

MIE Lecture Notes

Probability and Statistics

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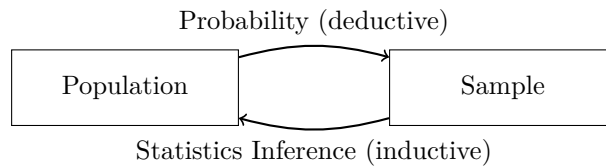
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Chapter 1

Statistics Definitions

Global definition: Statistics involves collecting, organizing, summarizing, presenting, and analyzing data, as well as making inferences, conclusions, and decisions based on data.

Statistical definition: A statistic is a numerical value calculated from data (e.g. mean, proportion, standard deviation).



Basic Terminology

Individuals: Objects on which data are collected (people, animals, plots of land, etc.).

Variable: Any characteristic of an individual.

Population: The entire group of individuals of interest.

Sample: A subset of individuals taken from the population.

Statistical Inference: Drawing conclusions about a population based on a sample.

Sampling Methods

Simple Random Sample (SRS):

- Every possible group of size n has an equal chance of being selected.
- Helps avoid bias in sampling.
- Can be selected using random number tables or software.

Stratified Random Sampling:

- The population is divided into homogeneous groups (*individuals are similar with respect to the variable being studied*) called strata.
- A simple random sample is taken from each stratum. (*one subgroup of the population created*)
- Ensures that important subgroups are neither over nor under represented.

Data, Variables, and Distributions

Types of Variables

Categorical Variable: Places individuals into categories (e.g. gender, major). These are qualitative.

Quantitative Variable: Takes numerical values for which arithmetic operations are meaningful.

- Discrete
- Continuous

Distributions

Distribution: Describes what values a variable takes and how often those values occur. When examining a distribution, look for:

- **Shape**
- **Center**
- **Spread**
- **Outliers**

Outlier: An individual value that falls outside the overall pattern of the data.

Describing Distributions with Numbers

Central Tendency: Describes where the data cluster or center.

Central Tendency: Describes where the data cluster or center.

- Mean: average value
- Median: middle value

Mean (Arithmetic Mean):

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

Median:

$$\tilde{x} = \begin{cases} x_{(\frac{n+1}{2})}, & \text{if } n \text{ is odd} \\ \frac{x_{(\frac{n}{2})} + x_{(\frac{n}{2}+1)}}{2}, & \text{if } n \text{ is even} \end{cases}$$

Theorem 1.1

1. The mean is more sensitive to extreme values than the median.
2. Changing a single data value will always change the mean, but may not change the median.
3. If a distribution is exactly symmetric, the mean and median are equal.

Trimmed Mean: The mean computed after removing extreme values.

$$\bar{x}_{\text{trim}} = \frac{1}{n - 2k} \sum_{i=k+1}^{n-k} x_{(i)}$$

where k values are removed from both ends of the ordered data. (normally given in question like 10%)

Measures of Spread

Range: Maximum minus minimum. Very sensitive to extreme values.

Sample Variance: Measures the average squared deviation from the mean.

$$s^2 = \frac{1}{n - 1} \sum_{i=1}^n (x_i - \bar{x})^2$$

Standard Deviation: The square root of the sample variance.

$$s = \sqrt{s^2}$$

Degrees of Freedom: The number of independent pieces of information available to estimate variability. For sample variance: $df = n - 1$.

Graphical Representations of Data

Scatter Plot: Used to display the relationship between two quantitative variables (x, y) . A scatter plot helps identify trends, patterns, and associations between variables.

Stem-and-Leaf Plot: An intermediate step between raw data and a frequency table. Preserves the original data values while showing the distribution.

Stem	Leaf
1	2 4 7
2	1 3 5 8
3	0 4 6

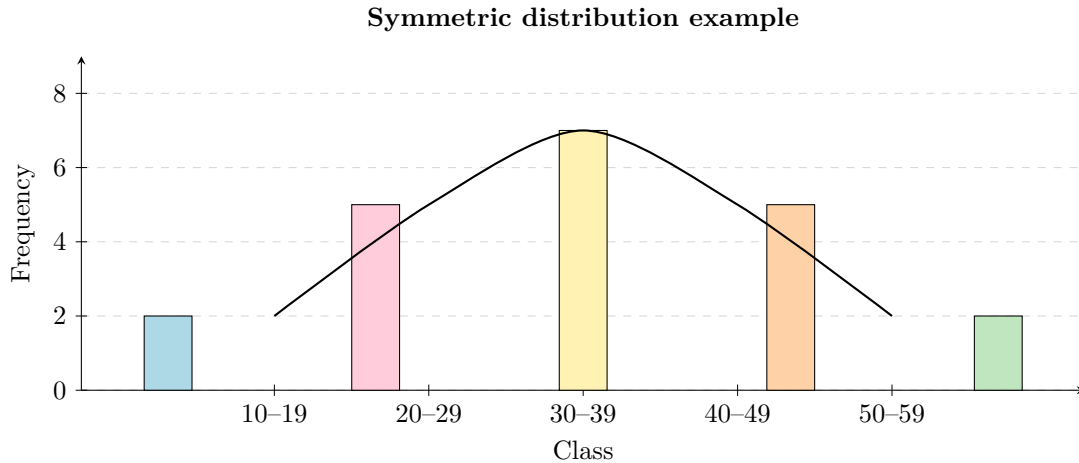
Relative Frequency Table: Shows the proportion of observations in each class.

Class Interval	Class Midpoint	Frequency	Relative Frequency
10–19	14.5	3	0.30
20–29	24.5	4	0.40
30–39	34.5	3	0.30

Histogram: A graphical representation of a frequency or relative frequency table using contiguous bars.

When describing the shape of a histogram, we commonly classify it as:

- Symmetric
- Skewed right (positively skewed)
- Skewed left (negatively skewed)



Chapter 2, Jan 9th

Experiments, Sample Spaces, and Events

Experiment: A process that generates an outcome.

Sample Space (S): The set of all possible outcomes of an experiment.

Example 1:

Select 3 items from a production line. Each item can be classified as either defective (D) or non-defective (N).

$$S = \{DDD, DDN, DND, NDD, DNN, NDN, NND, NNN\}$$

Since each item has 2 possible outcomes,

$$|S| = 2^3 = 8$$

Example 2:

$$S = \{(x, y) \mid x^2 + y^2 \leq 4\}$$

Event (A): A subset of the sample space S .

Examples of events:

$$A = \{DDD, DDN, DND, NDD\}$$

$$B = \{NNN\}$$

$$C = \{(x, y) \mid x^2 + y^2 \leq 4\}$$

Event Operations and Probability Rules

Event Operations:

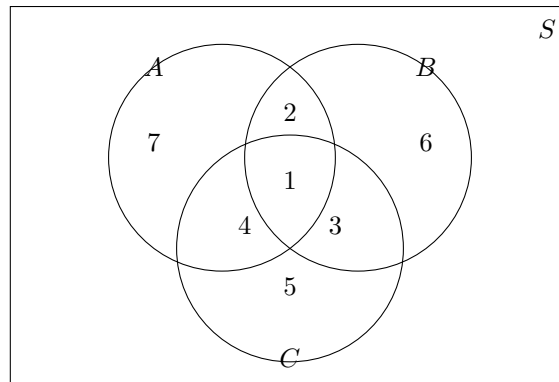
- Complement: A^c (or A')
- Intersection: $A \cap B$
- Union: $A \cup B$
- Null Event: \emptyset

If

$$A \cap B = \emptyset,$$

then A and B are mutually exclusive.

