THE ANNIHILATING-IDEAL GRAPH OF A COMMUTATIVE RING WITH OR WITHOUT S-VERTICES

A study of the interplay between Graph Theory and Ring Theory

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What is a Graph?

- A graph G is defined as an ordered pair (V, E).
- In a Graph, two vertices $a, b \in G$ **adjacent** if they share an edge $e \in E(G)$.
- A simple graph has **no direction, no multiple edges and no loops.**
- The **order** of a graph refers to the number of **vertices** (also called nodes) in the graph.
- A **tree** $\boldsymbol{T}$ is a connected graph with no cycles.

What is a Ring?

- A ring R is an algebraic structure consisting of a set equipped by two binary operations usually called addition $+$ and multiplication $\bullet$.

- The structure $(\boldsymbol{R}, +)$ is an additive abelian group with additive identity $\boldsymbol{0}$.

- While $(\boldsymbol{R}, \bullet)$ is associative semigroup which is distributes $\bullet$ over $+$.

- **Essentially, a ring is an abelian group with the above added properties.**

Some types of Rings

- A ring $\boldsymbol{R\ with\ 1}$ has multiplicative identity **1.**

- A **commutative ring** has the property for all $a, b\ in\ R, ab = ba$.

    - An element $a \in R$ is **a zero-divisor** if there exists some $b\ \neq 0\ \in\ R\ such\ that\ ab = 0$.

- A commutative ring $\boldsymbol{R\ with\ 1}$ without (non-zero) **zero divisors** is an $\boldsymbol{integral\ domain}$.

- A ring $\boldsymbol{R\ with\ 1}$ is a **division ring** if and only if every non-zero element is invertible.

- A Field is a $\boldsymbol{commutative\ division\ ring.}$

Zero divisors, Units and integral domains

- An element $a \in R$ is **a zero-divisor** if there exists some $b \neq 0 \in R$ $such\ that\ ab = 0.$
- $0 \neq a \in R$ is a unit if there is $0 \neq b \in R\ s.t\ ab = 1$**.**

- A field is a commutative ring in which every non-zero element is a unit.

- **Integral domains** don't have **zero-divisors**: i.e if $ab = 0, then\ a = 0\ or\ b = 0.$

- In ring theory, an element a of a ring R is said to be **nilpotent** if there exists a positive integer n such that $a^n = 0$**.**

- If a ring has only 0 as nilpotent element, then it is called a ***reduced ring***. *Ex: Field* $\mathbb{Z}_5$ *and* $\mathbb{Z}$.

Ideals

- A nonempty subset I of a ring R is called a *subring* ($\boldsymbol{I} \subseteq \boldsymbol{R}$) if:
  - (i) $\forall\, i, j \in I,\ \boldsymbol{i} - \boldsymbol{j} \in \boldsymbol{I}$;
  - (ii) $\forall\, i, j \in I,\ \boldsymbol{ij} \in \boldsymbol{I}$ **.**

- A subring is a left (right) ideal of R ($\boldsymbol{I} \trianglelefteq \boldsymbol{R}$) if $\forall\, \boldsymbol{r} \in \boldsymbol{R}\ \boldsymbol{and}\ \boldsymbol{i} \in \boldsymbol{I},\ \boldsymbol{ri} \in \boldsymbol{I}\ (\boldsymbol{ir} \in \boldsymbol{I})$**.**

- **In commutative rings** left ideal = right ideal

- In a commutative ring with 1**, if** $\boldsymbol{a} \in \boldsymbol{R}$**,** then **the principal Ideal which** is an Ideal generated by a is $< \boldsymbol{a} > = \{\boldsymbol{ra} \mid \boldsymbol{r} \in \boldsymbol{R}\}$**.**

