maximal slicing

exercise 1.1: Maximal slicing of Schwarzschild

The 3+1 decomposition can be performed with different choices for the slicing of spacetime.
The choice of time slicing amounts to making a choice for the lapse
α by either prescribing a function or an equation. Since α can vary depending on the position on the spatial slice, the proper time is allowed to advance at different rates at different points on a given slice.
A common choice is the *maximal slicing* condition:

$$K = 0. \tag{eq:maxslicecond}$$

If we want maximal slicing to hold at all times, then we should impose
$\partial_t K = 0$ as well.
In this case, the evolution equation for the extrinsic curvature scalar reduces to an elliptic equation for the lapse function

$$D_i D^i \alpha = \alpha\big(K_{ij} K^{ij}\big).$$

- [**Step 1**]: In Schwarzschild coordinates, consider $\bar{t} = t + h(r)$ and enforce $K = 0$. Find the family of time independent slicings.
- [**Step 2**]: From the general result in [**Step 1**], recover the specific solutions for Schwarzschild and isotropic metric.
- [**Step 3**]: Draw the Kruskal-Szekeres diagrams for the Schwarzschild and the isotropic slicings.
- [**Step 4**]: Draw the embedding diagram.

(soln) exercise 1.1

There are many ways to get the required equation. In order to reduce some of the effort involved in the computation we proceed as follows.

As a preliminary note that in $3+1$ decomposition the ambient metric can be written in the form:

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + \beta^i \mathrm{d}t)(\mathrm{d}x^j + \beta^j \mathrm{d}t);$$

or:

$$g_{ab} = \begin{bmatrix} -\alpha^2 + \beta_l \beta^l & \beta_i \\ \beta_j & \gamma_{ij} \end{bmatrix}. \tag{eq:metrDecEx3}$$

Furthermore:

$$n^a = (\alpha^{-1},\ -\alpha^{-1}\beta^i), \qquad n_a = (-\alpha,\ 0,\ 0,\ 0);$$

and

$$\beta^a = (0,\ \beta^i).$$

Put $\mu(r) := 1 - 2m/r$, then, the standard form of the Schwarzschild solution takes the form:

$$\mathrm{d}s^2 = -\mu(r)\mathrm{d}t^2 + \frac{1}{\mu(r)}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2. \tag{eq:solnStdSS}$$

As written, we are working with the so-called "areal" radius. The nomenclature is due to r being related to the area $\mathcal{A}$ of a spherical surface at r centred on the black hole according to the Euclidean expression $r = (\mathcal{A}/(4\pi))^{1/2}$.

The time coordinate is to be mapped as:

$$t \mapsto \bar{t} = t + h(r).$$

The interpretation here is that $h(r)$ is in a sense a kind of "height-function" measuring how far $\bar{t} = \mathrm{const}$ surfaces "lift-off" the usual $t = \mathrm{const}$ surfaces.

With the reparametrization Eq.(eq:solnStdSS) becomes:

$$\begin{aligned}\mathrm{d}s^2 =& -\mu(r)\mathrm{d}\bar{t}^2 + 2\mu(r)h'(r)\mathrm{d}\bar{t}\mathrm{d}r \\ &+ \left(\frac{1}{\mu(r)} - \mu(r)(h'(r))^2\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2.\end{aligned} \tag{eq:solnStdSSRepar}$$

Comparing to the standard form of $3+1$ decomposition in Eq.(eq:metrDecEx3) we see:

$$\begin{aligned}\beta_i =& \mu(r)h'(r)(1,\,0,\,0), \qquad -\alpha^2 + \beta_l\beta^l = -\mu(r);\\ \gamma_{ij} =& \mathrm{diag}\left(\frac{1}{\mu(r)}\left(1-\mu(r)^2(h'(r))^2\right),\, r^2,\, r^2\sin^2\theta\right);\end{aligned}$$

and:

$$\beta^j = \gamma^{ij}\beta_i = \frac{\mu(r)^2h'(r)}{1-\mu(r)^2(h'(r)^2)};$$

with:

$$\alpha^2 = \frac{\mu(r)}{1-\mu(r)^2(h'(r))^2}.$$

We may immediately relate the mean extrinsic curvature through the divergence relation:

$$\begin{aligned}-K =& \nabla_a[n^a] = \frac{1}{\sqrt{|g|}}\partial_a\left[\sqrt{|g|}n^a\right] = 0,\\ \implies 0 =& \partial_a\left[\alpha\gamma^{1/2}n^a\right],\end{aligned}$$

