x + k_{\ell 1} t \leq \tau(t)\lceil x \leq \tau(0)\lceil x + k_{\ell 2} t$
- $\mathcal{D}$ is a set of transitions such that
  - Guards are described by rational clock constraings
  - $\langle x, l \rangle \rightarrow_a \langle x', l' \rangle$ implies $x' = x\ or x' \in [c_1, c_2]$

CSR Decidable for RHA?

- Given an RHA, check if a particular location is reachable from the initial states?
- Is this problem decidable? No
  - [Henz95] Thomas Henzinger, Peter Kopke, Anuj Puri, and Pravin Varaiya. What's Decidable About Hybrid Automata?. Journal of Computer and System Sciences, pages 373–382. ACM Press, 1995.
  - CSR for RHA reduction to Halting problem for 2 counter machines
  - Halting problem for 2CM known to be undecidable
  - Reduction in next lecture

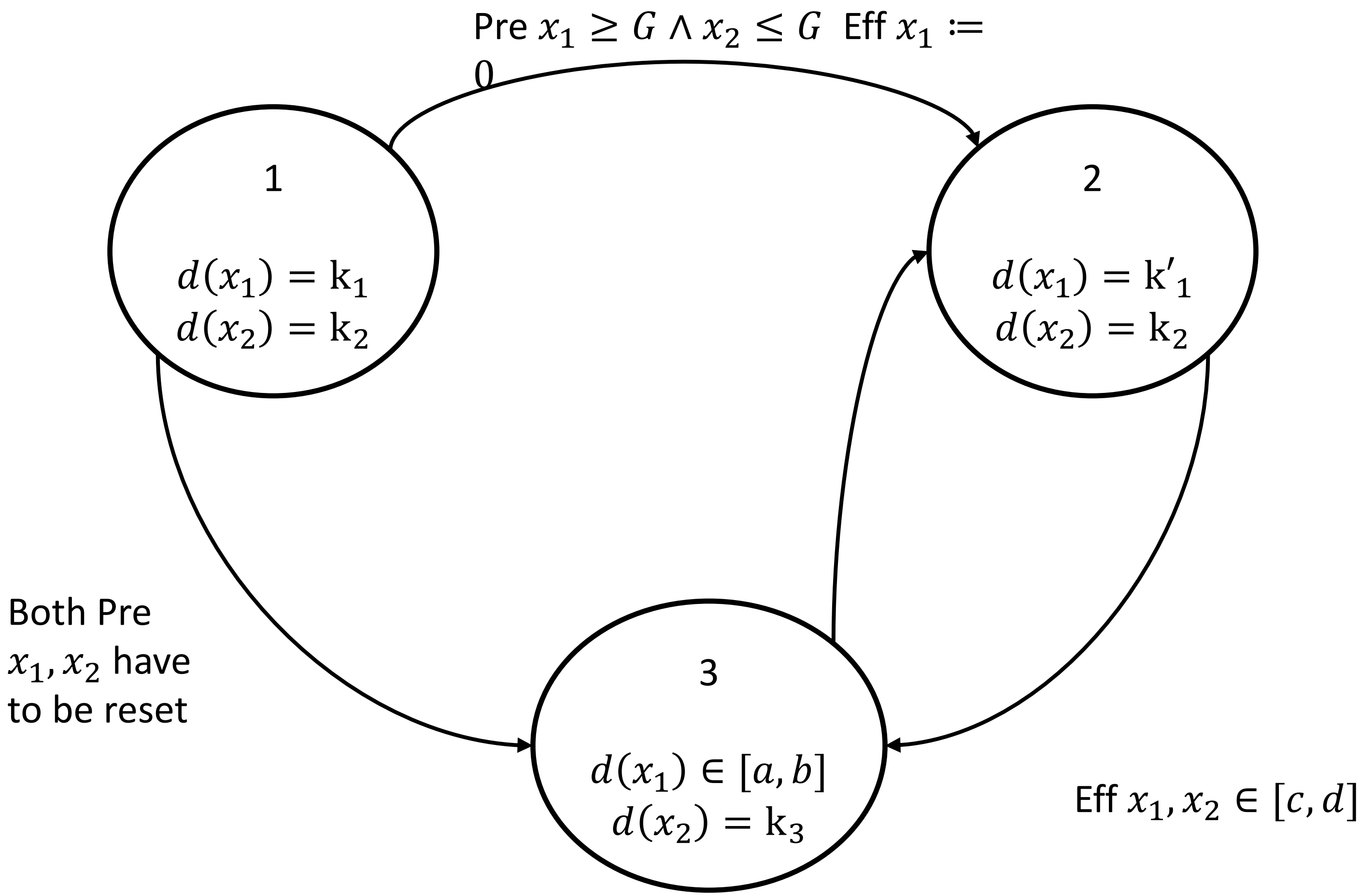
Step 4. Initialized Rectangular HA

Definition. *An initialized rectangular hybrid automaton* (IRHA) is a RHA $\boldsymbol{\mathcal{A}}$ where

- V = X ∪ $\{loc\}$, where X is a set of n continuous variables and $\{loc\}$ is a discrete state variable of finite type Ł
