Matched Filter

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1 Introduction

The matched filter is designed to detect a template within an unknown signal. In our particular case, used to look for point sources, the template will be the beam of the experiment, since point sources are spread out by the observing beam and will look like it. The equation for the matched filter map in real space is

$$T_{\mathrm{MF}}(\boldsymbol{\theta}) = \int \exp(2\pi j\mathbf{k}\cdot\boldsymbol{\theta})\Phi_{\mathrm{MF}}(\mathbf{k})\tilde{T}(\mathbf{k})d\mathbf{k}, \tag{1}$$

where $\tilde{T}(\mathbf{k})$ is the harmonic transfer of the real space map of the observed signal and $\Phi_{\mathrm{MF}}(\mathbf{k})$ is the matched filter. Therefore, the matched filter is the inverse harmonic transform of the filtered harmonic transform of the map. The matched filter is defined by

$$\Phi_{\mathrm{MF}}(\mathbf{k}) = \frac{\tilde{B}^{\dagger}(\mathbf{k})|\tilde{T}_{\mathrm{other}}(\mathbf{k})|^{-2}}{\int \tilde{B}^{\dagger}(\mathbf{k}')|\tilde{T}_{\mathrm{other}}(\mathbf{k}')|^{-2}\tilde{B}(\mathbf{k}')d\mathbf{k}'}, \tag{2}$$

where $\tilde{B}(\mathbf{k})$ is the harmonic transfer of the effective instrument beam. Usually, it will be azimuthally symmetric, i.e. $\tilde{B}(\mathbf{k}) = \tilde{B}(k)$. $\tilde{T}_{\mathrm{other}}(\mathbf{k})$ is the harmonic transform of the map minus the point sources, so it will include the correlated atmospheric noise, as well as instrument noise and the CMB + foregrounds signal. Therefore, $|\tilde{T}_{\mathrm{other}}(\mathbf{k}')|^{-2}$ is the inverse (note the $-$ sign) of the power spectrum. In practice, it can be estimated by calculating the power spectrum of the input map $T(\boldsymbol{\theta})$, masking known point sources if possible. There is relatively small power in the individual point sources, so we can do this. Also, we can approximate this spectrum with some analytic function.

Because the matched filter map is essentially a convolution between the beam and the input map $T(\boldsymbol{\theta})$, the value on each pixel will be the beam-integrated spectral radiance of the input map $T(\boldsymbol{\theta})$, centered at that pixel.

2 Flat-sky example with Atacama Cosmology Telescope (ACT) observations

In this case, we will take a small square map observed with ACT. The harmonic transform of the beam is azimuthally symmetric, so it's a 1D array $\tilde{B}(\ell)$. The $|\tilde{T}_{\mathrm{other}}(\ell)|^2$ spectrum will be approximated by the function

$$|\tilde{T}_{\mathrm{other}}(\ell)|^2 = 1 + (\ell/\ell_{\mathrm{knee}})^{-3}, \tag{3}$$

with $\ell_{\mathrm{knee}} = 2000$ for 90 GHz, 3000 for 150 GHz and 4000 for 220 GHz. The amplitude of this spectrum won't matter for this case, since it is being normalized by the integral in the denominator of eq. 2.

2.1 Normalization of the beam

In this case, the Fourier transform of the beam $\tilde{B}(\boldsymbol{\ell})$ will be normalized in such a way that $B(\theta = 0) = 1$. Given that $\tilde{B}(\boldsymbol{\ell})$ is the Fourier transform of $B(\boldsymbol{\theta})$, we can use the definition of the inverse Fourier transform to get a constraining equation. We will put a normalization constant in the Fourier transform of the beam, that is, the correct beam is given by $\tilde{B}(\boldsymbol{\ell}) = A\tilde{b}(\boldsymbol{\ell})$, where $\tilde{b}(\boldsymbol{\ell})$ is the Fourier transform of the beam in arbitrary units.

