# MIE286 Lecture Notes

Probability and Statistics

Cheryl Shi

## Contents

# Chapter 1

## Statistics Definitions

**Global definition:** Statistics involves collecting, organizing, summarizing, presenting, and analyzing data, as well as making inferences, conclusions, and decisions based on data.

**Statistical definition:** A <u>statistic</u> is a numerical value calculated from data (e.g. mean, proportion, standard deviation).

Probability (deductive)

| Population | | Sample |

Statistics Inference (inductive)

## Basic Terminology

<u>Individuals</u>: Objects on which data are collected (people, animals, plots of land, etc.).
<u>Variable</u>: Any characteristic of an individual.
<u>Population</u>: The entire group of individuals of interest.
<u>Sample</u>: A subset of individuals taken from the population.
<u>Statistical Inference</u>: Drawing conclusions about a population based on a sample.

# Sampling Methods

<u>Simple Random Sample (SRS)</u>:

- Every possible group of size $n$ has an equal chance of being selected.

- Helps avoid bias in sampling.

- Can be selected using random number tables or software.

<u>Stratified Random Sampling</u>:

- The population is divided into <u>homogeneous groups</u> *(individuals are similar with respect to the variable being studied)* called <u>strata</u>.

- A simple random sample is taken from each <u>stratum</u>. *(one subgroup of the population created)*

- Ensures that important subgroups are neither over nor under represented.

# Data, Variables, and Distributions

## Types of Variables

<u>Categorical Variable</u>: Places individuals into categories (e.g. gender, major). These are qualitative.

Quantitative Variable: Takes numerical values for which arithmetic operations are meaningful.

- Discrete

- Continuous

## Distributions

Distribution: Describes what values a variable takes and how often those values occur. When examining a distribution, look for:

- **Shape**

- **Center**

- **Spread**

- **Outliers**

Outlier: An individual value that falls outside the overall pattern of the data.

## Describing Distributions with Numbers

Central Tendency: Describes where the data cluster or center.

Central Tendency: Describes where the data cluster or center.

- Mean: average value

- Median: middle value

Mean (Arithmetic Mean):

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Median:

$$\tilde{x} = \begin{cases} x_{\left(\frac{n+1}{2}\right)}, & \text{if } n \text{ is odd} \\ \dfrac{x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)}}{2}, & \text{if } n \text{ is even} \end{cases}$$

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**Theorem 1.1**

1. The mean is more sensitive to extreme values than the median.

2. Changing a single data value will always change the mean, but may not change the median.

3. If a distribution is exactly symmetric, the mean and median are equal.

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Trimmed Mean: The mean computed after removing extreme values.

$$\bar{x}_{\text{trim}} = \frac{1}{n - 2k} \sum_{i=k+1}^{n-k} x_{(i)}$$

where $k$ values are removed from both ends of the ordered data. (normally given in question like 10% )

## Measures of Spread

Range: Maximum minus minimum. Very sensitive to extreme values.

Sample Variance: Measures the average squared deviation from the mean.

$$s^2 = \frac{1}{n - 1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

Standard Deviation: The square root of the sample variance.

$$s = \sqrt{s^2}$$

Degrees of Freedom: The number of independent pieces of information available to estimate variability. For sample variance: $df = n - 1$.

# Graphical Representations of Data

Scatter Plot: Used to display the relationship between two quantitative variables $(x, y)$. A scatter plot helps identify trends, patterns, and associations between variables.

Stem-and-Leaf Plot: An intermediate step between raw data and a frequency table. Preserves the original data values while showing the distribution.

| Stem | Leaf |
|---|---|
| 1 | 2 4 7 |
| 2 | 1 3 5 8 |
| 3 | 0 4 6 |

Relative Frequency Table: Shows the proportion of observations in each class.
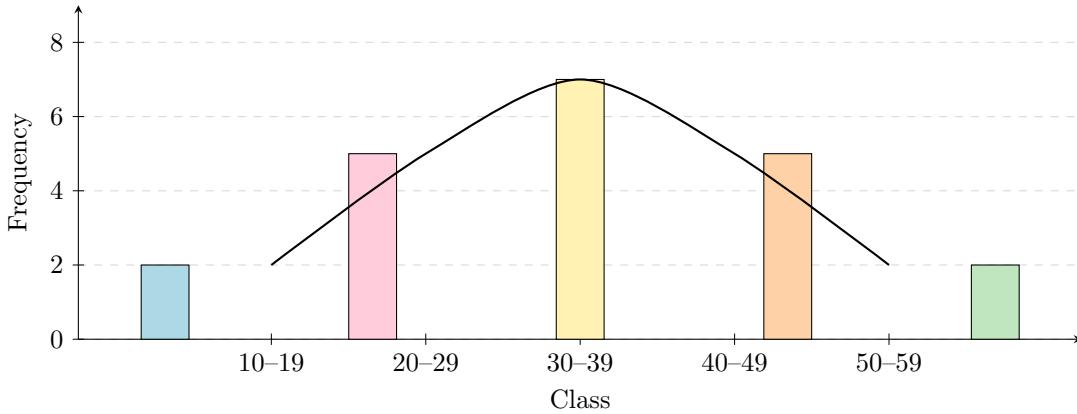
| Class Interval | Class Midpoint | Frequency | Relative Frequency |
|---|---|---|---|
| 10–19 | 14.5 | 3 | 0.30 |
| 20–29 | 24.5 | 4 | 0.40 |
| 30–39 | 34.5 | 3 | 0.30 |

Histogram: A graphical representation of a frequency or relative frequency table using contiguous bars.

When describing the shape of a histogram, we commonly classify it as:

- **Symmetric**

- **Skewed right** (positively skewed)

- **Skewed left** (negatively skewed)

**Symmetric distribution example**



# Chapter 2, Jan 9th

## Experiments, Sample Spaces, and Events

Experiment: A process that generates an outcome.

Sample Space ($S$): The set of all possible outcomes of an experiment.

**Example 1:**

Select 3 items from a production line. Each item can be classified as either defective ($D$) or non-defective ($N$).

$$S = \{DDD, DDN, DND, NDD, DNN, NDN, NND, NNN\}$$

Since each item has 2 possible outcomes,

$$|S| = 2^3 = 8$$

**Example 2:**

$$S = \{(x, y) \mid x^2 + y^2 \leq 4\}$$

Event ($A$): A subset of the sample space $S$.

**Examples of events:**

$$A = \{DDD, DDN, DND, NDD\}$$

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$$B = \{NNN\}$$

$$C = \{(x, y) \mid x^2 + y^2 \le 4\}$$

# Event Operations and Probability Rules

Event Operations:
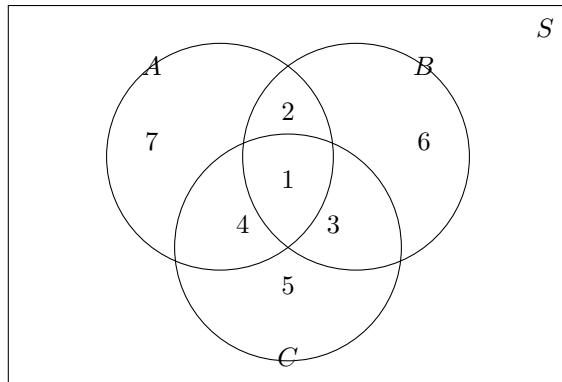
- Complement: $A^c$ (or $A'$)

- Intersection: $A \cap B$

- Union: $A \cup B$

- Null Event: $\varnothing$

If

$$A \cap B = \varnothing,$$

then $A$ and $B$ are mutually exclusive.

**Example (Venn Diagram):**



$$A = \{DDD, DDN, DND, NDD\}, \quad B = \{NNN\}$$

$$A \cup B = \{DDD, DDN, DND, NDD, NNN\}$$

$$A \cap B = \varnothing$$

# Chapter 2: January 12

## Review

1. *Experiment: A process that generates an outcome.*

2. *Sample Space (S): The set of all possible outcomes of an experiment.*

3. *Event Operations:*

   - *Complement: $A'$ ($A^c$)*
   - *Intersection: $A \cap B$*
   - *Union: $A \cup B$*
   - *Null Event: $\varnothing$*

4. *If $A \cap B = \varnothing$, then $A$ and $B$ are called mutually exclusive.*

$$(A \cap B)' = A' \cup B'$$

$$(A \cup B)' = A' \cap B'$$

$$A \cap \varnothing = \varnothing$$

$$A \cup \varnothing = A$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

## Probability

$P(A) = \underline{\text{probability}}$ of event $A$: the proportion of times the event occurs in infinitely many repetitions of the experiment.

> **Theorem 2.1**
>
> $$0 \le P(A) \le 1$$

$$P(A) + P(A') = 1$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+ P(A \cap B \cap C)$$

## Mutually Exclusive Events

<u>Definition</u>: If $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

If

$$A_1 \cup A_2 \cup \cdots \cup A_n = S,$$

then $\{A_1, A_2, \ldots, A_n\}$ is a <u>partition</u> of $S$.
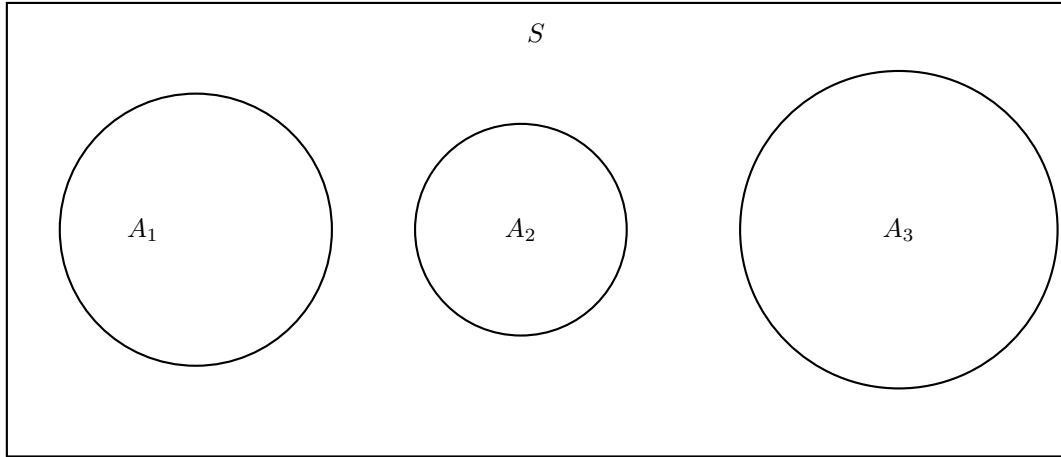


Figure 1: Partition of the sample space $S$ into $A_1, A_2, A_3$

## Example

In a class of 33 students:

- 17 earned an A on the midterm

- 14 earned an A on the final

- 11 earned no A on either exam

Find the probability that a randomly selected student earned A's on <u>both</u> exams.