- A is a finite set
- $\mathcal{T} = \cup_{\ell} \mathcal{T}_{\ell}$ set of trajectories for X
  - For each $\tau \in \mathcal{T}_{\ell}, x \in X$ either (i) $d(x) = k_{\ell}$ or (ii) $d(x) \in [k_{\ell 1}, k_{\ell 2}]$
  - Equivalently, (i) $\tau(t)\lceil x = \tau(0)\lceil x + k_{\ell} t$

    (ii) $\tau(0)\lceil x + k_{\ell 1} t \leq \tau(t)\lceil x \leq \tau(0)\lceil x + k_{\ell 2} t$
- $\mathcal{D}$ is a set of transitions such that
  - Guards are described by rational clock constrings
  - $\langle x, l\rangle \rightarrow_a \langle x', l'\rangle$ implies if dynamics changes from ℓ to ℓ' then $x' \in [c_1, c_2]$, otherwise $x' = x$

Example: Rectangular Initialized HA



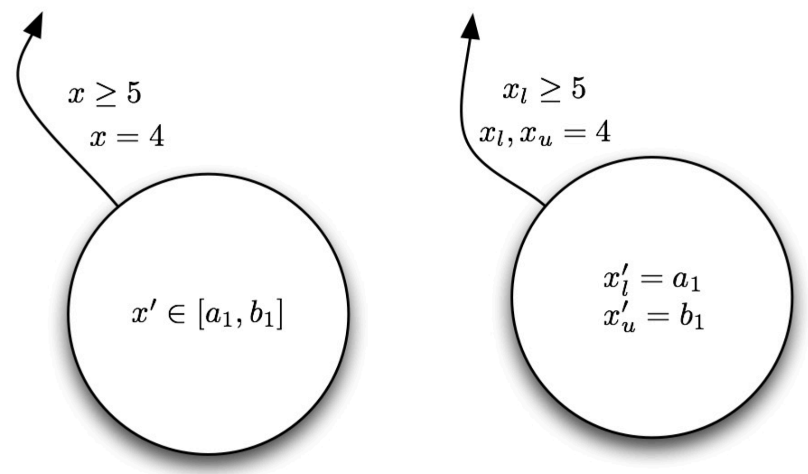
CSR Decidable for IRHA?

- Given an IRHA, check if a particular location is reachable from the initial states
- Is this problem decidable? Yes
- Key idea:
  - Construct a 2n-dimensional **initialized m**ulti-rate automaton that is bisimilar to the given IRHA
  - Construct a ITA that is bisimilar to the Singular TA

