Take Home Exam

Introduction to Plasma Dynamcics

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1 Part 1: Theory Questions

1.1 Constants

Constants of reference are [1]:

- Vacum Electric Permittivity: $\epsilon_0 = 8.85 \cdot 10^{-12}[Fm^{-1}]$
- Vacum Magnetic Permittivity: $\mu_0 = 1.26 \cdot 10^{-6}[Hm^{-1}]$
- Boltzmann Constant: $k_b = 1.38 \cdot 10^{-23}[JK^{-1}]$
- Electron Charge: $q_e = 1.62 \cdot 10^{-19}[C]$
- Electron Mass: $m_e = 9.11 \cdot 10^{31}[Kg]$
- Proton Mass: $m_p = 1.67 \cdot 10^{-27}[Kg]$
- Tritium Mass: $m_t = 3 \cdot 1.67 \cdot 10^{-27}[Kg]$
- Deuterium Mass: $m_d = 2 \cdot 1.67 \cdot 10^{-27}[Kg]$
- Tritium Mass: $m_t = 3 \cdot 1.67 \cdot 10^{-27}[Kg]$
- Deuterium Mass: $m_d = 2 \cdot 1.67 \cdot 10^{-27}[Kg]$
- Proton Charge: $q_p = 1.62 \cdot 10^{-19}[C]$
- Speed of Light: $c = 3.0 \cdot 10^{8}[ms^{-1}]$
- Electrons adiabatic Index: $\gamma_e = 1.0$
- Protons adiabatic Index: $\gamma_p = 3.0$

[1] Quantities are expressed in SI units.

1.2 Thermonuclear Plasma: ITER

ITER [2] (International Thermonuclear Experimental Reactor) will be the world's biggest magnetic confinement plasma physics experiment. Inside, under the influence of extreme heat and pressure, gaseous hydrogen fuel becomes a plasma. Like in stars Tritium and Deuterium fuse to reach Helium and, theoretically for the moment, a large amount of energy.

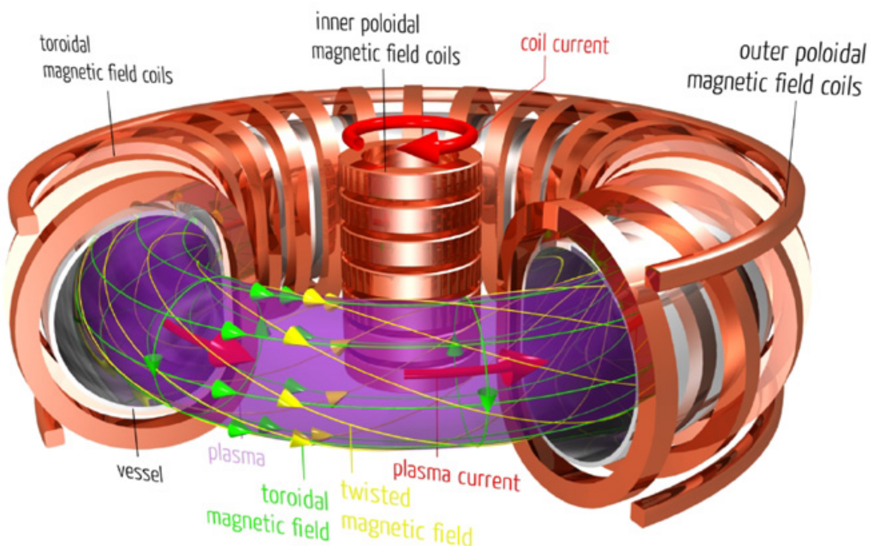

Figure 1: Tokamak scheme.

To simplify this system I consider a simple 1-D ring of radius $R_{max} = 6.2[m]$.
Typical plasma parameter are:

- Density: $n_e = 1.0 \cdot 10^{19}[m^{-3}]$ [3]
- Temperature: $T = 1.0 \cdot 10^8[K]$
- Magnetic Field Intensity: $B = 11.8[T]$

I'm considering an Hydrogen isothermal plasma ($T_i \sim T_e$) composed by electrons, Tritium and Deuterium.

