MIE286 Lecture Notes

Probability and Statistics

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Chapter 1

Statistics Definitions

Global definition: Statistics involves collecting, organizing, summarizing, presenting, and analyzing data, as well as making inferences, conclusions, and decisions based on data.

Statistical definition: A statistic is a numerical value calculated from data (e.g. mean, proportion, standard deviation).

$$\begin{tikzcd} \boxed{\text{Population}} \arrow[r, bend left, "\text{Probability (deductive)}"] & \boxed{\text{Sample}} \arrow[l, bend left, "\text{Statistics Inference (inductive)}"] \end{tikzcd}$$

Basic Terminology

Individuals: Objects on which data are collected (people, animals, plots of land, etc.).
Variable: Any characteristic of an individual.
Population: The entire group of individuals of interest.
Sample: A subset of individuals taken from the population.
Statistical Inference: Drawing conclusions about a population based on a sample.

Sampling Methods

Simple Random Sample (SRS):

- Every possible group of size n has an equal chance of being selected.
- Helps avoid bias in sampling.
- Can be selected using random number tables or software.

Stratified Random Sampling:

- The population is divided into homogeneous groups *(individuals are similar with respect to the variable being studied)* called strata.
- A simple random sample is taken from each stratum. *(one subgroup of the population created)*
- Ensures that important subgroups are neither over nor under represented.

Data, Variables, and Distributions

Types of Variables

Categorical Variable: Places individuals into categories (e.g. gender, major). These are qualitative.

Quantitative Variable: Takes numerical values for which arithmetic operations are meaningful.

- Discrete
- Continuous

Distributions

Distribution: Describes what values a variable takes and how often those values occur. When examining a distribution, look for:

- **Shape**
- **Center**
- **Spread**
- **Outliers**

Outlier: An individual value that falls outside the overall pattern of the data.

Describing Distributions with Numbers

Central Tendency: Describes where the data cluster or center.

Central Tendency: Describes where the data cluster or center.

- Mean: average value
- Median: middle value

Mean (Arithmetic Mean):

$$\boxed{\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

Median:

$$\boxed{\tilde{x} = \begin{cases} x_{\left(\frac{n+1}{2}\right)}, & \text{if } n \text{ is odd} \\ \dfrac{x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)}}{2}, & \text{if } n \text{ is even} \end{cases}}$$

Theorem 1.1

1. The mean is more sensitive to extreme values than the median.
2. Changing a single data value will always change the mean, but may not change the median.
3. If a distribution is exactly symmetric, the mean and median are equal.

Trimmed Mean: The mean computed after removing extreme values.

$$\boxed{\bar{x}_{\text{trim}} = \frac{1}{n-2k} \sum_{i=k+1}^{n-k} x_{(i)}}$$

where k values are removed from both ends of the ordered data. (normally given in question like 10%)

Measures of Spread

Range: Maximum minus minimum. Very sensitive to extreme values.

Sample Variance: Measures the average squared deviation from the mean.

$$\boxed{s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

Standard Deviation: The square root of the sample variance.

$$\boxed{s = \sqrt{s^2}}$$

Degrees of Freedom: The number of independent pieces of information available to estimate variability. For sample variance: $df = n - 1$.

Graphical Representations of Data

Scatter Plot: Used to display the relationship between two quantitative variables (x, y). A scatter plot helps identify trends, patterns, and associations between variables.

Stem-and-Leaf Plot: An intermediate step between raw data and a frequency table. Preserves the original data values while showing the distribution.

Stem	Leaf
1	2 4 7
2	1 3 5 8
3	0 4 6

Relative Frequency Table: Shows the proportion of observations in each class.

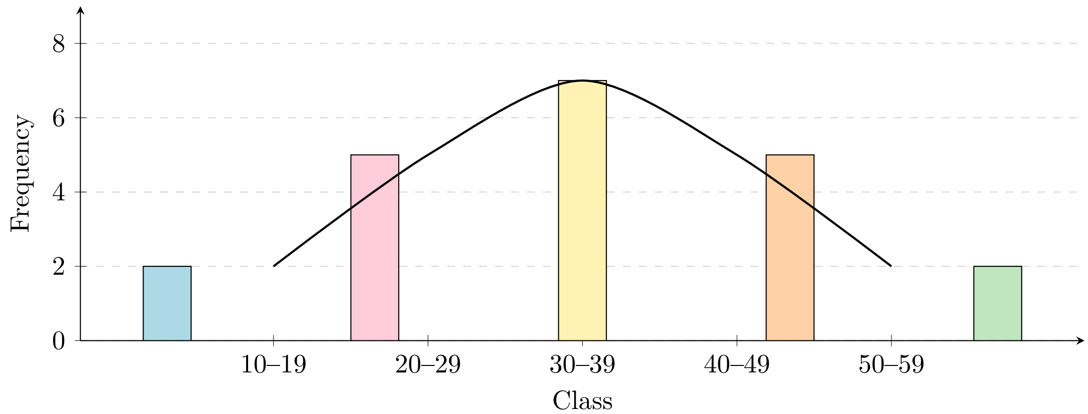
Class Interval	Class Midpoint	Frequency	Relative Frequency
10–19	14.5	3	0.30
20–29	24.5	4	0.40
30–39	34.5	3	0.30

Histogram: A graphical representation of a frequency or relative frequency table using contiguous bars.

When describing the shape of a histogram, we commonly classify it as:

- **Symmetric**
- **Skewed right** (positively skewed)
- **Skewed left** (negatively skewed)

Symmetric distribution example



Chapter 2, Jan 9th

Experiments, Sample Spaces, and Events

Experiment: A process that generates an outcome.

Sample Space (S): The set of all possible outcomes of an experiment.

Example 1:

Select 3 items from a production line. Each item can be classified as either defective (D) or non-defective (N).

$$S = \{DDD, DDN, DND, NDD, DNN, NDN, NND, NNN\}$$

Since each item has 2 possible outcomes,

$$|S| = 2^3 = 8$$

Example 2:

$$S = \{(x, y) \mid x^2 + y^2 \leq 4\}$$

Event (A): A subset of the sample space S.

Examples of events:

$$A = \{DDD, DDN, DND, NDD\}$$

$$B = \{NNN\}$$

$$C = \{(x, y) \mid x^2 + y^2 \leq 4\}$$

Event Operations and Probability Rules

Event Operations:

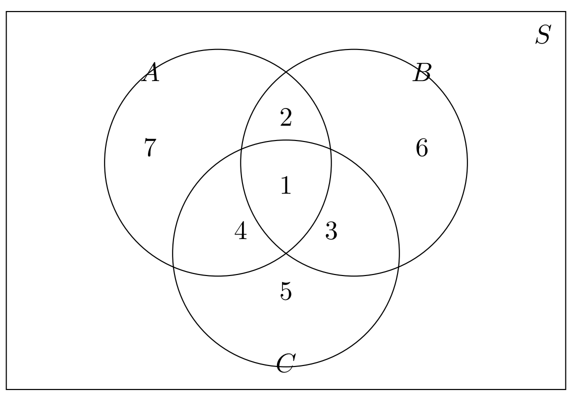
- Complement: A^c (or A')
- Intersection: $A \cap B$
- Union: $A \cup B$
- Null Event: $\varnothing$

If

$$A \cap B = \varnothing,$$

then A and B are mutually exclusive.

Example (Venn Diagram):



$$A = \{DDD, DDN, DND, NDD\}, \quad B = \{NNN\}$$

$$A \cup B = \{DDD, DDN, DND, NDD, NNN\}$$

$$A \cap B = \varnothing$$

Chapter 2: January 12

Review

1. *Experiment:* *A process that generates an outcome.*
2. *Sample Space* (S)*: The set of all possible outcomes of an experiment.*
3. *Event Operations:*
   - *Complement:* A' (A^c)
   - *Intersection:* $A \cap B$
   - *Union:* $A \cup B$
   - *Null Event:* $\varnothing$
4. *If* $A \cap B = \varnothing$*, then* A *and* B *are called mutually exclusive.*

$$(A \cap B)' = A' \cup B'$$

$$(A \cup B)' = A' \cap B'$$

$$A \cap \varnothing = \varnothing$$

$$A \cup \varnothing = A$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Probability

$P(A)$ = probability of event A: the proportion of times the event occurs in infinitely many repetitions of the experiment.

Theorem 2.1

$$0 \leq P(A) \leq 1$$

$$P(A) + P(A') = 1$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\begin{aligned} P(A \cup B \cup C) = P(A) &+ P(B) + P(C) \\ &- P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &+ P(A \cap B \cap C) \end{aligned}$$

Mutually Exclusive Events

Definition: If $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

If

$$A_1 \cup A_2 \cup \cdots \cup A_n = S,$$

then $\{A_1, A_2, \ldots, A_n\}$ is a partition of S.

