Chapter 5
Transport of Energy by Radiation and Conduction

The energy the star radiates away so profusely from its surface is generally replenished from reservoirs situated in the very hot central region. This requires an effective transfer of energy through the stellar material, which is possible owing to the existence of a non-vanishing temperature gradient in the star. Depending on the local physical situation, the transfer can occur mainly via radiation, conduction, and convection. In any case, certain "particles" (photons, atoms, electrons, "blobs" of matter) are exchanged between hotter and cooler parts, and their mean free path together with the temperature gradient of the surroundings will play a decisive role. The equation for the energy transport, written as a condition for the temperature gradient necessary for the required energy flow, will supply our next basic equation for the stellar structure.

5.1 Radiative Transport of Energy

5.1.1 Basic Estimates

Rough estimates show important features of the radiative transfer in stellar interiors and justify an enormous simplification of the formalism.

Let us first estimate the mean free path ℓ_{ph} of a photon at an "average" point inside a star like the Sun:

$$\ell_{\mathrm{ph}} = \frac{1}{\kappa \varrho} \; , \tag{5.1}$$

where κ is a mean absorption coefficient, i.e. a radiative cross section per unit mass averaged over frequency. Typical values for stellar material are of order $\kappa \approx 1\,\mathrm{cm}^2\,\mathrm{g}^{-1}$; for the ionized hydrogen in stellar interiors, a lower limit is certainly the value for electron scattering, $\kappa \approx 0.4\,\mathrm{cm}^2\,\mathrm{g}^{-1}$ (see Chap. 17). Using this and the

R. Kippenhahn et al., *Stellar Structure and Evolution*, Astronomy and Astrophysics Library, DOI 10.1007/978-3-642-30304-3_5, © Springer-Verlag Berlin Heidelberg 2012

mean density of matter in the Sun, $\bar{\varrho}_\odot = 3M_\odot/4\pi R_\odot^3 = 1.4\,\mathrm{g\,cm^{-3}}$, we obtain a mean free path of only

$$\ell_{\mathrm{ph}} \approx 2\,\mathrm{cm}\ , \tag{5.2}$$

i.e. stellar matter is very opaque.

The typical temperature gradient in the star can be roughly estimated by averaging between centre ($T_\mathrm{c} \approx 10^7\,\mathrm{K}$) and surface ($T_0 \approx 10^4\,\mathrm{K}$):

$$\frac{\Delta T}{\Delta r} \approx \frac{T_\mathrm{c} - T_0}{R_\odot} \approx 1.4 \times 10^{-4}\,\mathrm{K\,cm^{-1}}\ . \tag{5.3}$$

The radiation field at a given point is emitted from a small, nearly isothermal surrounding, the differences of temperature being only of order $\Delta T = \ell_{\mathrm{ph}}(dT/dr) \approx 3\times10^{-4}\,\mathrm{K}$. Since the energy density of radiation is $u \sim T^4$, the relative anisotropy of the radiation at a point with $T = 10^7\,\mathrm{K}$ is $4\Delta T/T \sim 10^{-10}$. The situation in stellar interiors must obviously be very close to thermal equilibrium, and the radiation very close to that of a black body. Nevertheless, the small remaining anisotropy can easily be the carrier of the stars' huge luminosity: this fraction of 10^{-10} of the flux emitted from $1\,\mathrm{cm}^2$ of a black body of $T = 10^7\,\mathrm{K}$ is still 10^3 times larger than the flux at the solar surface ($6 \times 10^{10}\,\mathrm{erg\,cm^{-2}\,s^{-1}}$). Radiative transport of energy occurs via the non-vanishing net flux, i.e. via the surplus of the outwards-going radiation (emitted from somewhat hotter material below) over the inwards-going radiation (emitted from less-hot material above).