1.3 Question 1

Considering the plasma defined above are obtained the following parameters. Length scales:

- Debye length: $\lambda_D = \sqrt{\frac{\epsilon_0 k_b T}{n_e q_e^2}} = 2.16 \cdot 10^{-4}[m]$
- Electrons inertial length: $d_e = \frac{c}{\omega_e} = 1.66 \cdot 10^{-3}[m]$
- Ions inertial length: $d_i = \frac{c}{\omega_i} = 5.54 \cdot 10^{-2}[m]$
- Electrons gyroradii: $r_e = \frac{v_{th,e}}{\omega_e} = 1.85 \cdot 10^{-5}[m]$
- Ions gyroradii: $r_i = \frac{v_{th,i}}{\omega_i} = 1.22 \cdot 10^{-3}[m]$

[2]https://www.iter.org/
[3]Electron and ions density are the same because we are working with Hydrogen isotopes ($n_e = n_i$).

Time scales:

- Electrons plasma frequency: $\omega_{p,e} = \sqrt{\frac{n_e q_e^2}{\epsilon_0 m_e}} = 1.80 \cdot 10^{11}[Hz]$
- Ions plasma frequency: $\omega_{p,i} = \sum_s \sqrt{\frac{n_s Z^2 q_p^2}{\epsilon_0 m_s}} = \sqrt{\frac{n_e q_e^2}{\epsilon_0 m_t}} + \sqrt{\frac{n_e q_e^2}{\epsilon_0 m_d}} = 4.21 \cdot 10^9 [Hz]$
- Electrons cyclotron frequencies: $\omega_{c,e} = \frac{q_e B}{m_e} = 2.1010^{12}[Hz]$
- Ions cyclotron frequencies: $\omega_{c,i} = \sum_s \frac{Z q_e B}{m_s} = \frac{q_e B}{m_t} + \frac{q_e B}{m_d} = 9.54 \cdot 10^8 [Hz]$ [4]

Velocity scales:

- Sound speed: $c_s = \sum_s \sqrt{\frac{(\gamma_e+\gamma_p)k_b T}{m_s}} = \sqrt{\frac{(\gamma_e+\gamma_p)k_b T}{m_t}} + \sqrt{\frac{(\gamma_e+\gamma_p)k_b T}{m_d}} = 2.34 \cdot 10^6 [ms^{-1}]$
- Electrons thermal speed: $v_{th,e} = \sqrt{\frac{k_b T}{m_e}} = 3.89 \cdot 10^7 [ms^{-1}]$
- Ions thermal speed: $v_{th,i} = \sum_s \sqrt{\frac{k_b T}{m_s}} = \sqrt{\frac{k_b T}{m_t}} + \sqrt{\frac{k_b T}{m_t}} = 1.17 \cdot 10^6 [m \cdot s^{-1}]$
- Alfvén speed: $v_{A,i} = \sum_s \frac{B}{\sqrt{\mu_0 n_s m_s}} = \frac{B}{\sqrt{\mu_0 n_e m_t}} + \frac{B}{\sqrt{\mu_0 n_e m_d}} = 1.64 \cdot 10^8 [ms^{-1}]$

Remarks the fact that in nature the most efficient way to get the fusion is with a mixture of tritium and deuterium, but on earth is extremely difficult to find tritium moreover it could be substitute with helium three. In our case Alfvèn velocity is bigger than the sound speed. Indeed in the presence of very intense magnetic fields or small ionic densities, Alfvén wave velocity can be approximates to that of light; consequently, the Alfvén wave assumes the connotations of a real electromagnetic wave.