where the determinant relation $|g|^{1/2} = \alpha\gamma^{1/2} = r^2\sin\theta$ has been used. Expanding, we find:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\left(\frac{\mu(r)}{1-\mu(r)^2(h'(r)^2)}\right)^{1/2}\mu(r)h'(r)\right] = 0,\\ \implies& r^2\left(\frac{\mu(r)}{1-\mu(r)^2(h'(r)^2)}\right)^{1/2}\mu(r)h'(r) = C,\end{aligned}$$

for some constant of integration C. It is useful to rearrange this into:

$$\mu(r)^2(h'(r)^2) = \frac{C^2}{\mu(r)r^4 + C^2},$$

as we may then write (plugging back into previous expressions):

$$\gamma_{ij} = \frac{1}{\mu(r;\,C)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2,$$

together with:

$$\alpha = \mu(r;\,C), \qquad \beta^a = (0, C\mu(r;\,C)^{1/2}/r^2,\,0,\,0);$$

where:

$$\mu(r;\,C) = 1 - \frac{2m}{r} + \frac{C^2}{r^4}.$$

Clearly C parametrizes the family of solutions with $C\to 0$ resulting in recovery of $t = \mathrm{const}$ slices of Schwarzschild space-time.

Recall: An alternate description of the Schwarzschild solution is provided in so-called "isotropic" coordinates. Here the isotropy is made manifest by transforming to a form where:

$$\mathrm{d}s^2 = A(\rho)\mathrm{d}t^2 - B(\rho)\mathrm{d}\Sigma^2, \tag{eq:isomapSS}$$

and $\mathrm{d}\Sigma^2$ represents flat 3-space:

$$\mathrm{d}\Sigma^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2 + \rho^2\sin^2\theta\,\mathrm{d}\phi^2.$$

Comparing Eq.(eq:solnStdSS) and Eq.(eq:isomapSS) we can immediately write:

$$\mathrm{d}s^2 = -\mu(r)\mathrm{d}t^2 + f(\rho)^2\left[\mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2 + \rho^2\sin^2\theta\,\mathrm{d}\phi^2\right],$$

and it follows that $r^2 = f^2\rho^2$. Furthermore, we impose:

$$\frac{1}{\mu(r)}\mathrm{d}r^2 = f(\rho)^2\mathrm{d}\rho^2,$$

and demand that $\rho \to \infty$ when $r \to \infty$ which yields:

$$\frac{\mathrm{d}r}{\sqrt{r^2 - 2mr}} = \frac{\mathrm{d}\rho}{\rho};$$

Integrating and rearranging yields the transformation from areal r to isotropic ρ coordinates:

$$r = \rho(1 + m/(2\rho))^2.$$

This leads to the isotropic form of Schwarzschild as:

$$\mathrm{d}s^2 = -\left(\frac{1 - m/(2\rho)}{1 + m/(2\rho)}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{m}{2\rho}\right)^4 \left[\mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2 + \rho^2\sin^2\theta\,\mathrm{d}\phi^2\right]. \qquad \text{(eq:solnIsoSS)}$$

The form of Eq.(eq:solnIsoSS) describes only the region of Schwarzschild geometry with $r \geq 2m$. The horizon is located at $\rho = m/2$ in these coordinates.

Following this sort of idea, we may consider mapping Eq.(eq:solnStdSSRepar) such that the spatial part takes an isotropic form in some new coordinates:

$$\left(\frac{1}{\mu(r)} - \mu(r)(h'(r))^2\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 = \psi^4(\rho)\left[\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega^2\right],$$

and we regard $\rho = \rho(r)$, say. We omit further details of this calculation but remark that this leads to so-called "trumpet" geometries; (see Class. Quantum Grav. 31 (2014) 117001).

For construction of **Kruskal-Szekeres (KS)** diagrams and properties see e.g. Wald. KS diagrams displaying properties of different kinds of maximal-slicings for Schwarzschild is provided in Alcubierre Fig. 4.2 together with a very nice discussion. Gourgoulhon's book ($\S 10.2.2$) is also a good place to look.