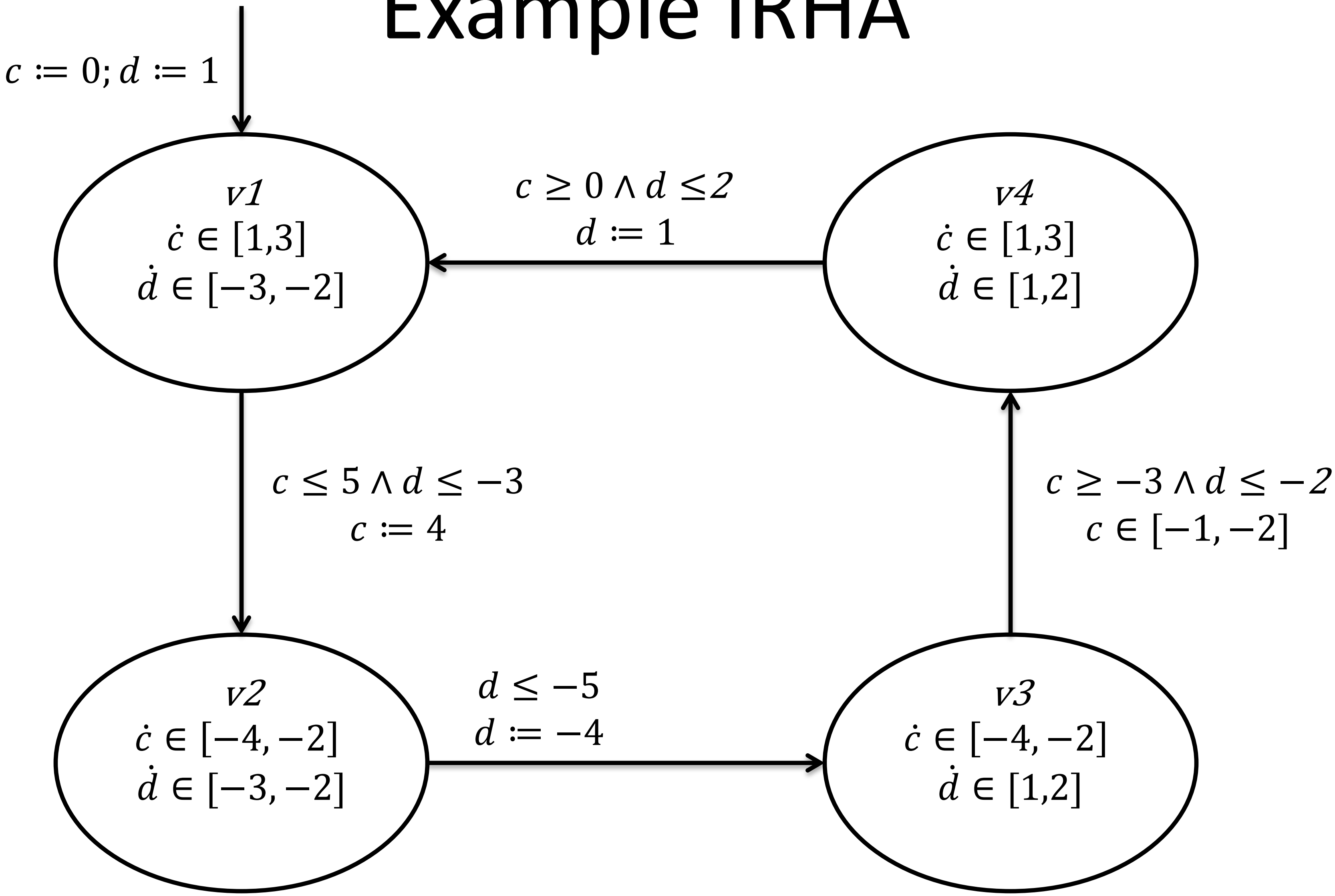
From IRHA to Singular HA conversion

For every variable create two variables---tracking the upper and lower bounds

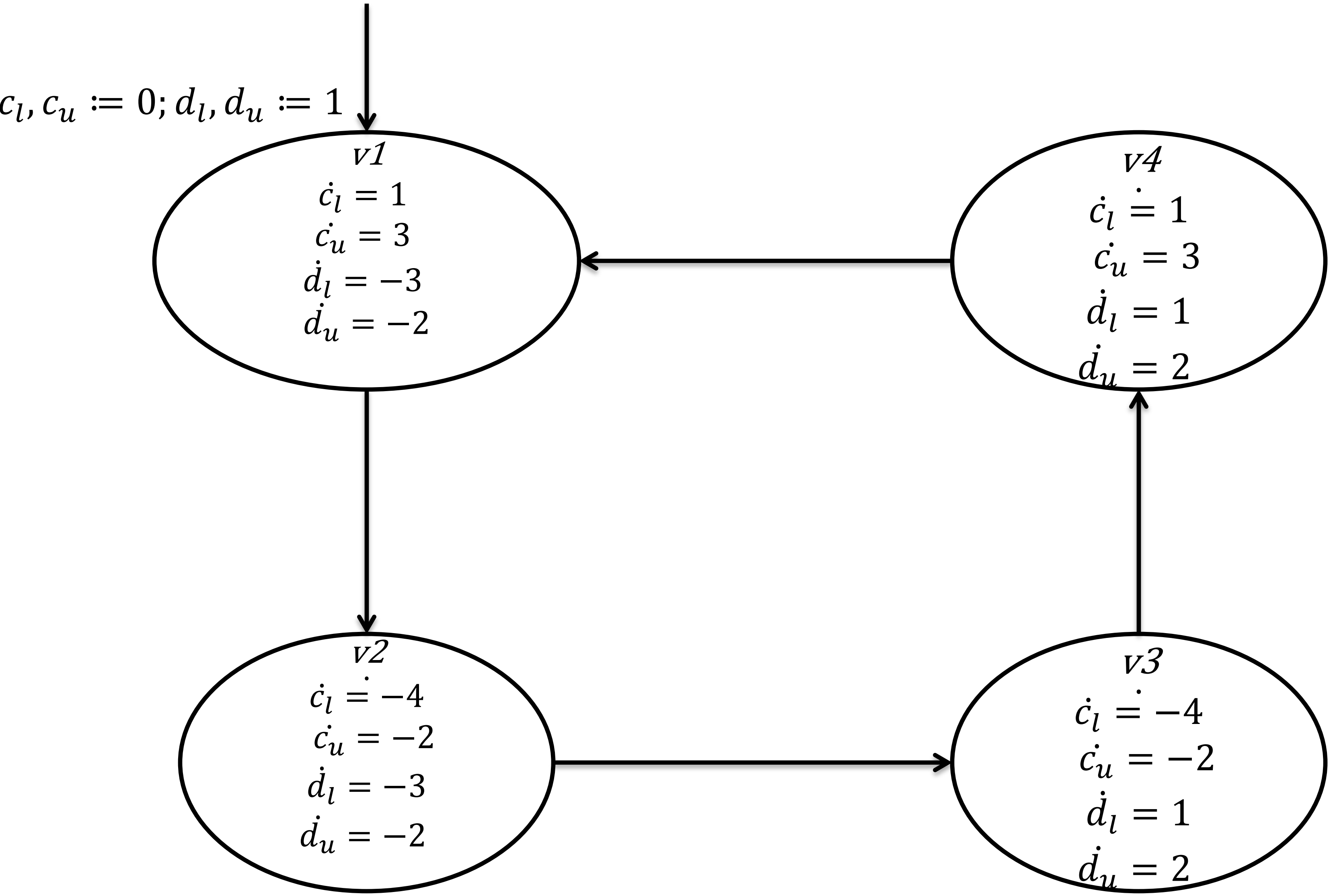
IRHA	MRA
x	x_ℓ ; x_u
Evolve: $d(x) \in [a_1, b_1]$	Evolve: $d(x_\ell) = a_1; d(x_u) = b_1$
Eff: $x' \in [a_1, b_1]$	Eff: $x_\ell = a_1; x_u = b_1$
$x' = c$	$x_\ell = x_u = c$
Guard: $x \geq 5$	$x_l \geq 5$
	$x_l < 5 \wedge x_u \geq 5$ Eff $x_l = 5$