1.4 Question 2

Imagine to build an experiment to study the two stream instability, in our plasma of reference, an important parameter is the minimum length should the experiment have to be able to develop instability. Suppose you can create two beams traveling in opposing direction along a straight line, but under a condition where the ions remain an immobile background and only the electrons are a beam. So the distribution function has the form as follow:

$$f(v) = \frac{n_0}{2}\delta(v - v_0) + \frac{n_0}{2}\delta(v + v_0) \tag{1}$$

Thus the dispersion relation:

$$\epsilon(k, \omega) = 1 - \frac{\omega_p^2}{2}\left(\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}\right) \tag{2}$$

Given this dispersion relation it is easy to find the typical modes of the plasma. Those are the values satisfying the relation above. Now we choose the condition for instability $\omega_p^2 > k^2 v_0^2$ that leads to the relation that we want:

$$L_e > \frac{2\pi v_0}{\omega_{p,e}} = 1.35 \cdot 10^{-1}[m]^{5} \tag{3}$$

Indeed the length of our system should is one hundred times bigger [6].

[4]For Tritium and Deuterium, as isotopes of Hydrogen, $Z = 1$

[5]Using $v_0 = 100 v_{th,e}$.

[6]ITER radius is $Rmax = 6.2m$ therefore its circumference is $L = 2\pi Rmax = 38.95m \simeq 4.0 \cdot 10^1 m$.

1.5 Question 3

In a strongly magnetized medium where $v_A > c_s$, the fast and slow modes propagate at the Alfvèn and sound speed, respectively. We can detect this in the diagram because the two curves touch each other.

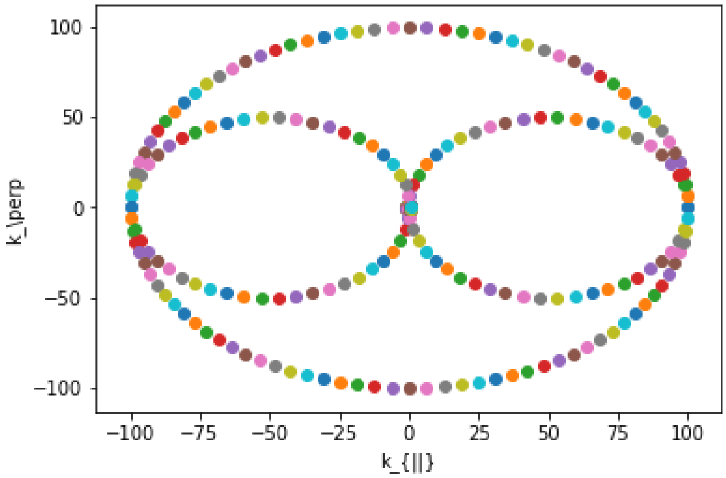

Figure 2: MHD Wave diagram computed with $c_s = 1$ and $v_A = 100$.

Consider now the case of a Langmuir wave. We compute the damping for a wave with wave vector $k = d_e^{-1}$ for your plasma, using the exact solution.

$$\frac{\omega}{\omega_{p,e}} = 1.0259867008 - 8.96771176541 \cdot 10^{-12} j \tag{4}$$

Then we compare it with the damping obtained from the following approximate formula:

$$\omega_r^2 = \omega_{p,e}^2 + 3k^2 v_{th,e}^2 \tag{5}$$

It's obtained:

$$\frac{\omega}{\omega_{p,e}} = 1.0250824452 - 9.49260470849 \cdot 10^{-12} j \tag{6}$$

1.6 Question 4

We consider a 1D plasma with only one component of the velocity, directed along x, ions at rest and electrostatic field. We want to study the stability of this plasma, so we start from the dispersion relation under the approximation that initial electron distribution is uniform in space and in velocity:

$$f_0(v) = \frac{1}{2}\delta(v) + \frac{1}{4}\delta(v - v_0) + \frac{1}{4}\delta(v + v_0) \tag{7}$$

The following condition must be satisfied for the assumed harmonic solution to be valid:

$$1 + \frac{q^2}{\epsilon_0 m k^2} \int \frac{\partial f_0(v)/\partial v}{\omega/k - v} dv \tag{8}$$

