HUIYI WANG
ID: 261-016-753
PHYS 512
Oct 6th 2021

Problem Set 3

1 Problem 1

The original RK4 method is discussed in class. This method gives out the value:

```
The error using the normal RK4 method is: 1.178e-04
```

Now we derive the new RK4 method, by taking a step size of h and two step sizes of $h/2$, we have two different estimations of y at each step. We define y_1 with step h and y_2 with step $h/2$. Hence we have, with $\phi = y^{(5)}(x)/5!$:

$$y(x+h) = y_1 + (h)^5\phi + O(h^6) \tag{1}$$

$$y(x+h) = y_2 + 2((h/2)^5)\phi + O(h^6) \tag{2}$$

Hence, if taking Equation (2) $\times 16$ minus Equation (1), we have:

$$y(x+2h) = y_2 + \frac{\Delta}{15} + O(h^6) \tag{3}$$

in which $\Delta = y_2 - y_1$. Hence we could use Equation (3) to implement our new RK4 method with the fifth order error term canceling out.

Now, the number of evaluations used in this new RK4 method is three separate original RK4 method. Since both step h and step $h/2$ shares the same starting points. A total number of $\boxed{\text{11 evaluations}}$ are used. Comparing to the original method, which uses only 4 evaluations. We should limit our step size to $200 \times 4/11 \sim \boxed{\text{72 steps}}$. Using this method, python output gives:

```
The error using the new RK4 method is: 8.219e-04
```

which is actually worse then our original RK4 step size. Hence, I think it is more reasonable to compare it to the smaller step size, which takes 8 evaluations. And we should limit out step size to: $200 \times 8/11 \sim \boxed{\text{145 steps}}$. This way, the python output gives:

```
The error using the new RK4 method with 145 steps is:
5.323e-05
```

Which gives out a more accurate estimatino than our original RK4 method.

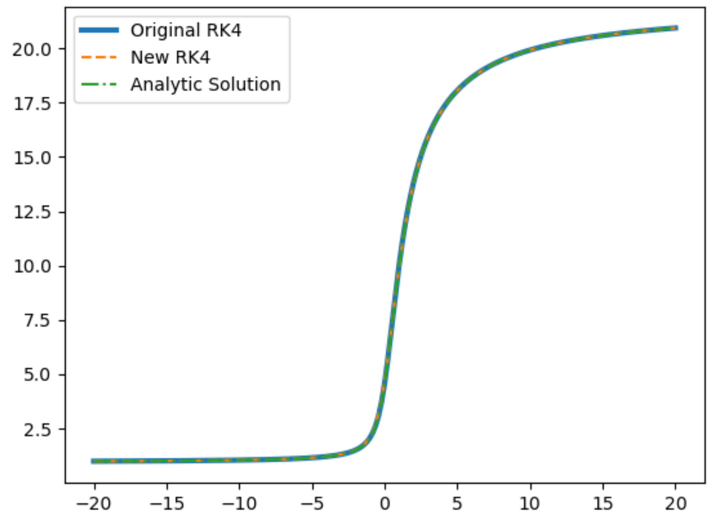


Figure 1: Two approximation using two RK4 methods.

2 Problem 2

Here, I use the Bateman equation for 14 sets of decays. I use `scipy.solve_ivp` with the Radau method. In addition, I have specify 1000 timesteps using `np.logspace`. Here, we set out time interval to 2×10^{21} timesteps.

We can see that from Figure (2a), the ratio of U238 decreases to about half around 10^{21} timesteps and almost to 0.25 around the end of timesteps. This behavior is exactly what we would expect from its half-life. The ratio of Pb206 to U238 is shown in blue and only increases starting from the half-life of U238.

In Figure (2b), we see that between timesteps of 10^{13} and 10^{18} is our area of interest, which U234 are converted into Th230, and reaches a stable ratio afterwards.

3 Problem 3

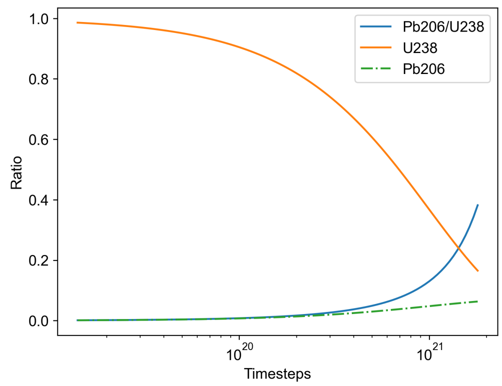
3.1 Linearize Paraboloid

We could rewrite the rotationally symmetric paraboloid as:

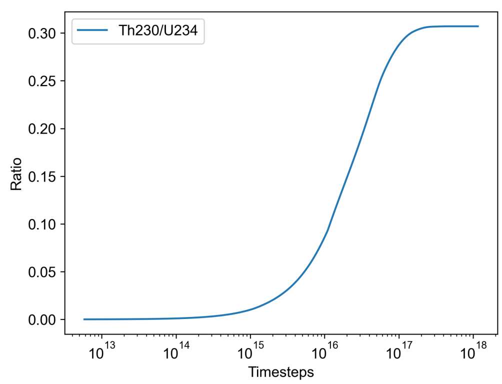
$$z = z_0 + a(x - x_0)^2 + a(y - y0)^2 \tag{4}$$

$$z = z_0 + ax^2 + ax_0^2 - 2axx_0 + ay^2 + ay_0^2 - 2yy_0a \tag{5}$$

$$z = z_0 + ax_0^2 + ay_0^2 + a(x^2 + y^2) - 2ax_0x - 2ay_0y \tag{6}$$



(a)



(b)

Figure 2: (a) Ratio of Pb206 to U238 over last 100 timesteps (b) Ratio of Th230 to U234 over timesteps of interest

Hence, we can expand it in terms of the matrix formula and write:

$$\begin{bmatrix} x0^2 + y0^2 & -x_0 0 & -y_0 0 & 1 \\ \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & 1 \\ xn^2 + yn^2 & -x_0 n & -y_0 n & 1 \end{bmatrix} \begin{bmatrix} q \\ b \\ c \\ p \end{bmatrix} = \begin{bmatrix} z0 \\ \ldots \\ \ldots \\ zn \end{bmatrix}$$

where we have $q = a$, $b = 2ax_0$, and $p = z_0 + ax_0^2 + ay_0^2$.

3.2 Best Fit Parameters

Carrying out the fit using SVD, we have:

```
The best fit parameters are: p = 1.667e-04,
b = -4.536e-04, c = 1.941e-02, d = -1.512e+03
The best fit parameters are: a = 1.667e-04,
x_0 = -1.360, y_0 = 58.221, z_0 = -1512.877
```

3.3 Noise, Uncertainty, and error bar

The noise of the system has a standard deviation of :

```
The noise of the system is: 3.768
```

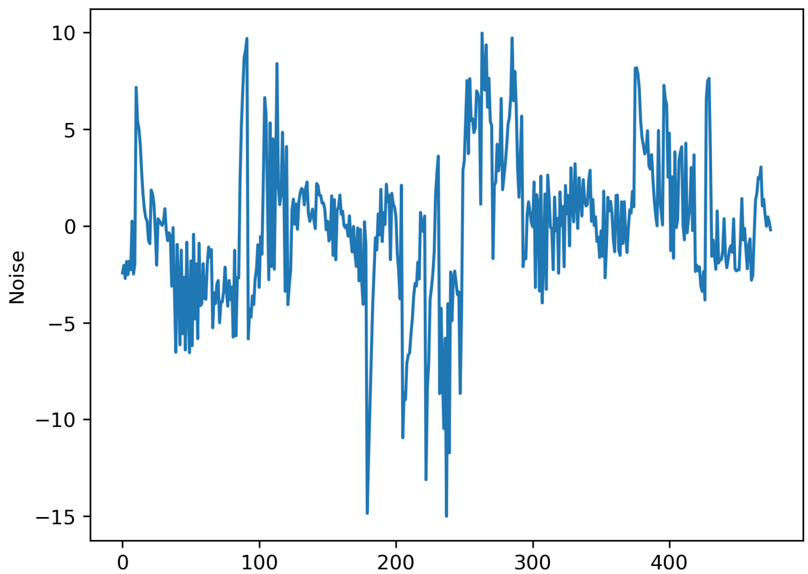


Figure 3: Noise of the z data

Since $A^T N^{-1} A m = A^T N^{-1} d$, we take the inverse of $A^T N^{-1} A$ and the square root of its diagonal to acquire the uncertainties in our best fit parameters. Since $q = a$ directly, we have the uncertainty in a as:

```
The uncertainty of a is: 6.452e-08
```

Now, taking $z = 0$, $x = x_0$, we have the radius of the paraboloid as $r = \sqrt{z_0/a} + y_0$. Plugging into the focal length equation:

$$f = \frac{x^2}{4*y} = \frac{r^2}{4*z_0} = \frac{(\sqrt{z_0/a} + y_0)^2}{4z_0} \tag{7}$$

The error bar of the focal length could be acquire from the Taylor Expansion of f in x and y coordinates. We have:

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \tag{8}$$

Hence:

$$\begin{aligned} \text{Error bar} &= f_x(a,b)(x-a) + f_y(a,b)(y-b) \qquad (9) \\ &= \frac{r}{2z}(r-a) - \frac{r^2}{4z^2}(z-b) \qquad (10) \end{aligned}$$

Here, we take $r-a$ and $z-b$ to be the noise of the system. And we get:

```
The focal length is:  1.5582 +/- 0.008 m
```