$$|A \cup B| = 33 - 11 = 22$$

$$|A \cap B| = |A| + |B| - |A \cup B|$$
$$= 17 + 14 - 22 = 9$$

$$|A \setminus B| = |A| - |A \cap B| = 17 - 9 = 8$$

$$|B \setminus A| = |B| - |A \cap B| = 14 - 9 = 5$$

$$P(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{9}{33} \approx 0.273$$
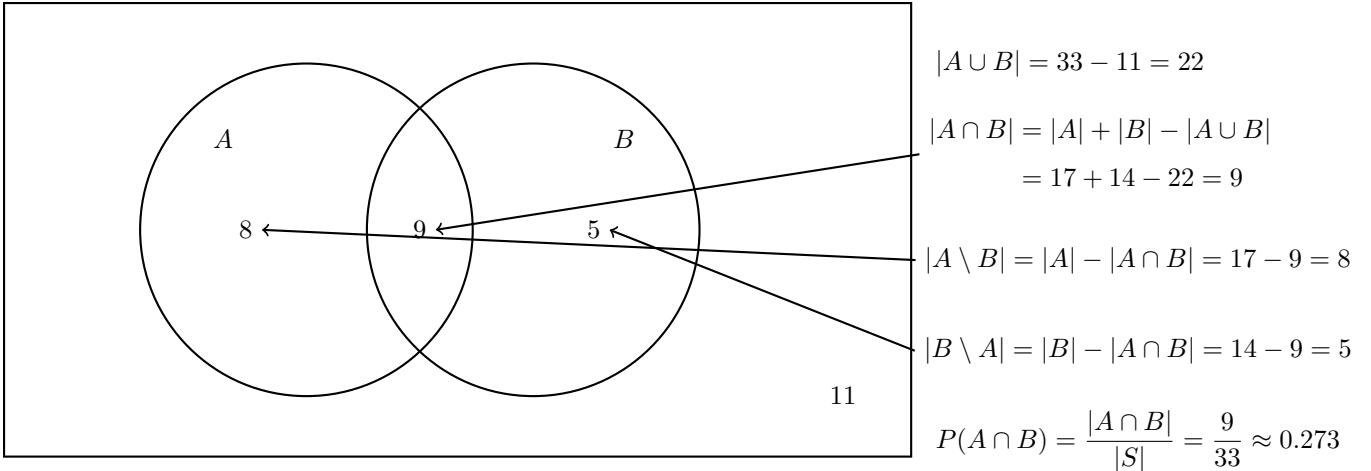
Figure 2: Events $A$: A on midterm, $B$: A on final, with region counts and calculations

# Counting Techniques and Equally Likely Outcomes

> **Theorem 2.2 (Equally Likely Outcomes)**
>
> If the sample space $S$ has a finite number of outcomes and all outcomes are equally likely, then for any event $A$,
> $$P(A) = \frac{|A|}{|S|}.$$
> where
>
> $|A|$ = number of outcomes in event $A$,     $|S|$ = number of outcomes in the sample space.

## Example 1: Poker Hands Basics

A standard deck has:
$$4 \text{ suits} \times 13 \text{ denominations (A,2,3,\ldots,Q,K)} = 52 \text{ cards.}$$

A poker hand consists of 5 cards chosen from 52:

$$|S| = \binom{52}{5} = 2{,}598{,}960.$$

## Combinations Reminder

If there are 3 objects $\{A, B, C\}$ and we choose 2:

$$\binom{3}{2} = \frac{3!}{(3-2)!2!}.$$

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Order does <u>not</u> matter.

## Example 2: Probability of 2 Aces and 1 Jack

A 5-card hand contains:

- exactly 2 aces,

- exactly 1 jack,

- 2 cards that are neither aces nor jacks.

$$P(\text{2 aces and 1 jack}) = \frac{\binom{4}{2}\binom{4}{1}\binom{44}{2}}{\binom{52}{5}}.$$

## Example 3: Probability of a Full House

A full house consists of:

- 3 cards of one denomination

- 2 cards of a different denomination

Number of full house hands:

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}.$$

Thus,

$$P(\text{full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}.$$

## Example 4: Probability of Four of a Kind

A four of a kind consists of:

- 4 cards of the same denomination

- 1 remaining card of a different denomination

Number of such hands:

$$\binom{13}{1}\binom{4}{4}\binom{48}{1}.$$

Thus,

$$P(\text{four of a kind}) = \frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}.$$

## Example 5: Probability of Exactly One Pair

An **excatly** one-pair hand consists of:

- 1 pair

- 3 cards of different denominations, none matching the pair

Number of such hands:

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^{3}.$$

Thus,

$$P(\text{exactly one pair}) = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^{3}}{\binom{52}{5}}.$$

## Note*: Counting Patterns

$$\binom{a}{b}$$

**Meaning:** Choose b different items from $a$ **at once**, order does not matter.

**Key features:**

- No repeats

- Grouped choice

- Used when items must be <u>distinct</u>

$$\binom{a}{1}^{b}$$

**Meaning:** Make $b$ <u>independent choices</u>, each time choosing 1 item from $c$.

**Key features:**

- Repeats allowed

- Choices are independent

- Used when selections do <u>not</u> restrict each other

**<span style="color:red">Rule to Remember:</span>**

$$\text{Different items, no repeats} \ \Rightarrow \ \binom{a}{b}$$

$$\text{Independent choices} \ \Rightarrow \ \binom{c}{1}^{b}$$

## Chapter 2 continue, Jan 14

### Review

1. _Probability_ is the proportion of times the event occurs in infinitely many repetitions of the experiment.

2. $0 \le P(A) \le 1$

3. $P(A) + P(A^c) = 1$

4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

5.
$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+ P(A \cap B \cap C)$$

6. _Permutation_: A permutation counts ordered arrangements.

$$\boxed{{}_nP_r = \frac{n!}{(n-r)!}}$$

### Example 1: Two fair dice

A pair of fair dice are rolled. Find the probability that the second die lands on a smaller value than the first.

The outcomes where the second die is smaller than the first are represented below.

| First Die (Stem) | Second Die (Leaf) |
|:---:|:---|
| 2 | 1 |
| 3 | 1 2 |
| 4 | 1 2 3 |
| 5 | 1 2 3 4 |
| 6 | 1 2 3 4 5 |

There are 15 favorable outcomes and 36 total outcomes.

$$P(\text{second} < \text{first}) = \frac{15}{36} = \frac{5}{12}.$$

## Conditional Probability and Independence

### Conditional Probability

The conditional probability of an event $B$ given that event $A$ has occurred is the probability that $B$ occurs when it is known that $A$ has occurred.

$$\boxed{P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0}$$

## Example 2: Drinking Survey

*A survey records the following data:*

|       | D  | N  | Total |
|-------|----|----|-------|
| M     | 19 | 41 | 60    |
| F     | 12 | 28 | 40    |
| Total | 31 | 69 | 100   |

The symbols used above are defined as follows:

- $M$: male

- $F$: female

- $D$: the individual drinks

- $N$: the individual does not drink

$$P(D|M) = \frac{19}{60} \qquad P(M|D) = \frac{19}{31}$$

## Law of Total Probability

**Theorem 2.3 (Law of Total Probability)**

If $B_1, B_2, \ldots, B_k$ form a partition of the sample space $S$ with $P(B_i) > 0$ for all $i$, then for any event $A$,
$$P(A) = \sum_{i=1}^{k} P(A \mid B_i) \, P(B_i).$$

## Example 3: Monty Hall (3 doors)

| Car location | Monty opens | Probability   | Stay | Switch |
|--------------|-------------|---------------|------|--------|
| Door 1       | Door 2      | $\frac{1}{6}$ | Car  | Goat   |
| Door 1       | Door 3      | $\frac{1}{6}$ | Car  | Goat   |
| Door 2       | Door 3      | $\frac{1}{3}$ | Goat | Car    |
| Door 3       | Door 2      | $\frac{1}{3}$ | Goat | Car    |

Staying wins only when the car is behind Door 1, so

$$P(\text{win by staying}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Switching wins when the car is behind Door 2 or Door 3, so

$$P(\text{win by switching}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

## Example 4: Birthday Problem

Assume the following:

- Leap years are ignored

- All 365 birthdays are equally likely

- Birthdays of different people are independent

**Question:** What is the probability that at least two people share the same birthday in a group of $n$ people?

Rather than computing this directly, we use the complement rule.

$$P(\text{at least one match}) = 1 - P(\text{no match})$$

**Probability of no shared birthdays**

- Person 1 can have any birthday: probability 1

- Person 2 must avoid that birthday: $\frac{364}{365}$

- Person 3 must avoid the first two birthdays: $\frac{363}{365}$

- . . .

- Person $n$ must avoid the previous $n-1$ birthdays: $\frac{365-(n-1)}{365}$

Therefore,

$$P(\text{no match}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-(n-1)}{365}$$

or equivalently,

$$P(\text{no match}) = \prod_{k=0}^{n-1} \frac{365-k}{365}$$

**Final result**

$$P(\text{at least one shared birthday}) = 1 - \prod_{k=0}^{n-1} \frac{365-k}{365}$$

**Important values**

- For $n = 23$:     $P(\text{at least one match}) \approx 0.507$

- For $n = 57$:     $P(\text{at least one match}) \approx 0.99$

# Chapter 2 — Jan 16

## Review: Conditional Probability

*Conditional Probability:*

*The probability of event B given that event A has occurred is*

$$\boxed{P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0}$$

*Read as: the probability of B given A.*

## Independence of Events

Definition (Independence): Events $A$ and $B$ are <u>independent</u> if and only if

$$\boxed{P(B \mid A) = P(B)}$$

Equivalently,

$$P(A \mid B) = P(A)$$

or

$$\boxed{P(A \cap B) = P(A)\,P(B)}$$

## Multiple Independent Events

<u>Definition:</u>

If events $A_1, A_2, \ldots, A_k$ are independent, then

$$\boxed{P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)\,P(A_2) \cdots P(A_k)}$$

## Mutual Independence

<u>Mutual Independence</u> : A collection of events $A_1, A_2, \ldots, A_n$ is <u>mutually independent</u> if and only if for *every* subcollection $\{A_{i_1}, \ldots, A_{i_k}\}$,

$$\boxed{P\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} P(A_{i_j})}$$

Example (Three Events):

Events $A_1, A_2, A_3$ are mutually independent if all of the following hold:

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1)P(A_3)$$
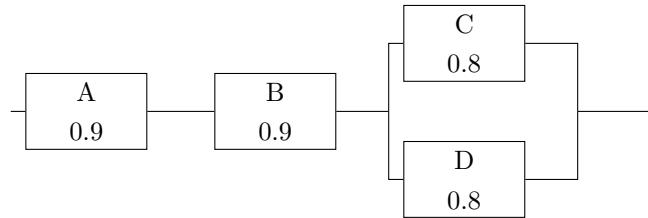
$$P(A_2 \cap A_3) = P(A_2)P(A_3)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

_Note: Mutually exclusive events are dependent. If one event occurs, the other cannot occur._

## Example: Component Reliability

An electrical system has four components $A, B, C, D$. The system works if $A$ and $B$ work and at least one of $C$ or $D$ works. Assume all components are independent.



$$P(A) = 0.9, \quad P(B) = 0.9, \quad P(C) = 0.8, \quad P(D) = 0.8$$

### (a) Probability the entire system works

The system works if $A$ and $B$ work and either $C$ or $D$ works.

$$P(\text{system works}) = P(\text{all work}) + P(A, B, C \text{ work}, D \text{ does not})$$
$$+ P(A, B, D \text{ work}, C \text{ does not})$$

$$= (0.9)(0.9)(0.8)(0.8) + (0.9)(0.9)(0.8)(1 - 0.8) + (0.9)(0.9)(0.8)(1 - 0.8)$$

$$\boxed{P(\text{system works}) = 0.7776}$$

**(b) Conditional probability**

$$P(C^c \mid \text{system works}) = \frac{P(C^c \cap \text{system works})}{P(\text{system works})}$$

$$P(C^c \cap \text{system works}) = (0.9)(0.9)(0.8)(1 - 0.8)$$

$$\boxed{P(C^c \mid \text{system works}) = \frac{(0.9)(0.9)(0.8)(1 - 0.8)}{0.7776} = 0.16}$$

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**Theorem of Total Probability**

Let $B_1, B_2, \ldots, B_k$ be a <u>partition</u> of the sample space $S$ such that $P(B_i) > 0$ for all $i$. Then for any event $A \subseteq S$,

$$\boxed{P(A) = \sum_{i=1}^{k} P(A \mid B_i)\, P(B_i) = \sum_{i=1}^{k} P(A \cap B_i)}$$

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**Theorem: Bayes' Rule (1701–1761)**