Using the following property of the Dirac's delta:

$$\int f(v)\delta(v - v_0)dv = f(v_0) \tag{9}$$

and by integration by part it follows also that

$$\int f(v)\delta'(v - v_0)dv = -f'(v_0) \tag{10}$$

Finally the dispersion relation has this form:

$$\epsilon(k, \omega) = 1 - \frac{1}{2}\left(\frac{\omega_{p,e}^2}{\omega^2} + \frac{1}{2}\frac{\omega_{p,e}^2}{(\omega - kv_0)^2} + \frac{1}{2}\frac{\omega_{p,e}^2}{(\omega + kv_0)^2}\right) \tag{11}$$

Proceed now with the study of the dispersion relation. The function can be split in two sub-function, changing variable to simplify the notation: $\omega = \omega_{p,e}x$ and $\alpha = \frac{kv_0}{\omega_{p,e}}$; and after some simple calculation we get:

$$y_1 = \frac{1}{2x^2} + \frac{1}{4}\frac{1}{(x - \alpha)^2} + \frac{1}{4}\frac{1}{(x + \alpha)^2} \tag{12}$$

$$y_2 = 1 \tag{13}$$

Solution of equation (11) are given now by the intersection of y_1 and y_2 like you can see in the plot:

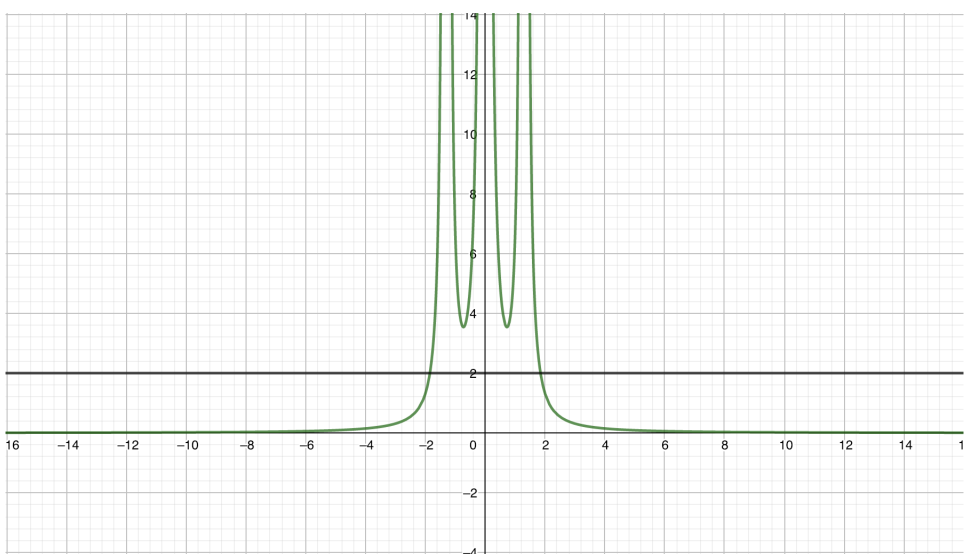

Figure 3: Plot of y_1 and y_2.

From α depends the intersection or not of the curves, indeed varying this parameter it means shifting the minimum along the y direction. So instability are reach if and only if $\epsilon(k, \omega) = 0$ have imaginary solution, thus the two function don't intersect each other. Let's calculate the minimum studying the first derivatives of y_1 in zero:

$$y_1'(x) = 0 \tag{14}$$

It's obtained the following equation of sixth order.

$$2x^6 + 3\alpha^4 x^2 - \alpha^6 = 0 \tag{15}$$

Rename x^2 to t and searching real solution we finally get the value of the local minimum in the domain $x \in (-\alpha, 0) \bigvee (0, \alpha)$:

$$2t^3 + 3\alpha^4 t - \alpha^6 = 0 \tag{16}$$

$$t = \left(\frac{(1+\sqrt{3})^{\frac{1}{3}}}{(2)^{\frac{2}{3}}} - \frac{1}{2(1+\sqrt{3})^{\frac{1}{3}}} \right) \alpha^2 \tag{17}$$

$$x_m = \pm \left(\frac{(1+\sqrt{3})^{\frac{1}{3}}}{(2)^{\frac{2}{3}}} - \frac{1}{2(1+\sqrt{3})^{\frac{1}{3}}} \right)^{\frac{1}{2}} \alpha = 0.55938 \quad \alpha \tag{18}$$

Stability criteria are reached plugging x_m in the dispersion relation, equation (11).

$$f(x_m) > 1 \tag{19}$$

Solving by α it comes out that instability forms if this condition is satisfied: $|\alpha|^2 < 3.027$. This means $\omega_p^2 > 0.304 k^2 v_0^2$.

