

# MIE Lecture Notes

## Probability and Statistics

Cheryl Shi

### Contents

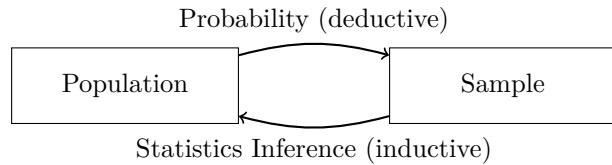
<b>Chapter 1: Probability Foundations</b> .....	1
1.1 Sampling Methods .....	1
1.2 Data, Variables, and Distributions .....	1
1.3 Graphical Representations of Data .....	3
<b>Chapter 2: Random Variables and Distributions</b> .....	4
2.1 Experiments, Sample Spaces, and Events .....	4
2.2 Event Operations and Probability Rules .....	5
2.3 Counting Techniques and Equally Likely Outcomes .....	8
2.4 Conditional Probability and Independence .....	11
<b>Chapter 3: Statistical Inference</b> .....	18
3.1 Random Variables and Their Interpretation .....	18
3.2 Discrete Random Variables .....	19
3.3 Probability Mass Functions .....	20
3.4 Cumulative Distribution Function(CDF) .....	23
3.5 Continuous Sample Space and Continuous Random Variables .....	25

# Chapter 1

## Statistics Definitions

**Global definition:** Statistics involves collecting, organizing, summarizing, presenting, and analyzing data, as well as making inferences, conclusions, and decisions based on data.

**Statistical definition:** A statistic is a numerical value calculated from data (e.g. mean, proportion, standard deviation).



## Basic Terminology

Individuals: Objects on which data are collected (people, animals, plots of land, etc.).

Variable: Any characteristic of an individual.

Population: The entire group of individuals of interest.

Sample: A subset of individuals taken from the population.

Statistical Inference: Drawing conclusions about a population based on a sample.

## Sampling Methods

### Simple Random Sample (SRS):

- Every possible group of size  $n$  has an equal chance of being selected.
- Helps avoid bias in sampling.
- Can be selected using random number tables or software.

### Stratified Random Sampling:

- The population is divided into homogeneous groups (*individuals are similar with respect to the variable being studied*) called strata.
- A simple random sample is taken from each stratum. (*one subgroup of the population created*)
- Ensures that important subgroups are neither over nor under represented.

## Data, Variables, and Distributions

### Types of Variables

Categorical Variable: Places individuals into categories (e.g. gender, major). These are qualitative.

Quantitative Variable: Takes numerical values for which arithmetic operations are meaningful.

- Discrete
- Continuous

## Distributions

Distribution: Describes what values a variable takes and how often those values occur. When examining a distribution, look for:

- **Shape**
- **Center**
- **Spread**
- **Outliers**

Outlier: An individual value that falls outside the overall pattern of the data.

## Describing Distributions with Numbers

Central Tendency: Describes where the data cluster or center.

Central Tendency: Describes where the data cluster or center.

- Mean: average value
- Median: middle value

Mean (Arithmetic Mean):

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

Median:

$$\tilde{x} = \begin{cases} x_{(\frac{n+1}{2})}, & \text{if } n \text{ is odd} \\ \frac{x_{(\frac{n}{2})} + x_{(\frac{n}{2}+1)}}{2}, & \text{if } n \text{ is even} \end{cases}$$

### Theorem 1.1

1. The mean is more sensitive to extreme values than the median.
2. Changing a single data value will always change the mean, but may not change the median.
3. If a distribution is exactly symmetric, the mean and median are equal.

Trimmed Mean: The mean computed after removing extreme values.

$$\bar{x}_{\text{trim}} = \frac{1}{n-2k} \sum_{i=k+1}^{n-k} x_{(i)}$$

where  $k$  values are removed from both ends of the ordered data. (normally given in question like 10% )

## Measures of Spread

Range: Maximum minus minimum. Very sensitive to extreme values.

Sample Variance: Measures the average squared deviation from the mean.

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

Standard Deviation: The square root of the sample variance.

$$s = \sqrt{s^2}$$

Degrees of Freedom: The number of independent pieces of information available to estimate variability. For sample variance:  $df = n - 1$ .

## Graphical Representations of Data

Scatter Plot: Used to display the relationship between two quantitative variables  $(x, y)$ . A scatter plot helps identify trends, patterns, and associations between variables.

Stem-and-Leaf Plot: An intermediate step between raw data and a frequency table. Preserves the original data values while showing the distribution.

Stem	Leaf
1	2 4 7
2	1 3 5 8
3	0 4 6

Relative Frequency Table: Shows the proportion of observations in each class.

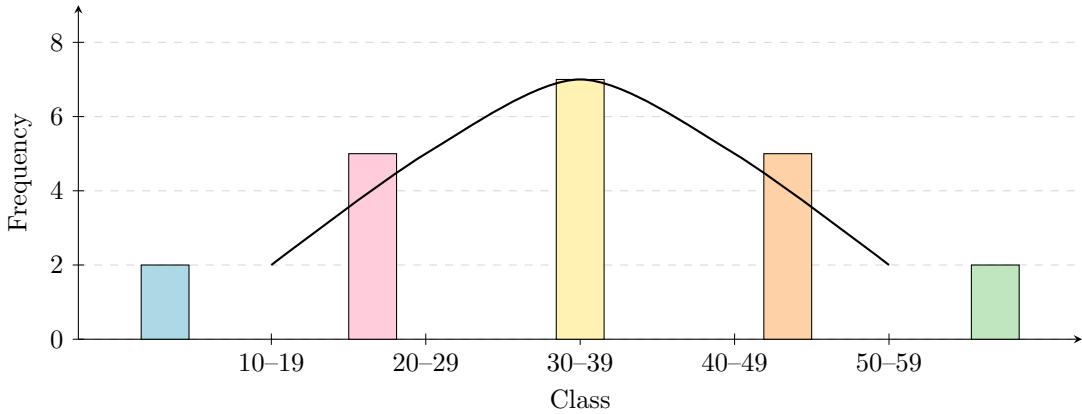
Class Interval	Class Midpoint	Frequency	Relative Frequency
10–19	14.5	3	0.30
20–29	24.5	4	0.40
30–39	34.5	3	0.30

Histogram: A graphical representation of a frequency or relative frequency table using contiguous bars.