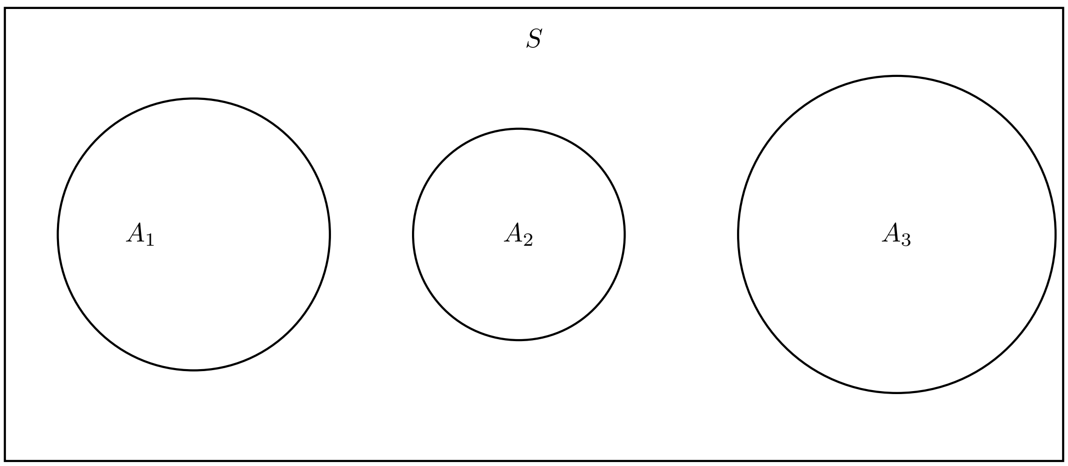


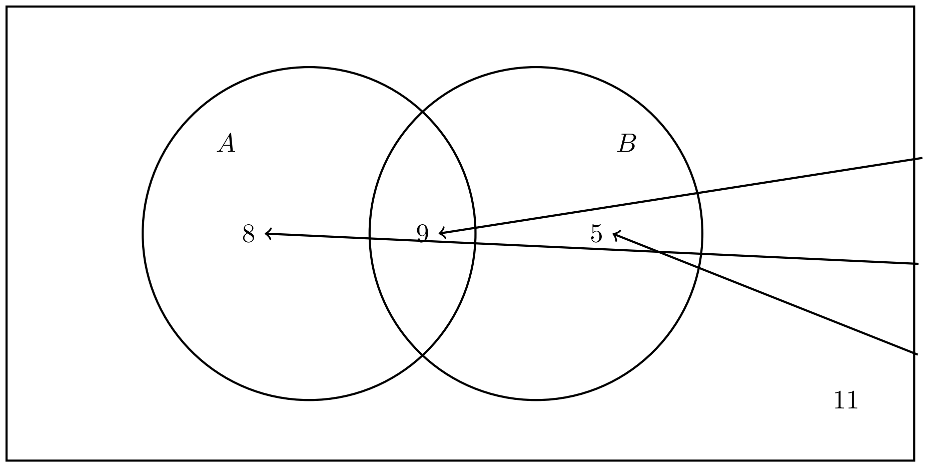
Figure 1: Partition of the sample space S into A_1, A_2, A_3

Example

In a class of 33 students:

- 17 earned an A on the midterm
- 14 earned an A on the final
- 11 earned no A on either exam

Find the probability that a randomly selected student earned A's on both exams.



$$|A \cup B| = 33 - 11 = 22$$

$$|A \cap B| = |A| + |B| - |A \cup B|$$
$$= 17 + 14 - 22 = 9$$

$$|A \setminus B| = |A| - |A \cap B| = 17 - 9 = 8$$

$$|B \setminus A| = |B| - |A \cap B| = 14 - 9 = 5$$

$$P(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{9}{33} \approx 0.273$$

Figure 2: Events A: A on midterm, B: A on final, with region counts and calculations

Counting Techniques and Equally Likely Outcomes

Theorem 2.2 (Equally Likely Outcomes)

If the sample space S has a finite number of outcomes and all outcomes are equally likely, then for any event A,

$$P(A) = \frac{|A|}{|S|}.$$

where

$|A|$ = number of outcomes in event A, $\qquad$ $|S|$ = number of outcomes in the sample space.

Example 1: Poker Hands Basics

A standard deck has:

$$4 \text{ suits} \times 13 \text{ denominations (A,2,3,\ldots,Q,K)} = 52 \text{ cards.}$$

A poker hand consists of 5 cards chosen from 52:

$$|S| = \binom{52}{5} = 2{,}598{,}960.$$

Combinations Reminder

If there are 3 objects $\{A, B, C\}$ and we choose 2:

$$\binom{3}{2} = \frac{3!}{(3-2)!2!}.$$

Order does <u>not</u> matter.

Example 2: Probability of 2 Aces and 1 Jack

A 5-card hand contains:

- exactly 2 aces,
- exactly 1 jack,
- 2 cards that are neither aces nor jacks.

$$P(2 \text{ aces and } 1 \text{ jack}) = \frac{\binom{4}{2}\binom{4}{1}\binom{44}{2}}{\binom{52}{5}}.$$

Example 3: Probability of a Full House

A full house consists of:

- 3 cards of one denomination
- 2 cards of a different denomination

Number of full house hands:

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}.$$

Thus,

$$P(\text{full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}.$$

Example 4: Probability of Four of a Kind

A four of a kind consists of:

- 4 cards of the same denomination
- 1 remaining card of a different denomination

Number of such hands:

$$\binom{13}{1}\binom{4}{4}\binom{48}{1}.$$

Thus,

$$P(\text{four of a kind}) = \frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}.$$

Example 5: Probability of Exactly One Pair

An **excatly** one-pair hand consists of:

- 1 pair
- 3 cards of different denominations, none matching the pair

Number of such hands:

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3.$$

Thus,

$$P(\text{exactly one pair}) = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{5}}.$$

Note*: Counting Patterns

$$\binom{a}{b}$$

Meaning: Choose b different items from a **at once**, order does not matter.

Key features:

- No repeats
- Grouped choice
- Used when items must be distinct

$$\binom{a}{1}^b$$

Meaning: Make b independent choices, each time choosing 1 item from c.

Key features:

- Repeats allowed
- Choices are independent
- Used when selections do not restrict each other

Rule to Remember:

$$\text{Different items, no repeats} \Rightarrow \binom{a}{b}$$

$$\text{Independent choices} \Rightarrow \binom{c}{1}^b$$

Chapter 2 continue, Jan 14

Review

1. *Probability is the proportion of times the event occurs in infinitely many repetitions of the experiment.*

2. $0 \leq P(A) \leq 1$

3. $P(A) + P(A^c) = 1$

4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

5. $$\begin{aligned} P(A \cup B \cup C) = &P(A) + P(B) + P(C) \\ &- P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &+ P(A \cap B \cap C) \end{aligned}$$

6. *Permutation: A permutation counts ordered arrangements.*

$$\boxed{{}_nP_r = \frac{n!}{(n-r)!}}$$

Example 1: Two fair dice

A pair of fair dice are rolled. Find the probability that the second die lands on a smaller value than the first.

The outcomes where the second die is smaller than the first are represented below.

First Die (Stem)	Second Die (Leaf)
2	1
3	1 2
4	1 2 3
5	1 2 3 4
6	1 2 3 4 5

There are 15 favorable outcomes and 36 total outcomes.

$$P(\text{second} < \text{first}) = \frac{15}{36} = \frac{5}{12}.$$

Conditional Probability and Independence

Conditional Probability

The conditional probability of an event B given that event A has occurred is the probability that B occurs when it is known that A has occurred.

$$\boxed{P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0}$$

Example 2: Drinking Survey

A survey records the following data:

	D	N	Total
M	19	41	60
F	12	28	40
Total	31	69	100

The symbols used above are defined as follows:

- M: male
- F: female
- D: the individual drinks
- N: the individual does not drink

$$P(D|M) = \frac{19}{60} \qquad P(M|D) = \frac{19}{31}$$

Law of Total Probability

Theorem 2.3 (Law of Total Probability)

If $B_1, B_2, \ldots, B_k$ form a partition of the sample space S with $P(B_i) > 0$ for all i, then for any event A,

$$P(A) = \sum_{i=1}^{k} P(A \mid B_i)\, P(B_i).$$

Example 3: Monty Hall (3 doors)

Car location	Monty opens	Probability	Stay	Switch
Door 1	Door 2	$\frac{1}{6}$	Car	Goat
Door 1	Door 3	$\frac{1}{6}$	Car	Goat
Door 2	Door 3	$\frac{1}{3}$	Goat	Car
Door 3	Door 2	$\frac{1}{3}$	Goat	Car

Staying wins only when the car is behind Door 1, so

$$P(\text{win by staying}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Switching wins when the car is behind Door 2 or Door 3, so

$$P(\text{win by switching}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Example 4: Birthday Problem

Assume the following:

- Leap years are ignored
- All 365 birthdays are equally likely
- Birthdays of different people are independent

Question: What is the probability that at least two people share the same birthday in a group of n people?

Rather than computing this directly, we use the complement rule.

$$P(\text{at least one match}) = 1 - P(\text{no match})$$

Probability of no shared birthdays

- Person 1 can have any birthday: probability 1
- Person 2 must avoid that birthday: $\frac{364}{365}$
- Person 3 must avoid the first two birthdays: $\frac{363}{365}$
- $\ldots$
- Person n must avoid the previous $n-1$ birthdays: $\frac{365-(n-1)}{365}$

Therefore,

$$P(\text{no match}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-(n-1)}{365}$$

or equivalently,

$$P(\text{no match}) = \prod_{k=0}^{n-1} \frac{365-k}{365}$$

Final result

$$P(\text{at least one shared birthday}) = 1 - \prod_{k=0}^{n-1} \frac{365-k}{365}$$

Important values

- For $n = 23$: $P(\text{at least one match}) \approx 0.507$
- For $n = 57$: $P(\text{at least one match}) \approx 0.99$

Chapter 2 — Jan 16

Review: Conditional Probability

Conditional Probability:

The probability of event B given that event A has occurred is

$$\boxed{P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0}$$

Read as: the probability of B given A.