Example (Venn Diagram):



$$A = \{DDD, DDN, DND, NDD\}, \quad B = \{NNN\}$$

$$A \cup B = \{DDD, DDN, DND, NDD, NNN\}$$

$$A \cap B = \emptyset$$

Chapter 2: January 12

Review

1. Experiment: A process that generates an outcome.
2. Sample Space (S): The set of all possible outcomes of an experiment.
3. Event Operations:
 - Complement: A' (A^c)
 - Intersection: $A \cap B$
 - Union: $A \cup B$
 - Null Event: \emptyset
4. If $A \cap B = \emptyset$, then A and B are called mutually exclusive.

$$(A \cap B)' = A' \cup B'$$

$$(A \cup B)' = A' \cap B'$$

$$A \cap \emptyset = \emptyset$$

$$A \cup \emptyset = A$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Probability

$P(A)$ = probability of event A: the proportion of times the event occurs in infinitely many repetitions of the experiment.

Theorem 2.1

$$0 \leq P(A) \leq 1$$

$$P(A) + P(A') = 1$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\quad + P(A \cap B \cap C) \end{aligned}$$

Mutually Exclusive Events

Definition: If A_1, A_2, \dots, A_n are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$$

If

$$A_1 \cup A_2 \cup \dots \cup A_n = S,$$

then $\{A_1, A_2, \dots, A_n\}$ is a partition of S .

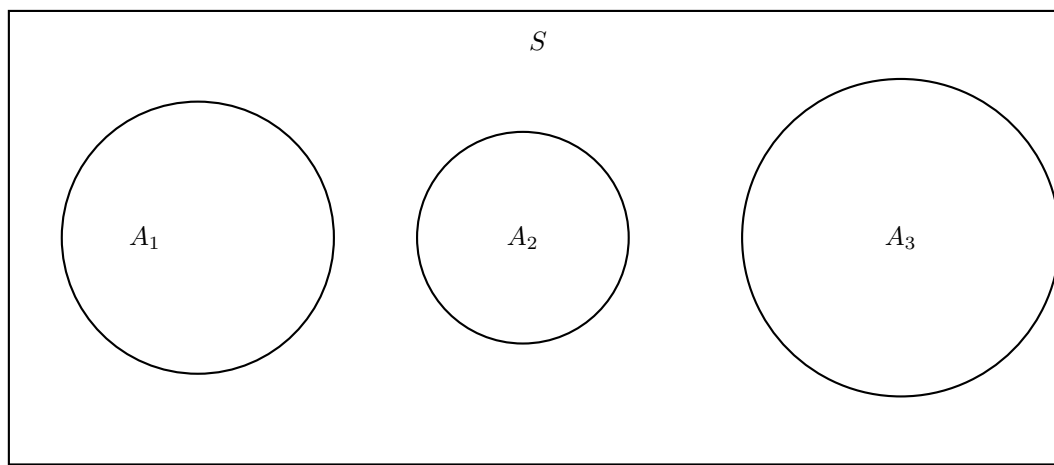


Figure 1: Partition of the sample space S into A_1, A_2, A_3

Example

In a class of 33 students:

- 17 earned an A on the midterm
- 14 earned an A on the final
- 11 earned no A on either exam

Find the probability that a randomly selected student earned A's on both exams.

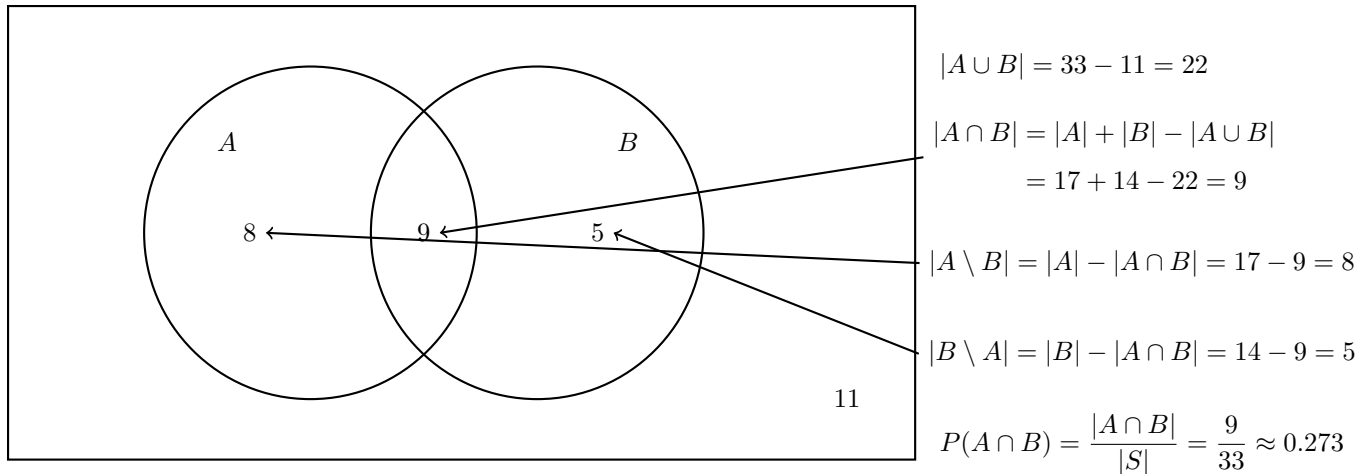


Figure 2: Events A : A on midterm, B : A on final, with region counts and calculations

Counting Techniques and Equally Likely Outcomes

Theorem 2.2 (Equally Likely Outcomes)

If the sample space S has a finite number of outcomes and all outcomes are equally likely, then for any event A ,

$$P(A) = \frac{|A|}{|S|}.$$

where

$|A|$ = number of outcomes in event A , $|S|$ = number of outcomes in the sample space.

Example 1: Poker Hands Basics

A standard deck has:

$$4 \text{ suits} \times 13 \text{ denominations (A,2,3,\dots,Q,K)} = 52 \text{ cards.}$$

A poker hand consists of 5 cards chosen from 52:

$$|S| = \binom{52}{5} = 2,598,960.$$

Combinations Reminder

If there are 3 objects $\{A, B, C\}$ and we choose 2:

$$\binom{3}{2} = \frac{3!}{(3-2)!2!}.$$

Order does not matter.