Let $B_1, B_2, \ldots, B_k$ be a <u>partition</u> of the sample space $S$ such that $P(B_i) > 0$ for $i = 1, \ldots, k$. For any event $A \subseteq S$ with $P(A) > 0$,

$$\boxed{P(B_r \mid A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)\, P(A \mid B_r)}{\sum_{i=1}^{k} P(B_i)\, P(A \mid B_i)}, \quad r = 1, \ldots, k}$$
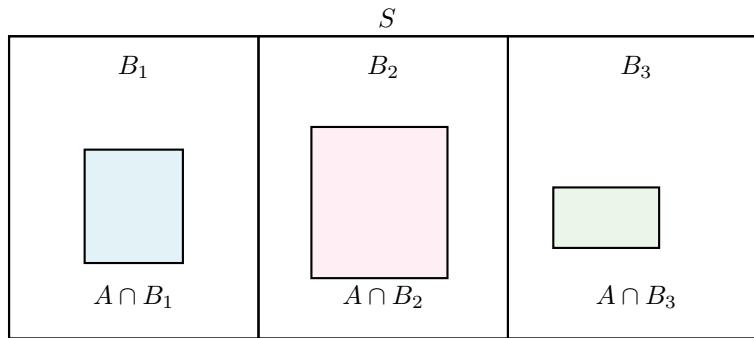


Figure 3: Partition of $S$ into $B_1, B_2, B_3$ with shaded regions $A \cap B_i$

## Example (Medical Test)

The fraction of people in a population who have a certain disease is 0.01.

$$P(D) = 0.01, \quad P(D^c) = 0.99$$

The test characteristics are:
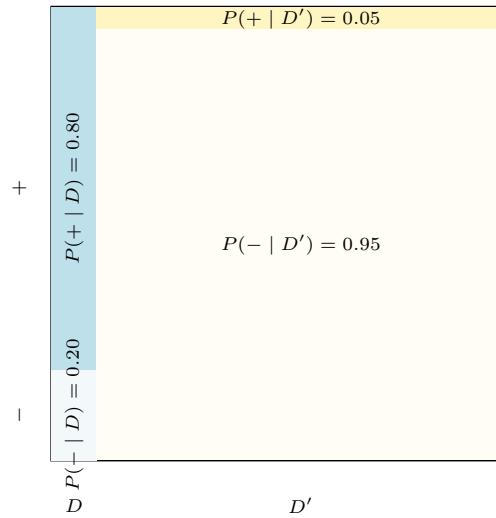
$$P(\text{test says } D \mid D^c) = 0.05 \quad \text{(false positive rate)}$$

$$P(\text{test says } D^c \mid D) = 0.20 \quad \text{(false negative rate)}$$

Thus,

$$P(\text{test says } D \mid D) = 1 - 0.20 = 0.80$$

_Note:_ $1 - P(\text{test says } D^c \mid D)$ _is called the_ _sensitivity_ _of the test, and_ $1 - P(\text{test says } D \mid D^c)$ _is called the_ _specificity_.



**(a) Probability the test says disease**

$$P(\text{test says } D) = P(D \cap \text{test says } D) + P(D^c \cap \text{test says } D)$$

$$= P(\text{test says } D \mid D)P(D) + P(\text{test says } D \mid D^c)P(D^c)$$

$$= (0.80)(0.01) + (0.05)(0.99) = 0.0575$$

**(b) Probability of disease given positive test**

$$P(D \mid \text{test says } D) = \frac{P(D \cap \text{test says } D)}{P(\text{test says } D)}$$

$$= \frac{P(\text{test says } D \mid D)P(D)}{0.0575} = \frac{(0.80)(0.01)}{0.0575}$$

$$\boxed{P(D \mid \text{test says } D) \approx 0.139}$$

**(c) Probability of disease given negative test**

$$P(D \mid \text{test says } D^c) = \frac{P(D \cap \text{test says } D^c)}{P(\text{test says } D^c)}$$

$$= \frac{P(\text{test says } D^c \mid D)P(D)}{1 - P(\text{test says } D)}$$

$$= \frac{(0.20)(0.01)}{1 - 0.0575}$$

$$\boxed{P(D \mid \text{test says } D^c) \approx 0.00212}$$

# Chapter 3 — January 19

## Random Variables and Their Interpretation

**Definition:** A <u>random variable (r.v.)</u> is a rule that assigns a **real number** to each outcome in the sample space.

**Alternative definition:** A <u>random variable</u> is a function that takes the outcome of an experiment and assigns it a number so that probabilities can be calculated.

### Example 1: Three Electronic Components

Each component is classified as either defective (D) or non-defective (N).

$$S = \{NNN, DNN, NDN, NND, DDN, DND, NDD, DDD\}$$

- **Defective (D):** the component does not meet required specifications and fails inspection.

- **Non-defective (N):** the component meets specifications and passes inspection.

Define the random variable

$$X = \text{number of defective components.}$$

Then:
$$X = 0 \quad \text{for } \{NNN\}$$
$$X = 1 \quad \text{for } \{DNN, NDN, NND\}$$
$$X = 2 \quad \text{for } \{DDN, DND, NDD\}$$
$$X = 3 \quad \text{for } \{DDD\}$$

Thus, the possible values of $X$ are:
$$\{0, 1, 2, 3\}.$$

## Example 2: One Component (Dummy Variable)

$$S = \{D, N\}$$

Define the random variable

$$X = \begin{cases} 1, & \text{if the component is defective (D)} \\ 0, & \text{if the component is non-defective (N)} \end{cases}$$

This is called a **dummy variable** because the outcome is categorical, but is encoded numerically.
A dummy variable is a special type of random variable that assigns numerical labels to categorical outcomes, where the numbers have no quantitative meaning beyond identification.

## Discrete Random Variables

**Definition:** A random variable is called discrete if its set of possible values is **countable** (finite or countably infinite).

## Example 3: Sampling Until First Defective

Components are tested one(independently) at a time until the first defective component is observed.

$$S = \{D, \ ND, \ NND, \ NNND, \ldots\}$$

Define
$$X = \text{number of components tested until the first defective.}$$

Then:
$$X = 1 \quad \text{for } \{D\}$$
$$X = 2 \quad \text{for } \{ND\}$$
$$X = 3 \quad \text{for } \{NND\}$$
$$\vdots$$

Hence,

$$X = 1, 2, 3, \ldots$$

Since the possible values can be listed, $X$ is a <u>discrete random variable</u>.

**Non-discrete version of the same experiment:**

Define

$$Y = \text{time (in seconds) until the first defective component is observed.}$$

Since $Y$ can take any real value in $[0, \infty)$, it cannot be listed and is therefore a <u>continuous (non-discrete) random variable</u>.

## Discrete vs. Continuous Random Variables

| Discrete Random Variable | Continuous Random Variable |
|---|---|
| Counts things | Measures things |
| Possible values are countable | Possible values fill an interval |
| $P(X = x)$ can be $> 0$ | $P(X = x) = 0$ for all $x$ |
| Uses a probability mass function (PMF) | Uses a probability density function (PDF) |

## Probability Mass Function (PMF)

**Definition:**

Let $X$ be a discrete random variable. The <u>probability mass function (PMF)</u> of $X$, denoted $f(x)$, is defined by:

$$
\begin{array}{ll}
1) & f(x) \geq 0 \quad \text{for all } x \\
2) & \sum_x f(x) = 1
\end{array}
$$

**Note:**

- Capital $X$: random variable

- Lowercase $x$: a specific value

## Bernoulli and Binomial Random Variables

**I. Bernoulli Random Variable (Single Trial)**

A <u>Bernoulli random variable: X</u> models a single experiment with only two possible outcomes: success or failure.

$$
X = \begin{cases} 1, & \text{success} \\ 0, & \text{failure} \end{cases}
$$

If $p = P(X = 1)$, then the PMF is

| $x$ | 0 | 1 |
|---|---|---|
| $P(X = x)$ | $1 - p$ | $p$ |

Here, $p$ is the probability of success (e.g. observing a defective component).

**Binomial Random Variable (Multiple Bernoulli Trials)**

The binomial random variable extends the Bernoulli case to multiple independent trials.

**Definition:**

A random variable $X$ is called a binomial random variable if it represents the number of successes in $n$ independent Bernoulli trials, each with success probability $p$.

$$X = \text{number of successes in } n \text{ trials}$$

In this case,

$$X \sim \text{Bin}(n, p)$$

and the probability mass function is

$$\boxed{P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, \ldots, n.}$$

**Conditions for a Binomial Model**

A binomial model applies only if:

- each trial has exactly two outcomes (success or failure),

- the probability of success $p$ is the same for every trial,

- the trials are independent,

- the number of trials $n$ is fixed.

**Example: Three Components Tested**

Assume each component is defective with probability

$$p = 0.1, \quad n = 3.$$
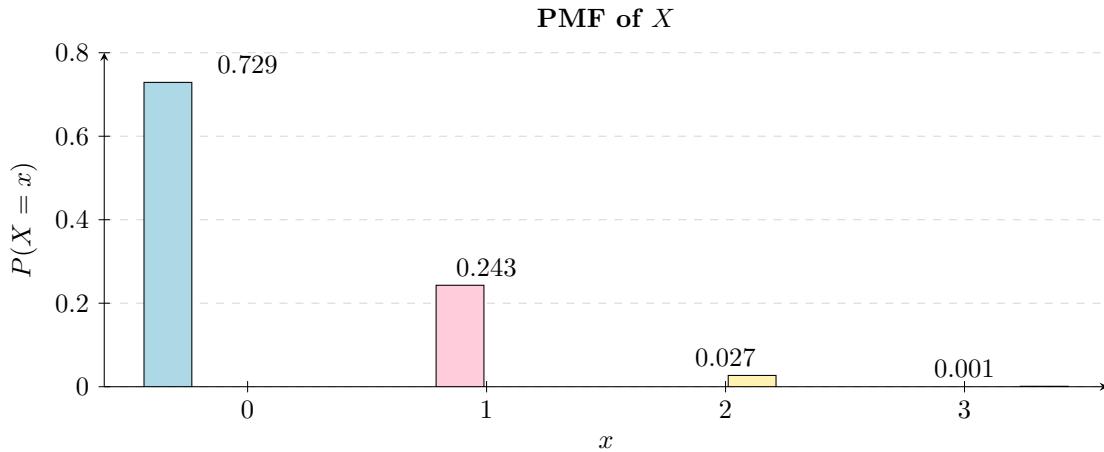
where p = P(a single component is defective), n = number of trials

Let

$$X = \text{number of defective components.}$$

| $x$ | $P(X = x)$ |
|---|---|
| 0 | $\binom{3}{0}(0.9)^3 = 0.729$ |
| 1 | $\binom{3}{1}(0.1)(0.9)^2 = 0.243$ |
| 2 | $\binom{3}{2}(0.1)^2(0.9) = 0.027$ |
| 3 | $\binom{3}{3}(0.1)^3 = 0.001$ |

$$0.729 + 0.243 + 0.027 + 0.001 = 1.$$

**PMF of $X$**



## Geometric Random Variable

### Example: Sampling Until First Defective

Components are sampled one at a time until the first defective component is observed. Assume the probability that a component is defective is

$$p = 0.1.$$

Define the random variable

$$X = \text{number of samples collected until the first defective.}$$

| $x$ | $P(X = x) = f(x)$ |
|---|---|
| 1 | 0.1 |
| 2 | 0.9(0.1) |
| 3 | $0.9^2(0.1)$ |
| ⋮ | ⋮ |

## Geometric Random Variable

**Definition:**

A random variable $X$ is called a <u>geometric random variable</u> if it represents the number of trials needed to obtain the first success in a sequence of independent Bernoulli trials with success probability $p$.

$$\boxed{P(X = x) = (1 - p)^{x-1}p, \quad x = 1, 2, 3, \ldots}$$

In this example,

$$P(X = x) = 0.9^{x-1}(0.1).$$

**Verification That Probabilities Sum to 1**

$$\sum_{x=1}^{\infty} 0.9^{x-1}(0.1) = 0.1 \sum_{x=0}^{\infty} 0.9^x = 0.1 \left( \frac{1}{1 - 0.9} \right) = 1.$$

## Cumulative Distribution Function (CDF)

**Definition:**

The <u>cumulative distribution function (CDF)</u> of a discrete random variable $X$ with PMF $f(x)$ is defined as

$$\boxed{F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad -\infty < x < \infty.}$$

## Example: Binomial Distribution (Three Components)

Let $X$ be the number of defective components when three components are tested, with

$$P(X = 0) = 0.729, \quad P(X = 1) = 0.243, \quad P(X = 2) = 0.027, \quad P(X = 3) = 0.001.$$

$$F(0) = P(X \leq 0) = 0.729$$
$$F(1) = P(X \leq 1) = 0.729 + 0.243 = 0.972$$
$$F(2) = P(X \leq 2) = 0.729 + 0.243 + 0.027 = 0.999$$
$$F(3) = P(X \leq 3) = 0.729 + 0.243 + 0.027 + 0.001 = 1$$

Thus, the CDF can be written as

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.729, & 0 \leq x < 1 \\ 0.972, & 1 \leq x < 2 \\ 0.999, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

## Properties of the CDF

- $F(x)$ is <u>monotone non-decreasing</u>.