Independence of Events

Definition (Independence): Events A and B are independent if and only if

$$\boxed{P(B \mid A) = P(B)}$$

Equivalently,

$$P(A \mid B) = P(A)$$

or

$$\boxed{P(A \cap B) = P(A)\,P(B)}$$

Multiple Independent Events

Definition:

If events $A_1, A_2, \ldots, A_k$ are independent, then

$$\boxed{P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)\,P(A_2) \cdots P(A_k)}$$

Mutual Independence

Mutual Independence : A collection of events $A_1, A_2, \ldots, A_n$ is mutually independent if and only if for *every* subcollection $\{A_{i_1}, \ldots, A_{i_k}\}$,

$$\boxed{P\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} P(A_{i_j})}$$

Example (Three Events):

Events A_1, A_2, A_3 are mutually independent if all of the following hold:

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1)P(A_3)$$

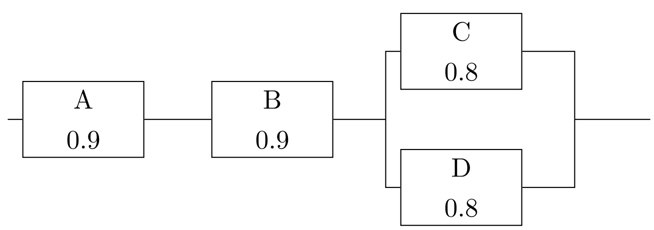
$$P(A_2 \cap A_3) = P(A_2)P(A_3)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

Note: Mutually exclusive events are dependent. If one event occurs, the other cannot occur.

Example: Component Reliability

An electrical system has four components A, B, C, D. The system works if A and B work and at least one of C or D works. Assume all components are independent.



$$P(A) = 0.9, \quad P(B) = 0.9, \quad P(C) = 0.8, \quad P(D) = 0.8$$

(a) Probability the entire system works

The system works if A and B work and either C or D works.

$$\begin{aligned} P(\text{system works}) &= P(\text{all work}) + P(A, B, C \text{ work}, D \text{ does not}) \\ &\quad + P(A, B, D \text{ work}, C \text{ does not}) \end{aligned}$$

$$= (0.9)(0.9)(0.8)(0.8) + (0.9)(0.9)(0.8)(1-0.8) + (0.9)(0.9)(0.8)(1-0.8)$$

$$\boxed{P(\text{system works}) = 0.7776}$$

(b) Conditional probability

$$P(C^c \mid \text{system works}) = \frac{P(C^c \cap \text{system works})}{P(\text{system works})}$$

$$P(C^c \cap \text{system works}) = (0.9)(0.9)(0.8)(1 - 0.8)$$

$$\boxed{P(C^c \mid \text{system works}) = \frac{(0.9)(0.9)(0.8)(1-0.8)}{0.7776} = 0.16}$$

Theorem of Total Probability

Let $B_1, B_2, \ldots, B_k$ be a partition of the sample space S such that $P(B_i) > 0$ for all i. Then for any event $A \subseteq S$,

$$\boxed{P(A) = \sum_{i=1}^{k} P(A \mid B_i)\, P(B_i) = \sum_{i=1}^{k} P(A \cap B_i)}$$

Theorem: Bayes' Rule (1701–1761)

Let $B_1, B_2, \ldots, B_k$ be a partition of the sample space S such that $P(B_i) > 0$ for $i = 1, \ldots, k$. For any event $A \subseteq S$ with $P(A) > 0$,

$$\boxed{P(B_r \mid A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)\, P(A \mid B_r)}{\sum_{i=1}^{k} P(B_i)\, P(A \mid B_i)}, \quad r = 1, \ldots, k}$$

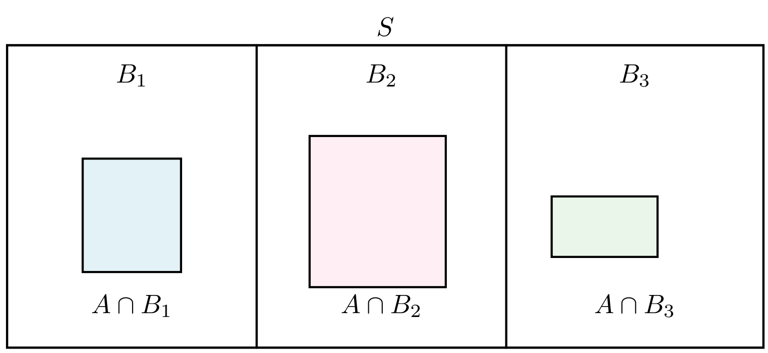


Figure 3: Partition of S into B_1, B_2, B_3 with shaded regions $A \cap B_i$

Example (Medical Test)

The fraction of people in a population who have a certain disease is 0.01.

$$P(D) = 0.01, \quad P(D^c) = 0.99$$

The test characteristics are:

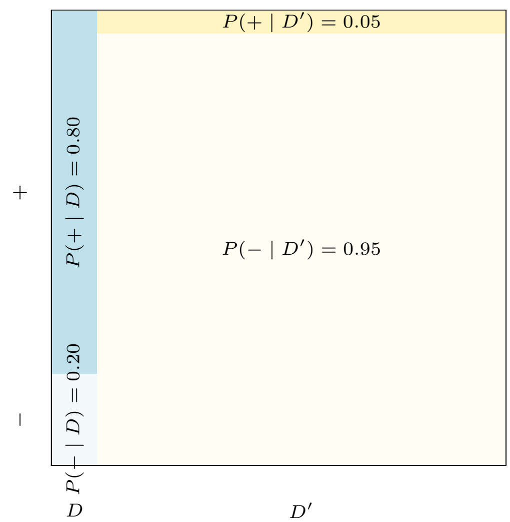
$$P(\text{test says } D \mid D^c) = 0.05 \quad \text{(false positive rate)}$$

$$P(\text{test says } D^c \mid D) = 0.20 \quad \text{(false negative rate)}$$

Thus,

$$P(\text{test says } D \mid D) = 1 - 0.20 = 0.80$$

<u>Note:</u> $1 - P(\text{test says } D^c \mid D)$ is called the <u>sensitivity</u> of the test, and $1 - P(\text{test says } D \mid D^c)$ is called the <u>specificity</u>.



(a) Probability the test says disease

$$P(\text{test says } D) = P(D \cap \text{test says } D) + P(D^c \cap \text{test says } D)$$

$$= P(\text{test says } D \mid D)P(D) + P(\text{test says } D \mid D^c)P(D^c)$$

$$= (0.80)(0.01) + (0.05)(0.99) = 0.0575$$

(b) Probability of disease given positive test

$$P(D \mid \text{test says } D) = \frac{P(D \cap \text{test says } D)}{P(\text{test says } D)}$$

$$= \frac{P(\text{test says } D \mid D)P(D)}{0.0575} = \frac{(0.80)(0.01)}{0.0575}$$

$$\boxed{P(D \mid \text{test says } D) \approx 0.139}$$

(c) Probability of disease given negative test

$$P(D \mid \text{test says } D^c) = \frac{P(D \cap \text{test says } D^c)}{P(\text{test says } D^c)}$$

$$= \frac{P(\text{test says } D^c \mid D)P(D)}{1 - P(\text{test says } D)}$$

$$= \frac{(0.20)(0.01)}{1 - 0.0575}$$

$$\boxed{P(D \mid \text{test says } D^c) \approx 0.00212}$$

Chapter 3 — January 19

Random Variables and Their Interpretation

Definition: A random variable (r.v.) is a rule that assigns a **real number** to each outcome in the sample space.
Alternative definition: A random variable is a function that takes the outcome of an experiment and assigns it a number so that probabilities can be calculated.

Example 1: Three Electronic Components

Each component is classified as either defective (D) or non-defective (N).

$$S = \{NNN, DNN, NDN, NND, DDN, DND, NDD, DDD\}$$

- **Defective (D):** the component does not meet required specifications and fails inspection.
- **Non-defective (N):** the component meets specifications and passes inspection.

Define the random variable

$$X = \text{number of defective components.}$$

Then:

$$\begin{aligned} X &= 0 \quad \text{for } \{NNN\} \\ X &= 1 \quad \text{for } \{DNN, NDN, NND\} \\ X &= 2 \quad \text{for } \{DDN, DND, NDD\} \\ X &= 3 \quad \text{for } \{DDD\} \end{aligned}$$

Thus, the possible values of X are:

$$\{0, 1, 2, 3\}.$$

Example 2: One Component (Dummy Variable)

$$S = \{D, N\}$$

Define the random variable

$$X = \begin{cases} 1, & \text{if the component is defective (D)} \\ 0, & \text{if the component is non-defective (N)} \end{cases}$$

This is called a **dummy variable** because the outcome is categorical, but is encoded numerically.
A <u>dummy variable</u> is a special type of random variable that assigns numerical labels to categorical outcomes, where the numbers have no quantitative meaning beyond identification.

Discrete Random Variables

Definition: A random variable is called <u>discrete</u> if its set of possible values is **countable** (finite or countably infinite).