Example IRHA



Initialized Singular HA



Transitions

v1

$\dot{c}_l = 1$

$\dot{c}_u = 3$

$\dot{d}_l = -3$

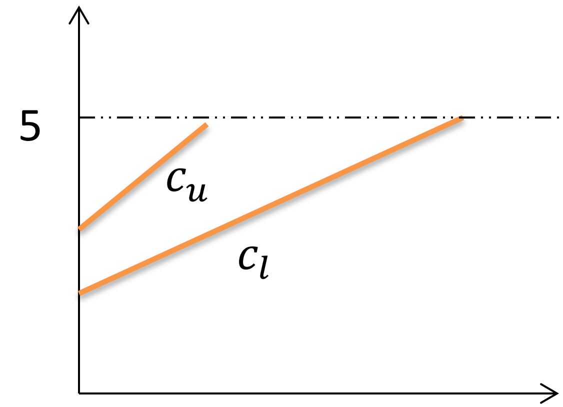
$\dot{d}_u = -2$

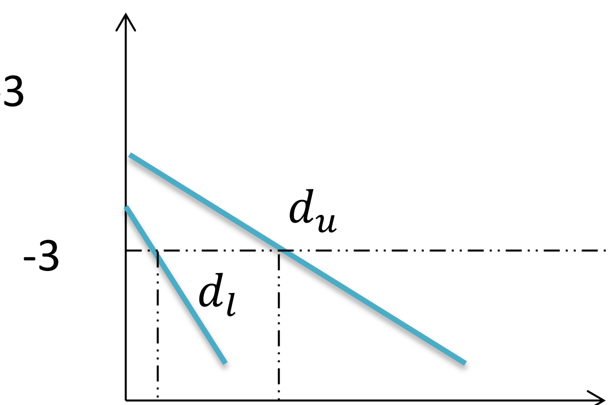
$c_l \leq 5$

$c_l, c_u \coloneqq 4$

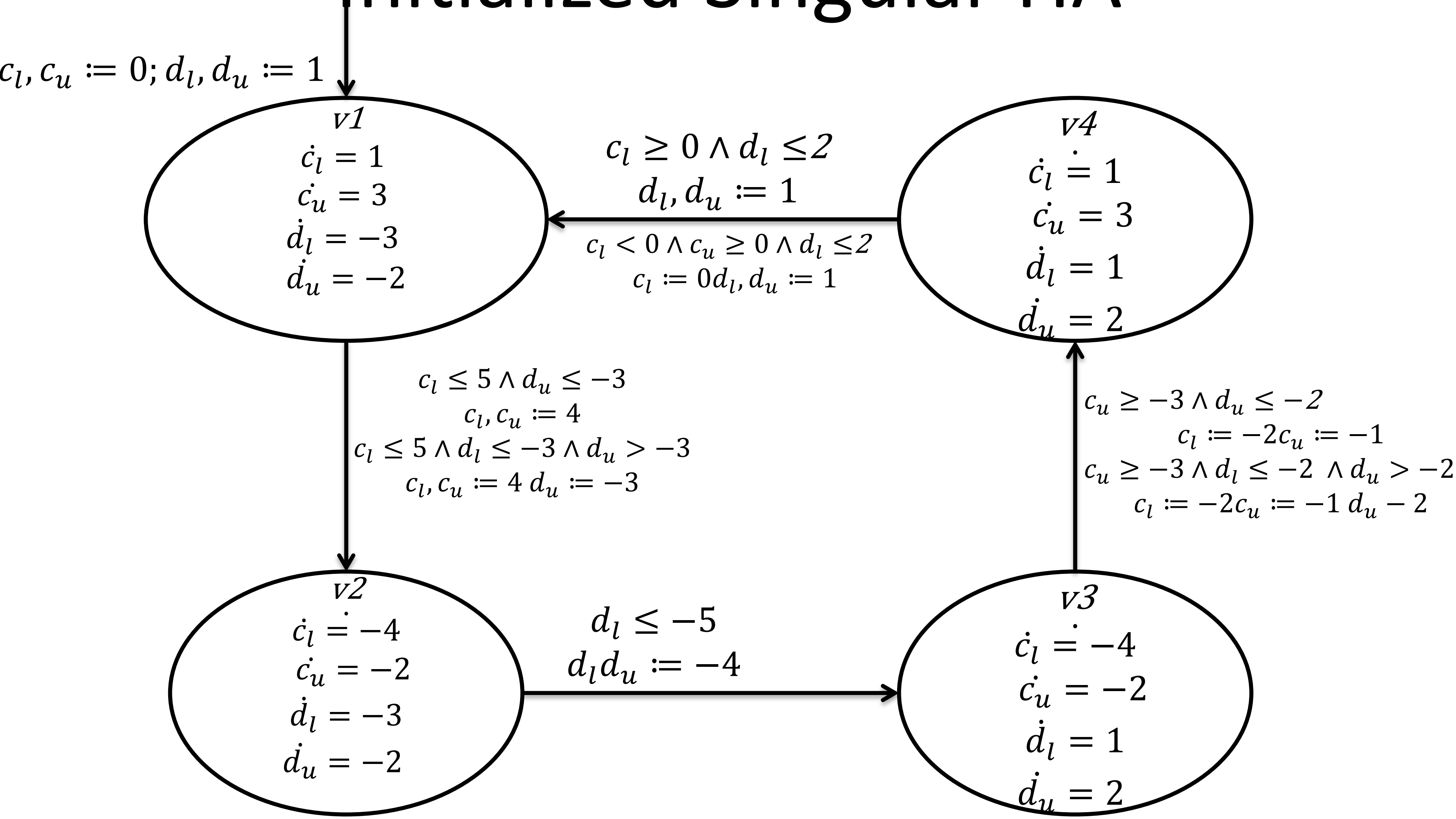
$d_u \leq -3$ *no reset*

$d_u > -3 \land d_l \leq -3$ $d_u \coloneqq$-3





Initialized Singular HA



Can this be further generalized ?

- For initialized Rectangular HA, control state reachability is decidable
  - Can we drop the initialization restriction?
    - No, problem becomes undecidable (next time)
  - Can we drop the rectangular restriction?
    - No, problem becomes undecidable

Verification in tools