- If $x < y$, then $F(x) \le F(y)$.

- $0 \le F(x) \le 1$.

**Note:** A function is **monotone** non-decreasing if its value never decreases as the input increases.

## Using the CDF to Compute Probabilities
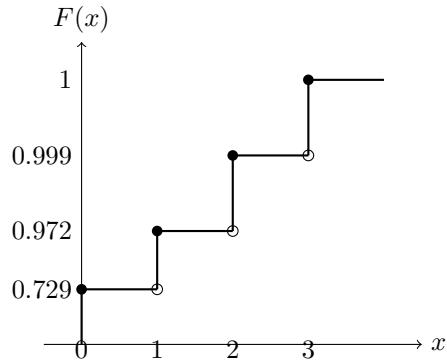
For $a < b$,
$$P(a < X \le b) = F(b) - F(a).$$

Example:
$$P(0 < X \le 2) = F(2) - F(0) = 0.999 - 0.729 = 0.27.$$

## CDF Histogram (Step Function)



**Note:** For a discrete random variable, the PMF is drawn as a bar chart since it shows probabilities at individual points, while the CDF is drawn as a step function since it represents cumulative probability and is monotone non-decreasing.

# Chapter 3 — January 21

## Review

1. *Random Variable (RV), $X$*: A random variable assigns a real number to each outcome.

2. *Discrete Random Variable*: If $X$ is discrete,

   - $P(X = x) = f(x)$
   - $f(x)$ is the *probability mass function (PMF)*
   - $f(x) \ge 0$
   - $\sum_x f(x) = 1$

3. *Cumulative Distribution Function (CDF), $F(x)$*:

- $F(x) = P(X \leq x)$

- If $X$ is discrete:

$$F(x) = \sum_{t \leq x} f(t), \quad -\infty < x < \infty$$

## Continuous Sample Space and Continuous Random Variables

If the sample space contains an infinite number of outcomes equal to the number of points on a line segment, it is called a continuous sample space.

A continuous random variable has

$$\boxed{P(X = x) = 0 \quad \text{for all } x,}$$

so probabilities are computed over intervals instead of single values.

**\*Alternative definition**: A continuous sample space contains infinitely many outcomes, like the points on a line segment. For a continuous random variable, the probability of taking any exact value is zero, i.e. $P(X = x) = 0$ for all $x$. Therefore, probabilities are computed over intervals rather than at single points.

## Probability Density Function (PDF)

A function $f(x)$ is called a probability density function (PDF) of a continuous random variable $X$, defined over $\mathbb{R}$, if:

1. $f(x) \geq 0$ for all $x \in \mathbb{R}$

2. $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$
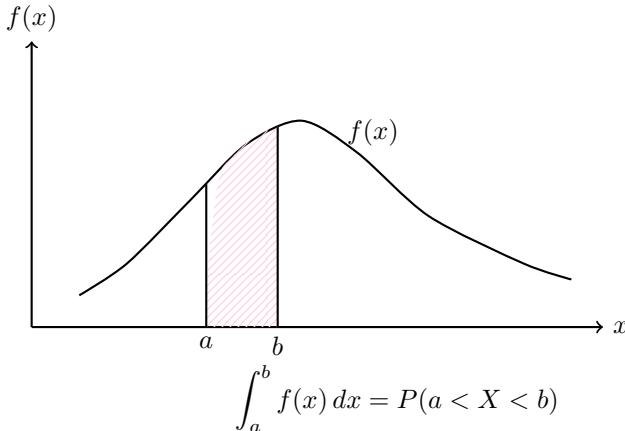
3. For any $a < b$,
$$P(a < X < b) = \int_a^b f(x)\, dx$$

For continuous random variables,

$$P(a < X < b) = P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b).$$

$$\boxed{P(X = a) = 0}$$

because a single point has zero area under the probability density function.

$$\int_a^b f(x)\, dx = P(a < X < b)$$

## Example: Uniform Distribution

Let the probability density function be

$$f(x) = \begin{cases} c, & 5 < x < 10, \\ 0, & \text{otherwise.} \end{cases}$$

**1) Determine the value of $c$**

Since $f(x)$ is a <u>probability density function</u>, the total area under the curve must equal 1:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

Because $f(x) = 0$ outside the interval $(5, 10)$,
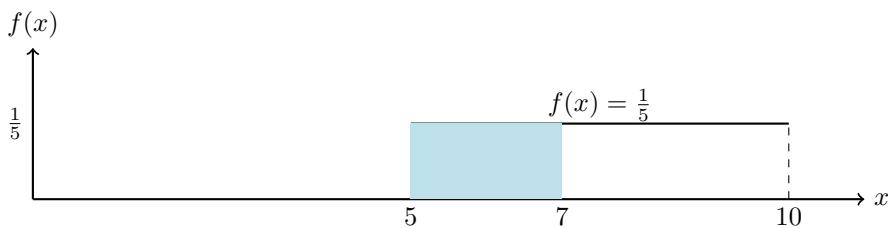
$$\int_5^{10} c\, dx = 1.$$

Evaluating the integral,

$$c(10 - 5) = 1 \quad \Rightarrow \quad c = \frac{1}{5}.$$

Thus,

$$f(x) = \begin{cases} \frac{1}{5}, & 5 < x < 10, \\ 0, & \text{otherwise.} \end{cases}$$



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**2): Compute $P(X < 7)$**

**Formula used:**

$$P(a < X < b) = \int_a^b f(x)\,dx.$$

Applying this formula,

$$P(X < 7) = \int_5^7 \frac{1}{5}\,dx = \frac{1}{5}(7 - 5) = \frac{2}{5}.$$

## CDF for Continuous Random Variables

**Def:** Let $X$ be a <u>continuous random variable</u> with pdf: <u>probability density function</u> $f(x)$. The <u>cumulative distribution function</u> is

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t)\,dt.$$

**Consequently:**

$$f(x) = \frac{d}{dx}F(x), \qquad P(a < X < b) = F(b) - F(a).$$

## Example

Let $X$ be the time until a chemical reaction is complete (in msec). Suppose the CDF is

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-0.01x}, & x \geq 0. \end{cases}$$

**(a) Find the pdf.**

Use $f(x) = \dfrac{d}{dx}F(x)$.

For $x < 0$, $F(x) = 0$, so

$$f(x) = 0.$$

For $x \geq 0$,

$$f(x) = \frac{d}{dx}\left(1 - e^{-0.01x}\right) = 0.01e^{-0.01x}.$$

Therefore,

$$f(x) = \begin{cases} 0, & x < 0, \\ 0.01e^{-0.01x}, & x \geq 0. \end{cases}$$

**(b) Find $P(X < 200)$.**

Use the CDF directly:

$$P(X < 200) = F(200) = 1 - e^{-0.01(200)} = 1 - e^{-2} \approx 0.8647.$$

**(c) Check if this is a valid CDF.**

> **Theorem: Properties of a Cumulative Distribution Function**
>
> A function $F(x)$ is a valid cumulative distribution function (CDF) if and only if:
>
> - $0 \leq F(x) \leq 1$ for all $x$
>
> - $F(x)$ is monotone non-decreasing, i.e.
>
> $$x \leq y \implies F(x) \leq F(y)$$
>
> - $\lim\limits_{x \to -\infty} F(x) = 0$
>
> - $\lim\limits_{x \to \infty} F(x) = 1$

For this $F(x)$:

$$\lim_{x \to -\infty} F(x) = 0, \qquad \lim_{x \to \infty} F(x) = 1,$$

and for $x \geq 0$, $1 - e^{-0.01x}$ increases as $x$ increases, so $F(x)$ is monotone non-decreasing.

Thus, $F(x)$ is a valid CDF.

## Joint Probability Distributions (Discrete)

**Def:** The function $f(x, y)$ is a joint probability mass function (PMF) of discrete random variables $X$ and $Y$ if:

1. $f(x, y) \geq 0$ for all $(x, y)$

2. $\sum\limits_{x} \sum\limits_{y} f(x, y) = 1$

3. $P(X = x, Y = y) = f(x, y)$

That is, $f(x, y)$ gives the probability that the two random variables $X$ and $Y$ take the values $x$ and $y$ *simultaneously*.

For any region $A$ in the $xy$-plane,

$$P\big((X, Y) \in A\big) = \sum_{(x,y) \in A} f(x, y)$$

## Example: Pen Refills

Two refills are selected at random and without replacement from a box containing:

- 3 blue refills

- 2 red refills

- 3 green refills

Define the random variables

$$X = \text{number of blue refills selected}, \qquad Y = \text{number of red refills selected}.$$

The total number of possible selections is

$$\binom{8}{2}.$$

## Joint PMF Table

For each pair $(x, y)$,

$$f(x, y) = \frac{\text{number of favorable outcomes}}{\binom{8}{2}}.$$

| $X \backslash Y$ | 0 | 1 | 2 | Row Total |
|---|---|---|---|---|
| 0 | $\dfrac{\binom{3}{2}}{\binom{8}{2}}$ | $\dfrac{\binom{2}{1}\binom{3}{1}}{\binom{8}{2}}$ | $\dfrac{\binom{2}{2}}{\binom{8}{2}}$ | $\dfrac{\binom{5}{2}}{\binom{8}{2}}$ |
| 1 | $\dfrac{\binom{3}{1}\binom{3}{1}}{\binom{8}{2}}$ | $\dfrac{\binom{3}{1}\binom{2}{1}}{\binom{8}{2}}$ | 0 | $\dfrac{15}{\binom{8}{2}}$ |
| 2 | $\dfrac{\binom{3}{2}}{\binom{8}{2}}$ | 0 | 0 | $\dfrac{3}{\binom{8}{2}}$ |
| Column Total | $\dfrac{15}{\binom{8}{2}}$ | $\dfrac{12}{\binom{8}{2}}$ | $\dfrac{1}{\binom{8}{2}}$ | 1 |

## Marginal Distributions

**Def:** Let $f(x, y)$ be the joint PMF of $X$ and $Y$.

The <u>marginal PMF of $X$</u> is obtained by summing over all values of $Y$:

$$g(x) = \sum_y f(x, y).$$

The <u>marginal PMF of $Y$</u> is obtained by summing over all values of $X$:

$$h(y) = \sum_x f(x, y).$$

**Marginal example:**

$$P(X = 1) = \sum_y P(X = 1, Y = y).$$

## Conditional Distributions (Discrete)

**Def:** The <u>conditional PMF of $Y$ given $X = x$</u> is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{g(x)}, \quad g(x) > 0.$$

Similarly, the <u>conditional PMF of $X$ given $Y = y$</u> is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{h(y)}, \quad h(y) > 0.$$

## Example: Conditional Probabilities

1.
$$P(Y = 1 \mid X = 1) = \frac{P(X = 1, Y = 1)}{P(X = 1)} = \frac{3/28}{15/28} = \frac{3}{15}.$$

2.
$$P(X = 0 \mid Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{9/28}{15/28} = \frac{9}{15}.$$

3.
$$P(Y = 2 \mid X = 1) = 0.$$

Check:
$$\sum_y P(Y = y \mid X = 1) = 1.$$

## Review: Chapter 3 — Random Variables

1. <u>*Random Variable*</u> A random variable is a function that maps outcomes of an experiment to real numbers.