5.1.2 Diffusion of Radiative Energy

The above estimates have shown that for radiative transport in stars, the mean free path ℓ_{ph} of the "transporting particles" (photons) is very small compared to the characteristic length R (stellar radius) over which the transport extends: $\ell_{\mathrm{ph}}/R_\odot \approx 3 \times 10^{-11}$. In this case, the transport can be treated as a diffusion process, which yields an enormous simplification of the formalism. We derive the corresponding equation by analogy to those for particle diffusion. A more rigorous derivation can be found in any textbook about radiation transport, for instance, in Chaps. 2 and 8 of Weiss et al. (2004).

The diffusive flux j of particles (per unit area and time) between places of different particle density n is given by

$$\boldsymbol{j} = -D\ \nabla n\ , \tag{5.4}$$

where D is the coefficient of diffusion,

$$D = \frac{1}{3} v\,\ell_\mathrm{p}\ , \tag{5.5}$$

determined by the average values of mean velocity v and mean free path ℓ_p of the particles.

In order to obtain the corresponding diffusive flux of radiative energy $\boldsymbol{F}$, we replace n by the energy density of radiation U,

$$U = aT^4 , \tag{5.6}$$

v by the velocity of light c, and ℓ_p by ℓ_{ph} according to (5.1).

In (5.6), $a = 7.57 \times 10^{-15}\,\mathrm{erg\,cm^{-3}\,K^{-4}}$ is the *radiation density constant.* Owing to the spherical symmetry of the problem, $\boldsymbol{F}$ has only a radial component $F_r = |\boldsymbol{F}| = F$ and ∇U reduces to the derivative in the radial direction

$$\frac{\partial U}{\partial r} = 4\,a\,T^3 \frac{\partial T}{\partial r} . \tag{5.7}$$

Then (5.4) and (5.5) give immediately that

$$F = -\frac{4ac}{3}\frac{T^3}{\kappa\varrho}\frac{\partial T}{\partial r} . \tag{5.8}$$

Note that this can be interpreted formally as an equation for heat conduction by writing

$$\boldsymbol{F} = -k_{rad}\nabla T , \tag{5.9}$$

where

$$k_{rad} = \frac{4ac}{3}\frac{T^3}{\kappa\varrho} \tag{5.10}$$

represents the coefficient of conduction for this radiative transport.

We solve (5.8) for the gradient of the temperature and replace F by the usual local luminosity $l = 4\pi r^2 F$; then

$$\frac{\partial T}{\partial r} = -\frac{3}{16\pi ac}\frac{\kappa\varrho l}{r^2 T^3} . \tag{5.11}$$

After transformation to the independent variable m (as in Sect. 2.1), the basic equation for radiative transport of energy is obtained in the form

$$\frac{\partial T}{\partial m} = -\frac{3}{64\pi^2 ac}\frac{\kappa l}{r^4 T^3} . \tag{5.12}$$

Of course, this neat and simple equation becomes invalid when one approaches the surface of the star. Because of the decreasing density, the mean free path of the photons will there become comparable with (and finally larger than) the remaining distance to the surface; hence the whole diffusion approximation breaks down, and one has to solve the far more complicated full set of transport equations for radiation in the stellar atmosphere (These equations indeed yield our simple diffusion approximation as the proper limiting case for large optical depths.). Fortunately, however, we have then left the stellar-interior regime with which this book deals, and we happily leave the complicated remainder to those of our colleagues who feel the call to treat the problem of stellar atmospheres.

5.1.3 The Rosseland Mean for κ_ν

The above equations are independent of the frequency ν; F and l are quantities integrated over all frequencies, so that the quantity κ must represent a "proper mean" over ν. We shall now prescribe a method for this averaging.

In general the absorption coefficient depends on the frequency ν. Let us denote this by adding a subscript ν to all quantities that thus become frequency dependent: $\kappa_\nu, \ell_\nu, D_\nu, U_\nu$, etc.