Algorithm: BasicReach
Input: $\mathbf{A} = \langle V, \Theta, A, \mathbf{D}, \mathbf{T} \rangle$, $d > 0$
 $Rt, Reach{:}val(V)$
 $Rt := \Theta$;
 $Reach := \emptyset$;
 While $(Rt \not\subseteq \text{Reach})$
  $Reach := Reach \cup Rt$;
  $Rt := Rt \cup Post_{\mathbf{D}}(Rt)$;
  $Rt := Post_{\mathbf{T}(d)}(Rt)$;
Output: $Reach$

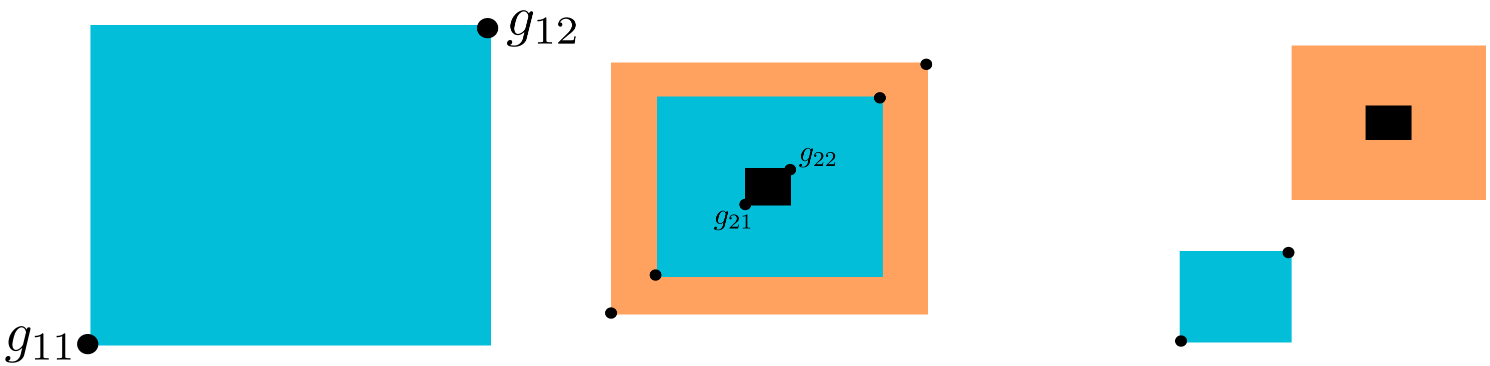
Algorithm: Post_{D}
\\ computes post of all transitions
Input: $A, \mathbf{D}, S_{in}$
 $S_{out} = \emptyset$
 For each $a \in A$
  For each $\langle g_1, g_2 \rangle \in S_{in}$
   If $[\![g_1, g_2]\!] \cap [\![g_{ga1}, g_{ga2}]\!] \neq \emptyset$
    $S_{out} := S_{out} \cup \langle g_{ra1}, g_{ra2} \rangle$
Output: S_{out}