When describing the shape of a histogram, we commonly classify it as:

- **Symmetric**
- **Skewed right** (positively skewed)
- **Skewed left** (negatively skewed)

**Symmetric distribution example**



## Chapter 2, Jan 9th

### Experiments, Sample Spaces, and Events

Experiment: A process that generates an outcome.

Sample Space ( $S$ ): The set of all possible outcomes of an experiment.

#### Example 1:

Select 3 items from a production line. Each item can be classified as either defective ( $D$ ) or non-defective ( $N$ ).

$$S = \{DDD, DDN, DND, NDD, DNN, NDN, NND, NNN\}$$

Since each item has 2 possible outcomes,

$$|S| = 2^3 = 8$$

#### Example 2:

$$S = \{(x, y) \mid x^2 + y^2 \leq 4\}$$

Event ( $A$ ): A subset of the sample space  $S$ .

**Examples of events:**

$$A = \{DDD, DDN, DND, NDD\}$$

$$B = \{NNN\}$$

$$C = \{(x, y) \mid x^2 + y^2 \leq 4\}$$

## Event Operations and Probability Rules

Event Operations:

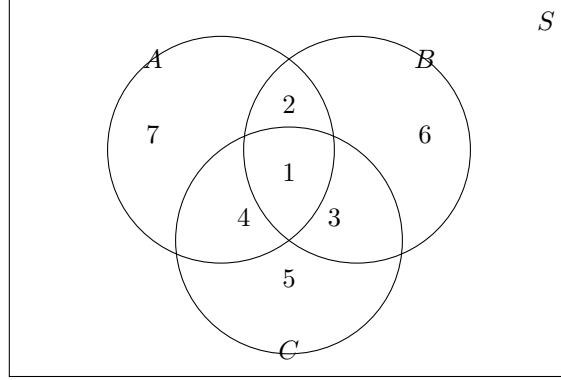
- Complement:  $A^c$  (or  $A'$ )
- Intersection:  $A \cap B$
- Union:  $A \cup B$
- Null Event:  $\emptyset$

If

$$A \cap B = \emptyset,$$

then  $A$  and  $B$  are mutually exclusive.

**Example (Venn Diagram):**



$$A = \{DDD, DDN, DND, NDD\}, \quad B = \{NNN\}$$

$$A \cup B = \{DDD, DDN, DND, NDD, NNN\}$$

$$A \cap B = \emptyset$$

## Chapter 2: January 12

### Review

1. Experiment: A process that generates an outcome.
2. Sample Space (S): The set of all possible outcomes of an experiment.
3. Event Operations:
  - Complement:  $A'$  ( $A^c$ )
  - Intersection:  $A \cap B$
  - Union:  $A \cup B$
  - Null Event:  $\emptyset$
4. If  $A \cap B = \emptyset$ , then  $A$  and  $B$  are called mutually exclusive.

$$(A \cap B)' = A' \cup B'$$

$$(A \cup B)' = A' \cap B'$$

$$A \cap \emptyset = \emptyset$$

$$A \cup \emptyset = A$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

### Probability

$P(A)$  = probability of event  $A$ : the proportion of times the event occurs in infinitely many repetitions of the experiment.

#### Theorem 2.1

$$0 \leq P(A) \leq 1$$

$$P(A) + P(A') = 1$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\quad + P(A \cap B \cap C) \end{aligned}$$

## Mutually Exclusive Events

Definition: If  $A_1, A_2, \dots, A_n$  are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$$

If

$$A_1 \cup A_2 \cup \dots \cup A_n = S,$$

then  $\{A_1, A_2, \dots, A_n\}$  is a partition of  $S$ .

