Charge

- Know the value and unit of e
- Understand quantization and conservation
- Be able to state attraction vs. repulsion
- Lecture 1 concepts support later laws (Coulomb, electric field)

1. **Electric Charge**

   Electric charge is a fundamental property of matter responsible for electric forces.

   - Two types: positive (protons) and negative (electrons)
   - SI unit: coulomb (C)

   $$q_p = +e = +1.602 \times 10^{-19}\,\text{C}, \qquad q_e = -e$$

2. **Elementary Charge**

   The smallest unit of free charge is the elementary charge e.

   - Protons carry $+e$
   - Electrons carry $-e$
   - Neutrons carry zero charge

3. **Charge Quantization**

   All observable charges appear as integer multiples of the elementary charge:

   $$q = ne, \qquad n \in \mathbb{Z}$$

   Fractional charges do not appear in isolation.

4. **Charge Conservation**

   In an isolated system, total electric charge remains constant.

   - Charge can move between objects
   - Charge cannot be created or destroyed

5. **Electric Force (Qualitative)**

   Electric forces arise due to charge interactions:

   - Like charges repel
   - Opposite charges attract

   The force acts along the line connecting the charges.

Coulomb's Law

- Know both **magnitude** and **vector form** of the force
- Direction errors are a common mistake
- Coulomb's Law applies only to **point charges**
- Leads directly to the definition of the **electric field**

1. **Statement of Coulomb's Law**

   The electric force between two stationary point charges is proportional to the product of the charges and inversely proportional to the square of the distance between them.

2. **Magnitude of the Force**

   $$F = \frac{1}{4\pi\varepsilon_0}\frac{|q_1 q_2|}{r^2}$$

   where:

   - q_1, q_2 are the **charges**
   - r is the **distance** between them
   - ε_0 is the **permittivity of free space**

3. **Permittivity of Free Space**

   $$\varepsilon_0 = 8.854 \times 10^{-12}\,\mathrm{C}^2/(\mathrm{N}\cdot\mathrm{m}^2)$$

   This constant determines the strength of electric interactions in vacuum.

4. **Direction of the Force**

   - Like charges → **repulsion**
   - Opposite charges → **attraction**

   The force always acts along the straight line connecting the two charges.

5. **Vector Form of Coulomb's Law**

   The force on charge q_2 due to charge q_1 is:

   $$\boxed{\mathbf{F}_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\mathbf{R}_2 - \mathbf{R}_1|^3}(\mathbf{R}_2 - \mathbf{R}_1)}$$

   where:

   - $\mathbf{R}_1, \mathbf{R}_2$ are the **position vectors**

- $(\mathbf{R}_2 - \mathbf{R}_1)$ gives the **direction**

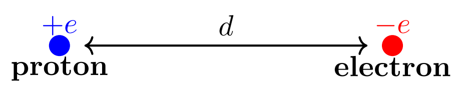
6. **Newton's Third Law**

   Electric forces come in equal and opposite pairs:

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

7. **Example: Electric vs. Gravitational Force**

   - **Proton** $(+e)$ and **electron** $(-e)$
   - Separation distance: $d = 10^{-12}\,\mathrm{m}$



Electric force magnitude:

$$|F_E| = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{d^2} = \frac{1}{4\pi(8.854 \times 10^{-12})}\frac{(1.602 \times 10^{-19})^2}{(10^{-12})^2} \approx 2.3 \times 10^{-4}\,\mathrm{N}$$

Gravitational force magnitude:

$$|F_G| = G\,\frac{m_p m_e}{d^2} = (6.67 \times 10^{-11})\frac{(1.67 \times 10^{-27})(9.11 \times 10^{-31})}{(10^{-12})^2} \approx 1.0 \times 10^{-43}\,\mathrm{N}$$

Comparison:

$$\frac{|F_E|}{|F_G|} \sim 10^{39}$$

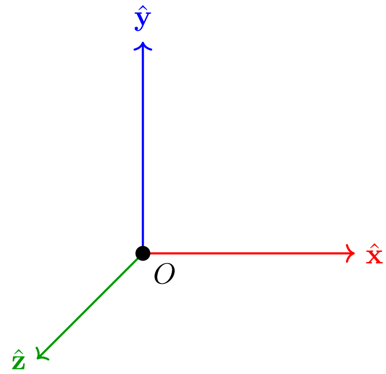
- **Electric force** is overwhelmingly stronger than gravity
- Explains why gravity is negligible at the **atomic scale**
- Justifies focusing on **electromagnetic forces** in E&M