Annihilator Ideals

- An Ideal $\boldsymbol{I} \trianglelefteq \boldsymbol{R}$ is an annihilator ideal if there exists a nonzero ideal $\boldsymbol{J} \trianglelefteq \boldsymbol{R}$ such that $\boldsymbol{I}\,\boldsymbol{J} = \{\mathbf{0}\}$. We use the notation $\mathbb{A}(\boldsymbol{R})$ for the set of **all annihilator ideals of R.**

- Example. In $\mathrm{R} = \mathbb{Z}_{12}$ which is not a field:

  ❖ The zero divisors are $\mathbf{Z}(\mathbb{Z}_{\mathbf{12}}) = \{\mathbf{0}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{8}, \mathbf{9}, \mathbf{10}\}$

  ❖ $\text{Divisors}(12) = \{12 \equiv 0, 1, 2, 3, 4, 6\}$ therefore the **proper** annihilator ideals are $\mathbb{A}(\mathrm{R}) = \{< 2 >, < 3 >, < 4 >, < 6 >\}$

  ❖ And $< 2 > = \{0,2,4,6,8,10\}$, $< 3 > = \{0,3,6,9\}$, $< 4 > = \{0,4,8\}$ and $< 6 > = \{0,6\}$

- Now checking which product gives the trivial ideal $< 0 > = \{0\}$:
  - ➢ $< 3 > \bullet < 4 > = < 0 >$
  - ➢ $< 4 > \bullet < 6 > = < 0 >$
  - ➢ $< 2 > \bullet < 6 > = < 0 >$
  - ➢ $< 6 > \bullet < 6 > = < 0 >$

Smarandache zero-divisors in Rings

- *In a commutative ring R with 1,* $a \neq 0$ *is a* ***Smarandache zero-divisor*** *if there exist three distinct nonzero elements (all different from a)* x, y*, and* b $\boldsymbol{\in R}$ $s.t$ $\boldsymbol{xa = ab = by = 0}, but$ $\boldsymbol{xy \neq 0}$.

- *Every Smarandache zero-divisor is a zero-divisor, but not every zero-divisor is a Smarandache zero-divisor.*

- *A ring containing a* ***Smarandache*** *zero-divisor must have at least* ***4*** *nontrivial zero-divisors, i.e.* $\boldsymbol{x, a, b\ and\ y.}$

Smarandache vertices (S-vertices)

- A vertex a in a graph G is a **Smarandache vertex** (or S-vertex for short) provided that there exist three distinct vertices $\boldsymbol{x, y, and\ b\ \neq a}$ in G such that $\boldsymbol{x — a, a — b, and\ b — y\ are\ edges\ in\ G}$, but there is **no edge** between x and y.

- The **degree** of each S-vertex must be **at least 2.**

Interplay Graphs and Ring Theory

- The interplay consists of assigning a graph to an algebraic structure to observe S-vertices and prove results about them in **Annihilating Ideal graphs:**

- In 1999, **Anderson and Livingston** defined $\mathbf{the\ zero\ divisor\ graph}\ \boldsymbol{\Gamma(R)}$ with **vertex set** $\boldsymbol{Z(R)}^{*} = \boldsymbol{Z(R)}/0$, where two distinct vertices are adjacent if $a \neq 0 \neq b$ and $ab = 0$.