$$B(\boldsymbol{\theta}) = \int \tilde{B}(\boldsymbol{\ell}) \exp(2\pi j \boldsymbol{\theta} \cdot \boldsymbol{\ell}) d\boldsymbol{\ell} \qquad = \int A\tilde{b}(\boldsymbol{\ell}) \exp(2\pi j \boldsymbol{\theta} \cdot \boldsymbol{\ell}) d\boldsymbol{\ell} \tag{4}$$

$$B(\boldsymbol{\theta} = 0) = 1 \qquad = \int A\tilde{b}(\boldsymbol{\ell}) d\boldsymbol{\ell}. \tag{5}$$

Therefore, $\frac{1}{A} = \int b(\boldsymbol{\ell}) d\boldsymbol{\ell}$. The correctly normalized matched filter will be

$$\Phi_{\mathrm{MF}}(\boldsymbol{\ell}) = \frac{A\tilde{b}^{\dagger}(\boldsymbol{\ell})|\tilde{T}_{\mathrm{other}}(\boldsymbol{\ell})|^{-2}}{A^2 \int \tilde{b}^{\dagger}(\boldsymbol{\ell'})|\tilde{T}_{\mathrm{other}}(\boldsymbol{\ell'})|^{-2}\tilde{b}(\boldsymbol{\ell'})d\boldsymbol{\ell'}}, \tag{6}$$

so it must be multiplied by $\frac{1}{A}$.

2.2 Units

A CMB map will usually have units of μK in thermodynamic units. With the definition of the Planck law, we can transform spectral radiance unit into a temperature with

$$dT = \frac{dI_\nu}{\frac{2k\nu^2}{c^2} \frac{x^2 \exp(x)}{(\exp(x)-1)^2}} \tag{7}$$

where $x = h\nu/kT_{\mathrm{CMB}}$. This is an expansion of the Planck law around the monopole temperature of the CMB. Since I_ν will be in SI units, you have to multiply by a 10^{26} factor to transform into Jy. To get the flux density in Jy, we need to multiply the map in K with the dI_{nu}/dT factor times 10^{26} times the beam area Ω_{B} in steradians. Look at this for details.

2.3 Code

The map is a 2D array with some WCS information on it. We can work with the FFT functions already built in in the *pixell* package. The simple code is

```
m = enmap.read_map(filename)[0]
ell = m.modlmap()
ell2 = m.lmap()
dell1 = ell2[0,300,300] - ell2[0,299,300]
dell2 = ell2[1,300,300] - ell2[1,300,299]
beam_2d_m = np.interp(ell.reshape(-1), np.arange(beam1d.size), beam1d).reshape(ell.shape)
factor = np.sum(beam_2d_m)*dell1*dell2
norm = np.sum(beam_2d_m**2 / (1+(ell/2000)**-3))*dell1*dell2
f = factor * beam_2d_m / (1+(ell/2000)**-3) / norm
m_filtered = enmap.ifft(f*enmap.fft(m)).real
fac = utils.flux_factor(b_area*1e-9,90*1e9)/1e3
m_mJy = fac*m_filtered
```

The temperature map is the variable `m`. `modlmap` will create a map of the absolute wavenumbers $\boldsymbol{\ell}$. `lmap` will create 2 maps with the values of the wavenumbers in each pixel, one map for ℓ_X and one for ℓ_Y. `dell1` then corresponds to the difference in ℓ for one direction, and `dell2` will be difference in the other direction. The wavenumbers are equidistant, so we can sample it between any pixels. `beam_2d_m` is the Fourier transform of the beam $\tilde{B}(\boldsymbol{\ell})$ interpolated azimuthally symmetric at the wavenumbers defined in `ell` from the 1 dimension beam transform `beam1d`. `factor` corresponds to the $\frac{1}{A}$ normalization defined in eq. 4, which is just the integrated beam transform. `norm` is the denominator in eq. 2. The full matched filter from eq. 2 is `f`. The matched filter map $T_{\mathrm{MF}}(\boldsymbol{\theta})$ is `m_filtered`, which is the inverse Fourier transform of the filter multiplied by the Fourier transform of the original map. `fac` is the factor to transform the map, which is in μK, to mJy. The beam area `b_area` is in nsr and the frequency is 90×10^9 Hz. The division by 1000 is because the `flux_factor` function will give the factor from K to Jy, and dividing by 1000 is the factor from μK to mJy.

3 Full-sky example with a Healpy map

References