For the diffusive energy flux $\boldsymbol{F}_\nu$ of radiation in the interval $[\nu, \nu + d\nu]$, we write now, as in Sect. 5.1.2,

$$\boldsymbol{F}_\nu = -D_\nu \nabla U_\nu, \qquad \text{with} \tag{5.13}$$

$$D_\nu = \frac{1}{3} c\, \ell_\nu = \frac{c}{3\kappa_\nu \varrho}\ , \tag{5.14}$$

while the energy density in the same interval is given by

$$U_\nu = \frac{4\pi}{c} B(\nu, T) = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1}\ . \tag{5.15}$$

$B(\nu, T)$ denotes here the Planck function for the *intensity* of black-body radiation (differing from the usual formula for the energy density simply by the factor $4\pi/c$). For simplicity, we will not always write the arguments ν and T explicitly in the following formulae. From (5.15) we have

$$\nabla U_\nu = \frac{4\pi}{c} \frac{\partial B}{\partial T} \nabla T\ , \tag{5.16}$$

which together with (5.14) is inserted into (5.13), the latter then being integrated over all frequencies to obtain the total flux $\boldsymbol{F}$:

$$\boldsymbol{F} = -\left[\frac{4\pi}{3\varrho} \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B}{\partial T} d\nu \right] \nabla T\ . \tag{5.17}$$

We have thus regained (5.9), but with

$$k_{\text{rad}} = \frac{4\pi}{3\varrho} \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B}{\partial T} d\nu\ . \tag{5.18}$$

Equating this expression for k_{rad} with that in the averaged form of (5.10), we have immediately the proper formula for averaging the absorption coefficient:

$$\frac{1}{\kappa} = \frac{\pi}{acT^3} \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B}{\partial T} d\nu \ . \tag{5.19}$$

This is the so-called *Rosseland mean* (after Sven Rosseland).

Since

$$\int_0^\infty \frac{\partial B}{\partial T} d\nu = \frac{acT^3}{\pi} \ , \tag{5.20}$$

the Rosseland mean is formally the harmonic mean of κ_ν with the weighting function $\partial B/\partial T$, and it can simply be calculated, once the function κ_ν is known from atomic physics.

In order to see the physical interpretation of the Rosseland mean, we rewrite (5.13) with the help of (5.14)–(5.16):

$$\boldsymbol{F}_\nu = -\left(\frac{1}{\kappa_\nu} \frac{\partial B(\nu, T)}{\partial T} \right) \frac{4\pi}{3\varrho} \nabla T \ . \tag{5.21}$$

This shows that, for a given point in the star (ϱ and ∇T given), the integrand in (5.19) is at all frequencies proportional to the net flux $\boldsymbol{F}_\nu$ of energy. The Rosseland mean therefore favours the frequency ranges of maximum energy flux. One could say that an average *transparency* is evaluated rather than an *opacity*–which is plausible, since it is to be used in an equation describing the transfer of energy rather than its blocking.

One can also easily evaluate the frequency where the weighting function $\partial B/\partial T$ has its maximum. From (5.15) one finds that, for given a temperature, $\partial B/\partial T \sim x^4 \mathrm{e}^x (\mathrm{e}^x - 1)^{-2}$ with the usual definition $x = h\nu/kT$. Differentiation with respect to x shows that the maximum of $\partial B/\partial T$ is close to $x = 4$.

The way we have defined the Rosseland mean κ, which is a kind of weighted harmonic mean value, has the uncomfortable consequence that the opacity κ of a mixture of two gases having the opacities κ_1, κ_2 is not the sum of the opacities: $\kappa \neq \kappa_1 + \kappa_2$.

Therefore, in order to find κ for a mixture containing the weight fractions X of hydrogen and Y of helium, the mean opacities of the two single gases are of no use. Rather one has to add the frequency-dependent opacities $\kappa_\nu = X\kappa_{\nu\mathrm{H}} + Y\kappa_{\nu\mathrm{He}}$ before calculating the Rosseland mean. For any new abundance ratio X/Y the averaging over the frequency has to be carried out separately.

In the above we have characterized the energy flux due to the diffusion of photons by $\boldsymbol{F}$. Since in the following we shall encounter other mechanisms for energy transport, from now, on we shall specify this radiative flux by the vector $\boldsymbol{F}_{\mathrm{rad}}$. Correspondingly we shall use κ_{rad} instead of κ, etc.