Example 3: Sampling Until First Defective

Components are tested one(independently) at a time until the first defective component is observed.

$$S = \{D,\ ND,\ NND,\ NNND, \ldots\}$$

Define

$$X = \text{number of components tested until the first defective.}$$

Then:

$$\begin{aligned} X &= 1 \quad \text{for } \{D\} \\ X &= 2 \quad \text{for } \{ND\} \\ X &= 3 \quad \text{for } \{NND\} \\ &\vdots \end{aligned}$$

Hence,

$$X = 1, 2, 3, \ldots$$

Since the possible values can be listed, X is a discrete random variable.

Non-discrete version of the same experiment:

Define

$$Y = \text{time (in seconds) until the first defective component is observed.}$$

Since Y can take any real value in $[0, \infty)$, it cannot be listed and is therefore a continuous (non-discrete) random variable.

Discrete vs. Continuous Random Variables

Discrete Random Variable	**Continuous Random Variable**
Counts things	Measures things
Possible values are countable	Possible values fill an interval
$P(X = x)$ can be > 0	$P(X = x) = 0$ for all x
Uses a probability mass function (PMF)	Uses a probability density function (PDF)

Probability Mass Function (PMF)

Definition:

Let X be a discrete random variable. The probability mass function (PMF) of X, denoted $f(x)$, is defined by:

$$\boxed{\begin{array}{ll} 1) & f(x) \geq 0 \quad \text{for all } x \\ 2) & \displaystyle\sum_x f(x) = 1 \end{array}}$$

Note:

- Capital X: random variable
- Lowercase x: a specific value

Bernoulli and Binomial Random Variables

I. Bernoulli Random Variable (Single Trial)

A Bernoulli random variable: X models a single experiment with only two possible outcomes: success or failure.

$$X = \begin{cases} 1, & \text{success} \\ 0, & \text{failure} \end{cases}$$

If $p = P(X = 1)$, then the PMF is

x	0	1
$P(X = x)$	$1 - p$	p

Here, p is the probability of success (e.g. observing a defective component).

Binomial Random Variable (Multiple Bernoulli Trials)

The binomial random variable extends the Bernoulli case to multiple independent trials.

Definition:

A random variable X is called a binomial random variable if it represents the number of successes in n independent Bernoulli trials, each with success probability p.

$$X = \text{number of successes in } n \text{ trials}$$

In this case,

$$X \sim \text{Bin}(n, p)$$

and the probability mass function is

$$\boxed{P(X = x) = \binom{n}{x} p^x (1 - p)^{n - x}, \quad x = 0, 1, \ldots, n.}$$

Conditions for a Binomial Model

A binomial model applies only if:

- each trial has exactly two outcomes (success or failure),
- the probability of success p is the same for every trial,
- the trials are independent,
- the number of trials n is fixed.

Example: Three Components Tested

Assume each component is defective with probability

$$p = 0.1, \quad n = 3.$$

where p = P(a single component is defective), n = number of trials

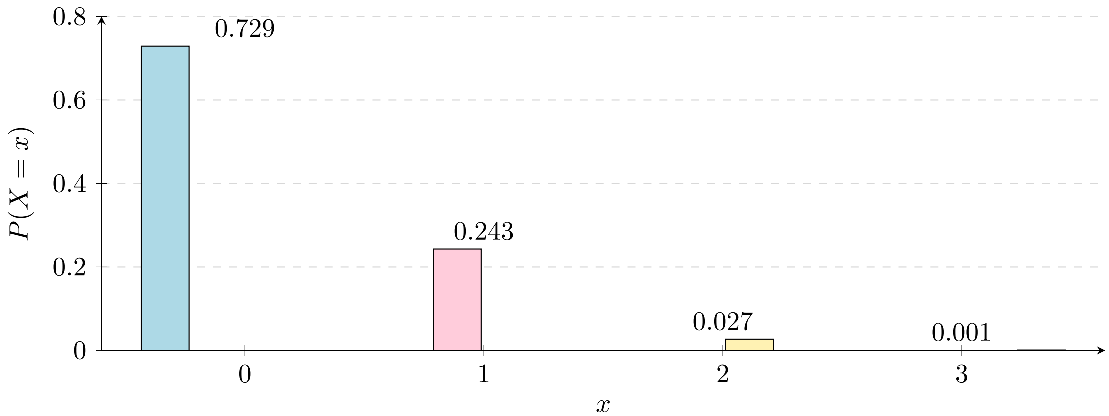
Let

$$X = \text{number of defective components.}$$

x	$P(X = x)$
0	$\binom{3}{0}(0.9)^3 = 0.729$
1	$\binom{3}{1}(0.1)(0.9)^2 = 0.243$
2	$\binom{3}{2}(0.1)^2(0.9) = 0.027$
3	$\binom{3}{3}(0.1)^3 = 0.001$

$$0.729 + 0.243 + 0.027 + 0.001 = 1.$$

PMF of X



Geometric Random Variable

Example: Sampling Until First Defective

Components are sampled one at a time until the first defective component is observed. Assume the probability that a component is defective is

$$p = 0.1.$$

Define the random variable

$$X = \text{number of samples collected until the first defective.}$$

x	$P(X = x) = f(x)$
1	0.1
2	$0.9(0.1)$
3	$0.9^2(0.1)$
$\vdots$	$\vdots$

Geometric Random Variable

Definition:

A random variable X is called a geometric random variable if it represents the number of trials needed to obtain the first success in a sequence of independent Bernoulli trials with success probability p.

$$\boxed{P(X = x) = (1-p)^{x-1}p, \quad x = 1, 2, 3, \ldots}$$

In this example,

$$P(X = x) = 0.9^{x-1}(0.1).$$

Verification That Probabilities Sum to 1

$$\sum_{x=1}^{\infty} 0.9^{x-1}(0.1) = 0.1 \sum_{x=0}^{\infty} 0.9^{x} = 0.1\left(\frac{1}{1-0.9}\right) = 1.$$

Cumulative Distribution Function (CDF)

Definition:

The cumulative distribution function (CDF) of a discrete random variable X with PMF $f(x)$ is defined as

$$\boxed{F(x) = P(X \le x) = \sum_{t \le x} f(t), \quad -\infty < x < \infty.}$$

Example: Binomial Distribution (Three Components)

Let X be the number of defective components when three components are tested, with

$$P(X=0) = 0.729, \quad P(X=1) = 0.243, \quad P(X=2) = 0.027, \quad P(X=3) = 0.001.$$

$$\begin{aligned}
F(0) &= P(X \le 0) = 0.729 \\
F(1) &= P(X \le 1) = 0.729 + 0.243 = 0.972 \\
F(2) &= P(X \le 2) = 0.729 + 0.243 + 0.027 = 0.999 \\
F(3) &= P(X \le 3) = 0.729 + 0.243 + 0.027 + 0.001 = 1
\end{aligned}$$

Thus, the CDF can be written as

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.729, & 0 \le x < 1 \\ 0.972, & 1 \le x < 2 \\ 0.999, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

Properties of the CDF

- $F(x)$ is monotone non-decreasing.

- If $x < y$, then $F(x) \le F(y)$.
- $0 \le F(x) \le 1$.

Note: A function is **monotone** non-decreasing if its value never decreases as the input increases.

Using the CDF to Compute Probabilities

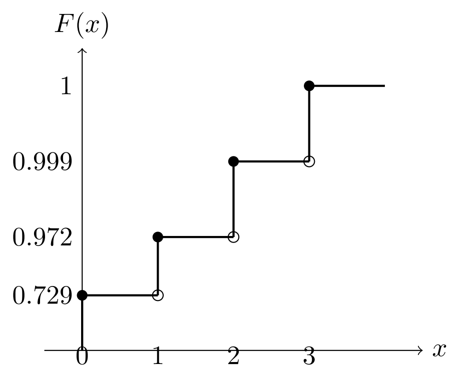
For $a < b$,

$$P(a < X \le b) = F(b) - F(a).$$

Example:

$$P(0 < X \le 2) = F(2) - F(0) = 0.999 - 0.729 = 0.27.$$

CDF Histogram (Step Function)



Note: For a discrete random variable, the PMF is drawn as a bar chart since it shows probabilities at individual points, while the CDF is drawn as a step function since it represents cumulative probability and is monotone non-decreasing.

Chapter 3 — January 21

Review

1. *Random Variable (RV), X*: A random variable assigns a real number to each outcome.
2. *Discrete Random Variable*: If X is discrete,
   - $P(X = x) = f(x)$
   - $f(x)$ is the *probability mass function (PMF)*
   - $f(x) \ge 0$
   - $\sum_x f(x) = 1$
3. *Cumulative Distribution Function (CDF), $F(x)$*:

- $F(x) = P(X \leq x)$
- If X is discrete:

$$F(x) = \sum_{t \leq x} f(t), \quad -\infty < x < \infty$$

Continuous Sample Space and Continuous Random Variables

If the sample space contains an infinite number of outcomes equal to the number of points on a line segment, it is called a continuous sample space.

A continuous random variable has

$$\boxed{P(X = x) = 0 \quad \text{for all } x,}$$

so probabilities are computed over intervals instead of single values.

***Alternative definition**: A continuous sample space contains infinitely many outcomes, like the points on a line segment. For a continuous random variable, the probability of taking any exact value is zero, i.e. $P(X = x) = 0$ for all x. Therefore, probabilities are computed over intervals rather than at single points.