Coordinate Systems

- Choice of system depends on **symmetry**

1. **Cartesian Coordinate System**

   - Unit vectors: $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$
   - Unit vectors are **constant**
   - Best for **rectangular geometry**



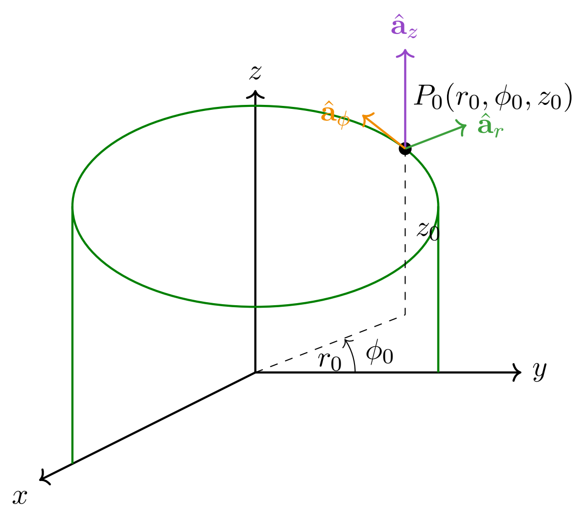
Coordinates:

$$(x, y, z)$$

Position vector:

$$\mathbf{R} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

2. **Cylindrical Coordinate System**



Coordinates:

$$(r, \phi, z)$$

Position vector:

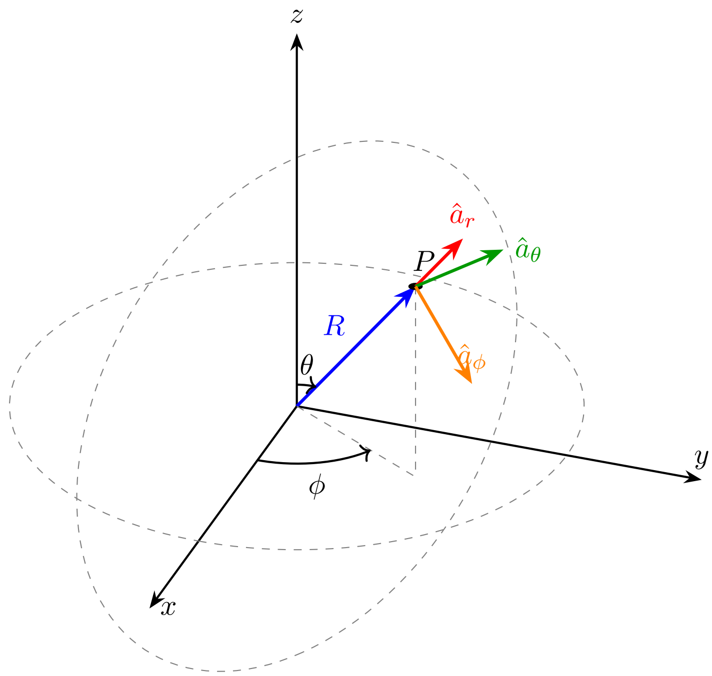
$$\mathbf{R} = r\,\hat{\mathbf{a}}_r + z\,\hat{\mathbf{a}}_z$$

Unit vectors:

$$\hat{\mathbf{a}}_r = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \qquad \hat{\mathbf{a}}_\phi = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

- Used for **lines**, **cylinders**, **rotational symmetry**
- Unit vectors **depend on position**

3. **Spherical Coordinate System**



Coordinates:

$$(R, \theta, \phi)$$

Position vector:

$$\mathbf{R} = R\,\hat{\mathbf{a}}_R(\theta, \phi)$$

- Used for **point charges** and **spherical symmetry**
- Simplifies Coulomb's law and electric field expressions

4. **Differential Length Elements**

The differential displacement vector $d\mathbf{l}$ depends on the coordinate system.

Cartesian:

$$d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$$

Cylindrical:

$$d\mathbf{l} = dr\,\hat{\mathbf{a}}_r + r\,d\phi\,\hat{\mathbf{a}}_\phi + dz\,\hat{\mathbf{a}}_z$$

Spherical:

$$d\mathbf{l} = dR\,\hat{\mathbf{a}}_R + R\,d\theta\,\hat{\mathbf{a}}_\theta + R\sin\theta\,d\phi\,\hat{\mathbf{a}}_\phi$$

The extra factors (r, R, $R\sin\theta$) come from **arc length**.