5.2 Conductive Transport of Energy

In heat conduction, energy transfer occurs via collisions during the random thermal motion of the particles (electrons and nuclei in completely ionized matter, otherwise atoms or molecules). A basic estimate similar to that in Sect. 5.1.1 shows that in "ordinary" stellar matter (i.e. in a non-degenerate gas), conduction has no chance of taking over an appreciable part of the total energy transport. Although the collisional cross sections of these charged particles are rather small at the high temperatures in stellar interiors ($10^{-18} \cdots 10^{-20}\,\mathrm{cm}^2$ per particle), the large density ($\bar{\varrho} = 1.4\,\mathrm{g\,cm}^{-3}$ in the Sun) results in mean free paths several orders of magnitude less than those for photons; and the velocity of the particles is only a few per cent of c. Therefore the coefficient of diffusion (5.5) is much smaller than that for photons.

The situation becomes quite different, however, for the cores of evolved stars (see Chap. 33), where the electron gas is highly degenerate. The density can be as large as $10^6\,\mathrm{g\,cm}^{-3}$. But degeneracy makes the electrons much faster, since they are pushed up close to the Fermi energy; and degeneracy increases the mean free path considerably, since the quantum cells of phase space are filled up such that collisions in which the momentum is changed become rather improbable. Then the coefficient of diffusion (which is proportional to the product of mean free path and particle velocity) is large, and heat conduction can become so efficient that it short-circuits the radiative transfer (see Sect. 17.6).

The energy flux $\boldsymbol{F}_{\mathrm{cd}}$ due to heat conduction may be written as

$$\boldsymbol{F}_{\mathrm{cd}} = -k_{\mathrm{cd}} \nabla T \; . \tag{5.22}$$

The sum of the conductive flux $\boldsymbol{F}_{\mathrm{cd}}$ and the radiative flux $\boldsymbol{F}_{\mathrm{rad}}$ as defined in (5.9) is

$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{rad}} + \boldsymbol{F}_{\mathrm{cd}} = -(k_{\mathrm{rad}} + k_{\mathrm{cd}}) \nabla T \; , \tag{5.23}$$

which shows immediately the benefit of writing the radiative flux in (5.9) formally as an equation of heat conduction. On the other hand, we can just as well write the conductive coefficient k_{cd} formally in analogy to (5.10) as

$$k_{\mathrm{cd}} = \frac{4ac}{3} \frac{T^3}{\kappa_{\mathrm{cd}} \varrho} \; , \tag{5.24}$$

hence defining the "conductive opacity" κ_{cd}. Then (5.23) becomes

$$\boldsymbol{F} = -\frac{4ac}{3} \frac{T^3}{\varrho} \left(\frac{1}{\kappa_{\mathrm{rad}}} + \frac{1}{\kappa_{\mathrm{cd}}} \right) \nabla T \; , \tag{5.25}$$

which shows that we arrive formally at the same type of equation (5.11) as in the pure radiative case, if we replace $1/\kappa$ there by $1/\kappa_{\mathrm{rad}} + 1/\kappa_{\mathrm{cd}}$. Again the result is plausible, since the mechanism of transport that provides the largest flux will dominate the sum, i.e. the mechanism for which the stellar matter has the highest "transparency".

Equation (5.12), which, if we define κ properly, holds for radiative and conductive energy transport, can be rewritten in a form which will be convenient for the following sections.

Assuming hydrostatic equilibrium, we divide (5.12) by (2.5) and obtain

$$\frac{(\partial T/\partial m)}{(\partial P/\partial m)} = \frac{3}{16\pi acG}\frac{\kappa l}{mT^3} \,. \tag{5.26}$$

We call the ratio of the derivatives on the left $(dT/dP)_{\mathrm{rad}}$, and we mean by this the variation of T in the star with depth, where the depth is expressed by the pressure, which increases monotonically inwards. In this sense, in a star which is in hydrostatic equilibrium and transports the energy by radiation (and conduction), $(dT/dP)_{\mathrm{rad}}$ is a gradient describing the temperature variation with depth. If we use the customary abbreviation

$$\nabla_{\mathrm{rad}} := \left(\frac{d\ln T}{d\ln P}\right)_{\mathrm{rad}} \,, \tag{5.27}$$