Example 2: Probability of 2 Aces and 1 Jack

A 5-card hand contains:

- exactly 2 aces,
- exactly 1 jack,
- 2 cards that are neither aces nor jacks.

$$P(2 \text{ aces and 1 jack}) = \frac{\binom{4}{2} \binom{4}{1} \binom{44}{2}}{\binom{52}{5}}.$$

Example 3: Probability of a Full House

A full house consists of:

- 3 cards of one denomination
- 2 cards of a different denomination

Number of full house hands:

$$\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2}.$$

Thus,

$$P(\text{full house}) = \frac{\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2}}{\binom{52}{5}}.$$

Example 4: Probability of Four of a Kind

A four of a kind consists of:

- 4 cards of the same denomination
- 1 remaining card of a different denomination

Number of such hands:

$$\binom{13}{1} \binom{4}{4} \binom{48}{1}.$$

Thus,

$$P(\text{four of a kind}) = \frac{\binom{13}{1} \binom{4}{4} \binom{48}{1}}{\binom{52}{5}}.$$

Example 5: Probability of Exactly One Pair

An **excatly** one-pair hand consists of:

- 1 pair
- 3 cards of different denominations, none matching the pair

Number of such hands:

$$\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1}^3.$$

Thus,

$$P(\text{exactly one pair}) = \frac{\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1}^3}{\binom{52}{5}}.$$

Note*: Counting Patterns

$$\binom{a}{b}$$

Meaning: Choose b different items from a **at once**, order does not matter.

Key features:

- No repeats
- Grouped choice
- Used when items must be distinct

$$\binom{a}{1}^b$$

Meaning: Make b independent choices, each time choosing 1 item from c .

Key features:

- Repeats allowed
- Choices are independent
- Used when selections do not restrict each other

Rule to Remember:

$$\text{Different items, no repeats} \Rightarrow \binom{a}{b}$$

$$\text{Independent choices} \Rightarrow \binom{c}{1}^b$$

Chapter 2 continue, Jan 14

Review

1. Probability is the proportion of times the event occurs in infinitely many repetitions of the experiment.
2. $0 \leq P(A) \leq 1$
3. $P(A) + P(A^c) = 1$
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
 $P(A \cup B \cup C) = P(A) + P(B) + P(C)$
 $- P(A \cap B) - P(A \cap C) - P(B \cap C)$
 $+ P(A \cap B \cap C)$
- 5.
6. Permutation: A permutation counts ordered arrangements.

$${}_nP_r = \frac{n!}{(n-r)!}$$

Example 1: Two fair dice

A pair of fair dice are rolled. Find the probability that the second die lands on a smaller value than the first. The outcomes where the second die is smaller than the first are represented below.

First Die (Stem)	Second Die (Leaf)
2	1
3	1 2
4	1 2 3
5	1 2 3 4
6	1 2 3 4 5

There are 15 favorable outcomes and 36 total outcomes.

$$P(\text{second} < \text{first}) = \frac{15}{36} = \frac{5}{12}.$$

Conditional Probability and Independence

Conditional Probability

The conditional probability of an event B given that event A has occurred is the probability that B occurs when it is known that A has occurred.

$$P(B | A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

Example 2: Drinking Survey

A survey records the following data:

	<i>D</i>	<i>N</i>	Total
<i>M</i>	19	41	60
<i>F</i>	12	28	40
Total	31	69	100

The symbols used above are defined as follows:

- *M*: male
- *F*: female
- *D*: the individual drinks
- *N*: the individual does not drink

$$P(D|M) = \frac{19}{60} \quad P(M|D) = \frac{19}{31}$$

Law of Total Probability

Theorem 2.3 (Law of Total Probability)

If B_1, B_2, \dots, B_k form a partition of the sample space S with $P(B_i) > 0$ for all i , then for any event A ,

$$P(A) = \sum_{i=1}^k P(A | B_i) P(B_i).$$

Example 3: Monty Hall (3 doors)

Car location	Monty opens	Probability	Stay	Switch
Door 1	Door 2	$\frac{1}{6}$	Car	Goat
Door 1	Door 3	$\frac{1}{6}$	Car	Goat
Door 2	Door 3	$\frac{1}{3}$	Goat	Car
Door 3	Door 2	$\frac{1}{3}$	Goat	Car

Staying wins only when the car is behind Door 1, so

$$P(\text{win by staying}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Switching wins when the car is behind Door 2 or Door 3, so

$$P(\text{win by switching}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Example 4: Birthday Problem

Assume the following:

- Leap years are ignored
- All 365 birthdays are equally likely
- Birthdays of different people are independent

Question: What is the probability that at least two people share the same birthday in a group of n people?

Rather than computing this directly, we use the complement rule.

$$P(\text{at least one match}) = 1 - P(\text{no match})$$

Probability of no shared birthdays

- Person 1 can have any birthday: probability 1
- Person 2 must avoid that birthday: $\frac{364}{365}$
- Person 3 must avoid the first two birthdays: $\frac{363}{365}$
- ...
- Person n must avoid the previous $n - 1$ birthdays: $\frac{365 - (n - 1)}{365}$

Therefore,

$$P(\text{no match}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365 - (n - 1)}{365}$$

or equivalently,

$$P(\text{no match}) = \prod_{k=0}^{n-1} \frac{365 - k}{365}$$

Final result

$$P(\text{at least one shared birthday}) = 1 - \prod_{k=0}^{n-1} \frac{365 - k}{365}$$

Important values

- For $n = 23$: $P(\text{at least one match}) \approx 0.507$
- For $n = 57$: $P(\text{at least one match}) \approx 0.99$

Chapter 2 — Jan 16

Review: Conditional Probability

Conditional Probability:

The probability of event B given that event A has occurred is

$$P(B | A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

Read as: the probability of B given A .

Independence of Events

Definition (Independence): Events A and B are independent if and only if

$$P(B | A) = P(B)$$

Equivalently,

$$P(A | B) = P(A)$$

or

$$P(A \cap B) = P(A) P(B)$$

Multiple Independent Events

Definition:

If events A_1, A_2, \dots, A_k are independent, then

$$P(A_1 \cap A_2 \cap \dots \cap A_k) = P(A_1) P(A_2) \cdots P(A_k)$$

Mutual Independence

Mutual Independence : A collection of events A_1, A_2, \dots, A_n is mutually independent if and only if for *every* subcollection $\{A_{i_1}, \dots, A_{i_k}\}$,

$$P\left(\bigcap_{j=1}^k A_{i_j}\right) = \prod_{j=1}^k P(A_{i_j})$$

Example (Three Events):

Events A_1, A_2, A_3 are mutually independent if all of the following hold:

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1)P(A_3)$$

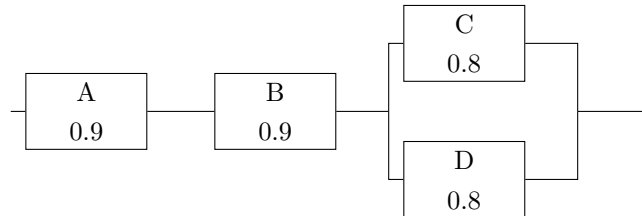
$$P(A_2 \cap A_3) = P(A_2)P(A_3)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

Note: Mutually exclusive events are dependent. If one event occurs, the other cannot occur.