5. **Integration Elements**

   Line integrals use $d\mathbf{l}$.

   Surface area elements:

   - Cylindrical surface ($r = \text{const}$):
   $$dS = r\, d\phi\, dz$$
   - Spherical surface ($R = \text{const}$):
   $$dS = R^2 \sin\theta\, d\theta\, d\phi$$

   Volume elements:

   $$dV_{\text{Cartesian}} = dx\, dy\, dz$$
   $$dV_{\text{Cylindrical}} = r\, dr\, d\phi\, dz$$
   $$dV_{\text{Spherical}} = R^2 \sin\theta\, dR\, d\theta\, d\phi$$

6. **Which Coordinate System to Use**

   Choose the coordinate system that matches the **symmetry** of the problem:

   - Point charge $\rightarrow$ **spherical**
   - Infinite line charge $\rightarrow$ **cylindrical**
   - Rectangular geometry $\rightarrow$ **Cartesian**

Superposition

- Always identify symmetry FIRST
- Choose coordinates matching geometry
- State which components cancel
- Check units and limiting cases

Core Concept

Total electric field = vector sum of individual fields:

$$\boxed{\mathbf{E} = \sum_{i=1}^{N} \mathbf{E}_i \quad \text{or} \quad \mathbf{E} = \int d\mathbf{E}}$$

Point Charge

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{Q(\mathbf{R} - \mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|^3}$$

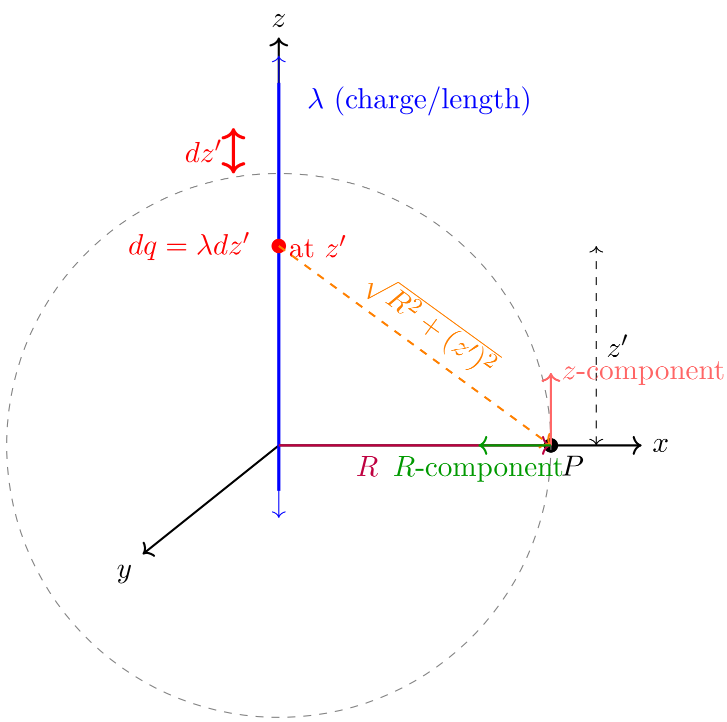
Continuous Distributions

Line charge:	$dq = \lambda\, dl'$
Surface charge:	$dq = \sigma\, dS'$
Volume charge:	$dq = \rho\, dV'$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{dq\,(\mathbf{R} - \mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|^3}$$

Example 1: Infinite Line of Charge

Consider an infinite line charge with uniform linear charge density λ along the z-axis. Find $\mathbf{E}$ at distance R from the line.