(5.26) can be written in the form

$$\nabla_{\mathrm{rad}} = \frac{3}{16\pi acG}\frac{\kappa l P}{mT^4} \,, \tag{5.28}$$

in which conduction effects are now included. Note the difference in definition and meaning of ∇_{rad} and of ∇_{ad} introduced in (4.21), which concerns not only their (in general different) numerical values. As just explained, ∇_{rad} means a spatial derivative (connecting P and T in two neighbouring mass shells), while ∇_{ad} describes the thermal variation of one and the same mass element during its adiabatic compression. Only in special cases $(d\ln T/d\ln P)$ and ∇_{ad} will have the same value, and we then speak of an "adiabatic stratification".

We will use ∇_{rad} also in connection with more general cases (other modes of energy transport like convection as in Chap. 7, deviation from hydrostatic equilibrium). It then means the gradient to which a radiative, hydrostatic layer would adjust at a corresponding point (same values of P, T, l, m), or simply an abbreviation for the expression on the right-hand side of (5.28), which is valid only for hydrostatic equilibrium and as long as an effective κ as in (5.25) can be defined.

5.3 The Thermal Adjustment Time of a Star

We can write (5.12), which holds for radiative and conductive energy transport, in the form

$$l = -\sigma^* \frac{\partial T}{\partial m} \,, \qquad \sigma^* = \frac{64\pi^2 acT^3 r^4}{3\kappa} \,. \tag{5.29}$$

Now, combining this with (4.45) and replacing the internal energy u by its value $c_v T$ for the perfect gas, it follows that

$$\frac{\partial}{\partial m}\left(\sigma^* \frac{\partial T}{\partial m}\right) - c_v \frac{\partial T}{\partial t} = -\left[\varepsilon + \frac{P}{\varrho^2}\frac{\partial \varrho}{\partial t}\right] . \tag{5.30}$$

If we put the right-hand side equal to zero, then (5.30) has the form of the equation of heat transfer with variable conductivity σ^*. Indeed variation of the temperature with time along a rod of conductivity σ and specific heat c is governed by the equation

$$\frac{\partial}{\partial x}\left(\sigma \frac{\partial T}{\partial x}\right) = c \frac{\partial T}{\partial t} , \tag{5.31}$$

where x is the spatial coordinate along the rod (see Landau and Lifshitz, vol. 6, 1987) . There exists a vast amount of mathematical theory associated with this equation, especially for the case where σ is constant. For example, one can define an initial-value problem with given $T = T(x)$ at $t = 0$. How, then, does this initial temperature profile evolve in time? There are classical methods for determining $T = T(x,t)$ for $t > 0$. One of the basic results is that one can start with an exciting temperature profile $T(x)$, for instance, one which resembles the skyline of Manhattan or the panorama of the Alps, and after some time, the temperature profile always looks like the landscape of Nebraska: $T(x,t)$ approaches the limit solution $T =$ constant after sufficient time.

One can easily estimate the timescale over which (5.31) demands considerable changes of an initially given temperature profile, *the timescale of thermal adjustment*, by replacing in (5.31) ∂T by $\triangle T$, ∂x by a characteristic length d, and ∂t by τ_{adj}:

$$\tau_{\mathrm{adj}} = \frac{c}{\sigma} d^2 , \tag{5.32}$$

where d is a characteristic length over which the (initially given) temperature variation changes. Obviously, only temperature profiles with variations over small distances can change rapidly in time.