Example: Component Reliability

An electrical system has four components A, B, C, D . The system works if A and B work and at least one of C or D works. Assume all components are independent.



$$P(A) = 0.9, \quad P(B) = 0.9, \quad P(C) = 0.8, \quad P(D) = 0.8$$

(a) Probability the entire system works

The system works if A and B work and either C or D works.

$$\begin{aligned}
 P(\text{system works}) &= P(\text{all work}) + P(A, B, C \text{ work}, D \text{ does not}) \\
 &\quad + P(A, B, D \text{ work}, C \text{ does not}) \\
 &= (0.9)(0.9)(0.8)(0.8) + (0.9)(0.9)(0.8)(1 - 0.8) + (0.9)(0.9)(0.8)(1 - 0.8)
 \end{aligned}$$

$$P(\text{system works}) = 0.7776$$

(b) **Conditional probability**

$$P(C^c \mid \text{system works}) = \frac{P(C^c \cap \text{system works})}{P(\text{system works})}$$

$$P(C^c \cap \text{system works}) = (0.9)(0.9)(0.8)(1 - 0.8)$$

$$P(C^c \mid \text{system works}) = \frac{(0.9)(0.9)(0.8)(1 - 0.8)}{0.7776} = 0.16$$

Theorem of Total Probability

Let B_1, B_2, \dots, B_k be a partition of the sample space S such that $P(B_i) > 0$ for all i . Then for any event $A \subseteq S$,

$$P(A) = \sum_{i=1}^k P(A \mid B_i) P(B_i) = \sum_{i=1}^k P(A \cap B_i)$$

Theorem: Bayes' Rule (1701–1761)

Let B_1, B_2, \dots, B_k be a partition of the sample space S such that $P(B_i) > 0$ for $i = 1, \dots, k$. For any event $A \subseteq S$ with $P(A) > 0$,

$$P(B_r \mid A) = \frac{P(B_r \cap A)}{\sum_{i=1}^k P(B_i \cap A)} = \frac{P(B_r) P(A \mid B_r)}{\sum_{i=1}^k P(B_i) P(A \mid B_i)}, \quad r = 1, \dots, k$$

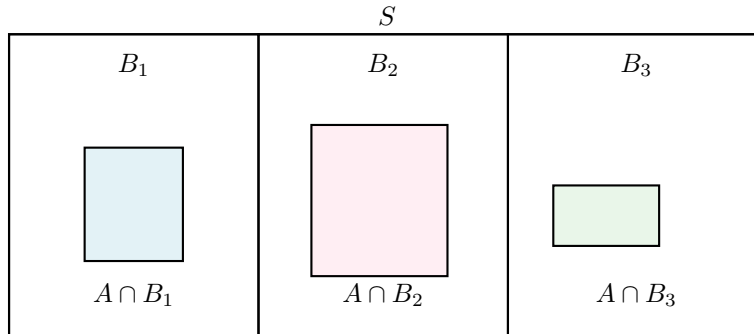


Figure 3: Partition of S into B_1, B_2, B_3 with shaded regions $A \cap B_i$

Example (Medical Test)

The fraction of people in a population who have a certain disease is 0.01.

$$P(D) = 0.01, \quad P(D^c) = 0.99$$

The test characteristics are:

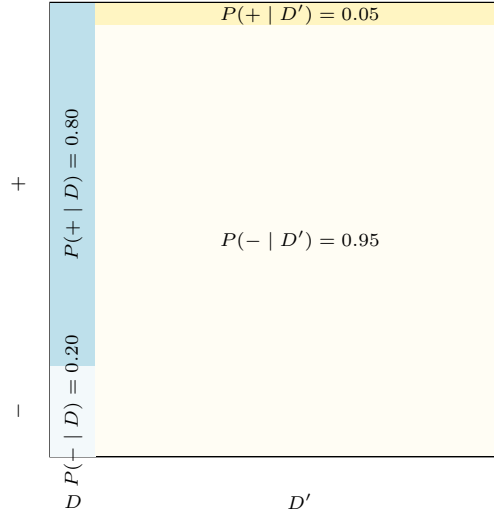
$$P(\text{test says } D \mid D^c) = 0.05 \quad (\text{false positive rate})$$

$$P(\text{test says } D^c \mid D) = 0.20 \quad (\text{false negative rate})$$

Thus,

$$P(\text{test says } D \mid D) = 1 - 0.20 = 0.80$$

Note: $1 - P(\text{test says } D^c \mid D)$ is called the sensitivity of the test, and $1 - P(\text{test says } D \mid D^c)$ is called the specificity.



(a) Probability the test says disease

$$P(\text{test says } D) = P(D \cap \text{test says } D) + P(D^c \cap \text{test says } D)$$

$$= P(\text{test says } D \mid D)P(D) + P(\text{test says } D \mid D^c)P(D^c)$$

$$= (0.80)(0.01) + (0.05)(0.99) = 0.0575$$

(b) Probability of disease given positive test

$$P(D \mid \text{test says } D) = \frac{P(D \cap \text{test says } D)}{P(\text{test says } D)}$$

$$= \frac{P(\text{test says } D \mid D)P(D)}{0.0575} = \frac{(0.80)(0.01)}{0.0575}$$

$$P(D \mid \text{test says } D) \approx 0.139$$

(c) Probability of disease given negative test

$$P(D \mid \text{test says } D^c) = \frac{P(D \cap \text{test says } D^c)}{P(\text{test says } D^c)}$$

$$= \frac{P(\text{test says } D^c \mid D)P(D)}{1 - P(\text{test says } D)}$$

$$= \frac{(0.20)(0.01)}{1 - 0.0575}$$

$$P(D \mid \text{test says } D^c) \approx 0.00212$$

Chapter 3 — January 19

Random Variables and Their Interpretation

Definition: A random variable (r.v.) is a rule that assigns a **real number** to each outcome in the sample space.

Alternative definition: A random variable is a function that takes the outcome of an experiment and assigns it a number so that probabilities can be calculated.

Example 1: Three Electronic Components

Each component is classified as either defective (D) or non-defective (N).

$$S = \{NNN, DNN, NDN, NND, DDN, DND, NDD, DDD\}$$

- **Defective (D):** the component does not meet required specifications and fails inspection.
- **Non-defective (N):** the component meets specifications and passes inspection.

Define the random variable

X = number of defective components.

Then:

$$\begin{aligned} X = 0 & \text{ for } \{NNN\} \\ X = 1 & \text{ for } \{DNN, NDN, NND\} \\ X = 2 & \text{ for } \{DDN, DND, NDD\} \\ X = 3 & \text{ for } \{DDD\} \end{aligned}$$

Thus, the possible values of X are:

$$\{0, 1, 2, 3\}.$$

Example 2: One Component (Dummy Variable)

$$S = \{D, N\}$$

Define the random variable

$$X = \begin{cases} 1, & \text{if the component is defective (D)} \\ 0, & \text{if the component is non-defective (N)} \end{cases}$$

This is called a **dummy variable** because the outcome is categorical, but is encoded numerically.

A dummy variable is a special type of random variable that assigns numerical labels to categorical outcomes, where the numbers have no quantitative meaning beyond identification.

Discrete Random Variables

Definition: A random variable is called discrete if its set of possible values is **countable** (finite or countably infinite).