Setup using cylindrical coordinates:

- Source element: $dq = \lambda\, dz'$ at $(r' = 0, \phi' = \text{x}, z')$
- Field point: P at $(R, \phi, z = 0)$
- Distance: $|\mathbf{R} - \mathbf{R}'| = \sqrt{R^2 + (z')^2}$

Solution:

The electric field contribution from element dz':

$$d\mathbf{E} = \frac{\lambda\, dz'}{4\pi\varepsilon_0} \frac{1}{R^2 + (z')^2} \frac{(R\hat{\mathbf{a}}_R - z'\hat{\mathbf{a}}_z)}{\sqrt{R^2 + (z')^2}}$$

$$= \frac{\lambda\, dz'}{4\pi\varepsilon_0} \frac{1}{[R^2 + (z')^2]^{3/2}} (R\hat{\mathbf{a}}_R - z'\hat{\mathbf{a}}_z)$$

By symmetry, the z-components cancel when integrating over all z'. Only the radial component survives:

$$E_R = \int_{-\infty}^{\infty} \frac{\lambda}{4\pi\varepsilon_0} \frac{R\, dz'}{[R^2 + (z')^2]^{3/2}}$$

Using the integral:

$$\int_{-\infty}^{\infty} \frac{dz'}{[R^2 + (z')^2]^{3/2}} = \frac{2}{R^2}$$

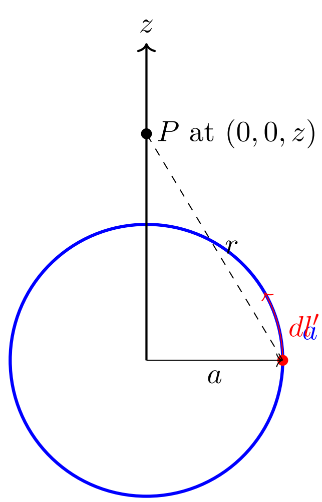
We get:

$$\boxed{\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 R}\hat{\mathbf{a}}_R}$$

Key result: Field points radially outward (for $\lambda > 0$) and falls off as $1/R$.

Example 2: Uniformly Charged Ring

A ring of radius a carries uniform charge density λ. Find $\mathbf{E}$ on the axis at distance z from the center.



Setup:

- Ring in xy-plane with radius a
- Charge element: $dq = \lambda\, dl' = \lambda\, a\, d\phi'$
- Distance from element to field point: $r = \sqrt{a^2 + z^2}$

Solution:

Each charge element dq produces a field:

$$d\mathbf{E} = \frac{\lambda\, a\, d\phi'}{4\pi\varepsilon_0(a^2+z^2)} \frac{(-a\cos\phi'\hat{\mathbf{x}} - a\sin\phi'\hat{\mathbf{y}} + z\hat{\mathbf{z}})}{\sqrt{a^2+z^2}}$$

By symmetry, the horizontal components (x and y) cancel. Only the z-component survives:

$$E_z = \int_0^{2\pi} \frac{\lambda\, a}{4\pi\varepsilon_0} \frac{z\, d\phi'}{(a^2+z^2)^{3/2}}$$

$$= \frac{\lambda\, a\, z}{4\pi\varepsilon_0(a^2+z^2)^{3/2}} \int_0^{2\pi} d\phi' = \frac{\lambda\, a\, z}{4\pi\varepsilon_0(a^2+z^2)^{3/2}} \cdot 2\pi$$

Total charge: $Q = \lambda \cdot 2\pi a$

$$\boxed{\mathbf{E} = \frac{Qz}{4\pi\varepsilon_0(a^2+z^2)^{3/2}}\hat{\mathbf{z}}}$$

Limiting cases:

- At center ($z = 0$): $\mathbf{E} = 0$ (by symmetry)
- Far from ring ($z \gg a$): $\mathbf{E} \approx \frac{Q}{4\pi\varepsilon_0 z^2}\hat{\mathbf{z}}$ (point charge behavior)

Gauss's Law

- Relates electric flux through a **closed surface** to enclosed charge
- Works for **any** closed surface, but choose one with symmetry for easy calculation
- Only charge distributions with enough symmetry make Gauss's Law useful
- Can be used to identify a **Gaussian surface**

Integral Form

$$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{enc}}}{\varepsilon_0}}$$

When to use: Only with high symmetry (spherical, cylindrical, planar)

Choosing Gaussian Surface

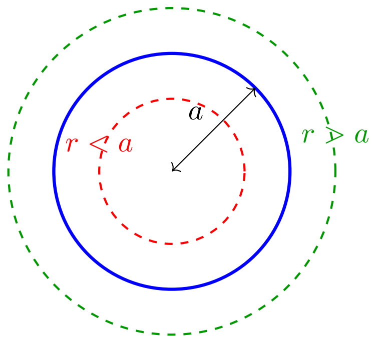
Requirements:

1. $\mathbf{E} \perp d\mathbf{S}$ everywhere (or $\mathbf{E} \parallel d\mathbf{S} = 0$ flux)

2. $|\mathbf{E}| =$ constant on surface

Common choices: - Sphere for spherical symmetry - Cylinder for cylindrical symmetry - Pillbox for planar symmetry

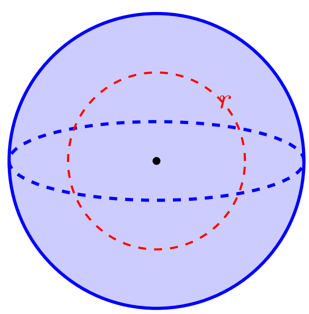
Example 1: Spherical Shell (radius a, charge Q)



Inside ($r < a$)**:** $Q_{\text{enc}} = 0 \implies \boxed{\mathbf{E} = 0}$

Outside ($r > a$)**:** $Q_{\text{enc}} = Q \implies \boxed{\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}}$ (acts like point charge!)

Example 2: Uniformly Charged Solid Sphere



Sphere radius a, density ρ

A solid sphere of radius a has uniform volume charge density ρ. Find the electric field $\mathbf{E}$ everywhere.

Total charge:

$$Q = \rho \cdot \frac{4}{3}\pi a^3$$

Symmetry:

By spherical symmetry, the electric field must be purely radial and depend only on the distance from the center:

$$\mathbf{E}(\mathbf{r}_{\text{obs}}) = E_r(r_{\text{obs}})\,\hat{\mathbf{a}}_r$$

Inside the sphere ($r_{\mathrm{obs}} < a$):

Choose a **spherical Gaussian surface** of radius r_{obs}, centered at the origin.

The charge enclosed is the charge contained within radius r_{obs}:

$$Q_{\mathrm{enc}} = \rho \cdot \frac{4}{3}\pi r_{\mathrm{obs}}^3$$

Gauss's law:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\mathrm{enc}}}{\varepsilon_0}$$

On a spherical surface, $\mathbf{E}$ is radial and constant in magnitude, so

$$E_r(r_{\mathrm{obs}})\,(4\pi r_{\mathrm{obs}}^2) = \frac{\rho \frac{4}{3}\pi r_{\mathrm{obs}}^3}{\varepsilon_0}$$

Solving,

$$\boxed{\mathbf{E}(\mathbf{r}_{\mathrm{obs}}) = \frac{\rho}{3\varepsilon_0}\, r_{\mathrm{obs}}\,\hat{\mathbf{a}}_r \qquad (r_{\mathrm{obs}} < a)}$$

Outside the sphere ($r_{\mathrm{obs}} > a$):

The Gaussian surface now encloses the **entire charge distribution**:

$$Q_{\mathrm{enc}} = Q = \rho \cdot \frac{4}{3}\pi a^3$$

Applying Gauss's law:

$$E_r(r_{\mathrm{obs}})\,(4\pi r_{\mathrm{obs}}^2) = \frac{\rho \frac{4}{3}\pi a^3}{\varepsilon_0}$$

Solving,

$$\boxed{\mathbf{E}(\mathbf{r}_{\mathrm{obs}}) = \frac{\rho a^3}{3\varepsilon_0\, r_{\mathrm{obs}}^2}\,\hat{\mathbf{a}}_r \qquad (r_{\mathrm{obs}} > a)}$$

Key physical interpretation:

- For $r_{\mathrm{obs}} < a$, only charge at $r_{\mathrm{src}} < r_{\mathrm{obs}}$ contributes to the field.
- For $r_{\mathrm{obs}} > a$, the sphere behaves like a point charge located at the origin.

Gauss's Law - Differential Form

The differential form relates $\mathbf{E}$ to charge density at each point:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}}$$

This is one of Maxwell's equations.

Connection to integral form:

Using the divergence theorem:

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{E})\, dV = \int_V \frac{\rho}{\varepsilon_0}\, dV = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

Work in Electrostatics

Work done by electric field:

To move a charge q from point A to point B:

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{l} = q \int_A^B \mathbf{E} \cdot d\mathbf{l}$$

Work done by external force:

To move a charge q **against** the electric field (at constant velocity):

$$W_{\text{ext}} = -q \int_A^B \mathbf{E} \cdot d\mathbf{l}$$

Conservative field:

The electric field is **conservative**, meaning:

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

for any closed path. This means work is **path independent**.