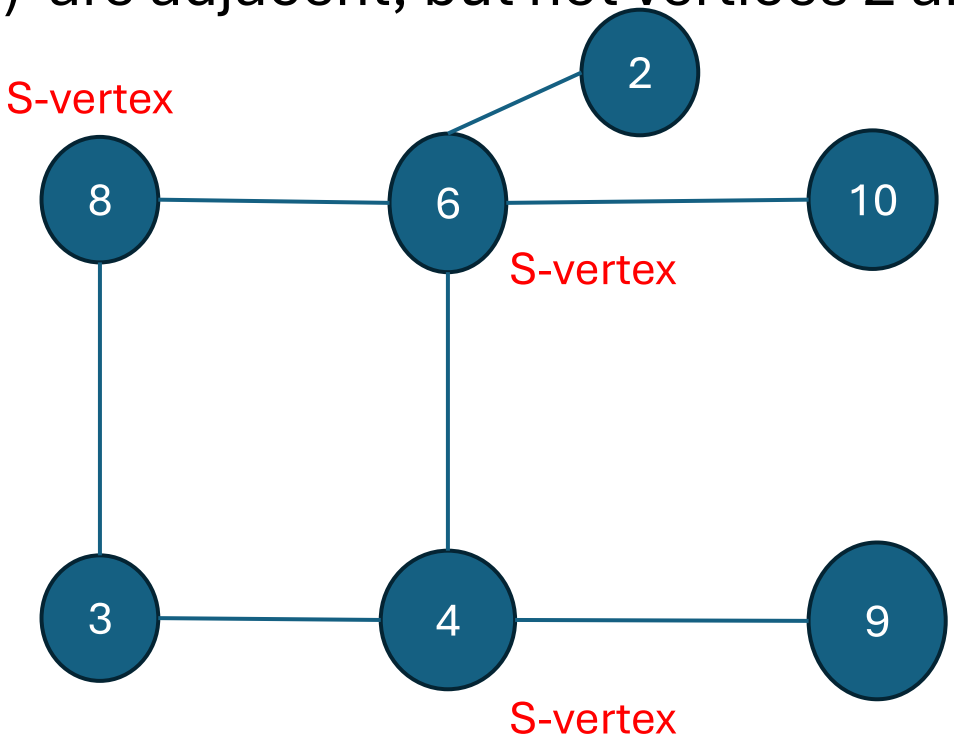
- In 2011, **Behboodi** and **Rakeei** introduced $\mathbb{AG}(R)$, the **annihilating ideal graph** with **vertex set** $\mathbb{A}(R)^{*} = \mathbb{A}(R) / < \mathbf{0} >$, where two distinct vertices are adjacent if and only if $IJ =< 0 >$.

Interplay Graphs and Ring Theory: example

Visualization $\Gamma(\mathbb{Z}_{12})$:

- In this graph (3,8), (8,6), (6,10) are edges, but the vertices 3 and 10 are not adjacent. In this case, 8 and 6 are s-vertices.
- Also, 4 and 6 are s-vertices. Because (9,4),(4,6) and (6,10) are adjacent but not 9 and 10.
- Finally, (2,6), (6,8), (8,3) are adjacent, but not vertices 2 and 3.

$\Gamma(\mathbb{Z}_{12})$:

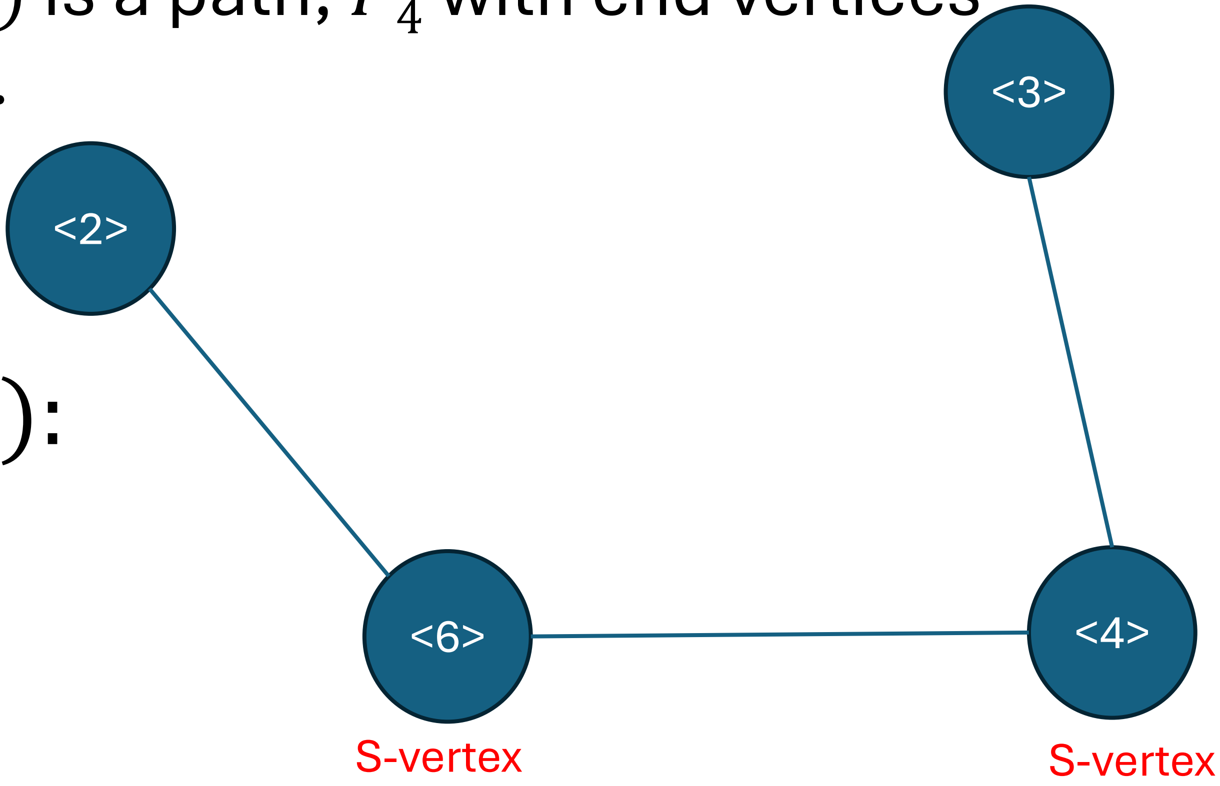


Interplay Graphs and Ring Theory

Visualization $\mathbb{AG}(\mathbb{Z}_{12})$:

- Here, the ideals <6> and <4> are s-vertices since $\mathbb{AG}(\mathbb{Z}_{12})$ is a path, P_4 with end vertices <2> and <3> .

$\mathbb{AG}(\mathbb{Z}_{12})$:



In this interdisciplinary subject, we investigate the **interplay between the algebraic and graph-theoretic** properties of $\mathbb{A}\mathbb{G}(\boldsymbol{R})$ using the notion of the Smarandache vertices in connection with the existence or nonexistence of S-vertices in the graph $\mathbb{A}\mathbb{G}(\boldsymbol{R})$,

We will show that a conjecture related to the **weakly perfectness of** $\mathbb{A}\mathbb{G}(\mathbf{R})$ is true when the graph has **no S-vertices**. It is shown how the existence of an S-vertex in $\boldsymbol{\Gamma}(R)$, **the zero-divisor graph of R**, implies the existence of an S-vertex in $\mathbb{A}\mathbb{G}(\boldsymbol{R})$.

We characterize rings R when $gr(\mathbb{A}\mathbb{G}(\boldsymbol{R})) \geq 4$, and so we characterize rings whose annihilating-ideal graphs are bipartite.

There is also a discussion on a relationship between the diameter, girth, and S-vertices of $\boldsymbol{\Gamma}(R)$, and $\mathbb{A}\mathbb{G}(\boldsymbol{R})$

Definitions 1

- A **path** $\boldsymbol{P_n}$ in G is a sequence of adjacent distinct vertices.
  - ❖ A closed path on $\boldsymbol{n}$ **vertices** is called a **cycle** $\boldsymbol{C_n}$.
    - o A path, P_4 contains 2 s-vertices

- A graph G is **connected** if a path joins every pair of vertices in G.

- The **girth** of a graph $gr(G)$ is the length of its **shortest cycle**.