2 Part 2: PiC Exercise

2.1 Question 1

If we try to solve the dispersion relation with a Maxwellian-type function we get an interesting feature of plasma. Landau damping is one of the most important collisionless dissipation mechanisms for longitudinal plasma waves. The integral cannot be computed easily so we use the Faddeeva function to make the integral feasible.

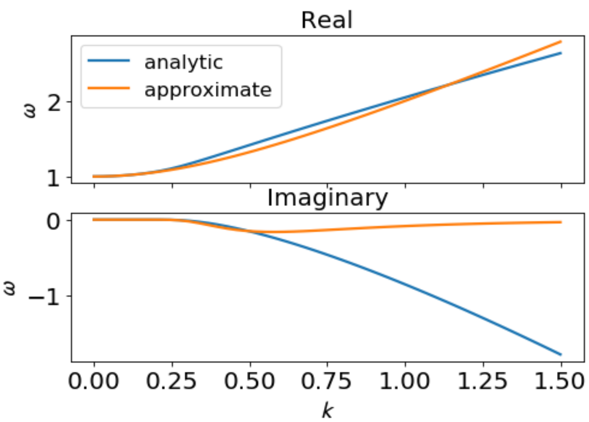

Figure 4: Plot of the approximate and exact expressions for the damping rate $\omega_i/\omega_p, e$ for $k\lambda_D \in (0, 1.5)$.

Remarks the fact that the approximation is valid only if $k\lambda_d << 1$, indeed as you note in the graphics for imaginary part we don't have convergence.

2.2 Question 2

After set the code parameter as follow, we launch the simulation and check what happen in the phase space each 50 time step.

- $grid - lenght = 2\pi$
- $v_0 = 0.2$
- $dt = 0.05$
- $npart = 10000$
- $nsteps = 100$
- $ngrid = 100$

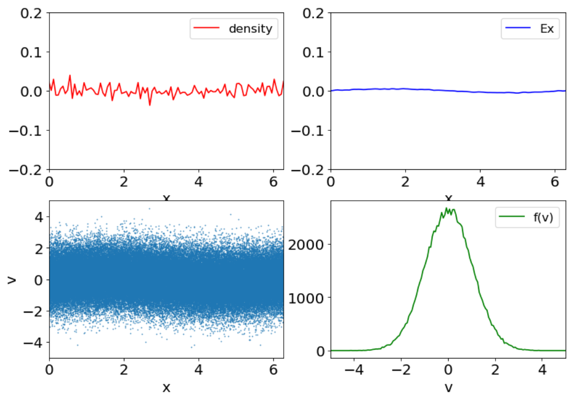

Figure 5: Phase space at $t = 0\omega^{-1}$

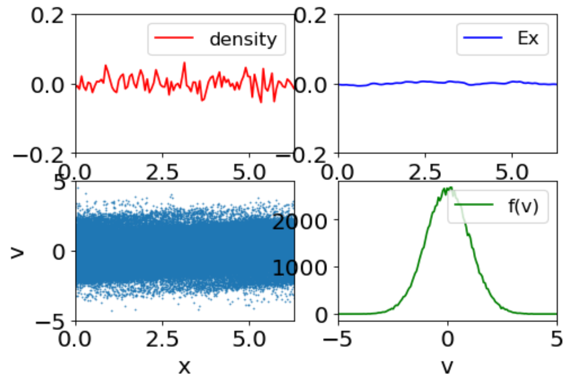

Figure 6: Phase space at $t = 100\omega^{-1}$

As you can notice, particles have only different velocity and position.