Probability Density Function (PDF)

A function $f(x)$ is called a probability density function (PDF) of a continuous random variable X, defined over $\mathbb{R}$, if:

1. $f(x) \geq 0$ for all $x \in \mathbb{R}$
2. $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$
3. For any $a < b$,

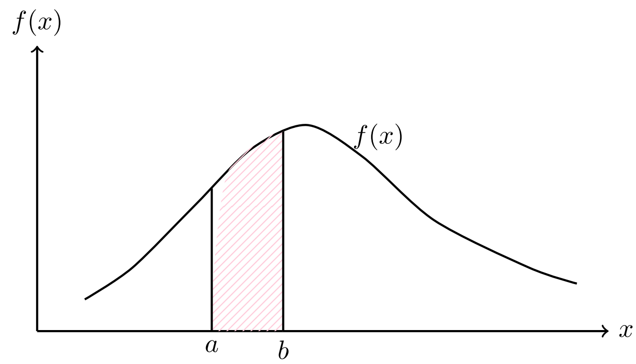
$$P(a < X < b) = \int_a^b f(x)\, dx$$

For continuous random variables,

$$P(a < X < b) = P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b).$$

$$\boxed{P(X = a) = 0}$$

because a single point has zero area under the probability density function.



$$\int_a^b f(x)\,dx = P(a < X < b)$$

Example: Uniform Distribution

Let the probability density function be

$$f(x) = \begin{cases} c, & 5 < x < 10, \\ 0, & \text{otherwise.} \end{cases}$$

1) Determine the value of c

Since $f(x)$ is a probability density function, the total area under the curve must equal 1:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Because $f(x) = 0$ outside the interval $(5, 10)$,

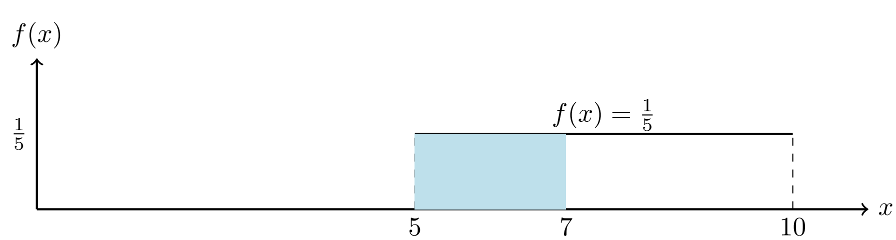
$$\int_5^{10} c\,dx = 1.$$

Evaluating the integral,

$$c(10-5) = 1 \quad \Rightarrow \quad c = \frac{1}{5}.$$

Thus,

$$f(x) = \begin{cases} \frac{1}{5}, & 5 < x < 10, \\ 0, & \text{otherwise.} \end{cases}$$



2): Compute $P(X < 7)$

Formula used:

$$P(a < X < b) = \int_a^b f(x)\, dx.$$

Applying this formula,

$$P(X < 7) = \int_5^7 \frac{1}{5}\, dx = \frac{1}{5}(7 - 5) = \frac{2}{5}.$$

CDF for Continuous Random Variables

Def: Let X be a continuous random variable with pdf: probability density function $f(x)$. The cumulative distribution functio
is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\, dt.$$

Consequently:

$$\boxed{f(x) = \frac{d}{dx} F(x), \qquad P(a < X < b) = F(b) - F(a).}$$

Example

Let X be the time until a chemical reaction is complete (in msec). Suppose the CDF is

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-0.01x}, & x \ge 0. \end{cases}$$

(a) Find the pdf.

Use $f(x) = \dfrac{d}{dx} F(x)$.
For $x < 0$, $F(x) = 0$, so

$$f(x) = 0.$$

For $x \ge 0$,

$$f(x) = \frac{d}{dx}\left(1 - e^{-0.01x}\right) = 0.01e^{-0.01x}.$$

Therefore,

$$f(x) = \begin{cases} 0, & x < 0, \\ 0.01e^{-0.01x}, & x \ge 0. \end{cases}$$

(b) Find $P(X < 200)$**.**

Use the CDF directly:

$$P(X < 200) = F(200) = 1 - e^{-0.01(200)} = 1 - e^{-2} \approx 0.8647.$$

(c) Check if this is a valid CDF.

Theorem: Properties of a Cumulative Distribution Function

A function $F(x)$ is a valid cumulative distribution function (CDF) if and only if:

- $0 \le F(x) \le 1$ for all x
- $F(x)$ is monotone non-decreasing, i.e.

$$x \le y \implies F(x) \le F(y)$$

- $\lim\limits_{x \to -\infty} F(x) = 0$
- $\lim\limits_{x \to \infty} F(x) = 1$

For this $F(x)$:

$$\lim_{x \to -\infty} F(x) = 0, \qquad \lim_{x \to \infty} F(x) = 1,$$

and for $x \ge 0$, $1 - e^{-0.01x}$ increases as x increases, so $F(x)$ is monotone non-decreasing.

Thus, $F(x)$ is a valid CDF.

Joint Probability Distributions (Discrete)

Def: The function $f(x, y)$ is a joint probability mass function (PMF) of discrete random variables X and Y if:

1. $f(x, y) \ge 0$ for all (x, y)
2. $\sum\limits_{x} \sum\limits_{y} f(x, y) = 1$
3. $P(X = x, Y = y) = f(x, y)$

That is, $f(x, y)$ gives the probability that the two random variables X and Y take the values x and y *simultaneously*.

For any region A in the xy-plane,

$$\boxed{P\big((X, Y) \in A\big) = \sum_{(x,y) \in A} f(x, y)}$$

Example: Pen Refills

Two refills are selected at random and without replacement from a box containing:

- 3 blue refills
- 2 red refills
- 3 green refills

Define the random variables

$$X = \text{number of blue refills selected}, \qquad Y = \text{number of red refills selected}.$$

The total number of possible selections is

$$\binom{8}{2}.$$

Joint PMF Table

For each pair (x, y),

$$f(x,y) = \frac{\text{number of favorable outcomes}}{\binom{8}{2}}.$$

$X \backslash Y$	0	1	2	Row Total
0	$\frac{\binom{3}{2}}{\binom{8}{2}}$	$\frac{\binom{2}{1}\binom{3}{1}}{\binom{8}{2}}$	$\frac{\binom{2}{2}}{\binom{8}{2}}$	$\frac{\binom{5}{2}}{\binom{8}{2}}$
1	$\frac{\binom{3}{1}\binom{3}{1}}{\binom{8}{2}}$	$\frac{\binom{3}{1}\binom{2}{1}}{\binom{8}{2}}$	0	$\frac{15}{\binom{8}{2}}$
2	$\frac{\binom{3}{2}}{\binom{8}{2}}$	0	0	$\frac{3}{\binom{8}{2}}$
Column Total	$\frac{15}{\binom{8}{2}}$	$\frac{12}{\binom{8}{2}}$	$\frac{1}{\binom{8}{2}}$	1

Marginal Distributions

Def: Let $f(x, y)$ be the joint PMF of X and Y.

The marginal PMF of X is obtained by summing over all values of Y:

$$g(x) = \sum_y f(x,y).$$

The marginal PMF of Y is obtained by summing over all values of X:

$$h(y) = \sum_x f(x,y).$$

Marginal example:

$$P(X=1) = \sum_y P(X=1, Y=y).$$

Conditional Distributions (Discrete)

Def: The conditional PMF of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{g(x)}, \quad g(x) > 0.$$

Similarly, the conditional PMF of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{h(y)}, \quad h(y) > 0.$$

Example: Conditional Probabilities

1. $$P(Y = 1 \mid X = 1) = \frac{P(X = 1, Y = 1)}{P(X = 1)} = \frac{3/28}{15/28} = \frac{3}{15}.$$

2. $$P(X = 0 \mid Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{9/28}{15/28} = \frac{9}{15}.$$

3. $$P(Y = 2 \mid X = 1) = 0.$$

Check:

$$\sum_y P(Y = y \mid X = 1) = 1.$$

Review: Chapter 3 — Random Variables

1. *Random Variable* A random variable is a function that maps outcomes of an experiment to real numbers.
   - Domain: sample space outcomes
   - Range: real numbers
   - Can be **discrete** or **continuous**

2. *Probability Mass Function (PMF)* The PMF of a discrete random variable X is

   $$f_X(x) = P(X = x)$$

   - Only for discrete random variables
   - $f_X(x) \geq 0$
   - $\sum_x f_X(x) = 1$

3. *Probability Density Function (PDF)* For a continuous random variable X, probability is defined by

   $$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

- Only for continuous random variables
- Area under the curve gives probability
- $P(X = x) = 0$

4. *Cumulative Distribution Function (CDF)* The CDF is defined as

$$F_X(x) = P(X \leq x)$$

- Discrete: $F(x) = \sum_{t \leq x} f(t)$
- Continuous: $F(x) = \int_{-\infty}^{x} f(t)\, dt$
- $0 \leq F(x) \leq 1$, non-decreasing

5. *Joint Distribution* The joint distribution describes probabilities involving two random variables X and Y.

- Discrete: $f_{X,Y}(x, y) = P(X = x, Y = y)$
- Continuous: joint PDF $f_{X,Y}(x, y)$

6. *Marginal Distribution* A marginal distribution is obtained by eliminating the other variable.

- $f_X(x) = \sum_y f_{X,Y}(x, y)$ or $f_X(x) = \int f_{X,Y}(x, y)\, dy$
- $f_Y(y) = \sum_x f_{X,Y}(x, y)$ or $f_Y(y) = \int f_{X,Y}(x, y)\, dx$

Joint Probability Density Function (Continuous)

Def: A function $f(x, y)$ is a joint probability density function (PDF) of continuous random variables X and Y if:

1. $f(x, y) \geq 0$ for all (x, y)
2. $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$
3. For any region A in the xy-plane,

$$P\big((X, Y) \in A\big) = \iint_A f(x, y)\, dx\, dy$$

Geometric interpretation: The joint PDF is a surface above the xy-plane. Probabilities correspond to the **volume under the surface** over a specified region.