- A tree has no cycles, therefore $\boldsymbol{gr(T)} = \infty$

Definitions 1(Continued)

- For two vertices $u, v \in G$, the distance $\boldsymbol{d(u,v)}$ is the shortest $u-v$ path.

- The eccentricity $\boldsymbol{ecc(v)}$ is the greater distance from $v \in G$ to any vertices in G

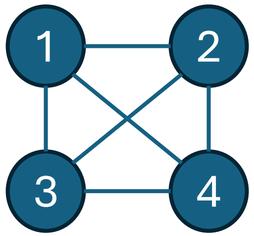
- The diameter $\boldsymbol{diam(G)}$ is the greatest eccentricity in G, while the radius $rad(G)$ is the smallest eccentricity,
  - ❖ $If\ diam(G) = 1,$ then $G\ is\ a\ complete\ graph, etc.$

Some properties of $\boldsymbol{\Gamma(R)} \boldsymbol{and}\ \mathbb{AG}(\boldsymbol{R})$

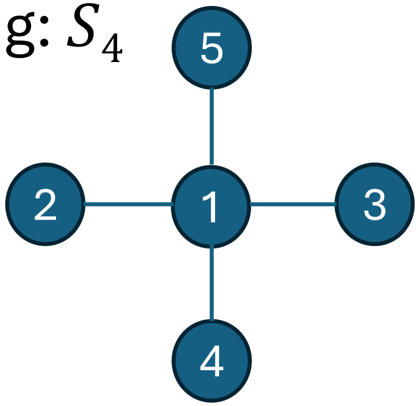
- $\boldsymbol{\Gamma(R)}$ is connected with $\mathrm{diam}(\boldsymbol{\Gamma}(\mathbf{R})) \leq 3$. Thus, diam($\boldsymbol{\Gamma}(R)$) $= 0,1,2\ or\ 3$
- If $\boldsymbol{\Gamma(R)}$ contains a cycle, then $\mathrm{gr}(\boldsymbol{\Gamma(R)}) \leq 4$. Thus, $\mathrm{gr}(\boldsymbol{\Gamma(R)}) = 4,3\ or\ \infty$
- $\mathbb{AG}(R)$ is connected with $diam(\mathbb{AG}(R)) \leq 3$. Thus, $diam(\mathbb{AG}(R)) = 3,2,1\ or\ 0$
- If $\mathbb{AG}(R$)) contains a cycle, then $gr(\mathbb{AG}(R)) \leq 4$. Thus, $gr(\mathbb{AG}(R)) = 4,3\ or\ \infty$

Example1 : Simple graphs with no s-vertices

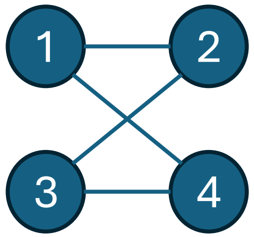
1. A complete graph does not have any S-vertices. e.g: K_4



2. A star graph does not have any S-vertices. e.g: S_4



3. A complete bipartite graph has no S-vertices. e.g: $K_{2,2}$ or C_4



Definitions 2

- An **empty graph** E_n is a graph with an empty edge set $E(G)$.

- A **complete graph** K_n has an **edge between every pair of distinct vertices.**

- A **proper coloring** of a graph G is an assignment of colors to the vertices of G such that **adjacent** vertices are **colored differently.**

- The **chromatic number** $\boldsymbol{\chi(G)}$ of a graph is the smallest number of colors needed to color its vertices so that no two adjacent vertices share the same color. e.g. $\boldsymbol{\chi(K_n) = n}$

Definitions 2 (Continued)

- If $G(V,E)$ is a graph, then a subgraph $H(V', E')$ is a graph with the property $V' \subseteq V \; and \; E' \subseteq E$.