Example 3: Sampling Until First Defective

Components are tested one(independently) at a time until the first defective component is observed.

$$S = \{D, ND, NND, NNND, \dots\}$$

Define

$$X = \text{number of components tested until the first defective.}$$

Then:

$$\begin{aligned} X = 1 & \text{ for } \{D\} \\ X = 2 & \text{ for } \{ND\} \\ X = 3 & \text{ for } \{NND\} \\ & \vdots \end{aligned}$$

Hence,

$$X = 1, 2, 3, \dots$$

Since the possible values can be listed, X is a discrete random variable.

Non-discrete version of the same experiment:

Define

Y = time (in seconds) until the first defective component is observed.

Since Y can take any real value in $[0, \infty)$, it cannot be listed and is therefore a continuous (non-discrete) random variable.

Discrete vs. Continuous Random Variables

Discrete Random Variable	Continuous Random Variable
Counts things	Measures things
Possible values are countable	Possible values fill an interval
$P(X = x)$ can be > 0	$P(X = x) = 0$ for all x
Uses a probability mass function (PMF)	Uses a probability density function (PDF)

Probability Mass Function (PMF)

Definition:

Let X be a discrete random variable. The probability mass function (PMF) of X , denoted $f(x)$, is defined by:

$$\begin{array}{l} 1) \quad f(x) \geq 0 \quad \text{for all } x \\ 2) \quad \sum_x f(x) = 1 \end{array}$$

Note:

- Capital X : random variable
- Lowercase x : a specific value

Bernoulli and Binomial Random Variables

I. Bernoulli Random Variable (Single Trial)

A Bernoulli random variable: X models a single experiment with only two possible outcomes: success or failure.

$$X = \begin{cases} 1, & \text{success} \\ 0, & \text{failure} \end{cases}$$

If $p = P(X = 1)$, then the PMF is

x	0	1
$P(X = x)$	$1 - p$	p

Here, p is the probability of success (e.g. observing a defective component).

Binomial Random Variable (Multiple Bernoulli Trials)

The binomial random variable extends the Bernoulli case to multiple independent trials.

Definition:

A random variable X is called a binomial random variable if it represents the number of successes in n independent Bernoulli trials, each with success probability p .

X = number of successes in n trials

In this case,

$$X \sim \text{Bin}(n, p)$$

and the probability mass function is

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, \dots, n.$$

Conditions for a Binomial Model

A binomial model applies only if:

- each trial has exactly two outcomes (success or failure),
- the probability of success p is the same for every trial,
- the trials are independent,
- the number of trials n is fixed.

Example: Three Components Tested

Assume each component is defective with probability

$$p = 0.1, \quad n = 3.$$

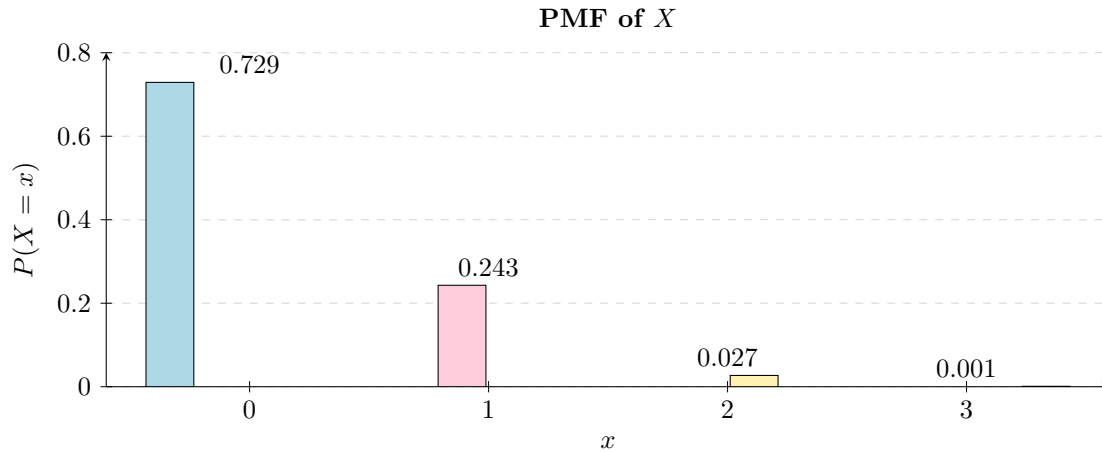
where $p = P(\text{a single component is defective})$, $n = \text{number of trials}$

Let

X = number of defective components.

x	$P(X = x)$
0	$\binom{3}{0}(0.9)^3 = 0.729$
1	$\binom{3}{1}(0.1)(0.9)^2 = 0.243$
2	$\binom{3}{2}(0.1)^2(0.9) = 0.027$
3	$\binom{3}{3}(0.1)^3 = 0.001$

$$0.729 + 0.243 + 0.027 + 0.001 = 1.$$



Geometric Random Variable

Example: Sampling Until First Defective

Components are sampled one at a time until the first defective component is observed. Assume the probability that a component is defective is

$$p = 0.1.$$

Define the random variable

X = number of samples collected until the first defective.

x	$P(X = x) = f(x)$
1	0.1
2	$0.9(0.1)$
3	$0.9^2(0.1)$
\vdots	\vdots

Geometric Random Variable

Definition:

A random variable X is called a geometric random variable if it represents the number of trials needed to obtain the first success in a sequence of independent Bernoulli trials with success probability p .

$$P(X = x) = (1 - p)^{x-1}p, \quad x = 1, 2, 3, \dots$$

In this example,

$$P(X = x) = 0.9^{x-1}(0.1).$$

Verification That Probabilities Sum to 1

$$\sum_{x=1}^{\infty} 0.9^{x-1}(0.1) = 0.1 \sum_{x=0}^{\infty} 0.9^x = 0.1 \left(\frac{1}{1 - 0.9} \right) = 1.$$

Cumulative Distribution Function (CDF)

Definition:

The cumulative distribution function (CDF) of a discrete random variable X with PMF $f(x)$ is defined as

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad -\infty < x < \infty.$$

Example: Binomial Distribution (Three Components)

Let X be the number of defective components when three components are tested, with

$$P(X = 0) = 0.729, \quad P(X = 1) = 0.243, \quad P(X = 2) = 0.027, \quad P(X = 3) = 0.001.$$

$$F(0) = P(X \leq 0) = 0.729$$

$$F(1) = P(X \leq 1) = 0.729 + 0.243 = 0.972$$

$$F(2) = P(X \leq 2) = 0.729 + 0.243 + 0.027 = 0.999$$

$$F(3) = P(X \leq 3) = 0.729 + 0.243 + 0.027 + 0.001 = 1$$

Thus, the CDF can be written as

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.729, & 0 \leq x < 1 \\ 0.972, & 1 \leq x < 2 \\ 0.999, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

Properties of the CDF

- $F(x)$ is monotone non-decreasing.

- If $x < y$, then $F(x) \leq F(y)$.
- $0 \leq F(x) \leq 1$.

Note: A function is **monotone** non-decreasing if its value never decreases as the input increases.

