Bézier Curves — de Casteljau Algorithm

Examples: $n=3$ "Cubic Bézier Curve"

$$\mathbb{C}(t) = \sum_{i=0}^{3} \mathbb{b}_i B_i^3(t) = \sum_{i=0}^{3} \mathbb{b}_i \binom{n}{i}(1-t)^{n-i} t^i$$

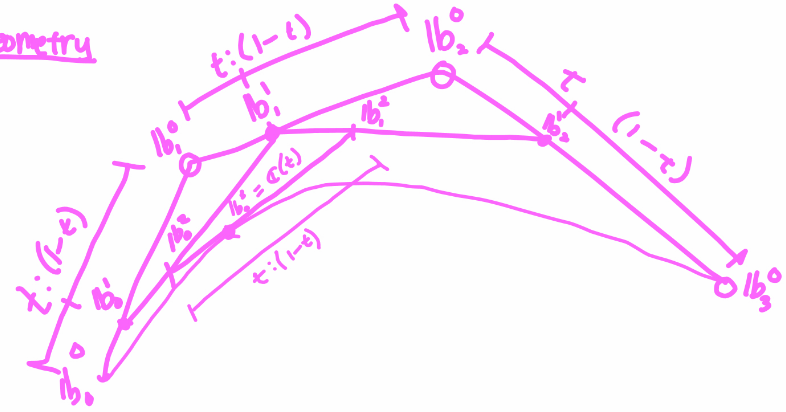
$$= (1-t)^3 \mathbb{b}_0 + 3(1-t)^2 t\, \mathbb{b}_1 + 3(1-t)t^2 \mathbb{b}_2 + t^3 \mathbb{b}_3$$

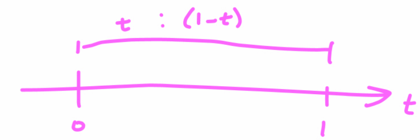
$$= (1-t)\{(1-t)^2 \mathbb{b}_0 + 2(1-t)t\, \mathbb{b}_1 + t^2 \mathbb{b}_2\}$$
$$+ t\{(1-t)^2 \mathbb{b}_1 + 2(1-t)t\, \mathbb{b}_2 + t^2 \mathbb{b}_3\}$$

$$= (1-t)\{(1-t)\langle (1-t)\mathbb{b}_0 + t\mathbb{b}_1\rangle$$
$$+ t\langle (1-t)\mathbb{b}_1 + t\mathbb{b}_2\rangle$$
$$+ t\{(1-t)\langle (1-t)\mathbb{b}_1 + t\mathbb{b}_2\rangle$$
$$+ t\langle (1-t)\mathbb{b}_2 + t\mathbb{b}_3\rangle\}$$

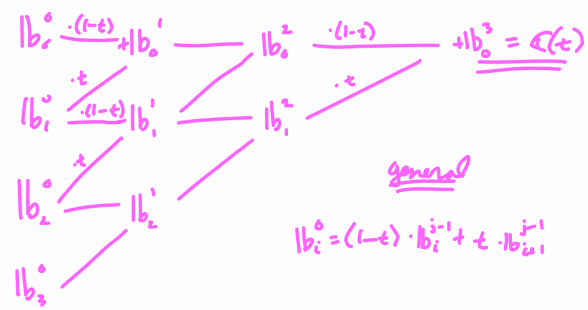
$$= (1-t)\{(1-t)\,\mathbb{b}_0^1 + t\,\mathbb{b}_1^1\}$$
$$+ t\{(1-t)\,\mathbb{b}_1^1 + t\,\mathbb{b}_2^1\}$$
$$= (1-t)\,\mathbb{b}_0^2 + t\,\mathbb{b}_1^2$$
$$= \mathbb{b}_0^3 = \mathbb{C}(t)$$

Geometry





resulting "de Casteljau scheme"



general

$$\mathbb{b}_i^j = (1-t)\cdot \mathbb{b}_i^{j-1} + t \cdot \mathbb{b}_{i+1}^{j-1}$$

"de Casteljau Algorithm"