   - Domain: sample space outcomes
   - Range: real numbers
   - Can be **discrete** or **continuous**

2. <u>*Probability Mass Function (PMF)*</u> The PMF of a discrete random variable $X$ is

$$f_X(x) = P(X = x)$$

   - Only for discrete random variables
   - $f_X(x) \geq 0$
   - $\sum_x f_X(x) = 1$

3. <u>*Probability Density Function (PDF)*</u> For a continuous random variable $X$, probability is defined by

$$P(a \leq X \leq b) = \int_a^b f_X(x)\, dx$$

- Only for continuous random variables

- Area under the curve gives probability

- $P(X = x) = 0$

4. *Cumulative Distribution Function (CDF)* The CDF is defined as

$$F_X(x) = P(X \leq x)$$

- Discrete: $F(x) = \sum_{t \leq x} f(t)$
- Continuous: $F(x) = \int_{-\infty}^{x} f(t)\, dt$
- $0 \leq F(x) \leq 1$, non-decreasing

5. *Joint Distribution* The joint distribution describes probabilities involving two random variables $X$ and $Y$.

- Discrete: $f_{X,Y}(x, y) = P(X = x, Y = y)$
- Continuous: joint PDF $f_{X,Y}(x, y)$

6. *Marginal Distribution* A marginal distribution is obtained by eliminating the other variable.

- $f_X(x) = \sum_y f_{X,Y}(x, y)$ or $f_X(x) = \int f_{X,Y}(x, y)\, dy$
- $f_Y(y) = \sum_x f_{X,Y}(x, y)$ or $f_Y(y) = \int f_{X,Y}(x, y)\, dx$

## Joint Probability Density Function (Continuous)

**Def:** A function $f(x, y)$ is a joint probability density function (PDF) of continuous random variables $X$ and $Y$ if:

1. $f(x, y) \geq 0$ for all $(x, y)$

2.
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

3. For any region $A$ in the $xy$-plane,

$$P\big((X, Y) \in A\big) = \iint_A f(x, y)\, dx\, dy$$

**Geometric interpretation:** The joint PDF is a surface above the $xy$-plane. Probabilities correspond to the **volume under the surface** over a specified region.

**Example 1:**

$$f(x, y) = \begin{cases} \dfrac{12}{7}(x^2 + xy), & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

**(a) Verify it is a valid joint PDF:**

$$\int_0^1 \int_0^1 \frac{12}{7}(x^2 + xy) \, dx \, dy = 1$$
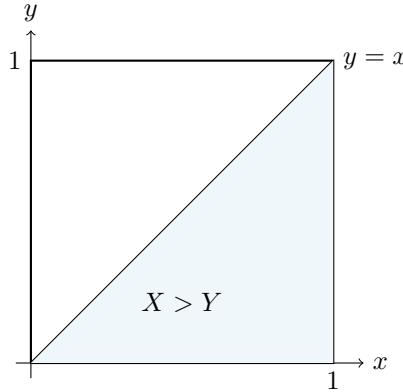
**(b) Find $P(0 < X < 0.2, \ 0 < Y < 1)$:**

$$P = \int_0^1 \int_0^{0.2} \frac{12}{7}(x^2 + xy) \, dx \, dy$$

**(c) Find $P(X > Y)$:**

The region $X > Y$ corresponds to the area below the line $y = x$ in the unit square $0 \le x \le 1, \ 0 \le y \le 1$.

$$P(X > Y) = \int_0^1 \int_0^x \frac{12}{7}(x^2 + xy) \, dy \, dx = \frac{9}{14}$$



**(d) Find $P(X = Y)$:**

$$P(X = Y) = \int_0^1 \int_y^y \frac{12}{7}(x^2 + xy) \, dx \, dy = 0$$

(Probability along a line is zero for continuous random variables.)

**Example:**

Let the joint PDF be

$$f(x, y) = \begin{cases} e^{-y}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

We want to compute

$$P(X + Y \ge 1).$$

The support of the joint PDF is the region $0 < x < y$, which lies above the line $y = x$ in the first quadrant.

The boundary of the event $X + Y \ge 1$ is the line $x + y = 1$.

Rather than integrating over the unbounded region $X + Y \geq 1$, we compute the complement:
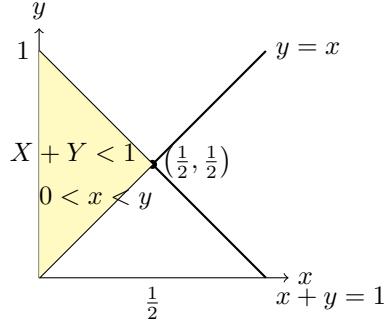
$$P(X + Y \geq 1) = 1 - P(X + Y < 1).$$

The region $X + Y < 1$ that lies within the support is bounded by:

$$0 \leq x \leq \tfrac{1}{2}, \qquad x \leq y \leq 1 - x.$$

Therefore,

$$P(X + Y \geq 1) = 1 - \int_0^{1/2} \int_x^{1-x} e^{-y} \, dy \, dx = 2e^{-1/2} - e^{-1}.$$



## Marginal and Conditional PDFs

**Def:** The marginal PDFs of $X$ and $Y$ are defined as

$$g_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy, \qquad h_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx$$

**Example (continued):**

$$f(x, y) = \begin{cases} \dfrac{12}{7}(x^2 + xy), & 0 \leq x \leq 1, \ 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

**Marginal PDF of $X$:**

$$g_X(x) = \int_0^1 \frac{12}{7}(x^2 + xy) \, dy = \frac{12}{7}\left(x^2 + \frac{x}{2}\right), \qquad 0 \leq x \leq 1$$

**Marginal PDF of $Y$:**

$$h_Y(y) = \int_0^1 \frac{12}{7}(x^2 + xy) \, dx = \frac{12}{7}\left(\frac{1}{3} + \frac{y}{2}\right), \qquad 0 \leq y \leq 1$$

**Conditional PDF of $Y$ given $X = x$:**

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{\frac{12}{7}(x^2 + xy)}{\frac{12}{7}\left(x^2 + \frac{x}{2}\right)}, \qquad 0 \le y \le 1,\ 0 < x \le 1$$

**Why the bounds are $0 < x \le 1$ and not $0 \le x \le 1$:**

The marginal PDF

$$g(x) = \frac{12}{7}\left(x^2 + \frac{x}{2}\right)$$

satisfies $g(0) = 0$.

Since the conditional PDF is defined as

$$f_{Y|X}(y|x) = \frac{f(x,y)}{g(x)},$$

it is **undefined at** $x = 0$ due to division by zero.

Therefore, the conditional density is only defined for values of $x$ such that

$$g(x) > 0 \ \Rightarrow\ 0 < x \le 1.$$

**Key takeaway:** The bounds of a conditional PDF exclude points where the conditioning density is zero.

## Statistical Independence

**Def:** Random variables $X$ and $Y$ (discrete or continuous) are <u>statistically independent</u> if and only if

$$f(x,y) = g(x)\,h(y) \quad \text{for all } (x,y) \text{ in their range}$$

**Consequences:**

- $f(x|y) = g(x)$

- $f(y|x) = h(y)$

## Recall Example: Pen Refills

Two refills are selected at random and without replacement from a box containing:

- 3 blue refills

- 2 red refills

- 3 green refills

Define the random variables

$$X = \text{number of blue refills selected}, \qquad Y = \text{number of red refills selected}.$$

Total outcomes:

$$\binom{8}{2} = 28.$$

## Joint PMF Table

For each pair $(x, y)$,

$$f(x, y) = \frac{\text{number of favorable outcomes}}{28}.$$

| $X \backslash Y$ | 0 | 1 | 2 | $g(x) = P(X = x)$ |
|---|---|---|---|---|
| 0 | $\dfrac{3}{28}$ | $\dfrac{3}{14}$ | $\dfrac{1}{28}$ | $\dfrac{5}{14}$ |
| 1 | $\dfrac{9}{28}$ | $\dfrac{3}{14}$ | 0 | $\dfrac{15}{28}$ |
| 2 | $\dfrac{3}{28}$ | 0 | 0 | $\dfrac{3}{28}$ |
| $h(y) = P(Y = y)$ | $\dfrac{15}{28}$ | $\dfrac{3}{7}$ | $\dfrac{1}{28}$ | 1 |

**Marginals:** $g(x) = P(X = x)$ is the <u>marginal PMF of $X$</u> and is given by the **row totals**. $h(y) = P(Y = y)$ is the <u>marginal PMF of $Y$</u> and is given by the **column totals**.

$$g(0) = \frac{5}{14}, \quad g(1) = \frac{15}{28}, \quad g(2) = \frac{3}{28}$$

$$h(0) = \frac{15}{28}, \quad h(1) = \frac{3}{7}, \quad h(2) = \frac{1}{28}$$

**Independence check:** If $X$ and $Y$ were <u>statistically independent</u>, then $f(x, y) = g(x)h(y)$ for all $(x, y)$. Check $(x, y) = (0, 1)$:

$$f(0, 1) = \frac{3}{14}, \qquad g(0)h(1) = \left(\frac{5}{14}\right)\left(\frac{3}{7}\right) = \frac{15}{98}$$

$$\frac{3}{14} \neq \frac{15}{98} \quad \Rightarrow \quad X \text{ and } Y \text{ are } \underline{\text{not independent}}.$$

## Example: Independence via Factorization (Discrete Case)

Let $X$ and $Y$ be <u>discrete random variables</u> whose values in the nonnegative integers.

Suppose the joint PMF is

$$f(x, y) = \frac{1}{x!\, y!} \lambda^x \mu^y e^{-(\lambda + \mu)}, \qquad x, y = 0, 1, 2, \ldots$$

**Factorization:**

We can write

$$f(x, y) = \left(\frac{\lambda^x e^{-\lambda}}{x!}\right)\left(\frac{\mu^y e^{-\mu}}{y!}\right) = g(x)\, h(y)$$

**Marginal PMF of $X$:**

$$g(x) = \sum_{y=0}^{\infty} f(x, y) = \sum_{y=0}^{\infty} \frac{1}{x!\, y!}\, \lambda^x \mu^y\, e^{-(\lambda+\mu)}$$

Factor out terms that do not depend on $y$:

$$g(x) = \frac{1}{x!}\, \lambda^x e^{-(\lambda+\mu)} \sum_{y=0}^{\infty} \frac{\mu^y}{y!}$$

Using $\displaystyle\sum_{y=0}^{\infty} \frac{\mu^y}{y!} = e^{\mu}$:

$$g(x) = \frac{1}{x!}\, \lambda^x e^{-(\lambda+\mu)} e^{\mu} = \frac{1}{x!}\, \lambda^x e^{-\lambda}, \qquad x = 0, 1, 2, \ldots$$

**Marginal PMF of $Y$:**

$$h(y) = \sum_{x=0}^{\infty} f(x, y) = \sum_{x=0}^{\infty} \frac{1}{x!\, y!}\, \lambda^x \mu^y\, e^{-(\lambda+\mu)}$$

Factor out terms that do not depend on $x$:

$$h(y) = \frac{1}{y!}\, \mu^y e^{-(\lambda+\mu)} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

Using $\displaystyle\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}$:

$$h(y) = \frac{1}{y!}\, \mu^y e^{-(\lambda+\mu)} e^{\lambda} = \frac{1}{y!}\, \mu^y e^{-\mu}, \qquad y = 0, 1, 2, \ldots$$

**Conclusion:**

Since the joint PMF can be written as

$$f(x, y) = g(x)\, h(y)$$

for all $x, y$, the random variables $X$ and $Y$ are <u>statistically independent</u>. ✓

**Important notes:**

- Factorization of the joint PMF is **sufficient** to prove independence.