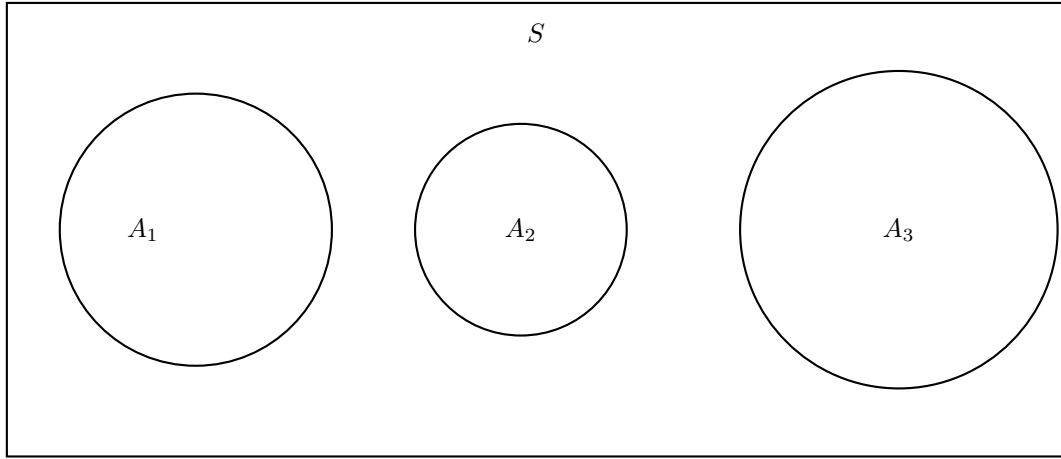


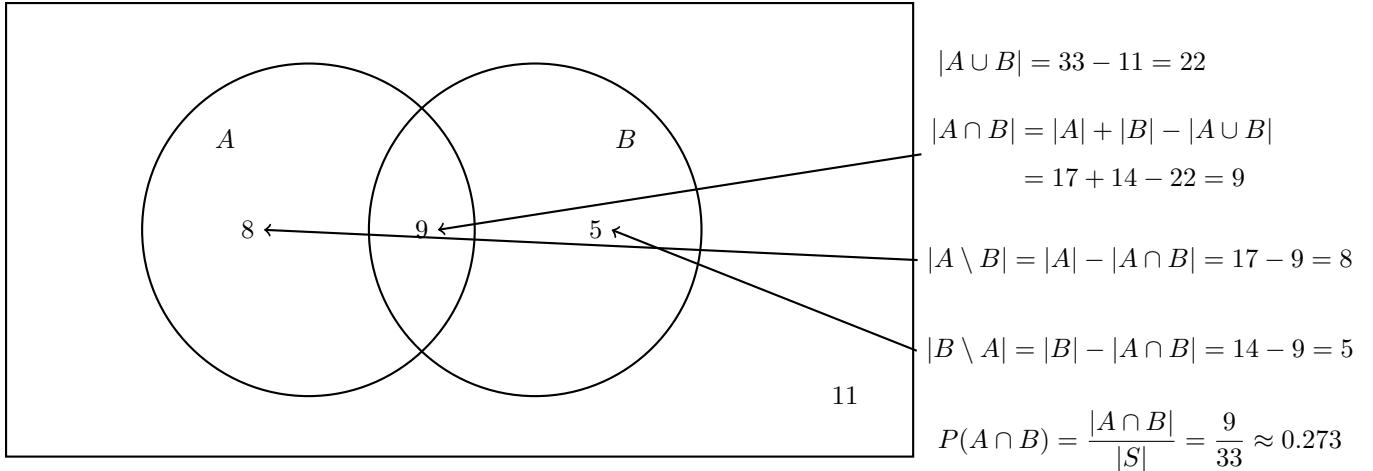
Figure 1: Partition of the sample space  $S$  into  $A_1, A_2, A_3$

## Example

In a class of 33 students:

- 17 earned an A on the midterm
- 14 earned an A on the final
- 11 earned no A on either exam

Find the probability that a randomly selected student earned A's on both exams.



## Counting Techniques and Equally Likely Outcomes

### Theorem 2.2 (Equally Likely Outcomes)

If the sample space  $S$  has a finite number of outcomes and all outcomes are equally likely, then for any event  $A$ ,

$$P(A) = \frac{|A|}{|S|}.$$

where

$|A|$  = number of outcomes in event  $A$ ,  $|S|$  = number of outcomes in the sample space.

### Example 1: Poker Hands Basics

A standard deck has:

$$4 \text{ suits} \times 13 \text{ denominations (A,2,3,\dots,Q,K)} = 52 \text{ cards.}$$

A poker hand consists of 5 cards chosen from 52:

$$|S| = \binom{52}{5} = 2,598,960.$$

### Combinations Reminder

If there are 3 objects  $\{A, B, C\}$  and we choose 2:

$$\binom{3}{2} = \frac{3!}{(3-2)!2!}.$$

Order does not matter.

### Example 2: Probability of 2 Aces and 1 Jack

A 5-card hand contains:

- exactly 2 aces,
- exactly 1 jack,
- 2 cards that are neither aces nor jacks.

$$P(\text{2 aces and 1 jack}) = \frac{\binom{4}{2} \binom{4}{1} \binom{44}{2}}{\binom{52}{5}}.$$

### Example 3: Probability of a Full House

A full house consists of:

- 3 cards of one denomination
- 2 cards of a different denomination

Number of full house hands:

$$\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2}.$$

Thus,

$$P(\text{full house}) = \frac{\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2}}{\binom{52}{5}}.$$

### Example 4: Probability of Four of a Kind

A four of a kind consists of:

- 4 cards of the same denomination
- 1 remaining card of a different denomination

Number of such hands:

$$\binom{13}{1} \binom{4}{4} \binom{48}{1}.$$

Thus,

$$P(\text{four of a kind}) = \frac{\binom{13}{1} \binom{4}{4} \binom{48}{1}}{\binom{52}{5}}.$$

### Example 5: Probability of Exactly One Pair

An exactly one-pair hand consists of:

- 1 pair
- 3 cards of different denominations, none matching the pair

Number of such hands:

$$\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1}^3.$$

Thus,

$$P(\text{exactly one pair}) = \frac{\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1}^3}{\binom{52}{5}}.$$

### Note\*: Counting Patterns

$$\binom{a}{b}$$

**Meaning:** Choose  $b$  different items from  $a$  **at once**, order does not matter.

**Key features:**

- No repeats
- Grouped choice
- Used when items must be distinct

$$\binom{a}{1}^b$$

**Meaning:** Make  $b$  independent choices, each time choosing 1 item from  $c$ .

**Key features:**

- Repeats allowed
- Choices are independent
- Used when selections do not restrict each other

### Rule to Remember:

Different items, no repeats  $\Rightarrow \binom{a}{b}$

Independent choices  $\Rightarrow \binom{c}{1}^b$

## Chapter 2 continue, Jan 14

### Review

1. Probability is the proportion of times the event occurs in infinitely many repetitions of the experiment.
2.  $0 \leq P(A) \leq 1$
3.  $P(A) + P(A^c) = 1$
4.  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

$$\begin{aligned} 5. \quad & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C) \end{aligned}$$

6. Permutation: A permutation counts ordered arrangements.

$$nP_r = \frac{n!}{(n-r)!}$$

### Example 1: Two fair dice

A pair of fair dice are rolled. Find the probability that the second die lands on a smaller value than the first.

The outcomes where the second die is smaller than the first are represented below.

First Die (Stem)	Second Die (Leaf)
2	1
3	1 2
4	1 2 3
5	1 2 3 4
6	1 2 3 4 5

There are 15 favorable outcomes and 36 total outcomes.

$$P(\text{second} < \text{first}) = \frac{15}{36} = \frac{5}{12}.$$

## Conditional Probability and Independence

### Conditional Probability

The conditional probability of an event  $B$  given that event  $A$  has occurred is the probability that  $B$  occurs when it is known that  $A$  has occurred.

$$P(B | A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

## Example 2: Drinking Survey

A survey records the following data:

	<i>D</i>	<i>N</i>	Total
<i>M</i>	19	41	60
<i>F</i>	12	28	40
Total	31	69	100

The symbols used above are defined as follows:

- $M$ : male
- $F$ : female
- $D$ : the individual drinks
- $N$ : the individual does not drink

$$P(D|M) = \frac{19}{60} \quad P(M|D) = \frac{19}{31}$$

## Law of Total Probability

### Theorem 2.3 (Law of Total Probability)

If  $B_1, B_2, \dots, B_k$  form a partition of the sample space  $S$  with  $P(B_i) > 0$  for all  $i$ , then for any event  $A$ ,

$$P(A) = \sum_{i=1}^k P(A | B_i) P(B_i).$$

## Example 3: Monty Hall (3 doors)

Car location	Monty opens	Probability	Stay	Switch
Door 1	Door 2	$\frac{1}{6}$	Car	Goat
Door 1	Door 3	$\frac{1}{6}$	Car	Goat
Door 2	Door 3	$\frac{1}{3}$	Goat	Car
Door 3	Door 2	$\frac{1}{3}$	Goat	Car

Staying wins only when the car is behind Door 1, so

$$P(\text{win by staying}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Switching wins when the car is behind Door 2 or Door 3, so

$$P(\text{win by switching}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

### Example 4: Birthday Problem

Assume the following:

- Leap years are ignored
- All 365 birthdays are equally likely
- Birthdays of different people are independent

**Question:** What is the probability that at least two people share the same birthday in a group of  $n$  people?

Rather than computing this directly, we use the complement rule.

$$P(\text{at least one match}) = 1 - P(\text{no match})$$

#### Probability of no shared birthdays

- Person 1 can have any birthday: probability 1
- Person 2 must avoid that birthday:  $\frac{364}{365}$
- Person 3 must avoid the first two birthdays:  $\frac{363}{365}$
- ...
- Person  $n$  must avoid the previous  $n - 1$  birthdays:  $\frac{365 - (n - 1)}{365}$

Therefore,

$$P(\text{no match}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365 - (n - 1)}{365}$$

or equivalently,

$$P(\text{no match}) = \prod_{k=0}^{n-1} \frac{365 - k}{365}$$

#### Final result

$$P(\text{at least one shared birthday}) = 1 - \prod_{k=0}^{n-1} \frac{365 - k}{365}$$

## Important values

- For  $n = 23$ :  $P(\text{at least one match}) \approx 0.507$
- For  $n = 57$ :  $P(\text{at least one match}) \approx 0.99$

## Chapter 2 — Jan 16

### Review: Conditional Probability

Conditional Probability:

The probability of event  $B$  given that event  $A$  has occurred is

$$P(B | A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

Read as: the probability of  $B$  given  $A$ .

### Independence of Events

Definition (Independence): Events  $A$  and  $B$  are independent if and only if

$$P(B | A) = P(B)$$

Equivalently,

$$P(A | B) = P(A)$$

or

$$P(A \cap B) = P(A) P(B)$$

### Multiple Independent Events

Definition:

If events  $A_1, A_2, \dots, A_k$  are independent, then

$$P(A_1 \cap A_2 \cap \dots \cap A_k) = P(A_1) P(A_2) \dots P(A_k)$$

### Mutual Independence

Mutual Independence : A collection of events  $A_1, A_2, \dots, A_n$  is mutually independent if and only if for *every* subcollection  $\{A_{i_1}, \dots, A_{i_k}\}$ ,

$$P\left(\bigcap_{j=1}^k A_{i_j}\right) = \prod_{j=1}^k P(A_{i_j})$$

Example (Three Events):

Events  $A_1, A_2, A_3$  are mutually independent if all of the following hold:

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1)P(A_3)$$

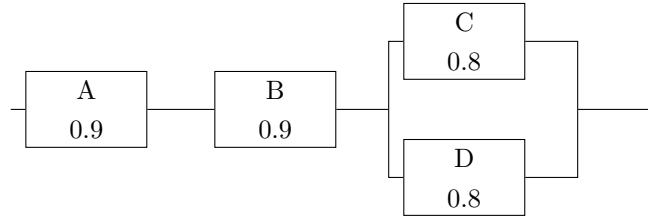
$$P(A_2 \cap A_3) = P(A_2)P(A_3)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

Note: Mutually exclusive events are dependent. If one event occurs, the other cannot occur.

### Example: Component Reliability

An electrical system has four components  $A, B, C, D$ . The system works if  $A$  and  $B$  work and at least one of  $C$  or  $D$  works. Assume all components are independent.



$$P(A) = 0.9, \quad P(B) = 0.9, \quad P(C) = 0.8, \quad P(D) = 0.8$$

#### (a) Probability the entire system works

The system works if  $A$  and  $B$  work and either  $C$  or  $D$  works.

$$\begin{aligned}
 P(\text{system works}) &= P(\text{all work}) + P(A, B, C \text{ work}, D \text{ does not}) \\
 &\quad + P(A, B, D \text{ work}, C \text{ does not}) \\
 &= (0.9)(0.9)(0.8)(0.8) + (0.9)(0.9)(0.8)(1 - 0.8) + (0.9)(0.9)(0.8)(1 - 0.8)
 \end{aligned}$$

$$P(\text{system works}) = 0.7776$$

**(b) Conditional probability**

$$P(C^c \mid \text{system works}) = \frac{P(C^c \cap \text{system works})}{P(\text{system works})}$$

$$P(C^c \cap \text{system works}) = (0.9)(0.9)(0.8)(1 - 0.8)$$

$$P(C^c \mid \text{system works}) = \frac{(0.9)(0.9)(0.8)(1 - 0.8)}{0.7776} = 0.16$$

**Theorem of Total Probability**

Let  $B_1, B_2, \dots, B_k$  be a partition of the sample space  $S$  such that  $P(B_i) > 0$  for all  $i$ . Then for any event  $A \subseteq S$ ,

$$P(A) = \sum_{i=1}^k P(A \mid B_i) P(B_i) = \sum_{i=1}^k P(A \cap B_i)$$

**Theorem: Bayes' Rule (1701–1761)**

Let  $B_1, B_2, \dots, B_k$  be a partition of the sample space  $S$  such that  $P(B_i) > 0$  for  $i = 1, \dots, k$ . For any event  $A \subseteq S$  with  $P(A) > 0$ ,

$$P(B_r \mid A) = \frac{P(B_r \cap A)}{\sum_{i=1}^k P(B_i \cap A)} = \frac{P(B_r) P(A \mid B_r)}{\sum_{i=1}^k P(B_i) P(A \mid B_i)}, \quad r = 1, \dots, k$$