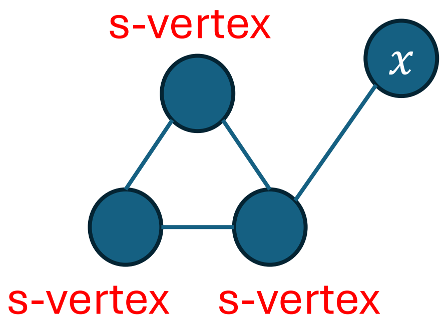
- A complete subgraph H of a graph G is called a **"clique".** The **clique number $\boldsymbol{\omega(G)}$** of a graph G is the number of vertices in a largest clique in G.

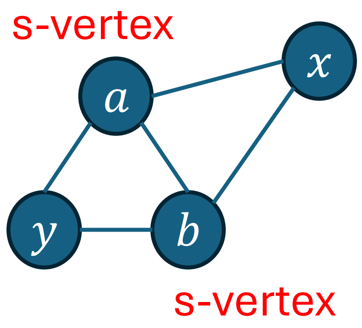
- $\boldsymbol{\omega(G) \le \chi(G)}$**.**

- A graph is **weakly perfect** if $\boldsymbol{\omega(G) = \chi(G) : e.g.\, C_{2n}, T_n, complete\ graphs\ K_n}$.

Lemma 1

- Let C be a clique in a graph G such that $|C| \geq 3$. Suppose that x is a vertex in $G \backslash C$ adjacent to at **least one vertex** or at **most** $|\boldsymbol{C}| - \mathbf{2}$ **vertices** of C, then every vertex of C is an S-vertex. In other case, if x makes links with $|C| - 1$ vertices of C, then all those $|C| - 1$ vertices are S-vertices.





- ❑ $|C| = n = 3$
- ❑ x adjacent to $|\boldsymbol{C}| - \mathbf{2} = \mathbf{1}$ vertex
- ❑ $n = 3$ s-vertices

- ❑ $|C| = n = 3$
- ❑ x adjacent to $|\boldsymbol{C}| - \mathbf{1} = \mathbf{2}$ vertex
- ❑ $\boldsymbol{n} - \mathbf{1}$ s-vertices

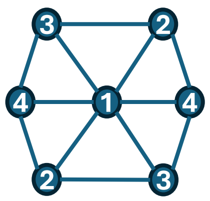
Proposition 1

Let G be a connected simple graph whose clique number is strictly larger than 2. If $\omega(G) \neq \chi(G)$, then G has an S-vertex. In other words, for any connected simple graph G with $\omega(G) \geq 3$ and no S-vertices, then $\omega(G) = \chi(G)$ (i.e., **G is weakly perfect**)

Proof:

Let C be a (**largest**) clique in G with $|C| \geq 3$. Since $\boldsymbol{\omega(G) \neq \chi(G)}$, then G is not a **complete** graph. Thus, there exists a vertex $\boldsymbol{x \in G \,/\, C}$ which makes edge(s) with at least one or at most $\omega(G) - 1$ element(s) of C. Now, the proof is immediate from **Lemma 1**. □

$\boldsymbol{W_6}$:



- $\boldsymbol{\omega(G) = 3}$
- $\boldsymbol{\chi(G) = 4}$

Constructing S-vertices in $\mathbb{AG}(R)$

Theorem 1: Let $\boldsymbol{C} = \{\boldsymbol{I_1}, \boldsymbol{I_2}, \dots, \boldsymbol{I_n}\}$ be a clique in $\mathbb{AG}(R)$ with $n \geq 3$. Then

(1) $\mathbb{AG}(\boldsymbol{R})$ contains n S-vertices provided that $\boldsymbol{I_i^2} \neq < \mathbf{0} >$ and $\boldsymbol{I_j^2} \neq < \mathbf{0} >$ for some $1 \leq i \neq j \leq n$.