Algorithm: $\text{Post}_{\mathbf{T}(\mathbf{d})}$
\\ computes post of all trajectories
Input: $A, \mathbf{T}, S_{in}, d$
 $S_{out} = \emptyset$
 For each $\ell \in L$
  For each $\langle g_1, g_2 \rangle \in S_{in}$
   $P := \cup_{t \leq d} [\![g_1, g_2]\!] \oplus [\![t g_{\ell 1}, t g_{\ell 2}]\!]$
   $S_{out} := S_{out} \cup Approx(P)$
Output: S_{out}

Data structures make reachability go around

- Hyperrectangles
  - $[[\mathrm{g}_1; \mathrm{g}_2]] = \{x \in R^n \mid \ ||\mathrm{x} - \mathrm{g}_1||_\infty \leq ||\mathrm{g}_2 - \mathrm{g}_1||_\infty\} = \Pi_i[g_{1i}, g_{2i}]$
- Polyhedra
- Zonotopes [Girard 2005]
- Ellipsoids [Kurzhanskiy 2001]
- Support functions [Guernic et al. 2009]
- Generalized star set [Duggirala and Viswanathan 2018]

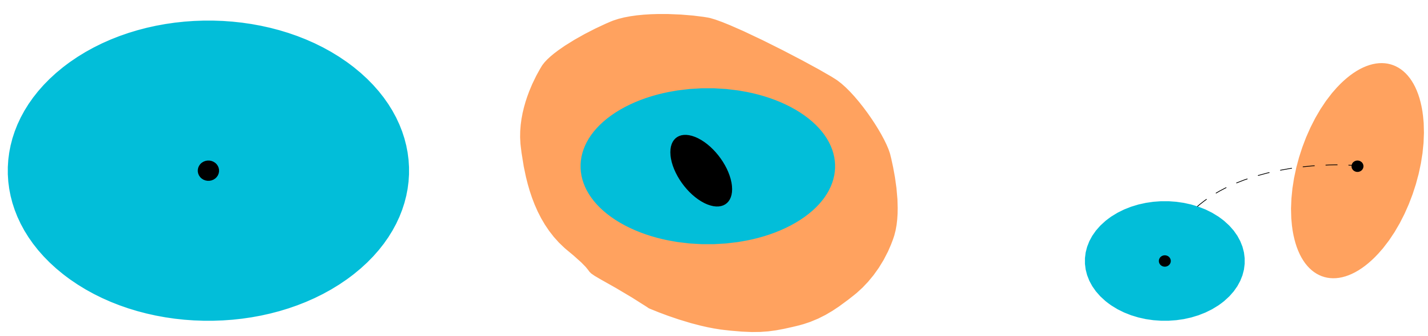
Data structures: rectangles and ellipsoids



$[[g_{11}, g_{12}]]$

$[[g_{11}, g_{12}]] \oplus [[g_{21}, g_{22}]]$
$= [[g_{11} + g_{12}, g_{21} + g_{22}]]$