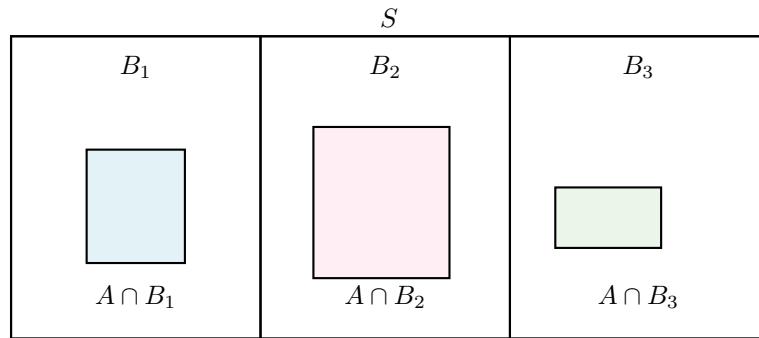


Figure 3: Partition of  $S$  into  $B_1, B_2, B_3$  with shaded regions  $A \cap B_i$

**Example (Medical Test)**

The fraction of people in a population who have a certain disease is 0.01.

$$P(D) = 0.01, \quad P(D^c) = 0.99$$

The test characteristics are:

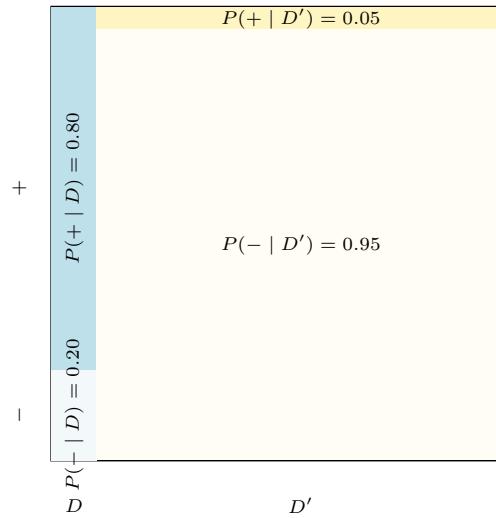
$$P(\text{test says } D \mid D^c) = 0.05 \quad (\text{false positive rate})$$

$$P(\text{test says } D^c \mid D) = 0.20 \quad (\text{false negative rate})$$

Thus,

$$P(\text{test says } D \mid D) = 1 - 0.20 = 0.80$$

Note:  $1 - P(\text{test says } D^c \mid D)$  is called the sensitivity of the test, and  $1 - P(\text{test says } D \mid D^c)$  is called the specificity.



### (a) Probability the test says disease

$$\begin{aligned}
 P(\text{test says } D) &= P(D \cap \text{test says } D) + P(D^c \cap \text{test says } D) \\
 &= P(\text{test says } D \mid D)P(D) + P(\text{test says } D \mid D^c)P(D^c) \\
 &= (0.80)(0.01) + (0.05)(0.99) = 0.0575
 \end{aligned}$$

### (b) Probability of disease given positive test

$$P(D \mid \text{test says } D) = \frac{P(D \cap \text{test says } D)}{P(\text{test says } D)}$$

$$= \frac{P(\text{test says } D \mid D)P(D)}{0.0575} = \frac{(0.80)(0.01)}{0.0575}$$

$P(D \mid \text{test says } D) \approx 0.139$

(c) Probability of disease given negative test

$$\begin{aligned} P(D \mid \text{test says } D^c) &= \frac{P(D \cap \text{test says } D^c)}{P(\text{test says } D^c)} \\ &= \frac{P(\text{test says } D^c \mid D)P(D)}{1 - P(\text{test says } D)} \\ &= \frac{(0.20)(0.01)}{1 - 0.0575} \end{aligned}$$

$P(D \mid \text{test says } D^c) \approx 0.00212$

## Chapter 3 — January 19

### Random Variables and Their Interpretation

**Definition:** A random variable (r.v.) is a rule that assigns a **real number** to each outcome in the sample space.

**Alternative definition:** A random variable is a function that takes the outcome of an experiment and assigns it a number so that probabilities can be calculated.

#### Example 1: Three Electronic Components

Each component is classified as either defective (D) or non-defective (N).

$$S = \{NNN, DNN, NDN, NND, DDN, DND, NDD, DDD\}$$

- **Defective (D):** the component does not meet required specifications and fails inspection.
- **Non-defective (N):** the component meets specifications and passes inspection.

Define the random variable

$X = \text{number of defective components.}$

Then:

$$\begin{aligned}
 X = 0 & \text{ for } \{NNN\} \\
 X = 1 & \text{ for } \{DNN, NDN, NND\} \\
 X = 2 & \text{ for } \{DDN, DND, NDD\} \\
 X = 3 & \text{ for } \{DDD\}
 \end{aligned}$$

Thus, the possible values of  $X$  are:

$$\{0, 1, 2, 3\}.$$

### Example 2: One Component (Dummy Variable)

$$S = \{D, N\}$$

Define the random variable

$$X = \begin{cases} 1, & \text{if the component is defective (D)} \\ 0, & \text{if the component is non-defective (N)} \end{cases}$$

This is called a **dummy variable** because the outcome is categorical, but is encoded numerically.

A dummy variable is a special type of random variable that assigns numerical labels to categorical outcomes, where the numbers have no quantitative meaning beyond identification.

## Discrete Random Variables

**Definition:** A random variable is called discrete if its set of possible values is **countable** (finite or countably infinite).

### Example 3: Sampling Until First Defective

Components are tested one(independently) at a time until the first defective component is observed.

$$S = \{D, ND, NND, NNND, \dots\}$$

Define

$X$  = number of components tested until the first defective.

Then:

$$\begin{aligned}
 X = 1 & \text{ for } \{D\} \\
 X = 2 & \text{ for } \{ND\} \\
 X = 3 & \text{ for } \{NND\} \\
 & \vdots
 \end{aligned}$$

Hence,

$$X = 1, 2, 3, \dots$$

Since the possible values can be listed,  $X$  is a discrete random variable.