(2) $\mathbb{AG}(\boldsymbol{R})$ contains n S-vertices provided that $\boldsymbol{I_i^2} \neq < \mathbf{0} >$, $\boldsymbol{I_j} \nsubseteq \boldsymbol{I_i}$ for some $1 \leq i \neq j \leq n$, and $I_j(I_i + I_i) \neq 0$ (otherwise, $\mathbb{AG}(R)$ contains $n-1$ S-vertices if $I_j(I_i + I_i) = 0$).

(3) $\mathbb{AG}(R)$ contains n S-vertices provided that R is a reduced ring.

Constructing S-vertices in $\mathbb{AG}(R)$ (Contd.)

Theorem 1: Let $C = \{\boldsymbol{I_1}, \boldsymbol{I_2}, \dots, \boldsymbol{I_n}\}$ be a clique in $\mathbb{AG}(R)$ with $n \geq 3$. Then

(1) $\mathbb{AG}(\boldsymbol{R})$ contains n S-vertices provided that $\boldsymbol{I_i^2} \neq < \boldsymbol{0} >$ **and** $\boldsymbol{I_j^2} \neq < \boldsymbol{0} >$ for some $1 \leq i \neq j \leq n$.

<u>**Proof**</u>:

We just prove Part (1) and leave the other parts to the reader. **Without loss of generality**, suppose that $\boldsymbol{I_1^2} \neq < \boldsymbol{0} >$ and $\boldsymbol{I_2^2} \neq < \boldsymbol{0} >$.

Now the proof follows from Lemma 1 and the fact that $\boldsymbol{I_1} + \boldsymbol{I_2}$ **is a vertex different from all vertices of the clique** and **makes a link with each of them except** $\boldsymbol{I_1}$ $\boldsymbol{and}$ $\boldsymbol{I_2}$.

For example if $\boldsymbol{I_1} + \boldsymbol{I_2} = \boldsymbol{I_3}$, then multiplying both sides by $\boldsymbol{I_1}$ gives:

$(\boldsymbol{0} \neq \boldsymbol{I_1}\,\boldsymbol{I_1}) + \boldsymbol{I_1}\boldsymbol{I_2} = \boldsymbol{I_1}\,\boldsymbol{I_3} = 0$ which is a contradiction. Then $\boldsymbol{I_1} + \boldsymbol{I_2}$ is a vertex different from all vertices of the clique. Note that $\boldsymbol{I_1} + \boldsymbol{I_2} \neq \boldsymbol{R}$. Otherwise, $\boldsymbol{I_3} = \boldsymbol{I_3}\boldsymbol{R} = \boldsymbol{I_3}(\boldsymbol{I_1} + \boldsymbol{I_2}) = \boldsymbol{I_3}\boldsymbol{I_1} + \boldsymbol{I_3}\boldsymbol{I_2} = < \boldsymbol{0} >$ which is a ***contradiction***.

Corollaries

1. Let R be a ring with $\boldsymbol{gr}(\mathbb{AG}(\boldsymbol{R})) = \mathbf{3}$. Suppose $\mathbb{AG}(R)$ contains a triangle with vertices $\boldsymbol{A}, \boldsymbol{B},$ **and** $\boldsymbol{C}$ **such that** $\boldsymbol{A}^{\mathbf{2}} \neq \mathbf{0}$ $\boldsymbol{and}$ $\boldsymbol{B}^{\mathbf{2}} \neq \mathbf{0}$. Then $\mathbb{AG}(R)$ contains **3** S-vertices; namely $\boldsymbol{A}, \boldsymbol{B},$ **and** $\boldsymbol{C}$ in $\mathbb{AG}(R)$**.**

2. Let R be a reduced ring such that $\mathbb{AG}(R)$ has a clique of size $n \geq 3$ (for example $\boldsymbol{\omega}(\mathbb{AG}(R)) = \boldsymbol{n}$). Then $\mathbb{AG}(R)$ contains n S-vertices.