Example 1:

$$f(x, y) = \begin{cases} \dfrac{12}{7}(x^2 + xy), & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Verify it is a valid joint PDF:

$$\int_0^1 \int_0^1 \frac{12}{7}(x^2 + xy)\, dx\, dy = 1$$

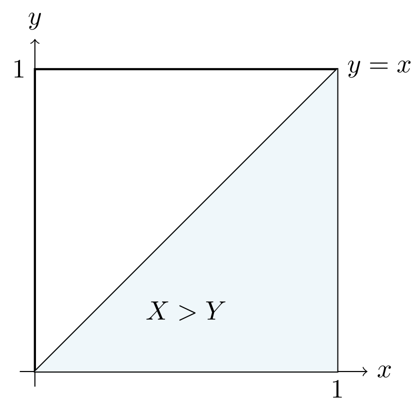
(b) Find $P(0 < X < 0.2,\ 0 < Y < 1)$**:**

$$P = \int_0^1 \int_0^{0.2} \frac{12}{7}(x^2 + xy)\, dx\, dy$$

(c) Find $P(X > Y)$**:**

The region $X > Y$ corresponds to the area below the line $y = x$ in the unit square $0 \leq x \leq 1,\ 0 \leq y \leq 1$.

$$P(X > Y) = \int_0^1 \int_0^x \frac{12}{7}(x^2 + xy)\, dy\, dx = \frac{9}{14}$$



(d) Find $P(X = Y)$**:**

$$P(X = Y) = \int_0^1 \int_y^y \frac{12}{7}(x^2 + xy)\, dx\, dy = 0$$

(Probability along a line is zero for continuous random variables.)

Example:

Let the joint PDF be

$$f(x, y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

We want to compute

$$P(X + Y \geq 1).$$

The support of the joint PDF is the region $0 < x < y$, which lies above the line $y = x$ in the first quadrant.

The boundary of the event $X + Y \geq 1$ is the line $x + y = 1$.

Rather than integrating over the unbounded region $X+Y \geq 1$, we compute the complement:

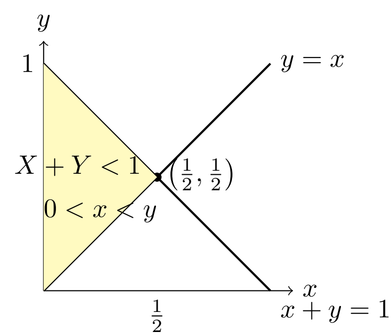
$$P(X+Y \geq 1) = 1 - P(X+Y < 1).$$

The region $X+Y<1$ that lies within the support is bounded by:

$$0 \leq x \leq \tfrac{1}{2}, \qquad x \leq y \leq 1-x.$$

Therefore,

$$P(X+Y \geq 1) = 1 - \int_0^{1/2} \int_x^{1-x} e^{-y}\, dy\, dx = 2e^{-1/2} - e^{-1}.$$



Marginal and Conditional PDFs

Def: The marginal PDFs of X and Y are defined as

$$g_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy, \qquad h_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

Example (continued):

$$f(x,y) = \begin{cases} \dfrac{12}{7}(x^2+xy), & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Marginal PDF of X:

$$g_X(x) = \int_0^1 \frac{12}{7}(x^2+xy)\, dy = \frac{12}{7}\left(x^2 + \frac{x}{2}\right), \qquad 0 \leq x \leq 1$$

Marginal PDF of Y:

$$h_Y(y) = \int_0^1 \frac{12}{7}(x^2+xy)\, dx = \frac{12}{7}\left(\frac{1}{3} + \frac{y}{2}\right), \qquad 0 \leq y \leq 1$$

Conditional PDF of Y given $X=x$:

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{\frac{12}{7}(x^2+xy)}{\frac{12}{7}\left(x^2+\frac{x}{2}\right)}, \qquad 0 \le y \le 1,\ 0 < x \le 1$$

Why the bounds are $0 < x \le 1$ and not $0 \le x \le 1$:

The marginal PDF

$$g(x) = \frac{12}{7}\left(x^2 + \frac{x}{2}\right)$$

satisfies $g(0) = 0$.

Since the conditional PDF is defined as

$$f_{Y|X}(y|x) = \frac{f(x,y)}{g(x)},$$

it is **undefined at $x = 0$** due to division by zero.

Therefore, the conditional density is only defined for values of x such that

$$g(x) > 0 \ \Rightarrow\ 0 < x \le 1.$$

Key takeaway: The bounds of a conditional PDF exclude points where the conditioning density is zero.

Statistical Independence

Def: Random variables X and Y (discrete or continuous) are statistically independent if and only if

$$f(x,y) = g(x)\,h(y) \quad \text{for all } (x,y) \text{ in their range}$$

Consequences:

- $f(x|y) = g(x)$
- $f(y|x) = h(y)$

Recall Example: Pen Refills

Two refills are selected at random and without replacement from a box containing:

- 3 blue refills
- 2 red refills
- 3 green refills

Define the random variables

$$X = \text{number of blue refills selected}, \qquad Y = \text{number of red refills selected}.$$

Total outcomes:

$$\binom{8}{2} = 28.$$

Joint PMF Table

For each pair (x, y),

$$f(x, y) = \frac{\text{number of favorable outcomes}}{28}.$$

$X \backslash Y$	0	1	2	$g(x) = P(X = x)$
0	$\frac{3}{28}$	$\frac{3}{14}$	$\frac{1}{28}$	$\frac{5}{14}$
1	$\frac{9}{28}$	$\frac{3}{14}$	0	$\frac{15}{28}$
2	$\frac{3}{28}$	0	0	$\frac{3}{28}$
$h(y) = P(Y = y)$	$\frac{15}{28}$	$\frac{3}{7}$	$\frac{1}{28}$	1

Marginals: $g(x) = P(X = x)$ is the marginal PMF of X and is given by the **row totals**. $h(y) = P(Y = y)$ is the marginal PMF of Y and is given by the **column totals**.

$$g(0) = \frac{5}{14}, \quad g(1) = \frac{15}{28}, \quad g(2) = \frac{3}{28}$$
$$h(0) = \frac{15}{28}, \quad h(1) = \frac{3}{7}, \quad h(2) = \frac{1}{28}$$

Independence check: If X and Y were statistically independent, then $f(x, y) = g(x)h(y)$ for all (x, y). Check $(x, y) = (0, 1)$:

$$f(0, 1) = \frac{3}{14}, \qquad g(0)h(1) = \left(\frac{5}{14}\right)\left(\frac{3}{7}\right) = \frac{15}{98}$$

$$\frac{3}{14} \neq \frac{15}{98} \qquad \Rightarrow \qquad X \text{ and } Y \text{ are not independent.}$$

Example: Independence via Factorization (Discrete Case)

Let X and Y be discrete random variables whose values in the nonnegative integers.

Suppose the joint PMF is

$$f(x, y) = \frac{1}{x!\, y!} \lambda^x \mu^y e^{-(\lambda+\mu)}, \qquad x, y = 0, 1, 2, \ldots$$

Factorization:

We can write

$$f(x, y) = \left(\frac{\lambda^x e^{-\lambda}}{x!}\right)\left(\frac{\mu^y e^{-\mu}}{y!}\right) = g(x)\, h(y)$$

Marginal PMF of X:

$$g(x) = \sum_{y=0}^{\infty} f(x,y) = \sum_{y=0}^{\infty} \frac{1}{x!\,y!}\,\lambda^x \mu^y\, e^{-(\lambda+\mu)}$$

Factor out terms that do not depend on y:

$$g(x) = \frac{1}{x!}\,\lambda^x e^{-(\lambda+\mu)} \sum_{y=0}^{\infty} \frac{\mu^y}{y!}$$

Using $\displaystyle\sum_{y=0}^{\infty} \frac{\mu^y}{y!} = e^{\mu}$:

$$g(x) = \frac{1}{x!}\,\lambda^x e^{-(\lambda+\mu)} e^{\mu} = \frac{1}{x!}\,\lambda^x e^{-\lambda}, \qquad x = 0, 1, 2, \ldots$$

Marginal PMF of Y:

$$h(y) = \sum_{x=0}^{\infty} f(x,y) = \sum_{x=0}^{\infty} \frac{1}{x!\,y!}\,\lambda^x \mu^y\, e^{-(\lambda+\mu)}$$

Factor out terms that do not depend on x:

$$h(y) = \frac{1}{y!}\,\mu^y e^{-(\lambda+\mu)} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

Using $\displaystyle\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$:

$$h(y) = \frac{1}{y!}\,\mu^y e^{-(\lambda+\mu)} e^{\lambda} = \frac{1}{y!}\,\mu^y e^{-\mu}, \qquad y = 0, 1, 2, \ldots$$

Conclusion:

Since the joint PMF can be written as

$$f(x,y) = g(x)\,h(y)$$

for all x, y, the random variables X and Y are statistically independent. ✓

Important notes:

- Factorization of the joint PMF is **sufficient** to prove independence.
- The constant terms (such as $e^{-\lambda}$, $e^{-\mu}$) must be included to obtain the **correct marginals**.
- Independence requires that every combination of values with positive marginal probability also has positive joint probability.
  This means that if $X = x$ is possible and $Y = y$ is possible, then the pair $(X = x, Y = y)$ must also be possible.

$$\{(x,y) : f(x,y) > 0\} = \{x : g(x) > 0\} \times \{y : h(y) > 0\}.$$

This means that every value of X with positive marginal probability can occur with every value of Y with positive marginal probability.