2.3 Question 3

Increasing the length domain we get Landau damping. Then Fitting the maximum with a line can be obtain its slope:

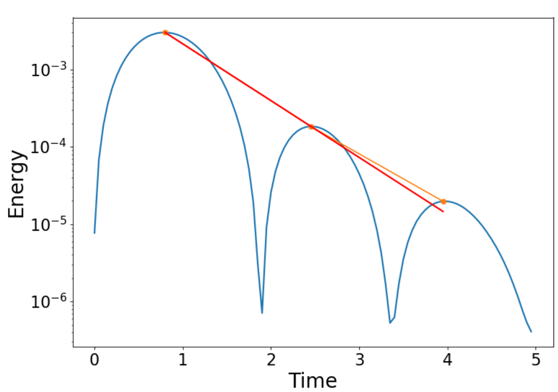

Figure 7: Landau Damping $k = 0.5$.

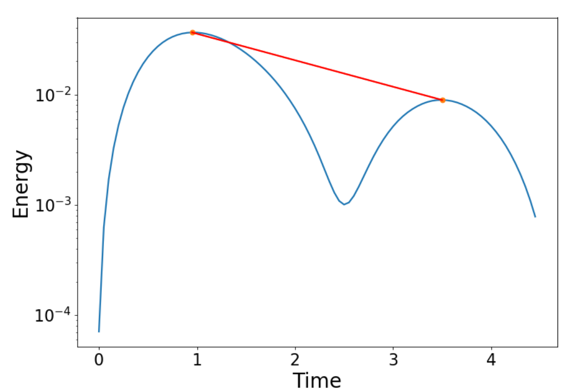

Figure 8: Landau Damping $k = 0.8$.

This is a typical PIC simulation of Landau damping, showing that the energy of the wave decreases exponentially (i.e. linearly in a logarithmic scale). Let's compare the result obtained in question one:

Fit parameter for study on velocity			
k value	**Damping**	$\frac{\omega_i}{\omega_p}$ **Exact**	$\frac{\omega_i}{\omega_p}$ **Approx.**
$k = 0.5$	- 0.0912	-1.0259	-1.0251
$k = 0.8$	-0.5529	-1.7798	-1.7088

The damping becomes exponentially stronger as the wavenumber is increased: short wavelength are dumped powerfully.

2.4 Question 4

Considering the distribution function for $k\lambda_d = 0.5$ and plotting it in a logaritmic scale. After few wave cycle we get the graphic below.

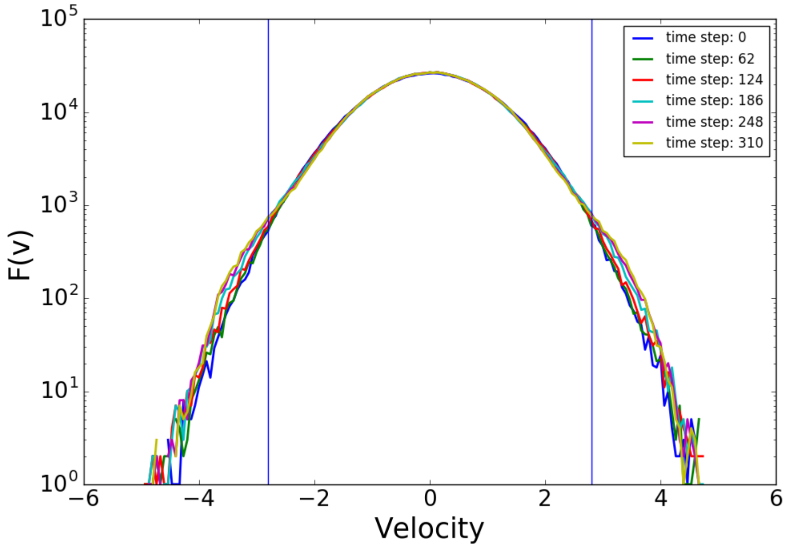

Figure 9: Overlap of the distribution function for $k\lambda_d = 0.5$

According with the Landau Damping effect particles with speed $-2.8 < v < 2.8$, the majority, are accelerated by the plasma waves, therefore as time evolve we will see more high velocity particles than the moment before.