Using the CDF to Compute Probabilities

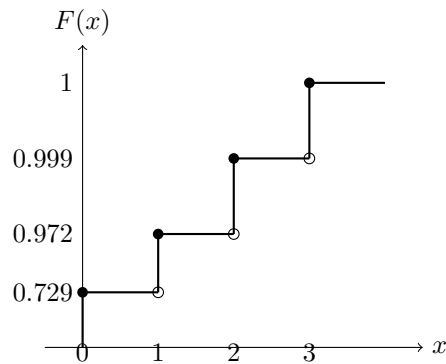
For $a < b$,

$$P(a < X \leq b) = F(b) - F(a).$$

Example:

$$P(0 < X \leq 2) = F(2) - F(0) = 0.999 - 0.729 = 0.27.$$

CDF Histogram (Step Function)



Note: For a discrete random variable, the PMF is drawn as a bar chart since it shows probabilities at individual points, while the CDF is drawn as a step function since it represents cumulative probability and is monotone non-decreasing.

Chapter 3 — January 21

Review

1. Random Variable (RV), X : A random variable assigns a real number to each outcome.
2. Discrete Random Variable: If X is discrete,
 - $P(X = x) = f(x)$
 - $f(x)$ is the probability mass function (PMF)
 - $f(x) \geq 0$
 - $\sum_x f(x) = 1$
3. Cumulative Distribution Function (CDF), $F(x)$:

- $F(x) = P(X \leq x)$
- If X is discrete:

$$F(x) = \sum_{t \leq x} f(t), \quad -\infty < x < \infty$$

Continuous Sample Space and Continuous Random Variables

If the sample space contains an infinite number of outcomes equal to the number of points on a line segment, it is called a continuous sample space.

A continuous random variable has

$$P(X = x) = 0 \quad \text{for all } x,$$

so probabilities are computed over intervals instead of single values.

***Alternative definition:** A continuous sample space contains infinitely many outcomes, like the points on a line segment. For a continuous random variable, the probability of taking any exact value is zero, i.e. $P(X = x) = 0$ for all x . Therefore, probabilities are computed over intervals rather than at single points.

Probability Density Function (PDF)

A function $f(x)$ is called a probability density function (PDF) of a continuous random variable X , defined over \mathbb{R} , if:

$$1. f(x) \geq 0 \text{ for all } x \in \mathbb{R}$$

$$2. \int_{-\infty}^{\infty} f(x) dx = 1$$

$$3. \text{ For any } a < b,$$

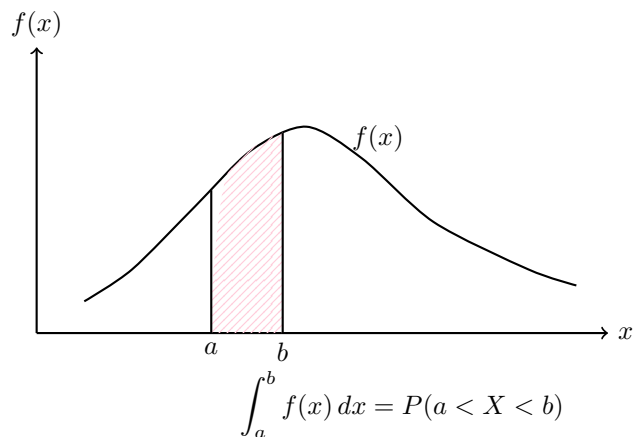
$$P(a < X < b) = \int_a^b f(x) dx$$

For continuous random variables,

$$P(a < X < b) = P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b).$$

$$P(X = a) = 0$$

because a single point has zero area under the probability density function.



Example: Uniform Distribution

Let the probability density function be

$$f(x) = \begin{cases} c, & 5 < x < 10, \\ 0, & \text{otherwise.} \end{cases}$$

1) Determine the value of c

Since $f(x)$ is a probability density function, the total area under the curve must equal 1:

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

Because $f(x) = 0$ outside the interval $(5, 10)$,

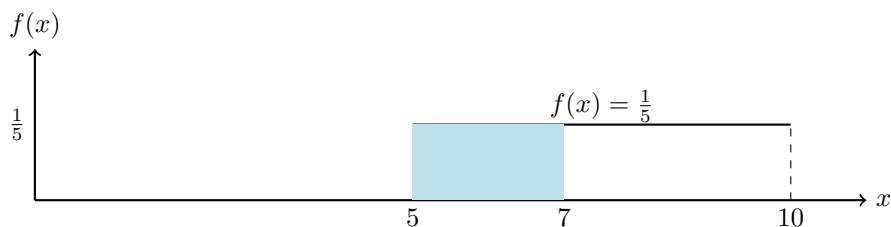
$$\int_5^{10} c dx = 1.$$

Evaluating the integral,

$$c(10 - 5) = 1 \quad \Rightarrow \quad c = \frac{1}{5}.$$

Thus,

$$f(x) = \begin{cases} \frac{1}{5}, & 5 < x < 10, \\ 0, & \text{otherwise.} \end{cases}$$



2): Compute $P(X < 7)$

Formula used:

$$P(a < X < b) = \int_a^b f(x) dx.$$

Applying this formula,

$$P(X < 7) = \int_5^7 \frac{1}{5} dx = \frac{1}{5}(7 - 5) = \frac{2}{5}.$$

CDF for Continuous Random Variables

Def: Let X be a continuous random variable with pdf: probability density function $f(x)$. The cumulative distribution function is

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt.$$

Consequently:

$$f(x) = \frac{d}{dx} F(x), \quad P(a < X < b) = F(b) - F(a).$$

Example

Let X be the time until a chemical reaction is complete (in msec). Suppose the CDF is

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-0.01x}, & x \geq 0. \end{cases}$$

(a) Find the pdf.

Use $f(x) = \frac{d}{dx} F(x)$.

For $x < 0$, $F(x) = 0$, so

$$f(x) = 0.$$

For $x \geq 0$,

$$f(x) = \frac{d}{dx} (1 - e^{-0.01x}) = 0.01e^{-0.01x}.$$

Therefore,

$$f(x) = \begin{cases} 0, & x < 0, \\ 0.01e^{-0.01x}, & x \geq 0. \end{cases}$$

(b) Find $P(X < 200)$.

Use the CDF directly:

$$P(X < 200) = F(200) = 1 - e^{-0.01(200)} = 1 - e^{-2} \approx 0.8647.$$

(c) Check if this is a valid CDF.

Theorem: Properties of a Cumulative Distribution Function

A function $F(x)$ is a valid cumulative distribution function (CDF) if and only if:

- $0 \leq F(x) \leq 1$ for all x
- $F(x)$ is monotone non-decreasing, i.e.

$$x \leq y \implies F(x) \leq F(y)$$

- $\lim_{x \rightarrow -\infty} F(x) = 0$
- $\lim_{x \rightarrow \infty} F(x) = 1$

For this $F(x)$:

$$\lim_{x \rightarrow -\infty} F(x) = 0, \quad \lim_{x \rightarrow \infty} F(x) = 1,$$

and for $x \geq 0$, $1 - e^{-0.01x}$ increases as x increases, so $F(x)$ is monotone non-decreasing.

Thus, $F(x)$ is a valid CDF.