Summary Table

Configuration	**Gaussian Surface**	**Electric Field $\mathbf{E}(\mathbf{r}_{\text{obs}})$**
Point charge Q	Sphere (radius r_{obs})	$\dfrac{Q}{4\pi\varepsilon_0\, r_{\text{obs}}^2}\, \hat{\mathbf{a}}_r$
Spherical shell (outside)	Sphere ($r_{\text{obs}} > a$)	$\dfrac{Q}{4\pi\varepsilon_0\, r_{\text{obs}}^2}\, \hat{\mathbf{a}}_r$
Spherical shell (inside)	Sphere ($r_{\text{obs}} < a$)	0
Solid sphere (outside)	Sphere ($r_{\text{obs}} > a$)	$\dfrac{Q}{4\pi\varepsilon_0\, r_{\text{obs}}^2}\, \hat{\mathbf{a}}_r$
Solid sphere (inside)	Sphere ($r_{\text{obs}} < a$)	$\dfrac{Q\, r_{\text{obs}}}{4\pi\varepsilon_0\, a^3}\, \hat{\mathbf{a}}_r$
Infinite line charge λ	Cylinder (radius r_{obs})	$\dfrac{\lambda}{2\pi\varepsilon_0\, r_{\text{obs}}}\, \hat{\mathbf{a}}_R$
Infinite plane charge σ	Pillbox	$\dfrac{\sigma}{2\varepsilon_0}\, \hat{\mathbf{n}}$

Electric Potential, Work, and Voltage

Two Charges You Must Never Confuse

Rule:

- q_{source}: creates the electric field and potential
- q_{test}: small charge used to probe the field
- $\mathbf{E}$ and V depend only on q_{source}
- Forces, work, and energy depend on q_{test}

Work Done by an Electric Field

Force on a test charge:

$$\mathbf{F} = q_{\text{test}}\mathbf{E}$$

Work done by the electric field moving q_{test} from A to B:

$$W_{\text{field}} = \int_A^B \mathbf{F} \cdot d\mathbf{l} = q_{\text{test}} \int_A^B \mathbf{E} \cdot d\mathbf{l}$$

Key idea:

- If the field does positive work, the charge loses potential energy

Electric Potential Difference (Voltage)

We define electric potential difference as:

$$V(B) - V(A) = -\int_A^B \mathbf{E} \cdot d\mathbf{l}$$

Why the minus sign?

- Electric fields naturally move positive charges downhill in potential

Work–potential relationship:

$$W_{\text{field}} = q_{\text{test}}\big(V(A) - V(B)\big)$$

Very important:

- Voltage depends only on the field, not on q_{test}
- Voltage is energy per unit charge

Electric Potential (Absolute)

Electric potential at a point is defined using a reference point:

$$V(\mathbf{r}) = -\int_{\text{ref}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

Standard reference:

$$V(\infty) = 0$$

Why this works:

- Electrostatic fields are <u>conservative</u>
- The integral is <u>path independent</u>

Point Charge Source

Let a source charge q_{source} be at the origin.

Electric potential:

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_{\text{source}}}{r}$$

intuition:

- Bringing a positive test charge closer requires work
- That stored work shows up as higher potential

Compare with electric field:

$$E \propto \frac{1}{r^2}, \qquad V \propto \frac{1}{r}$$

From Potential to Field

Electric field is the <u>spatial rate of change</u> of potential:

$$\mathbf{E} = -\nabla V$$

In Cartesian coordinates:

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)$$

Physical meaning:

- Field points from <u>high V</u> to <u>low V</u>
- Steeper V change $\Rightarrow$ stronger field

Multiple Source Charges

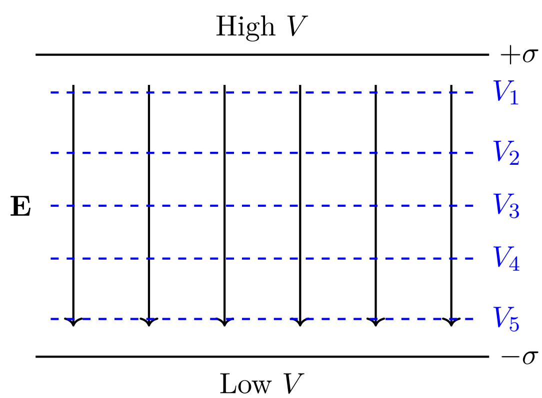
For multiple source charges $q_1, q_2, \ldots, q_n$:

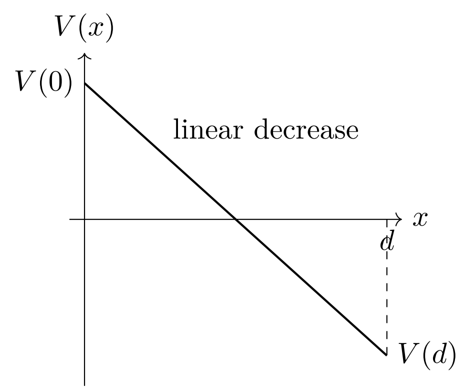
$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|}$$

Why potential is powerful:

- Potentials add as scalars
- Fields require vector addition

Example: Parallel Plates





Two infinite plates with surface charge densities $+\sigma$ and $-\sigma$.

Electric field:

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{x}$$

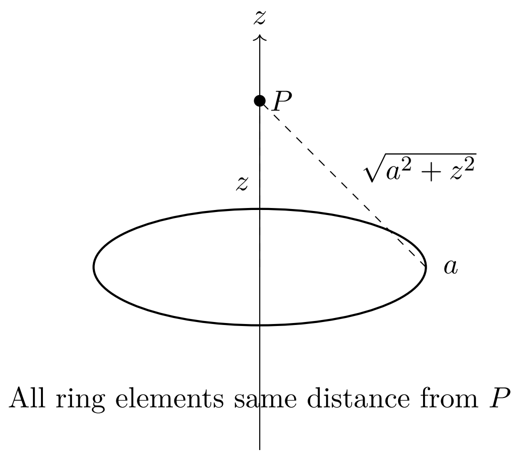
Potential difference:

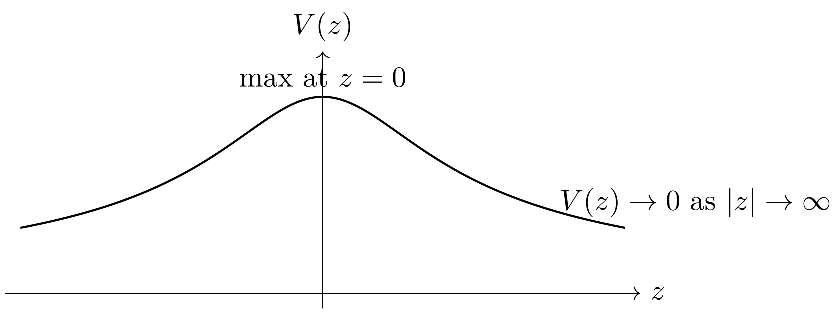
$$V(x) = -\frac{\sigma x}{\varepsilon_0}$$

Teaching insight:

- Uniform field ⇒ linear potential
- Motion along field direction lowers potential

Example: Charged Ring (Axis)





Ring radius a, total source charge Q.

Potential on axis:

$$V(z) = \frac{Q}{4\pi\varepsilon_0\sqrt{a^2 + z^2}}$$

Electric field from potential:

$$E_z = -\frac{dV}{dz} = \frac{Qz}{4\pi\varepsilon_0(a^2 + z^2)^{3/2}}$$

Why this works:

- All charge elements are same distance from point
- Symmetry cancels transverse field components

Equipotential Surfaces

Surfaces where:

$$V = \text{constant}$$

Properties:

- $\mathbf{E}$ is <u>perpendicular</u> to equipotentials
- No work is done moving along an equipotential
- Closer spacing $\Rightarrow$ stronger field

Example:

- Point charge $\Rightarrow$ spherical equipotentials

Continuous Charge Distributions

Line charge:

$$V(\mathbf{r}_{\text{obs}}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda \, dl_{\text{src}}}{|\mathbf{r}_{\text{obs}} - \mathbf{r}_{\text{src}}|}$$

Surface charge:

$$V(\mathbf{r}_{\text{obs}}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma \, dS_{\text{src}}}{|\mathbf{r}_{\text{obs}} - \mathbf{r}_{\text{src}}|}$$

Volume charge:

$$V(\mathbf{r}_{\text{obs}}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho \, dV_{\text{src}}}{|\mathbf{r}_{\text{obs}} - \mathbf{r}_{\text{src}}|}$$