We give a brief description of **embedding**:

Consider equatorial "plane" at a fixed moment in time $t = \mathrm{const}$, $\theta = \pi/2$:

$$^{[2]}\mathrm{d}s^2 = \frac{1}{\mu(r)}\mathrm{d}r^2 + r^2\mathrm{d}\phi^2.$$

We aim to embed in three dimensional Euclidean space a two dimensional surface describing the two dimensional geometry. Introduce cylindrical coordinates $(r,\, z,\, \phi)$:

$$^{[3]}\mathrm{d}s^2 = \mathrm{d}r^2 + \mathrm{d}z^2 + r^2\mathrm{d}\phi^2.$$

The surface to embed is cylindrically symmetric so should only need to regard as function of radius, i.e., $z = z(r)$.

Plugging this in we have:

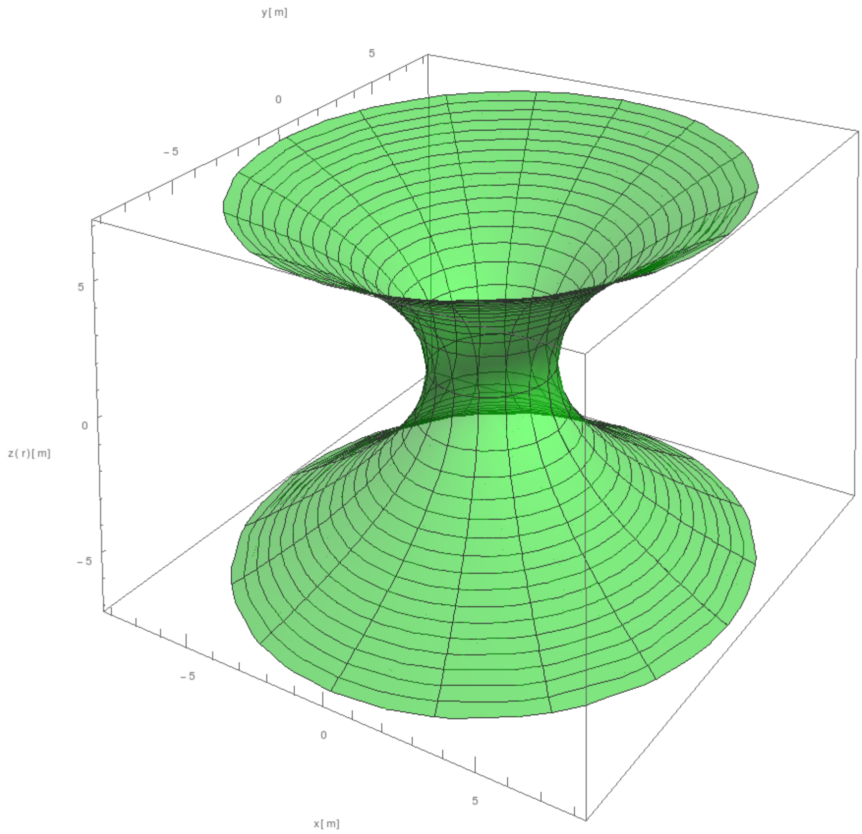
$$^{[3]}\mathrm{d}s^2 \to {}^{[2]}\mathrm{d}s^2 = \mathrm{d}r^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\mathrm{d}r^2 + r^2\mathrm{d}\phi^2.$$

Comparing with earlier $^{[2]}\mathrm{d}s^2$:

$$\left(1 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\right) = \frac{1}{\mu(r)} \implies \frac{\mathrm{d}z}{\mathrm{d}r} = \sqrt{\frac{1}{\mu(r)} - 1}.$$

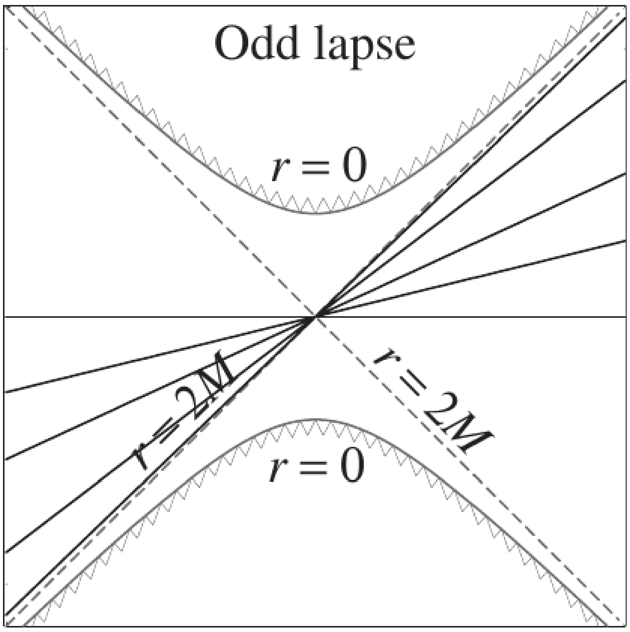
We may instead compare with the maximal slicing form which leads to $\mu(r) \to \mu(r;\, C)$ in the above.