Jan 30

Joint distribution

Describes probabilities involving two random variables X and Y.

- Discrete:
$$f(X,Y) = P(X = x, Y = y)$$
- Continuous: joint PDF $f(X,Y)$

Marginal distribution

Obtained by eliminating the other variable.

-
$$f(X) = \sum_y f(X,Y), \qquad f(Y) = \sum_x f(X,Y)$$
-
$$f(X) = \int_{-\infty}^{\infty} f(X,Y)\,dy, \qquad f(Y) = \int_{-\infty}^{\infty} f(X,Y)\,dx$$

Joint PDF validity

A function $f(X,Y)$ is a valid joint PDF if:

- $f(X,Y) \geq 0$
-
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(X,Y)\,dx\,dy = 1$$

Example (valid joint PDF):

$$f(X,Y) = \begin{cases} \frac{12}{7}(x^2 + xy), & 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

Geometric interpretation

Probabilities correspond to the **volume under the surface** $f(X,Y)$ over a region.

Conditional PDF

Distribution of one variable given the other.

- $$f(Y \mid X) = \frac{f(X,Y)}{f(X)}, \quad f(X) > 0$$

Statistical Independence

Definition: Random variables X and Y are statistically independent if

$$f(X,Y) = f(X)f(Y)$$

This means knowing the value of one variable gives no information about the other.

Support and Independence

Support: The support of a joint distribution is the set

$$\{(x,y) : f(X,Y) > 0\}$$

Key idea: If X and Y are independent, the support must factor as

$$\{x : f(X) > 0\} \times \{y : f(Y) > 0\}$$

That is, every allowed value of X can occur with every allowed value of Y.

Example: Non-Independent Random Variables

$$f(X,Y) = \begin{cases} 4(x+y^2), & xy > 0,\ x+y \le 1, \\ 0, & \text{otherwise} \end{cases}$$

The condition $x + y \le 1$ links x and y, so the support does not factor.

$$\Rightarrow X \text{ and } Y \text{ are not independent}$$

Why this implies dependence (key intuition):

Pick a value of X that is allowed:

$$x = 0.8 \quad (\text{positive and } < 1)$$

Pick a value of Y that is allowed:

$$y = 0.8 \quad (\text{positive and } < 1)$$

Individually, both values are valid. But together:

$$x + y = 0.8 + 0.8 = 1.6 > 1$$

This violates the condition $x + y \le 1$, so the pair $(0.8, 0.8)$ is impossible.

Conclusion: Knowing the value of X restricts which values Y can take. Therefore, X and Y are not independent.

Independent Random Variables

If random variables are independent, joint probabilities factor.

Example: Let X_1, X_2, X_3 be independent with

$$f(X) = e^{-x}, \quad x > 0$$

Then

$$f(X_1, X_2, X_3) = e^{-x_1} e^{-x_2} e^{-x_3}$$

$$P(X_1 < 2,\ 1 < X_2 < 3,\ X_3 > 2) = (1 - e^{-2})(e^{-1} - e^{-3})e^{-2}$$

Explanation (why this works):

Since X_1, X_2, X_3 are independent, the joint probability factors:

$$P(X_1 < 2,\ 1 < X_2 < 3,\ X_3 > 2) = P(X_1 < 2)\, P(1 < X_2 < 3)\, P(X_3 > 2)$$

For an exponential random variable with

$$f(X) = e^{-x}, \quad x > 0,$$

we have:

$$P(X < a) = 1 - e^{-a}, \qquad P(X > a) = e^{-a}$$

Thus,

$$P(X_1 < 2) = 1 - e^{-2}$$

$$P(1 < X_2 < 3) = P(X_2 < 3) - P(X_2 < 1) = e^{-1} - e^{-3}$$

$$P(X_3 > 2) = e^{-2}$$

Multiplying gives:

$$(1 - e^{-2})(e^{-1} - e^{-3})e^{-2}$$

Mutual Independence and Modeling Assumptions

Mutual independence: Random variables $X_1, \ldots, X_n$ are mutually independent if

$$f(X_1, \ldots, X_n) = \prod_{i=1}^{n} f(X_i)$$

Pairwise independence does **not** imply mutual independence.

Independent selection: Selections are independent if:

- each selection is random,
- distributions are identical,
- outcomes do not affect future selections.

Sampling without replacement generally produces dependent variables.

Chapter 4

Motivation

Example: If you roll a fair die repeatedly, what *average value* do you expect in the long run?

This motivates the idea of expectation: a theoretical long-run average.

Expected Value

Expected Value of a Random Variable

Def: For a random variable X, the expectation (or expected value or mean) is the long-run average value of X.

Discrete random variable:

$$\boxed{\mu = E(X) = \sum_{x} x\, f(X)}$$

Continuous random variable:

$$\boxed{\mu = E(X) = \int_{-\infty}^{\infty} x\, f(X)\, dx}$$

Interpretation: Expectation is a weighted average, where values of X are weighted by how likely they are.

Example: Fair Die

Let X be the outcome when a fair die is rolled.

$$E(X) = \sum_{x=1}^{6} x \cdot \frac{1}{6}$$

$$E(X) = \frac{1+2+3+4+5+6}{6} = 3.5$$

Even though 3.5 is not a possible outcome, it represents the long-run average.

Example: Number of Messages per Hour

Let X be the number of messages sent per hour, with PMF:

x	10	11	12	13	14	15
$f(X)$	0.08	0.15	0.30	0.20	0.20	0.07

Check:

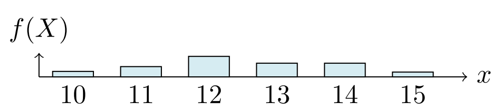
$$\sum f(X) = 1$$

Expected value:

$$\begin{aligned} E(X) &= 10(0.08) + 11(0.15) + 12(0.30) + 13(0.20) + 14(0.20) + 15(0.07) \\ &= 12.5 \end{aligned}$$

This means that over many hours, the average number of messages per hour is about 12.5.

PMF Visualization



Example: Deal or No Deal

Consider a game with two possible outcomes:

- \$1 with probability 1/2
- \$10,000 with probability 1/2

Expected value:

$$E(X) = \frac{1}{2}(1) + \frac{1}{2}(10{,}000) = 5000.5$$

Key idea: The expected value is the average payout in the long run, not the most likely outcome.

Example: Continuous RV (Device Lifetime)

Let X be a random variable that denotes the lifetime (in hours) of a certain device, with PDF

$$f(X) = \begin{cases} \dfrac{20000}{x^3}, & x > 100, \\ 0, & \text{otherwise.} \end{cases}$$

Check (valid PDF):

$$\int_{-\infty}^{\infty} f(X)\,dx = \int_{100}^{\infty} \frac{20000}{x^3}\,dx = 20000\left[\frac{-1}{2x^2}\right]_{100}^{\infty} = 1$$

Expected lifetime:

$$E(X) = \int_{100}^{\infty} x\frac{20000}{x^3}\,dx = \int_{100}^{\infty} \frac{20000}{x^2}\,dx = 20000\left[\frac{-1}{x}\right]_{100}^{\infty} = 200$$

So we expect this type of device to last on average $\boxed{200}$ hours.

Example: Discrete RV and Transformation

Let X be a discrete random variable with PMF:

x	-1	0	1	2
$f(X)$	0.3	0.2	0.3	0.2

Define a new random variable as a transformation of X:

$$g(X) = X^2$$

Possible values of $g(X)$:

$$g(-1) = 1, \quad g(0) = 0, \quad g(1) = 1, \quad g(2) = 4$$

So $g(X)$ can take values $\{0, 1, 4\}$.

Distribution of $g(X)$:

$$P(g(X) = 0) = P(X = 0) = 0.2$$

$$P(g(X) = 1) = P(X = -1) + P(X = 1) = 0.3 + 0.3 = 0.6$$

$$P(g(X) = 4) = P(X = 2) = 0.2$$

$g(X)$	0	1	4
$P(g(X))$	0.2	0.6	0.2

This is called a transformation of a random variable.

Expected value of the transformed RV:

$$\boxed{E(g(X)) = E(X^2) = \sum_x x^2 f(X) = \sum_x g(x)\,f(X)}$$

Numerically:

$$E(X^2) = 0^2(0.2) + (-1)^2(0.3) + (1)^2(0.3) + (2)^2(0.2) = 0 + 0.3 + 0.3 + 0.8 = 1.4$$

Expected Value of a Function of a RV

Let X be a random variable with distribution $f(X)$. The expected value of the random variable $g(X)$ is

$$\boxed{E(g(X)) = \begin{cases} \sum_x g(x)\,f(X), & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x)\,f(X)\,dx, & \text{if } X \text{ is continuous} \end{cases}}$$

Example: Chip Game (Expected Winnings)

A bowl contains 5 chips:

- 3 chips are worth \$1 each
- 2 chips are worth \$4 each

A player draws 2 chips at random (without replacement) and is paid the sum.