$[[g_{11}, g_{12}]] \oplus [[t.g_1, t.g_2]]$

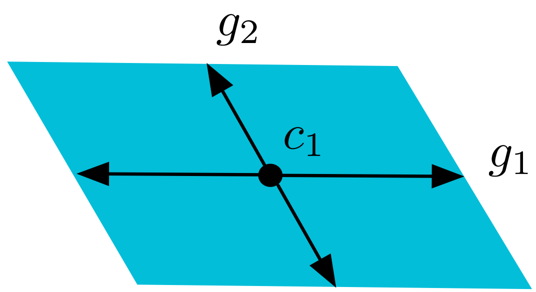


$[[c_1, Q]]$

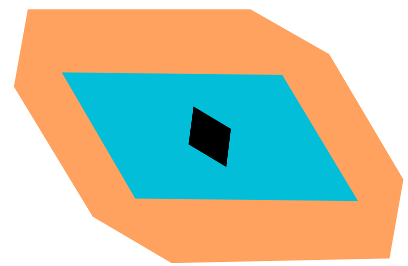
$[[c_1, Q_1]] \oplus [[c_2, Q_2]] \neq [[c_3, Q_3]]$

$[[Ac_1, AQA^T]]$

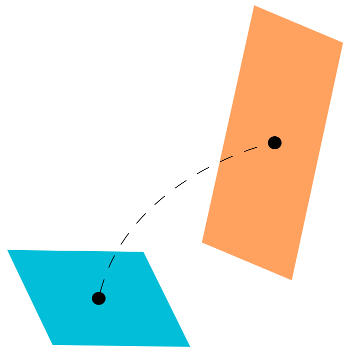
Zonotopes and polytopes



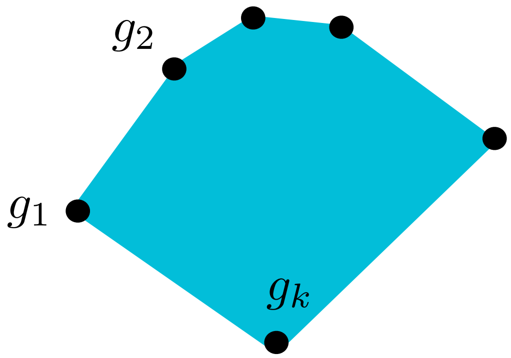
$[[c_1, \langle g_1, g_2 \rangle]]$



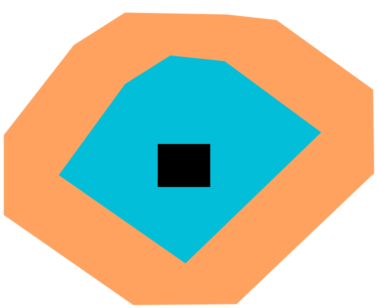
$[[c_1, \langle g_1, g_2 \rangle]] \oplus [[c_2, \langle g_1', g_2' \rangle]]$
$= [[c_1 + c_2, \langle g_1, g_1', g_2, g_2' \rangle]]$

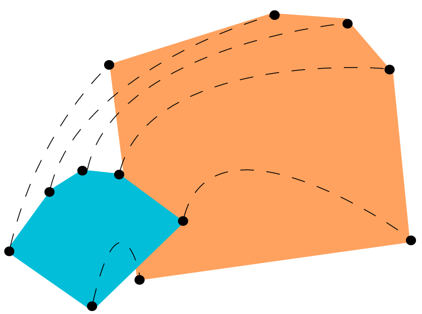


$[[Ac_1, \langle Ag_1, Ag_2 \rangle]]$



$[[A, b]]$
$[[g_1, \ldots, g_k]]$





$[[\xi(g_1, t), \ldots, \xi(g_k, t)]]$

Summary

- ITA: Restricted class of hybrid automata
  - Clocks, integer constraints
  - No clock comparison, linear
- Control state reachability with Alur-Dill's algorithm (region automaton construction)
- Rational coefficients; multirate Automata
- Initialized Rectangular Hybrid Automata
- HyTech, PHAVer use polyhedral reachability computations