Embedding diagram (both branches of the solution are included corresponding to extended Schwarzschild):



Maximal (standard Schwarzschild) slice:

From Alcubierre:



- Remark: It is also possible to construct other (maximal) slicings - see the previously mentioned references.

exercise 1.2

In the lecture notes it was demonstrated (see §6.6) that imposing the maximal slicing condition of Eq. (eq:maxslicecond) extremises volume.
Argue that in the case of a Lorentzian manifold the extremum constitutes a maximum.

(soln) exercise 1.2

Consider second variation while working in vacuum; use results from 3+1 decomposition and recall that we are working at an extremum.

Some related details: "Isolated Maximal Surfaces in Spacetime, Brill & Flaherty".

approximations

exercise 1.1: Newtonian limit

Recall that if the gravitational field is weak and static then it is always possible to find a coordinate system $(x^\alpha) = (x^0 = t,\, x^i)$ such that the metric components take the form:

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)f_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{eq:metricWeakField}$$

where Φ is the gravitational potential and $|\Phi| \ll 1$.

In performing $3+1$ and conformal decompositions, we have found that the mean curvature evolves according to:

$$\mathcal{L}_m[K] = -D_iD^i[\alpha] + \alpha(4\pi(E+S) + K_{ij}K^{ij}), \tag{eq:KtrEvo}$$

Working within the regime of validity for Eq.(eq:metricWeakField) show that Eq.(eq:KtrEvo) reduces to the Poisson equation:

$$\mathcal{D}_i\mathcal{D}^i[\Phi] = 4\pi\rho,$$

where $\mathcal{D}$ is the derivative operator associated with the flat metric f_{ij}.

(soln) exercise 1.1

We adopt spherical coordinates. As a preliminary note that approximate staticity entails $\Phi = \Phi(r,\, \theta,\, \phi)$. Furthermore, from Eq.(eq:metricWeakField) we may immediately write:

$$\begin{aligned}
\alpha^2 =& (1+2\Phi) \implies \alpha = 1 + \Phi + \mathcal{O}(\Phi^2),\\
\beta^a =& 0,\\
\gamma_{ij} =& (1-2\Phi)\mathrm{diag}\big(1,\, r^2,\, r^2\sin^2\theta\big).
\end{aligned}$$

Considering Φ as $|\Phi| \ll 1$ we find:

$$\partial^i[1+f] = \gamma^{ij}\partial_j[f] = \left((1+2\Phi)\partial_r[f],\, \frac{(1+2\Phi)}{r^2}\partial_\theta[f],\, \frac{(1+2\Phi)}{r^2}\csc^2\theta\,\partial_\phi[f]\right) + \mathcal{O}(\Phi^2);$$

where $f = f(r,\, \theta,\, \phi) \in \mathcal{F}(\Sigma)$ and thus replacement with $\partial \to \mathcal{D}$ on the LHS of this expression is permitted.

Recall that for $V^i \in \mathcal{T}(\Sigma)$:

$$D_i[V^i] = \frac{1}{\sqrt{\gamma}}\partial_i\big[\sqrt{\gamma}V^i\big] = \partial_i[V^i] + \frac{1}{\sqrt{\gamma}}\partial_i[\sqrt{\gamma}]V^i,$$

where in the present case:

$$\sqrt{\gamma} = (1-3\Phi)r^2\sin\theta + \mathcal{O}(\Phi^2).$$

If we set $V^i := \partial^i[1+f]$ and replace $f \to \Phi$ then we find:

$$D_i[D^i[\alpha]]/\alpha = \mathcal{D}_i[\mathcal{D}^i[\Phi]] + \mathcal{O}(\Phi^2),$$

where $\mathcal{D}_i[\cdot]$ is the covariant derivative operator associated with the flat 3-metric in spherical coordinates.

One can also verify directly from definition that $K_{ij} = 0 + \mathcal{O}(\Phi^2) \implies K = 0 + \mathcal{O}(\Phi^2)$. Thus, with $\rho = E + S$ we find that Eq.(eq:KtrEvo) becomes:

$$\mathcal{D}_i[\mathcal{D}^i[\Phi]] = 4\pi\rho,$$

exactly as was required.