Let X be the number of \$1 chips drawn. Then

$$X \in \{0, 1, 2\}.$$

PMF of X: (hypergeometric)

$$f(X) = \begin{cases} \dfrac{\binom{3}{x}\binom{2}{2-x}}{\binom{5}{2}}, & x = 0, 1, 2, \\ 0, & \text{otherwise.} \end{cases}$$

Define payout as a function of X: If you draw x one-dollar chips, then you draw $2 - x$ four-dollar chips, so the payout is

$$g(x) = 1(x) + 4(2 - x) = 8 - 3x.$$

So the payout random variable is $g(X) = 8 - 3X$.

Expected payout:

$$E(g(X)) = \sum_{x=0}^{2} g(x)\,f(X)$$

Compute $f(X)$ values:

$$f(0) = \frac{\binom{3}{0}\binom{2}{2}}{\binom{5}{2}} = \frac{1}{10}, \quad f(1) = \frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}} = \frac{6}{10}, \quad f(2) = \frac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}} = \frac{3}{10}.$$

Then

$$E(g(X)) = \sum_{x=0}^{2}(8-3x)\,f(X) = (8)\left(\frac{1}{10}\right) + (5)\left(\frac{6}{10}\right) + (2)\left(\frac{3}{10}\right) = 4.4$$

Decision: If it costs \$4.75 to play, your expected profit is

$$E(\text{profit}) = E(g(X)) - 4.75 = 4.4 - 4.75 = -0.35$$

So in the long run, you lose about $\boxed{\$0.35}$ per game on average, so you should <u>not</u> play.

Notation: $g(X)$ vs. $g(x)$

- X is a <u>random variable</u>; x is a specific value it can take.
- $g(X)$ is a <u>random variable</u>.
- $g(x)$ is a <u>number</u>.

Key rule:

$$\boxed{E(g(X)) = \sum_x g(x)\,f(X)}$$

Expectation averages the numerical values $g(x)$, weighted by their probabilities.

Exam rule:

$$\boxed{g(X) \text{ is random variable, } g(x) \text{ is a number.}}$$

February 2

Chapter 4 Review: Expectation

1. <u>*Expected value*</u>, $\mu = E(X)$
   - **Discrete:**
     $$E(X) = \sum_x x\,f(X)$$
   - **Continuous:**
     $$E(X) = \int_{-\infty}^{\infty} x\,f(X)\,dx$$
2. <u>*Expected value of a function*</u>, $E[g(X)]$
   - $g(X)$ is a function of X

- **Discrete:**

$$E[g(X)] = \sum_x g(x)\, f(X)$$

- **Continuous:**

$$E[g(X)] = \int_{-\infty}^{\infty} g(X)\, f(X)\, dx$$

3. **Example (continuous RV):**

$$f(X) = 4x^3, \quad 0 < x < 1$$

- $$E(X) = \int_0^1 x \cdot 4x^3\, dx$$

- $$E(X^2) = \int_0^1 x^2 \cdot 4x^3\, dx$$

Expected Value of a Function of a Random Variable

1. *Expected value of a function: expected average value in the long wrong*, $E[g(X)]$

- $g(X)$ is a function of X
- **Discrete:**

$$E[g(X)] = \sum_x g(x)\, f(X)$$

- **Continuous:**

$$E[g(X)] = \int_{-\infty}^{\infty} g(X)\, f(X)\, dx$$

2. **Example (continuous RV):**

$$f(X) = 4x^3, \quad 0 < x < 1$$

- $$E(X) = \int_0^1 x \cdot 4x^3\, dx$$

- $$E(X^2) = \int_0^1 x^2 \cdot 4x^3\, dx$$

Expected Value from a Joint Distribution

General formula

$$\boxed{E[g(X,Y)] = \int\int g(x,y)\, f(X,Y)\, dx\, dy}$$

Computing $E(X)$ from a joint PDF

There are two equivalent methods to compute $E(X)$.

Method 1: Using the marginal distribution

First find the marginal of X:

$$f(X) = \int_0^1 x(1+3y^2)\,dy$$

$$= x\int_0^1 (1+3y^2)\,dy$$

$$= x\left[y + y^3\right]_0^1 = x(1+1) = 2x$$

Now compute the expected value:

$$E(X) = \int_0^2 x f(X)\,dx = \int_0^2 x(2x)\,dx$$

$$= 2\int_0^2 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^2 = \frac{16}{3}$$

Method 2: Direct double integral

Apply the general formula with $g(X,Y) = X$:

$$E(X) = \int_0^1 \int_0^2 x\, f(X,Y)\,dx\,dy$$

$$= \int_0^1 \int_0^2 x \cdot x(1+3y^2)\,dx\,dy$$

$$= \int_0^1 (1+3y^2)\left(\int_0^2 x^2\,dx\right)dy$$

$$= \int_0^1 (1+3y^2)\left[\frac{x^3}{3}\right]_0^2 dy = \int_0^1 (1+3y^2)\frac{8}{3}\,dy$$

$$= \frac{8}{3}\int_0^1 (1+3y^2)\,dy = \frac{8}{3}\cdot 2 = \frac{16}{3}$$

Both methods give the same result:

$$\boxed{E(X) = \frac{16}{3}}$$

Properties of Expected Value

Properties of Expectation

Let X, Y be random variables and let $a, b \in \mathbb{R}$ be constants.

1. **Linearity (scaling and shifting):**

$$\boxed{E(aX+b) = aE(X) + b}$$

Continuous case:

$$E(aX+b) = \int_{-\infty}^{\infty} (ax+b)\, f(X)\, dx$$
$$= a\int_{-\infty}^{\infty} x f(X)\, dx + b\int_{-\infty}^{\infty} f(X)\, dx$$

Note that

$$\int_{-\infty}^{\infty} f(X)\, dx = 1$$

because $f(X)$ is a probability density function and the total probability over its entire support must equal 1.

Therefore,

$$E(aX+b) = aE(X) + b$$

2. **Expectation of a constant:**

$$\boxed{E(a) = a}$$

3. **Additivity:**

$$\boxed{E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]}$$

4. **Non-negativity:**

$$\boxed{X \geq 0 \;\Rightarrow\; E(X) \geq 0}$$

5. **Zero expectation:**

$$\boxed{X \geq 0 \text{ and } E(X) = 0 \iff P(X=0) = 1}$$

Variance

$E(X)$ is a measure of the **center** of a distribution.

The **variance**, denoted $\sigma^2 = \mathrm{Var}(X)$, measures how closely the distribution is concentrated around the mean μ.

Definition:

$$\boxed{\mathrm{Var}(X) = E[(X-\mu)^2], \quad \mu = E(X)}$$

$\sigma = \sqrt{\sigma^2}$ is called the **standard deviation**.

Theorem (Variance Formula)

$$\mathrm{Var}(X) = \sigma^2 = E(X^2) - \mu^2$$

Proof:

$$\begin{aligned} E[(X-\mu)^2] &= E[X^2 - 2\mu X + \mu^2] \\ &= E(X^2) - 2\mu E(X) + \mu^2 \\ &= E(X^2) - \mu^2 \end{aligned}$$

Example

Weekly demand for a drink (in thousand liters) is a continuous random variable X with PDF

$$f(X) = 2(x-1), \quad 1 < x < 2$$

Mean:

$$\mu = E(X) = \int_1^2 x \cdot 2(x-1)\, dx = \frac{5}{3}$$

Second moment:

$$E(X^2) = \int_1^2 x^2 \cdot 2(x-1)\, dx = \frac{17}{6}$$

Variance:

$$\boxed{\sigma^2 = \mathrm{Var}(X) = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{18}}$$

Variance of a function of a random variable

Let X be a random variable and let $g(X)$ be a function of X.

Mean of $g(X)$:

$$\mu_{g(X)} = E[g(X)]$$

Variance of $g(X)$:

$$\boxed{\mathrm{Var}(g(X)) = E[(g(X) - \mu_{g(X)})^2] = E[g(X)^2] - (E[g(X)])^2}$$

Properties of Variance

Let X, Y be random variables and let $a, b \in \mathbb{R}$.

1. $$\boxed{\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)}$$

Proof:

$$\begin{aligned}
\mathrm{Var}(aX + b) &= E[(aX + b)^2] - (E[aX + b])^2 \\
&= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2 \\
&= a^2E[X^2] + 2abE[X] + b^2 - a^2E[X]^2 - 2abE[X] - b^2 \\
&= a^2\big(E[X^2] - E[X]^2\big) = a^2\mathrm{Var}(X)
\end{aligned}$$

2. $$\boxed{\mathrm{Var}(X) \geq 0}$$

3. $$\boxed{\mathrm{Var}(a) = 0} \quad \text{for any constant } a$$

4. $$\boxed{\mathrm{Var}(X) = 0 \iff X \text{ is a constant}}$$

Theorem (Variance of a Linear Combination)

Let X, Y be random variables with joint distribution $f(X, Y)$. Then

$$\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X, Y)$$

where the **covariance** is defined as

$$\boxed{\mathrm{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]}$$

Interpretation of covariance

- $\mathrm{Cov}(X, Y) > 0$: X and Y tend to increase together
- $\mathrm{Cov}(X, Y) < 0$: one increases while the other decreases
- $\mathrm{Cov}(X, Y) = 0$: no linear relationship

Example (population intuition):

If X is height and Y is weight in a population,

$$(X - E[X])(Y - E[Y]) > 0$$

for tall–heavy and short–light individuals, so $\mathrm{Cov}(X, Y) > 0$.