Joint Probability Distributions (Discrete)

Def: The function $f(x, y)$ is a joint probability mass function (PMF) of discrete random variables X and Y if:

1. $f(x, y) \geq 0$ for all (x, y)
2. $\sum_x \sum_y f(x, y) = 1$
3. $P(X = x, Y = y) = f(x, y)$

That is, $f(x, y)$ gives the probability that the two random variables X and Y take the values x and y *simultaneously*.

For any region A in the xy -plane,

$$P((X, Y) \in A) = \sum_{(x, y) \in A} f(x, y)$$

Example: Pen Refills

Two refills are selected at random and without replacement from a box containing:

- 3 blue refills
- 2 red refills
- 3 green refills

Define the random variables

X = number of blue refills selected, Y = number of red refills selected.

The total number of possible selections is

$$\binom{8}{2}.$$

Joint PMF Table

For each pair (x, y) ,

$$f(x, y) = \frac{\text{number of favorable outcomes}}{\binom{8}{2}}.$$

$X \backslash Y$	0	1	2	Row Total
0	$\frac{\binom{3}{2}}{\binom{8}{2}}$	$\frac{\binom{2}{1}\binom{3}{1}}{\binom{8}{2}}$	$\frac{\binom{2}{2}}{\binom{8}{2}}$	$\frac{5}{\binom{8}{2}}$
1	$\frac{\binom{3}{1}\binom{3}{1}}{\binom{8}{2}}$	$\frac{\binom{3}{1}\binom{2}{1}}{\binom{8}{2}}$	0	$\frac{15}{\binom{8}{2}}$
2	$\frac{\binom{3}{2}}{\binom{8}{2}}$	0	0	$\frac{3}{\binom{8}{2}}$
Column Total	$\frac{15}{\binom{8}{2}}$	$\frac{12}{\binom{8}{2}}$	$\frac{1}{\binom{8}{2}}$	1

Marginal Distributions

Def: Let $f(x, y)$ be the joint PMF of X and Y .

The marginal PMF of X is obtained by summing over all values of Y :

$$g(x) = \sum_y f(x, y).$$

The marginal PMF of Y is obtained by summing over all values of X :

$$h(y) = \sum_x f(x, y).$$

Marginal example:

$$P(X = 1) = \sum_y P(X = 1, Y = y).$$

Conditional Distributions (Discrete)

Def: The conditional PMF of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f(x, y)}{g(x)}, \quad g(x) > 0.$$

Similarly, the conditional PMF of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{h(y)}, \quad h(y) > 0.$$

Example: Conditional Probabilities

1.

$$P(Y = 1 | X = 1) = \frac{P(X = 1, Y = 1)}{P(X = 1)} = \frac{3/28}{15/28} = \frac{3}{15}.$$

2.

$$P(X = 0 | Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{9/28}{15/28} = \frac{9}{15}.$$

3.

$$P(Y = 2 | X = 1) = 0.$$

Check:

$$\sum_y P(Y = y | X = 1) = 1.$$

Review: Chapter 3 — Random Variables

1. Random Variable A random variable is a function that maps outcomes of an experiment to real numbers.

- Domain: sample space outcomes
- Range: real numbers
- Can be **discrete** or **continuous**

2. Probability Mass Function (PMF) The PMF of a discrete random variable X is

$$f_X(x) = P(X = x)$$

- Only for discrete random variables
- $f_X(x) \geq 0$
- $\sum_x f_X(x) = 1$

3. Probability Density Function (PDF) For a continuous random variable X , probability is defined by

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$

- Only for continuous random variables
- Area under the curve gives probability
- $P(X = x) = 0$

4. Cumulative Distribution Function (CDF) The CDF is defined as

$$F_X(x) = P(X \leq x)$$

- Discrete: $F(x) = \sum_{t \leq x} f(t)$
- Continuous: $F(x) = \int_{-\infty}^x f(t) dt$
- $0 \leq F(x) \leq 1$, non-decreasing

5. Joint Distribution The joint distribution describes probabilities involving two random variables X and Y .

- Discrete: $f_{X,Y}(x, y) = P(X = x, Y = y)$
- Continuous: joint PDF $f_{X,Y}(x, y)$

6. Marginal Distribution A marginal distribution is obtained by eliminating the other variable.

- $f_X(x) = \sum_y f_{X,Y}(x, y)$ or $f_X(x) = \int f_{X,Y}(x, y) dy$
- $f_Y(y) = \sum_x f_{X,Y}(x, y)$ or $f_Y(y) = \int f_{X,Y}(x, y) dx$

Joint Probability Density Function (Continuous)

Def: A function $f(x, y)$ is a joint probability density function (PDF) of continuous random variables X and Y if:

1. $f(x, y) \geq 0$ for all (x, y)

2.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

3. For any region A in the xy -plane,

$$P((X, Y) \in A) = \iint_A f(x, y) dx dy$$

Geometric interpretation: The joint PDF is a surface above the xy -plane. Probabilities correspond to the **volume under the surface** over a specified region.

Example 1:

$$f(x, y) = \begin{cases} \frac{12}{7}(x^2 + xy), & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) **Verify it is a valid joint PDF:**

$$\int_0^1 \int_0^1 \frac{12}{7}(x^2 + xy) dx dy = 1$$

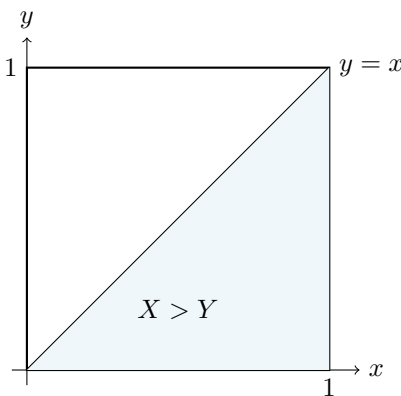
(b) **Find $P(0 < X < 0.2, 0 < Y < 1)$:**

$$P = \int_0^1 \int_0^{0.2} \frac{12}{7}(x^2 + xy) dx dy$$

(c) **Find $P(X > Y)$:**

The region $X > Y$ corresponds to the area below the line $y = x$ in the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$.

$$P(X > Y) = \int_0^1 \int_0^x \frac{12}{7}(x^2 + xy) dy dx = \frac{9}{14}$$



(d) **Find $P(X = Y)$:**

$$P(X = Y) = \int_0^1 \int_y^y \frac{12}{7}(x^2 + xy) dx dy = 0$$

(Probability along a line is zero for continuous random variables.)

Example:

Let the joint PDF be

$$f(x, y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

We want to compute

$$P(X + Y \geq 1).$$

The support of the joint PDF is the region $0 < x < y$, which lies above the line $y = x$ in the first quadrant.

The boundary of the event $X + Y \geq 1$ is the line $x + y = 1$.