- The constant terms (such as $e^{-\lambda}$, $e^{-\mu}$) must be included to obtain the **correct marginals**.

- Independence requires that every combination of values with positive marginal probability also has positive joint probability.
  This means that if $X = x$ is possible and $Y = y$ is possible, then the pair $(X = x, Y = y)$ must also be possible.

$$\{(x, y) : f(x, y) > 0\} = \{x : g(x) > 0\} \times \{y : h(y) > 0\}.$$

This means that every value of $X$ with positive marginal probability can occur with every value of $Y$ with positive marginal probability.

# Jan 30

## Joint distribution

Describes probabilities involving two random variables $X$ and $Y$.

- Discrete:
$$f(X, Y) = P(X = x, Y = y)$$

- Continuous: joint PDF $f(X, Y)$

## Marginal distribution

Obtained by eliminating the other variable.

- 
$$f(X) = \sum_y f(X, Y), \qquad f(Y) = \sum_x f(X, Y)$$

- 
$$f(X) = \int_{-\infty}^{\infty} f(X, Y)\, dy, \qquad f(Y) = \int_{-\infty}^{\infty} f(X, Y)\, dx$$

## Joint PDF validity

A function $f(X, Y)$ is a valid joint PDF if:

- $f(X, Y) \geq 0$

- 
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(X, Y)\, dx\, dy = 1$$

**Example (valid joint PDF):**

$$f(X, Y) = \begin{cases} \dfrac{12}{7}(x^2 + xy), & 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

## Geometric interpretation

Probabilities correspond to the **volume under the surface** $f(X, Y)$ over a region.

## Conditional PDF

Distribution of one variable given the other.

- 
$$f(Y \mid X) = \frac{f(X,Y)}{f(X)}, \quad f(X) > 0$$

## Statistical Independence

**Definition:** Random variables $X$ and $Y$ are <u>statistically independent</u> if

$$f(X,Y) = f(X)f(Y)$$

This means knowing the value of one variable gives no information about the other.

## Support and Independence

**Support:** The support of a joint distribution is the set

$$\{(x,y) : f(X,Y) > 0\}$$

**Key idea:** If $X$ and $Y$ are independent, the support must factor as

$$\{x : f(X) > 0\} \times \{y : f(Y) > 0\}$$

That is, every allowed value of $X$ can occur with every allowed value of $Y$.

## Example: Non-Independent Random Variables

$$f(X,Y) = \begin{cases} 4(x + y^2), & xy > 0, \ x + y \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

The condition $x + y \leq 1$ links $x$ and $y$, so the support does not factor.

$$\Rightarrow X \text{ and } Y \text{ are not independent}$$

**Why this implies dependence (key intuition):**

Pick a value of $X$ that is allowed:
$$x = 0.8 \quad (\text{positive and } < 1)$$

Pick a value of $Y$ that is allowed:
$$y = 0.8 \quad (\text{positive and } < 1)$$

Individually, both values are valid. But together:

$$x + y = 0.8 + 0.8 = 1.6 > 1$$

This violates the condition $x + y \leq 1$, so the pair $(0.8, 0.8)$ is impossible.

**Conclusion:** Knowing the value of $X$ restricts which values $Y$ can take. Therefore, $X$ and $Y$ are not independent.

## Independent Random Variables

If random variables are independent, joint probabilities factor.

**Example:** Let $X_1, X_2, X_3$ be independent with

$$f(X) = e^{-x}, \quad x > 0$$

Then

$$f(X_1, X_2, X_3) = e^{-x_1} e^{-x_2} e^{-x_3}$$

$$P(X_1 < 2, \ 1 < X_2 < 3, \ X_3 > 2) = (1 - e^{-2})(e^{-1} - e^{-3})e^{-2}$$

**Explanation (why this works):**

Since $X_1, X_2, X_3$ are independent, the joint probability factors:

$$P(X_1 < 2, \ 1 < X_2 < 3, \ X_3 > 2) = P(X_1 < 2)\,P(1 < X_2 < 3)\,P(X_3 > 2)$$

For an exponential random variable with

$$f(X) = e^{-x}, \quad x > 0,$$

we have:

$$P(X < a) = 1 - e^{-a}, \qquad P(X > a) = e^{-a}$$

Thus,

$$P(X_1 < 2) = 1 - e^{-2}$$

$$P(1 < X_2 < 3) = P(X_2 < 3) - P(X_2 < 1) = e^{-1} - e^{-3}$$

$$P(X_3 > 2) = e^{-2}$$

Multiplying gives:

$$(1 - e^{-2})(e^{-1} - e^{-3})e^{-2}$$

## Mutual Independence and Modeling Assumptions

**Mutual independence:** Random variables $X_1, \ldots, X_n$ are mutually independent if

$$f(X_1, \ldots, X_n) = \prod_{i=1}^{n} f(X_i)$$

Pairwise independence does **not** imply mutual independence.

**Independent selection:** Selections are independent if:

- each selection is random,

- distributions are identical,

- outcomes do not affect future selections.

Sampling without replacement generally produces dependent variables.

# Chapter 4

## Motivation

**Example:** If you roll a fair die repeatedly, what *average value* do you expect in the long run?

This motivates the idea of expectation: a theoretical long-run average.

# Expected Value

## Expected Value of a Random Variable

**Def:** For a random variable $X$, the expectation (or expected value or mean) is the long-run average value of $X$.

**Discrete random variable:**

$$\boxed{\mu = E(X) = \sum_x x\, f(X)}$$

**Continuous random variable:**

$$\boxed{\mu = E(X) = \int_{-\infty}^{\infty} x\, f(X)\, dx}$$

**Interpretation:** Expectation is a weighted average, where values of $X$ are weighted by how likely they are.

## Example: Fair Die

Let $X$ be the outcome when a fair die is rolled.

$$E(X) = \sum_{x=1}^{6} x \cdot \frac{1}{6}$$

$$E(X) = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = 3.5$$

Even though 3.5 is not a possible outcome, it represents the long-run average.

## Example: Number of Messages per Hour

Let $X$ be the number of messages sent per hour, with PMF:

| $x$ | 10 | 11 | 12 | 13 | 14 | 15 |
|------|------|------|------|------|------|------|
| $f(X)$ | 0.08 | 0.15 | 0.30 | 0.20 | 0.20 | 0.07 |
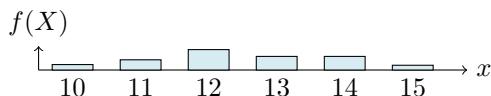
**Check:**
$$\sum f(X) = 1$$

**Expected value:**

$$E(X) = 10(0.08) + 11(0.15) + 12(0.30) + 13(0.20) + 14(0.20) + 15(0.07)$$
$$= 12.5$$

This means that over many hours, the average number of messages per hour is about 12.5.

## PMF Visualization



## Example: Deal or No Deal

Consider a game with two possible outcomes:

- $1 with probability 1/2

- $10,000 with probability 1/2

**Expected value:**
$$E(X) = \frac{1}{2}(1) + \frac{1}{2}(10{,}000) = 5000.5$$

**Key idea:** The expected value is the average payout in the long run, not the most likely outcome.

## Example: Continuous RV (Device Lifetime)

Let $X$ be a random variable that denotes the lifetime (in hours) of a certain device, with PDF

$$f(X) = \begin{cases} \dfrac{20000}{x^3}, & x > 100, \\ 0, & \text{otherwise.} \end{cases}$$

**Check (valid PDF):**

$$\int_{-\infty}^{\infty} f(X)\,dx = \int_{100}^{\infty} \frac{20000}{x^3}\,dx = 20000 \left[\frac{-1}{2x^2}\right]_{100}^{\infty} = 1$$

**Expected lifetime:**

$$E(X) = \int_{100}^{\infty} x\frac{20000}{x^3}\,dx = \int_{100}^{\infty} \frac{20000}{x^2}\,dx = 20000 \left[\frac{-1}{x}\right]_{100}^{\infty} = 200$$

So we expect this type of device to last on average $\boxed{200}$ hours.

## Example: Discrete RV and Transformation

Let $X$ be a discrete random variable with PMF:

| $x$ | $-1$ | $0$ | $1$ | $2$ |
|------|------|-----|-----|-----|
| $f(X)$ | 0.3 | 0.2 | 0.3 | 0.2 |

Define a new random variable as a <u>transformation</u> of $X$:

$$g(X) = X^2$$

**Possible values of $g(X)$:**

$$g(-1) = 1, \quad g(0) = 0, \quad g(1) = 1, \quad g(2) = 4$$

So $g(X)$ can take values $\{0, 1, 4\}$.

**Distribution of $g(X)$:**

$$P(g(X) = 0) = P(X = 0) = 0.2$$

$$P(g(X) = 1) = P(X = -1) + P(X = 1) = 0.3 + 0.3 = 0.6$$

$$P(g(X) = 4) = P(X = 2) = 0.2$$

| $g(X)$ | 0 | 1 | 4 |
|--------|-----|-----|-----|
| $P(g(X))$ | 0.2 | 0.6 | 0.2 |

This is called a <u>transformation</u> of a random variable.

**Expected value of the transformed RV:**

$$E(g(X)) = E(X^2) = \sum_x x^2 f(X) = \sum_x g(x)\, f(X)$$

Numerically:

$$E(X^2) = 0^2(0.2) + (-1)^2(0.3) + (1)^2(0.3) + (2)^2(0.2) = 0 + 0.3 + 0.3 + 0.8 = 1.4$$

> **Expected Value of a Function of a RV**
>
> Let $X$ be a random variable with distribution $f(X)$. The expected value of the random variable $g(X)$ is
>
> $$E(g(X)) = \begin{cases} \sum_x g(x)\, f(X), & \text{if } X \text{ is discrete} \\ \\ \int_{-\infty}^{\infty} g(x)\, f(X)\, dx, & \text{if } X \text{ is continuous} \end{cases}$$

## Example: Chip Game (Expected Winnings)

A bowl contains 5 chips:

- 3 chips are worth \$1 each

- 2 chips are worth \$4 each

A player draws 2 chips at random (without replacement) and is paid the sum.

Let $X$ be the number of \$1 chips drawn. Then

$$X \in \{0, 1, 2\}.$$

**PMF of $X$:** (hypergeometric)

$$f(X) = \begin{cases} \dfrac{\binom{3}{x}\binom{2}{2-x}}{\binom{5}{2}}, & x = 0, 1, 2, \\ \\ 0, & \text{otherwise.} \end{cases}$$

**Define payout as a function of $X$:** If you draw $x$ one-dollar chips, then you draw $2 - x$ four-dollar chips, so the payout is

$$g(x) = 1(x) + 4(2 - x) = 8 - 3x.$$

So the payout random variable is $g(X) = 8 - 3X$.

**Expected payout:**

$$E(g(X)) = \sum_{x=0}^{2} g(x)\, f(X)$$

Compute $f(X)$ values:

$$f(0) = \frac{\binom{3}{0}\binom{2}{2}}{\binom{5}{2}} = \frac{1}{10}, \quad f(1) = \frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}} = \frac{6}{10}, \quad f(2) = \frac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}} = \frac{3}{10}.$$

Then

$$E(g(X)) = \sum_{x=0}^{2}(8-3x)\, f(X) = (8)\left(\frac{1}{10}\right) + (5)\left(\frac{6}{10}\right) + (2)\left(\frac{3}{10}\right) = 4.4$$

**Decision:** If it costs \$4.75 to play, your expected profit is

$$E(\text{profit}) = E(g(X)) - 4.75 = 4.4 - 4.75 = -0.35$$

So in the long run, you lose about $\boxed{\$0.35}$ per game on average, so you should <u>not</u> play.

## Notation: $g(X)$ vs. $g(x)$

- $X$ is a <u>random variable</u>; $x$ is a specific value it can take.