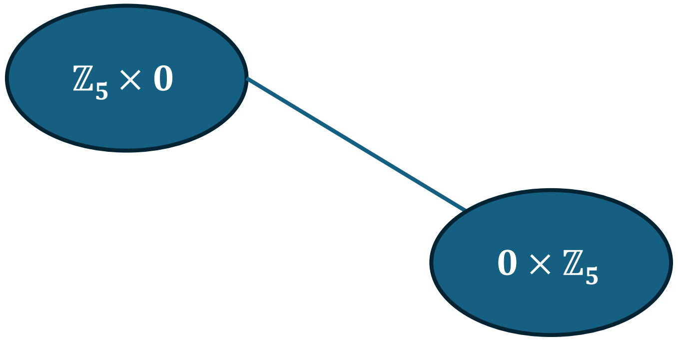
Conjecture

- **There is a conjecture in** the article: M. Behboodi, Z. Rakeei, The annihilating-ideal graph of commutative rings II, J. Algebra Appl. 10 (2011) 740-753. which is:
- **Conjecture**: $\mathbb{AG}(\boldsymbol{R})$ is **weakly perfect** for any ring R.
- Proposition 1 says the conjecture is true for any ring R with $\boldsymbol{\omega}(\mathbb{AG}(\boldsymbol{R})) \geq \mathbf{3}$ and $\mathbb{AG}(\boldsymbol{R})$ containing no S-vertices.

Example 2

- Let $R \cong R_1 \times R_2$ be the **direct product** of **two commutative integral domains.**
- Then it is not difficult to show that $\mathbb{AG}(\mathbf{R})$ **is a complete bipartite** graph with parts:
  - ❖ $\{(I \times 0) |\ I = a\ nonzero\ ideal\ of\ R_1\}$ and
  - ❖ $\{(0 \times J) |\ J = a\ nonzero\ ideal\ of\ R_2\}$

Example: $\mathbb{AG}(\mathbb{Z}_5 \times \mathbb{Z}_5)$:



Lemma 2

- $R \cong R_1 \times R_2 \times \cdots \times R_n$ be the **direct product** of $n \geq 2$ rings. If $\mathbb{AG}(\boldsymbol{R})$ has **no S-vertices**, then $n = 2$ and $R = R_1 \times R_2$, where each of the rings $\boldsymbol{R_1}$ $\boldsymbol{and}$ $\boldsymbol{R_2}$ is an integral domain.

Theorem 2

Let $n \geq 2$ be a fixed integer and $R \cong R_1 \times R_2 \times \cdots \times R_n$ the direct product of n commutative rings. ***Then $n = 2$ and $\mathbb{AG}(\boldsymbol{R})$ is a complete bipartite graph with each part of size greater than or equal to 2 if and only if it contains no S-vertices and its girth is 4***.

<u>**Proof**</u>: ⇒*The necessary part is clear*.

⇐ For the sufficient part, by Lemma 2, n = 2 and $R = R_1 \times R_2$ is isomorphic to the direct product of two commutative integral domains.

In this case, $\mathbb{AG}(\boldsymbol{R})$ is complete bipartite by **Example 2**. Consequently, each of R_1 and R_2 are non-field integral domains since $gr(\mathbb{AG}(R)) = 4$.

That is, each of R_1 and R_2 has at least one nonzero proper ideal, which implies that the cardinality of each part of the graph is greater than or equal to two. □

Theorem 3

The following statements are true for a commutative ring R.

(a) If $gr(\mathbb{AG}(\boldsymbol{R})) = 4$ and $A^2 \neq < 0 >$ for all nonzero annihilator ideals A of R (in particular, R could be a reduced ring), then $\mathbb{AG}(\boldsymbol{R})$ is a **complete bipartite graph and consequently has no S-vertices**.

(b) If $\mathbb{AG}(\boldsymbol{R})$ is a complete bipartite graph, then $gr(AG(R)) = 4$ or ∞.

Thank you!