Rather than integrating over the unbounded region $X + Y \geq 1$, we compute the complement:

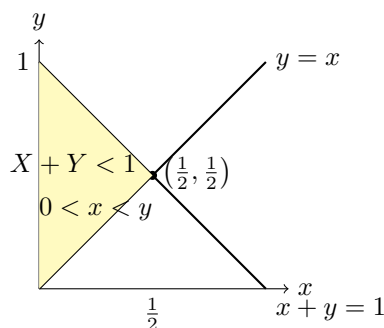
$$P(X + Y \geq 1) = 1 - P(X + Y < 1).$$

The region $X + Y < 1$ that lies within the support is bounded by:

$$0 \leq x \leq \frac{1}{2}, \quad x \leq y \leq 1 - x.$$

Therefore,

$$P(X + Y \geq 1) = 1 - \int_0^{1/2} \int_x^{1-x} e^{-y} dy dx = 2e^{-1/2} - e^{-1}.$$



Marginal and Conditional PDFs

Def: The marginal PDFs of X and Y are defined as

$$g_X(x) = \int_{-\infty}^{\infty} f(x, y) dy, \quad h_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

Example (continued):

$$f(x, y) = \begin{cases} \frac{12}{7}(x^2 + xy), & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Marginal PDF of X :

$$g_X(x) = \int_0^1 \frac{12}{7}(x^2 + xy) dy = \frac{12}{7} \left(x^2 + \frac{x}{2} \right), \quad 0 \leq x \leq 1$$

Marginal PDF of Y :

$$h_Y(y) = \int_0^1 \frac{12}{7}(x^2 + xy) dx = \frac{12}{7} \left(\frac{1}{3} + \frac{y}{2} \right), \quad 0 \leq y \leq 1$$

Conditional PDF of Y given $X = x$:

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{\frac{12}{7}(x^2 + xy)}{\frac{12}{7}\left(x^2 + \frac{x}{2}\right)}, \quad 0 \leq y \leq 1, \quad 0 < x \leq 1$$

Why the bounds are $0 < x \leq 1$ **and not** $0 \leq x \leq 1$:

The marginal PDF

$$g(x) = \frac{12}{7} \left(x^2 + \frac{x}{2} \right)$$

satisfies $g(0) = 0$.

Since the conditional PDF is defined as

$$f_{Y|X}(y|x) = \frac{f(x,y)}{g(x)},$$

it is **undefined at** $x = 0$ due to division by zero.

Therefore, the conditional density is only defined for values of x such that

$$g(x) > 0 \Rightarrow 0 < x \leq 1.$$

Key takeaway: The bounds of a conditional PDF exclude points where the conditioning density is zero.

Statistical Independence

Def: Random variables X and Y (discrete or continuous) are statistically independent if and only if

$$f(x,y) = g(x)h(y) \quad \text{for all } (x,y) \text{ in their range}$$

Consequences:

- $f(x|y) = g(x)$
- $f(y|x) = h(y)$

Recall Example: Pen Refills

Two refills are selected at random and without replacement from a box containing:

- 3 blue refills
- 2 red refills
- 3 green refills

Define the random variables

$$X = \text{number of blue refills selected}, \quad Y = \text{number of red refills selected}.$$

Total outcomes:

$$\binom{8}{2} = 28.$$

Joint PMF Table

For each pair (x, y) ,

$$f(x, y) = \frac{\text{number of favorable outcomes}}{28}.$$

$X \setminus Y$	0	1	2	$g(x) = P(X = x)$
0	$\frac{3}{28}$	$\frac{3}{14}$	$\frac{1}{28}$	$\frac{5}{14}$
1	$\frac{9}{28}$	$\frac{3}{14}$	0	$\frac{15}{28}$
2	$\frac{3}{28}$	0	0	$\frac{3}{28}$
$h(y) = P(Y = y)$	$\frac{15}{28}$	$\frac{3}{7}$	$\frac{1}{28}$	1

Marginals: $g(x) = P(X = x)$ is the marginal PMF of X and is given by the **row totals**. $h(y) = P(Y = y)$ is the marginal PMF of Y and is given by the **column totals**.

$$g(0) = \frac{5}{14}, \quad g(1) = \frac{15}{28}, \quad g(2) = \frac{3}{28}$$

$$h(0) = \frac{15}{28}, \quad h(1) = \frac{3}{7}, \quad h(2) = \frac{1}{28}$$

Independence check: If X and Y were statistically independent, then $f(x, y) = g(x)h(y)$ for all (x, y) .
Check $(x, y) = (0, 1)$:

$$f(0, 1) = \frac{3}{14}, \quad g(0)h(1) = \left(\frac{5}{14}\right)\left(\frac{3}{7}\right) = \frac{15}{98}$$

$$\frac{3}{14} \neq \frac{15}{98} \Rightarrow X \text{ and } Y \text{ are not independent}. \quad \underline{\hspace{1cm}}$$

Example: Independence via Factorization (Discrete Case)

Let X and Y be discrete random variables whose values in the nonnegative integers.

Suppose the joint PMF is

$$f(x, y) = \frac{1}{x!y!} \lambda^x \mu^y e^{-(\lambda+\mu)}, \quad x, y = 0, 1, 2, \dots$$

Factorization:

We can write

$$f(x, y) = \left(\frac{\lambda^x e^{-\lambda}}{x!}\right) \left(\frac{\mu^y e^{-\mu}}{y!}\right) = g(x) h(y)$$

Marginal PMF of X :

$$g(x) = \sum_{y=0}^{\infty} f(x, y) = \sum_{y=0}^{\infty} \frac{1}{x! y!} \lambda^x \mu^y e^{-(\lambda+\mu)}$$

Factor out terms that do not depend on y :

$$g(x) = \frac{1}{x!} \lambda^x e^{-(\lambda+\mu)} \sum_{y=0}^{\infty} \frac{\mu^y}{y!}$$

Using $\sum_{y=0}^{\infty} \frac{\mu^y}{y!} = e^{\mu}$:

$$g(x) = \frac{1}{x!} \lambda^x e^{-(\lambda+\mu)} e^{\mu} = \frac{1}{x!} \lambda^x e^{-\lambda}, \quad x = 0, 1, 2, \dots$$

Marginal PMF of Y :

$$h(y) = \sum_{x=0}^{\infty} f(x, y) = \sum_{x=0}^{\infty} \frac{1}{x! y!} \lambda^x \mu^y e^{-(\lambda+\mu)}$$

Factor out terms that do not depend on x :

$$h(y) = \frac{1}{y!} \mu^y e^{-(\lambda+\mu)} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

Using $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$:

$$h(y) = \frac{1}{y!} \mu^y e^{-(\lambda+\mu)} e^{\lambda} = \frac{1}{y!} \mu^y e^{-\mu}, \quad y = 0, 1, 2, \dots$$

Conclusion:

Since the joint PMF can be written as

$$f(x, y) = g(x) h(y)$$

for all x, y , the random variables X and Y are statistically independent. ✓

Important notes:

- Factorization of the joint PMF is **sufficient** to prove independence.
- The constant terms (such as $e^{-\lambda}$, $e^{-\mu}$) must be included to obtain the **correct marginals**.
- Independence requires that every combination of values with positive marginal probability also has positive joint probability.

This means that if $X = x$ is possible and $Y = y$ is possible, then the pair $(X = x, Y = y)$ must also be possible.

$$\{(x, y) : f(x, y) > 0\} = \{x : g(x) > 0\} \times \{y : h(y) > 0\}.$$

This means that every value of X with positive marginal probability can occur with every value of Y with positive marginal probability.