- **Input**: $\{\mathbb{b}_0^0, \dots, \mathbb{b}_n^0, t \in [0,1]\}$
- **Output**: $P(t)$
- **Algo**: **for** ($j=1$) **to** n **do**

  for ($i=0$) **to** ($n-j$) **do**

  $\mathbb{b}_i^j = (1-t)\mathbb{b}_i^{j-1} + t\,\mathbb{b}_{i+1}^{j-1}$

  return $\{\mathbb{b}_0^n\}$

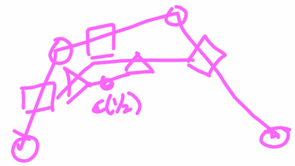
Summary

"situation"

$c(t) = \sum_{i=0}^{n} \mathbb{b}_i B_i^n(t)$

algebra

geometry



scheme

Complexity - time

→ count # of multiplications

$O(n^2)$ in terms of Multiplication

BUT: Can use $O(n)$--time algorithm for evaluation, e.g. HORNER's method

Properties of Bézier curves

1. End point interpolation: $c(0) = \dots = \mathbb{b}_0$, $c(1) = \dots \mathbb{b}_n$

2. Derivatives/velocities @ end points:

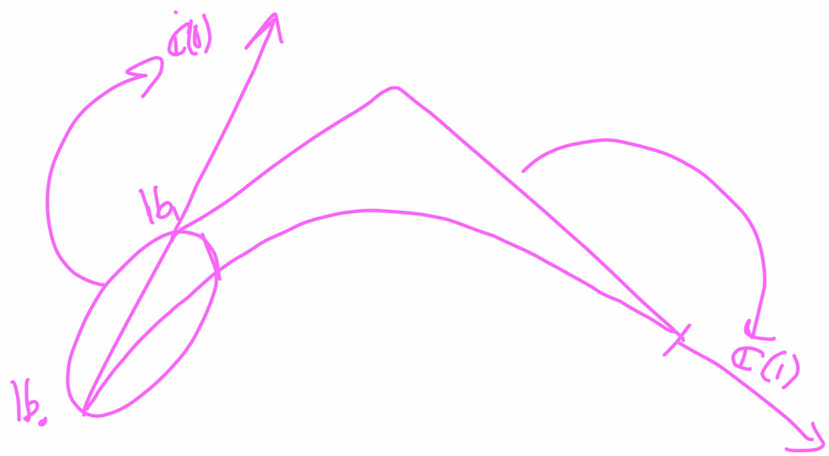
   → differentiate curve:

$$\frac{d}{dt} c(t) = \dot{c} = \left(\sum_{i=0}^{n} \mathbb{b}_i B_i^n(t)\right)' = \dots$$

$$= n \sum_{i=0}^{n-1} (\mathbb{b}_{i+1} - \mathbb{b}_i) B_i^{n-1}(t)$$

$\Rightarrow \dot{\mathbb{C}}(0) = n(\mathbb{b}_1 - \mathbb{b}_0)$

$\dot{\mathbb{C}}(1) = n(\mathbb{b}_n - \mathbb{b}_{n-1})$



3) Convex hull properties

"$\mathbb{C}(t)$ is a subset of / C_i is inside the CONVEX HULL of its defining control points."

Convex hull of $\{\mathbb{b}_0, \ldots \mathbb{b}_n\}$ is

CH $\{\mathbb{b}_0, \ldots \mathbb{b}_n\}$

$\{\mathbb{p} \mid \mathbb{p} = \sum_{i=0}^{n} w_i \mathbb{b}_i$ where

Midterm Topics

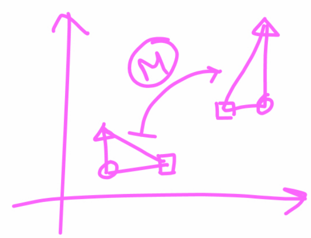
1) DDA, Bresenham...
2) Line & Polygon clipping – know names
3) 2D transforms
   - translate
   - scale
   - reflect
   - rotate

   apply sequence such operation to point, polygon,

usually "homogeneus"

Know Matrices

4) Projections Parallel, perspective



Oblique projection onto $z=0$ plane

5) Coord. / Coord. system transformations

6) Decomposing Difficult 2D/3D transformations into a sequence of SIMPLE one,

7) Phong

special cases

$I = I_a$

$I = I_a + I_d$

$I = I_a + I_d + I_s$

8) Gourand