- $g(X)$ is a <u>random variable</u>.

- $g(x)$ is a <u>number</u>.

**Key rule:**

$$\boxed{E(g(X)) = \sum_{x} g(x)\, f(X)}$$

Expectation averages the numerical values $g(x)$, weighted by their probabilities.

**Exam rule:**

$$\boxed{g(X) \text{ is random variable, } g(x) \text{ is a number.}}$$

## February 2

### Chapter 4 Review: Expected Value of a Function of a Random Variable

1. *Expected value of a function: expected average value in the long wrong, $E[g(X)]$*

   - $g(X)$ is a function of $X$

   - **Discrete:**
     $$E[g(X)] = \sum_{x} g(x)\, f(X)$$

   - **Continuous:**
     $$E[g(X)] = \int_{-\infty}^{\infty} g(X)\, f(X)\, dx$$

2. **Example (continuous RV):**

$$f(X) = 4x^3, \quad 0 < x < 1$$

- 

$$E(X) = \int_0^1 x \cdot 4x^3 \, dx$$

- 

$$E(X^2) = \int_0^1 x^2 \cdot 4x^3 \, dx$$

## Expected Value from a Joint Distribution

**General formula**

$$\boxed{E[g(X,Y)] = \int\!\!\int g(x,y)\, f(X,Y)\, dx\, dy}$$

**Computing $E(X)$ from a joint PDF**

There are two equivalent methods to compute $E(X)$.

**Method 1: Using the marginal distribution**

First find the marginal of $X$:

$$f(X) = \int_0^1 x(1 + 3y^2)\, dy$$

$$= x \int_0^1 (1 + 3y^2)\, dy$$

$$= x \left[ y + y^3 \right]_0^1 = x(1 + 1) = 2x$$

Now compute the expected value:

$$E(X) = \int_0^2 x f(X)\, dx = \int_0^2 x(2x)\, dx$$

$$= 2 \int_0^2 x^2 \, dx = 2 \left[ \frac{x^3}{3} \right]_0^2 = \frac{16}{3}$$

**Method 2: Direct double integral**

Apply the general formula with $g(X,Y) = X$:

$$E(X) = \int_0^1 \int_0^2 x\, f(X,Y)\, dx\, dy$$

$$= \int_0^1 \int_0^2 x \cdot x(1 + 3y^2) \, dx \, dy$$

$$= \int_0^1 (1 + 3y^2) \left( \int_0^2 x^2 \, dx \right) dy$$

$$= \int_0^1 (1 + 3y^2) \left[ \frac{x^3}{3} \right]_0^2 dy = \int_0^1 (1 + 3y^2) \frac{8}{3} \, dy$$

$$= \frac{8}{3} \int_0^1 (1 + 3y^2) \, dy = \frac{8}{3} \cdot 2 = \frac{16}{3}$$

Both methods give the same result:

$$\boxed{E(X) = \frac{16}{3}}$$

## Properties of Expected Value

### Properties of Expectation

Let $X, Y$ be random variables and let $a, b \in \mathbb{R}$ be constants.

1. **Linearity (scaling and shifting):**

$$\boxed{E(aX + b) = aE(X) + b}$$

   **Continuous case:**

$$E(aX + b) = \int_{-\infty}^{\infty} (ax + b) \, f(X) \, dx$$

$$= a \int_{-\infty}^{\infty} x f(X) \, dx + b \int_{-\infty}^{\infty} f(X) \, dx$$

   Note that

$$\int_{-\infty}^{\infty} f(X) \, dx = 1$$

   because $f(X)$ is a probability density function and the total probability over its entire support must equal 1.

   Therefore,

$$E(aX + b) = aE(X) + b$$

2. **Expectation of a constant:**

$$\boxed{E(a) = a}$$

3. **Additivity:**

$$\boxed{E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]}$$

4. **Non-negativity:**
$$\boxed{X \geq 0 \implies E(X) \geq 0}$$

5. **Zero expectation:**
$$\boxed{X \geq 0 \text{ and } E(X) = 0 \iff P(X = 0) = 1}$$

# Variance

$E(X)$ is a measure of the **center** of a distribution.

The **variance**, denoted $\sigma^2 = \text{Var}(X)$, measures how closely the distribution is concentrated around the mean $\mu$.

Definition:
$$\boxed{\text{Var}(X) = E[(X - \mu)^2], \quad \mu = E(X)}$$

$\sigma = \sqrt{\sigma^2}$ is called the **standard deviation**.

---

**Theorem (Variance Formula)**

$$\text{Var}(X) = \sigma^2 = E(X^2) - \mu^2$$

---

**Proof:**
$$E[(X - \mu)^2] = E[X^2 - 2\mu X + \mu^2]$$
$$= E(X^2) - 2\mu E(X) + \mu^2$$
$$= E(X^2) - \mu^2$$

**Example**

Weekly demand for a drink (in thousand liters) is a continuous random variable $X$ with PDF

$$f(X) = 2(x - 1), \quad 1 < x < 2$$

Mean:
$$\mu = E(X) = \int_1^2 x \cdot 2(x - 1)\, dx = \frac{5}{3}$$

Second moment:
$$E(X^2) = \int_1^2 x^2 \cdot 2(x - 1)\, dx = \frac{17}{6}$$

Variance:
$$\boxed{\sigma^2 = \text{Var}(X) = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{18}}$$

**Variance of a function of a random variable**

Let $X$ be a random variable and let $g(X)$ be a function of $X$.

Mean of $g(X)$:

$$\mu_{g(X)} = E[g(X)]$$

Variance of $g(X)$:

$$\boxed{\text{Var}(g(X)) = E[(g(X) - \mu_{g(X)})^2] = E[g(X)^2] - (E[g(X)])^2}$$

## Properties of Variance

Let $X, Y$ be random variables and let $a, b \in \mathbb{R}$.

1.
$$\boxed{\text{Var}(aX + b) = a^2 \text{Var}(X)}$$

   **Proof:**
   $$\text{Var}(aX + b) = E[(aX + b)^2] - (E[aX + b])^2$$
   $$= E[a^2 X^2 + 2abX + b^2] - (aE[X] + b)^2$$
   $$= a^2 E[X^2] + 2abE[X] + b^2 - a^2 E[X]^2 - 2abE[X] - b^2$$
   $$= a^2 (E[X^2] - E[X]^2) = a^2 \text{Var}(X)$$

2.
$$\boxed{\text{Var}(X) \geq 0}$$

3.
$$\boxed{\text{Var}(a) = 0} \quad \text{for any constant } a$$

4.
$$\boxed{\text{Var}(X) = 0 \iff X \text{ is a constant}}$$

---

**Theorem (Variance of a Linear Combination)**

Let $X, Y$ be random variables with joint distribution $f(X, Y)$. Then

$$\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \, \text{Cov}(X, Y)$$

---

where the **covariance** is defined as

$$\boxed{\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]}$$

**Interpretation of covariance**

- $\mathrm{Cov}(X, Y) > 0$: $X$ and $Y$ tend to increase together

- $\mathrm{Cov}(X, Y) < 0$: one increases while the other decreases

- $\mathrm{Cov}(X, Y) = 0$: no linear relationship

**Example (population intuition):**

If $X$ is height and $Y$ is weight in a population,

$$(X - E[X])(Y - E[Y]) > 0$$

for tall–heavy and short–light individuals, so $\mathrm{Cov}(X, Y) > 0$.

# Feb 4

## Review

1. *Expected Value*

    - **Discrete random variable:**
    $$E(X) = \sum_x x f(X)$$

    - **Continuous random variable:**
    $$E(X) = \int_{-\infty}^{\infty} x f(X) \, dx$$

2. *Variance*

    - Definition:
    $$\mathrm{Var}(X) = \sigma^2 = E\big[(X - \mu)^2\big]$$

    - Simplified form:
    $$\mathrm{Var}(X) = E(X^2) - \mu^2$$

    where $\mu = E(X)$.

3. *Variance of a Linear Combination*

    - For constants $a, b$:
    $$\mathrm{Var}(aX + bY) = a^2 \mathrm{Var}(X) + b^2 \mathrm{Var}(Y) + 2ab\,\sigma_{XY}$$

    - Covariance:
    $$\sigma_{XY} = \mathrm{Cov}(X, Y) = E\big[(X - E(X))(Y - E(Y))\big]$$

    - Equivalent form:
    $$\sigma_{XY} = E(XY) - E(X)E(Y)$$

**Theorem: Independence and Expected Value**

Let $X$ and $Y$ be independent random variables. Then

$$E(XY) = E(X)\,E(Y).$$

As a consequence,

$$\text{Cov}(X, Y) = 0.$$

**Continuous case:**

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(X, Y)\, dx\, dy$$

If $X$ and $Y$ are independent,

$$f(X, Y) = g(X)h(Y)$$

so

$$E(XY) = \left( \int_{-\infty}^{\infty} xg(X)\, dx \right) \left( \int_{-\infty}^{\infty} yh(Y)\, dy \right)$$

$$= E(X)\,E(Y)$$

## Example

**Given:** The joint PMF table below.

| $f(X,Y)$ | $X = -1$ | $X = 0$ | $X = 1$ | $h(Y)$ |
|----------|----------|---------|---------|--------|
| $Y = -1$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{3}{8}$ |
| $Y = 0$ | $\frac{1}{8}$ | $0$ | $\frac{1}{8}$ | $\frac{2}{8}$ |
| $Y = 1$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{3}{8}$ |
| $g(X)$ | $\frac{3}{8}$ | $\frac{2}{8}$ | $\frac{3}{8}$ | $1$ |

1. *Compute $E(X)$*

   Using the marginal PMF $g(X)$:

   $$E(X) = \sum_x x\, g(X)$$

   $$E(X) = (-1) \cdot \tfrac{3}{8} + 0 \cdot \tfrac{2}{8} + 1 \cdot \tfrac{3}{8} = -\tfrac{3}{8} + 0 + \tfrac{3}{8} = 0$$

2. *Compute $E(Y)$*

   Using the marginal PMF $h(Y)$:

   $$E(Y) = \sum_y y\, h(Y)$$

   $$E(Y) = (-1) \cdot \tfrac{3}{8} + 0 \cdot \tfrac{2}{8} + 1 \cdot \tfrac{3}{8} = -\tfrac{3}{8} + 0 + \tfrac{3}{8} = 0$$

3. *Compute $E(XY)$*

Using the joint PMF:
$$E(XY) = \sum_x \sum_y xy\, f(X,Y)$$

Compute by rows:

**Row $y = -1$:**
$$(-1)(-1)\tfrac{1}{8} + (0)(-1)\tfrac{1}{8} + (1)(-1)\tfrac{1}{8} = \tfrac{1}{8} + 0 - \tfrac{1}{8} = 0$$

**Row $y = 0$:** all terms are 0 because $y = 0$.

**Row $y = 1$:**
$$(-1)(1)\tfrac{1}{8} + (0)(1)\tfrac{1}{8} + (1)(1)\tfrac{1}{8} = -\tfrac{1}{8} + 0 + \tfrac{1}{8} = 0$$

Therefore,
$$E(XY) = 0$$

**Final:**
$$E(X) = 0, \qquad E(Y) = 0, \qquad E(XY) = 0$$

**Note:** Even if $E(XY) = E(X)E(Y)$, this <u>does not automatically mean</u> $X$ and $Y$ are independent. Independence requires:
$$f(X,Y) = g(X)\, h(Y).$$

## Covariance and Independence

**Recall:**
$$\mathrm{Cov}(X,Y) = 0 \quad \text{does NOT necessarily imply independence.}$$

**Check independence:**
$$f(X,Y) \overset{?}{=} g(X)h(Y)$$

In this example,
$$f(0,0) = 0$$

but
$$g(0)h(0) = \tfrac{2}{8} \cdot \tfrac{2}{8} \neq 0$$

Therefore,
$$f(0,0) \neq g(0)h(0)$$

$$\Rightarrow \quad X \text{ and } Y \text{ are } \underline{\text{not independent}}.$$

**Properties of Covariance**

1.
$$\mathrm{Cov}(X, X) = \mathrm{Var}(X) = E(X^2) - E(X)^2$$

2.
$$\mathrm{Cov}(aX + b,\ cY + d) = ac\,\mathrm{Cov}(X, Y)$$

3.
$$\mathrm{Cov}(X + Y,\ Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$$

4.
$$\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$$

**Important note:**

The covariance $\sigma_{XY}$ is <u>not scale-free</u>. Its magnitude does <u>not</u> directly indicate the strength of the linear relationship between $X$ and $Y$.