**Non-discrete version of the same experiment:**

Define

$$Y = \text{time (in seconds) until the first defective component is observed.}$$

Since  $Y$  can take any real value in  $[0, \infty)$ , it cannot be listed and is therefore a continuous (non-discrete) random variable.

## Discrete vs. Continuous Random Variables

Discrete Random Variable	Continuous Random Variable
Counts things	Measures things
Possible values are countable	Possible values fill an interval
$P(X = x)$ can be $> 0$	$P(X = x) = 0$ for all $x$
Uses a probability mass function (PMF)	Uses a probability density function (PDF)

## Probability Mass Function (PMF)

**Definition:**

Let  $X$  be a discrete random variable. The probability mass function (PMF) of  $X$ , denoted  $f(x)$ , is defined by:

$$\boxed{\begin{array}{l} 1) \quad f(x) \geq 0 \quad \text{for all } x \\ 2) \quad \sum_x f(x) = 1 \end{array}}$$

**Note:**

- Capital  $X$ : random variable
- Lowercase  $x$ : a specific value

## Bernoulli and Binomial Random Variables

### I. Bernoulli Random Variable (Single Trial)

A Bernoulli random variable:  $X$  models a single experiment with only two possible outcomes: success or failure.

$$X = \begin{cases} 1, & \text{success} \\ 0, & \text{failure} \end{cases}$$

If  $p = P(X = 1)$ , then the PMF is

$x$	0	1
$P(X = x)$	$1 - p$	$p$

Here,  $p$  is the probability of success (e.g. observing a defective component).

### Binomial Random Variable (Multiple Bernoulli Trials)

The binomial random variable extends the Bernoulli case to multiple independent trials.

#### Definition:

A random variable  $X$  is called a binomial random variable if it represents the number of successes in  $n$  independent Bernoulli trials, each with success probability  $p$ .

$X$  = number of successes in  $n$  trials

In this case,

$$X \sim \text{Bin}(n, p)$$

and the probability mass function is

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, \dots, n.$$

### Conditions for a Binomial Model

A binomial model applies only if:

- each trial has exactly two outcomes (success or failure),
- the probability of success  $p$  is the same for every trial,
- the trials are independent,
- the number of trials  $n$  is fixed.

### Example: Three Components Tested

Assume each component is defective with probability

$$p = 0.1, \quad n = 3.$$

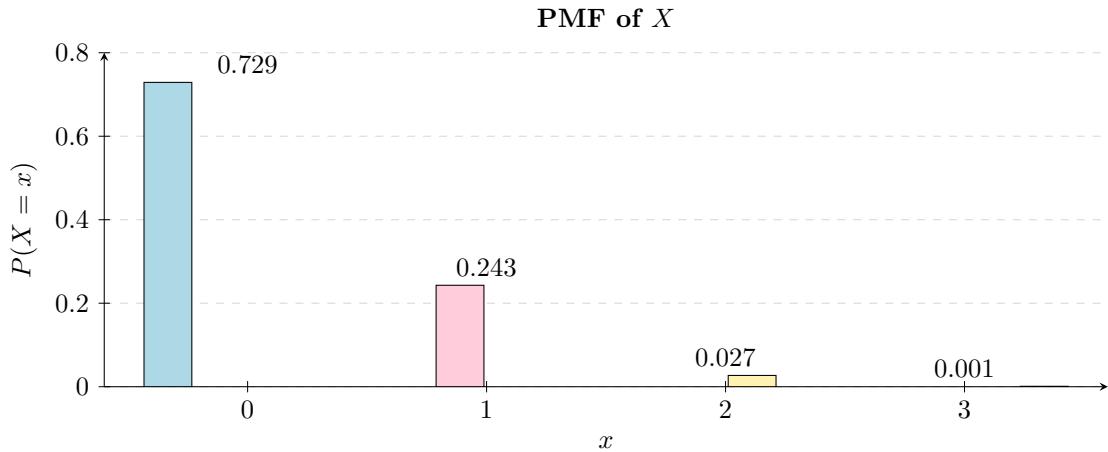
where  $p = P(\text{a single component is defective})$ ,  $n = \text{number of trials}$

Let

$$X = \text{number of defective components.}$$

$x$	$P(X = x)$
0	$\binom{3}{0}(0.9)^3 = 0.729$
1	$\binom{3}{1}(0.1)(0.9)^2 = 0.243$
2	$\binom{3}{2}(0.1)^2(0.9) = 0.027$
3	$\binom{3}{3}(0.1)^3 = 0.001$

$$0.729 + 0.243 + 0.027 + 0.001 = 1.$$



## Geometric Random Variable

### Example: Sampling Until First Defective

Components are sampled one at a time until the first defective component is observed. Assume the probability that a component is defective is

$$p = 0.1.$$

Define the random variable

$X$  = number of samples collected until the first defective.

$x$	$P(X = x) = f(x)$
1	0.1
2	$0.9(0.1)$
3	$0.9^2(0.1)$
$\vdots$	$\vdots$

## Geometric Random Variable

### Definition:

A random variable  $X$  is called a geometric random variable if it represents the number of trials needed to obtain the first success in a sequence of independent Bernoulli trials with success probability  $p$ .

$$P(X = x) = (1 - p)^{x-1} p, \quad x = 1, 2, 3, \dots$$

In this example,

$$P(X = x) = 0.9^{x-1} (0.1).$$

### Verification That Probabilities Sum to 1

$$\sum_{x=1}^{\infty} 0.9^{x-1} (0.1) = 0.1 \sum_{x=0}^{\infty} 0.9^x = 0.1 \left( \frac{1}{1 - 0.9} \right) = 1.$$

### Cumulative Distribution Function (CDF)

#### Definition:

The cumulative distribution function (CDF) of a discrete random variable  $X$  with PMF  $f(x)$  is defined as

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad -\infty < x < \infty.$$

### Example: Binomial Distribution (Three Components)

Let  $X$  be the number of defective components when three components are tested, with

$$P(X = 0) = 0.729, \quad P(X = 1) = 0.243, \quad P(X = 2) = 0.027, \quad P(X = 3) = 0.001.$$

$$F(0) = P(X \leq 0) = 0.729$$

$$F(1) = P(X \leq 1) = 0.729 + 0.243 = 0.972$$

$$F(2) = P(X \leq 2) = 0.729 + 0.243 + 0.027 = 0.999$$

$$F(3) = P(X \leq 3) = 0.729 + 0.243 + 0.027 + 0.001 = 1$$

Thus, the CDF can be written as

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.729, & 0 \leq x < 1 \\ 0.972, & 1 \leq x < 2 \\ 0.999, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

### Properties of the CDF

- $F(x)$  is monotone non-decreasing.