The inhomogeneous term on the right of (5.30) is a source term. It takes into account that energy can be added everywhere by nuclear reactions or by compression. In the case of the rod it would correspond to extra heat sources adding heat at different values of x. Similarly to (5.32) we can derive a characteristic time for a star:

$$\tau_{\mathrm{adj}} = \frac{c_v M^2}{\overline{\sigma}^*} , \tag{5.33}$$

where we have replaced the operator $\partial/\partial m$ by $1/M$ and introduced a mean value $\overline{\sigma}^*$, which we can estimate from (5.29). We find for the luminosity L of the star

$L \approx \overline{\sigma}^* \overline{T}/M$, where $\overline{T}$ is a mean temperature of the star. Therefore, for a rough estimate, we have from (5.33) that

$$\tau_{\text{adj}} \approx \frac{c_v \overline{T} M}{L} = \frac{E_{\text{i}}}{L} = \tau_{\text{KH}} \ . \tag{5.34}$$

This means that the Kelvin–Helmholtz timescale as defined in (3.17) can be considered a characteristic time of thermal adjustment of a star or – in other words – the time it takes a thermal fluctuation to travel from centre to surface.

In spite of the indicated equivalence of τ_{adj} and τ_{KH}, it is often advisable to consider τ_{adj} separately, in particular if it is to be applied to parts of a star only. For example, we will encounter evolved stars with isothermal cores of very high conductivity (Chap. 33). The luminosity there is zero so that formally the corresponding τ_{KH} becomes infinite. The decisive timescale that in fact enforces the isothermal situation is the very small τ_{adj}. The difference can be characterized as follows: how much energy may be transported after a temperature perturbation is often much more important than how much energy is flowing in the unperturbed configuration.

5.4 Thermal Properties of the Piston Model

We now investigate the thermal properties of the piston model discussed in Sects. 2.7 and 3.2 by first assuming that the gas of mass m^* in the container is thermally isolated from the surroundings. If the piston is moved, the gas changes adiabatically, i.e.

$$dQ = m^* du + P dV = 0 \ , \tag{5.35}$$

dQ being the heat added to the total mass of the gas. For a perfect gas the energy per unit mass is $u = c_v T$, and for adiabatic conditions, with $V = Ah$, this leads to

$$dQ = c_v m^* dT + PA\, dh = 0 \ . \tag{5.36}$$

We now relax the adiabatic condition in three ways. First, we allow a small leak through which heat (but no gas) can escape from the interior (gas at temperature T) to the surroundings (at temperature T_{s}) see Fig. 2.2. The corresponding heat flow will be $\chi(T - T_{\text{s}})$, where χ is a measure of the heat conduction at the leak indicated in Fig. 2.2. Second, in order to make the gas more similar to stellar matter, we assume the release of nuclear energy with a rate ε. Third, we assume that a radiative energy flux F penetrates the gas and that the energy $\kappa F m^*$ is absorbed per second. The energy balance of the gas in the stationary case then can be expressed by

$$\varepsilon m^* + \kappa F m^* = \chi(T - T_{\text{s}}) \ . \tag{5.37}$$

In general the heat dQ added to the gas within the time interval dt is

$$dQ = [\varepsilon m^* + \kappa F m^* - \chi(T - T_s)]dt \, , \tag{5.38}$$

and, if we compare (5.36) and (5.38), we find that

$$c_v m^* \frac{dT}{dt} + PA\frac{dh}{dt} = \varepsilon m^* + \kappa m^* F - \chi(T - T_s) \, . \tag{5.39}$$

This is the equation of energy conservation of the gas.

If we assume $\varepsilon = \kappa = 0$, then (5.39) has only one time-independent solution: $T = T_s$. What is the timescale of this adjustment of T?

The two time derivatives on the left-hand side of (5.39) give the same estimate for τ; indeed a change of h occurs only as a consequence of, and together with, the change of T. For our rough estimate we can therefore replace the left-hand side of (5.39) by $c_v \Delta T m^*/T$ where $\Delta T = |T - T_s|$:

$$c_v m^* \Delta T/\tau \approx \chi |T - T_s| \, . \tag{5.40}$$

For the timescale by which ΔT decays we obtain

$$\tau_{adj} \approx c_v m^*/\chi \, , \tag{5.41}$$

which is the time it takes the gas to adjust its temperature to that of the surroundings. This timescale for our piston model plays a role similar to the Kelvin–Helmholtz timescale in stars. For sufficiently small χ (sufficiently large τ_{adj}), we have $\tau_{hydr} \ll \tau_{adj}$, similar to the situation in stars, where $\tau_{hydr} \ll \tau_{KH}$.