**Definition: Correlation Coefficient**

The <u>correlation coefficient</u> of $X$ and $Y$ is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\,\sigma_Y}$$

where

$$\sigma_X = \sqrt{\mathrm{Var}(X)}, \qquad \sigma_Y = \sqrt{\mathrm{Var}(Y)}.$$

> **Theorem: Correlation Is Scale-Free**
>
> For constants $a, c \neq 0$ and any constants $b, d$,
>
> $$\rho(aX + b,\ cY + d) = \rho(X, Y)$$
>
> Therefore, the correlation coefficient is <u>scale-free</u>.

**Proof.**

By definition of correlation,

$$\rho(aX + b,\ cY + d) = \frac{\mathrm{Cov}(aX + b,\ cY + d)}{\sqrt{\mathrm{Var}(aX + b)\,\mathrm{Var}(cY + d)}}.$$

Using properties of covariance,

$$\mathrm{Cov}(aX + b,\ cY + d) = ac\,\mathrm{Cov}(X, Y)$$

Using properties of variance,

$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X), \qquad \mathrm{Var}(cY + d) = c^2\mathrm{Var}(Y)$$

Substituting,

$$\rho(aX + b, \ cY + d) = \frac{ac\,\mathrm{Cov}(X,Y)}{\sqrt{a^2\mathrm{Var}(X)\,c^2\mathrm{Var}(Y)}}.$$

The constants $a$ and $c$ appear in both the numerator and denominator and therefore cancel:

$$= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \rho(X,Y)$$

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**Theorem: Bounds on Correlation**

$$-1 \le \rho(X,Y) \le 1$$

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## Interpretation of the Correlation Coefficient
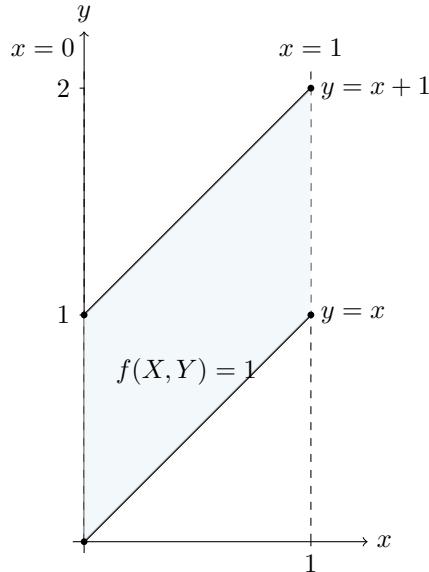
1. $\rho$ is a measure of the strength and direction of the linear relationship between $X$ and $Y$.

2. When there is an exact linear relationship,

$$\boxed{Y = aX + b} \quad \Rightarrow \quad \boxed{\rho = 1 \text{ or } \rho = -1}$$

## Example

**Given:**

$$f(X,Y) = \begin{cases} 1, & 0 < x < 1, \ x < y < x+1, \\ 0, & \text{otherwise.} \end{cases}$$



**Check total probability:**

$$\int_0^1 \int_x^{x+1} 1 \, dy \, dx = \int_0^1 (x + 1 - x) \, dx = \int_0^1 1 \, dx = 1$$

**Expected Values**

$$E(X) = \int_0^1 \int_x^{x+1} x\, f(X,Y)\, dy\, dx = \int_0^1 \int_x^{x+1} x\, dy\, dx$$

$$= \int_0^1 x(1)\, dx = \int_0^1 x\, dx = \frac{1}{2}$$

$$\boxed{E(X) = \frac{1}{2}}$$

$$E(Y) = \int_0^1 \int_x^{x+1} y\, f(X,Y)\, dy\, dx = \int_0^1 \int_x^{x+1} y\, dy\, dx$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_x^{x+1} dx = \int_0^1 \frac{(x+1)^2 - x^2}{2}\, dx$$

$$= \int_0^1 \frac{2x+1}{2}\, dx = \left[\frac{x^2}{2} + \frac{x}{2}\right]_0^1 = 1$$

$$\boxed{E(Y) = 1}$$

**Compute $E(XY)$**

$$E(XY) = \int_0^1 \int_x^{x+1} xy\, f(X,Y)\, dy\, dx = \int_0^1 \int_x^{x+1} xy\, dy\, dx$$

$$= \int_0^1 x \left[\frac{y^2}{2}\right]_x^{x+1} dx = \int_0^1 x \cdot \frac{(x+1)^2 - x^2}{2}\, dx$$

$$= \int_0^1 x \cdot \frac{2x+1}{2}\, dx = \int_0^1 \left(x^2 + \frac{x}{2}\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1 = \frac{7}{12}$$

$$\boxed{E(XY) = \frac{7}{12}}$$

**Variances**

$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_0^1 \int_x^{x+1} x^2\, dy\, dx = \int_0^1 x^2\, dx = \frac{1}{3}$$

$$\text{Var}(X) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

$$\boxed{\text{Var}(X) = \frac{1}{12}}$$

$$\text{Var}(Y) = E(Y^2) - E(Y)^2$$

$$E(Y^2) = \int_0^1 \int_x^{x+1} y^2 \, dy \, dx = \int_0^1 \left[\frac{y^3}{3}\right]_x^{x+1} dx$$

$$= \int_0^1 \frac{(x+1)^3 - x^3}{3} dx = \frac{7}{6}$$

$$\text{Var}(Y) = \frac{7}{6} - 1 = \frac{1}{6}$$

$$\boxed{\text{Var}(Y) = \frac{1}{6}}$$

**Variances (again but using $E[(X - \mu)^2]$**

**Variance of $X$:**

Since $E(X) = \dfrac{1}{2}$,

$$\text{Var}(X) = E\left[(X - \tfrac{1}{2})^2\right] = \int_0^1 \int_x^{x+1} \left(x - \tfrac{1}{2}\right)^2 f(X,Y) \, dy \, dx$$

Because $f(X, Y) = 1$,

$$\text{Var}(X) = \int_0^1 \int_x^{x+1} \left(x - \tfrac{1}{2}\right)^2 dy \, dx$$

$$= \int_0^1 \left(x - \tfrac{1}{2}\right)^2 (1) \, dx = \int_0^1 \left(x - \tfrac{1}{2}\right)^2 dx$$

$$= \int_0^1 \left(x^2 - x + \tfrac{1}{4}\right) dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right]_0^1 = \frac{1}{12}$$

$$\boxed{\text{Var}(X) = \frac{1}{12}}$$

**Variance of $Y$:**

Since $E(Y) = 1$,

$$\text{Var}(Y) = E\left[(Y - 1)^2\right] = \int_0^1 \int_x^{x+1} (y-1)^2 f(X,Y) \, dy \, dx$$

$$= \int_0^1 \int_x^{x+1} (y-1)^2 \, dy \, dx$$

$$= \int_0^1 \left[ \frac{(y-1)^3}{3} \right]_{y=x}^{y=x+1} dx = \int_0^1 \frac{(x)^3 - (x-1)^3}{3} dx$$

$$= \int_0^1 \left( x^2 - x + \tfrac{1}{3} \right) dx = \left[ \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{3} \right]_0^1 = \frac{1}{6}$$

$$\boxed{\text{Var}(Y) = \frac{1}{6}}$$

**Correlation**

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\frac{7}{12} - \frac{1}{2}}{\sqrt{\frac{1}{12} \cdot \frac{1}{6}}} = \frac{1}{\sqrt{2}}$$
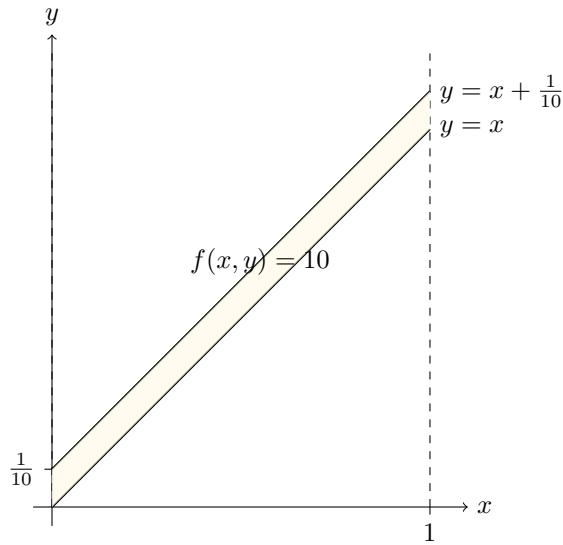
$$\boxed{\rho_{XY} = \frac{1}{\sqrt{2}}}$$

# Example

**Question:**

Let the joint PDF be

$$f(X,Y) = \begin{cases} 10, & 0 < x < 1, \ x < y < x + \dfrac{1}{10}, \\ 0, & \text{otherwise.} \end{cases}$$



**Idea:**

From the support $x < y < x + \frac{1}{10}$, subtract $x$:

$$0 < y - x < \tfrac{1}{10}.$$

Define

$$U = Y - X.$$

Then the support becomes

$$0 < X < 1, \qquad 0 < U < \tfrac{1}{10}.$$

So $X$ and $U$ are independent

**Calculations:**
**Variance of a Uniform random variable:**

$$\boxed{\operatorname{Var}(Z) = \frac{(b-a)^2}{12} \quad \text{for } Z \sim \operatorname{Uniform}(a, b)}$$

This follows from

$$E(Z) = \frac{a+b}{2}, \qquad E(Z^2) = \frac{a^2 + ab + b^2}{3},$$

and

$$\operatorname{Var}(Z) = E(Z^2) - [E(Z)]^2.$$

$$\operatorname{Var}(X) = \frac{(1-0)^2}{12} = \frac{1}{12}.$$

$$\operatorname{Var}(U) = \frac{(1/10 - 0)^2}{12} = \frac{1}{1200}.$$

Because $Y = X + U$ and $X$ and $U$ are independent,

$$\operatorname{Var}(Y) = \operatorname{Var}(X) + \operatorname{Var}(U) = \frac{1}{12} + \frac{1}{1200} = \frac{101}{1200}.$$

**Covariance:**

$$\operatorname{Cov}(X, Y) = \operatorname{Cov}(X, X + U) = \operatorname{Cov}(X, X) + \operatorname{Cov}(X, U).$$

Since $\operatorname{Cov}(X, U) = 0$,

$$\operatorname{Cov}(X, Y) = \operatorname{Var}(X) = \frac{1}{12}.$$

**Correlation coefficient:**

$$\rho_{X,Y} = \frac{\frac{1}{12}}{\sqrt{\left(\frac{1}{12}\right)\left(\frac{101}{1200}\right)}} = \sqrt{\frac{100}{101}}.$$

**Comparison and interpretation:**

In both examples, the support is a diagonal band of the form

$$x < y < x + w,$$

where $w$ is the band width.

- A **smaller band width** means $Y$ stays closer to the line $y = x$.

- This implies less variation in $Y - X$.

- Hence, $Y$ is more tightly determined by $X$.

In the second example,
$$w = \frac{1}{10},$$
so $Y = X + U$ with $U \sim \text{Unif}(0, \frac{1}{10})$ very small.

As a result,
$$\rho_{X,Y} = \sqrt{\frac{100}{101}} \approx 1,$$
indicating a very strong positive correlation.

**Key idea:** A thinner diagonal support band implies stronger linear dependence and higher correlation between $X$ and $Y$.