- If  $x < y$ , then  $F(x) \leq F(y)$ .
- $0 \leq F(x) \leq 1$ .

**Note:** A function is **monotone** non-decreasing if its value never decreases as the input increases.

## Using the CDF to Compute Probabilities

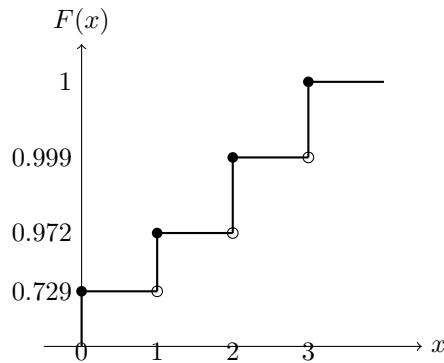
For  $a < b$ ,

$$P(a < X \leq b) = F(b) - F(a).$$

Example:

$$P(0 < X \leq 2) = F(2) - F(0) = 0.999 - 0.729 = 0.27.$$

## CDF Histogram (Step Function)



**Note:** For a discrete random variable, the PMF is drawn as a bar chart since it shows probabilities at individual points, while the CDF is drawn as a step function since it represents cumulative probability and is monotone non-decreasing.

## Chapter 3 — January 21

### Review

1. Random Variable (RV),  $X$ : A random variable assigns a real number to each outcome.
2. Discrete Random Variable: If  $X$  is discrete,
  - $P(X = x) = f(x)$
  - $f(x)$  is the *probability mass function (PMF)*
  - $f(x) \geq 0$
  - $\sum_x f(x) = 1$
3. Cumulative Distribution Function (CDF),  $F(x)$ :

- $F(x) = P(X \leq x)$

- If  $X$  is discrete:

$$F(x) = \sum_{t \leq x} f(t), \quad -\infty < x < \infty$$

## Continuous Sample Space and Continuous Random Variables

If the sample space contains an infinite number of outcomes equal to the number of points on a line segment, it is called a continuous sample space.

A continuous random variable has

$$P(X = x) = 0 \quad \text{for all } x,$$

so probabilities are computed over intervals instead of single values.

**\*Alternative definition:** A continuous sample space contains infinitely many outcomes, like the points on a line segment. For a continuous random variable, the probability of taking any exact value is zero, i.e.  $P(X = x) = 0$  for all  $x$ . Therefore, probabilities are computed over intervals rather than at single points.

## Probability Density Function (PDF)

A function  $f(x)$  is called a probability density function (PDF) of a continuous random variable  $X$ , defined over  $\mathbb{R}$ , if:

1.  $f(x) \geq 0$  for all  $x \in \mathbb{R}$

2.  $\int_{-\infty}^{\infty} f(x) dx = 1$

3. For any  $a < b$ ,

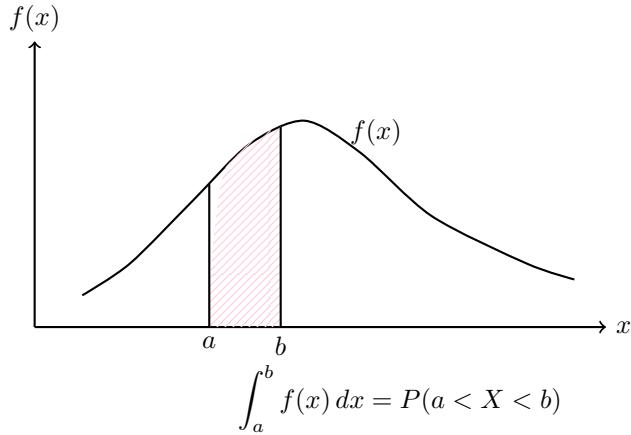
$$P(a < X < b) = \int_a^b f(x) dx$$

For continuous random variables,

$$P(a < X < b) = P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b).$$

$$P(X = a) = 0$$

because a single point has zero area under the probability density function.



### Example: Uniform Distribution

Let the probability density function be

$$f(x) = \begin{cases} c, & 5 < x < 10, \\ 0, & \text{otherwise.} \end{cases}$$

#### 1) Determine the value of $c$

Since  $f(x)$  is a probability density function, the total area under the curve must equal 1:

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

Because  $f(x) = 0$  outside the interval  $(5, 10)$ ,

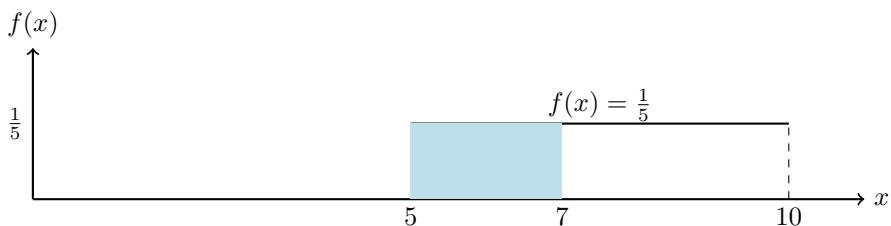
$$\int_5^{10} c dx = 1.$$

Evaluating the integral,

$$c(10 - 5) = 1 \quad \Rightarrow \quad c = \frac{1}{5}.$$

Thus,

$$f(x) = \begin{cases} \frac{1}{5}, & 5 < x < 10, \\ 0, & \text{otherwise.} \end{cases}$$



**2): Compute  $P(X < 7)$**

**Formula used:**

$$P(a < X < b) = \int_a^b f(x) dx.$$

Applying this formula,

$$P(X < 7) = \int_5^7 \frac{1}{5} dx = \frac{1}{5}(7 - 5) = \frac{2}{5}.$$

## CDF for Continuous Random Variables

**Def:** Let  $X$  be a continuous random variable with pdf: probability density function  $f(x)$ . The cumulative distribution function is

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt.$$

**Consequently:**

$$f(x) = \frac{d}{dx} F(x), \quad P(a < X < b) = F(b) - F(a).$$

## Example

Let  $X$  be the time until a chemical reaction is complete (in msec). Suppose the CDF is

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-0.01x}, & x \geq 0. \end{cases}$$

**(a) Find the pdf.**

Use  $f(x) = \frac{d}{dx} F(x)$ .

For  $x < 0$ ,  $F(x) = 0$ , so

$$f(x) = 0.$$

For  $x \geq 0$ ,

$$f(x) = \frac{d}{dx} (1 - e^{-0.01x}) = 0.01e^{-0.01x}.$$

Therefore,

$$f(x) = \begin{cases} 0, & x < 0, \\ 0.01e^{-0.01x}, & x \geq 0. \end{cases}$$

**(b) Find  $P(X < 200)$ .**

Use the CDF directly:

$$P(X < 200) = F(200) = 1 - e^{-0.01(200)} = 1 - e^{-2} \approx 0.8647.$$

**(c) Check if this is a valid CDF.**

### Theorem: Properties of a Cumulative Distribution Function

A function  $F(x)$  is a valid cumulative distribution function (CDF) if and only if:

- $0 \leq F(x) \leq 1$  for all  $x$
- $F(x)$  is monotone non-decreasing, i.e.

$$x \leq y \implies F(x) \leq F(y)$$

- $\lim_{x \rightarrow -\infty} F(x) = 0$
- $\lim_{x \rightarrow \infty} F(x) = 1$

For this  $F(x)$ :

$$\lim_{x \rightarrow -\infty} F(x) = 0, \quad \lim_{x \rightarrow \infty} F(x) = 1,$$

and for  $x \geq 0$ ,  $1 - e^{-0.01x}$  increases as  $x$  increases, so  $F(x)$  is monotone non-decreasing.

Thus,  $F(x)$  is a valid CDF.

### Joint Probability Distributions (Discrete)

**Def:** The function  $f(x, y)$  is a joint probability mass function (PMF) of discrete random variables  $X$  and  $Y$  if:

1.  $f(x, y) \geq 0$  for all  $(x, y)$
2.  $\sum_x \sum_y f(x, y) = 1$
3.  $P(X = x, Y = y) = f(x, y)$

That is,  $f(x, y)$  gives the probability that the two random variables  $X$  and  $Y$  take the values  $x$  and  $y$  *simultaneously*.

For any region  $A$  in the  $xy$ -plane,

$$P((X, Y) \in A) = \sum_{(x, y) \in A} f(x, y)$$

### Example: Pen Refills

Two refills are selected at random and without replacement from a box containing:

- 3 blue refills
- 2 red refills
- 3 green refills

Define the random variables

$$X = \text{number of blue refills selected}, \quad Y = \text{number of red refills selected.}$$

The total number of possible selections is

$$\binom{8}{2}.$$

### Joint PMF Table

For each pair  $(x, y)$ ,

$$f(x, y) = \frac{\text{number of favorable outcomes}}{\binom{8}{2}}.$$

$X \setminus Y$	0	1	2	Row Total
0	$\frac{\binom{3}{2}}{\binom{8}{2}}$	$\frac{\binom{2}{1}\binom{3}{1}}{\binom{8}{2}}$	$\frac{\binom{2}{2}}{\binom{8}{2}}$	$\frac{\binom{5}{2}}{\binom{8}{2}}$
1	$\frac{\binom{3}{1}\binom{3}{1}}{\binom{8}{2}}$	$\frac{\binom{3}{1}\binom{2}{1}}{\binom{8}{2}}$	0	$\frac{15}{\binom{8}{2}}$
2	$\frac{\binom{3}{2}}{\binom{8}{2}}$	0	0	$\frac{3}{\binom{8}{2}}$
Column Total	$\frac{15}{\binom{8}{2}}$	$\frac{12}{\binom{8}{2}}$	$\frac{1}{\binom{8}{2}}$	1

### Marginal Distributions

**Def:** Let  $f(x, y)$  be the joint PMF of  $X$  and  $Y$ .

The marginal PMF of  $X$  is obtained by summing over all values of  $Y$ :

$$g(x) = \sum_y f(x, y).$$

The marginal PMF of  $Y$  is obtained by summing over all values of  $X$ :

$$h(y) = \sum_x f(x, y).$$

**Marginal example:**

$$P(X = 1) = \sum_y P(X = 1, Y = y).$$

## Conditional Distributions (Discrete)

**Def:** The conditional PMF of  $Y$  given  $X = x$  is

$$f_{Y|X}(y|x) = \frac{f(x, y)}{g(x)}, \quad g(x) > 0.$$

Similarly, the conditional PMF of  $X$  given  $Y = y$  is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{h(y)}, \quad h(y) > 0.$$

### Example: Conditional Probabilities

1.

$$P(Y = 1 | X = 1) = \frac{P(X = 1, Y = 1)}{P(X = 1)} = \frac{3/28}{15/28} = \frac{3}{15}.$$

2.

$$P(X = 0 | Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{9/28}{15/28} = \frac{9}{15}.$$

3.

$$P(Y = 2 | X = 1) = 0.$$

Check:

$$\sum_y P(Y = y | X = 1) = 1.$$

## Joint Probability Density Function (Continuous)

**Def:** The function  $f(x, y)$  is a joint probability density function (PDF) of continuous random variables  $X$  and  $Y$  if:

1.  $f(x, y) \geq 0$  for all  $(x, y)$
2.  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$
3. For any region  $A$  in the  $xy$ -plane,

$$P((X, Y) \in A) = \iint_A f(x, y) dx dy$$

**Geometric interpretation:** The joint PDF is a surface above the  $xy$ -plane, and probabilities correspond to the **volume under the surface**.