Numerical relativity — Exercise sheet # 5

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Providing initial data

Exercise 1.1: Conformal transverse-traceless and Bowen-York data

In the conformal transverse-traceless approach one considers splitting the conformal traceless extrinsic curvature $\hat{A}^{ij}$ into longitudinal (L) and transverse-traceless (TT) parts:

$$\hat{A}^{ij} := \hat{A}_{\mathrm{L}}{}^{ij} + \hat{A}^{ij}{}_{\mathrm{TT}} = \mathbb{L}_{\tilde{\gamma}}[X]^{ij} + \hat{A}_{\mathrm{TT}}{}^{ij},$$

where $\tilde{\gamma}_{ij}\hat{A}_{\mathrm{TT}}{}^{ij} = 0 = \tilde{D}_j[\hat{A}_{\mathrm{TT}}{}^{ij}]$ and X^k is a vector potential. We have also introduced the (symmetric and traceless) *conformal Killing operator*:

$$\mathbb{L}_{\tilde{\gamma}}[X]^{ij} := \tilde{D}^i[X^j] + \tilde{D}^j[X^i] - \frac{2}{3}\tilde{D}_k X^k \tilde{\gamma}^{ij},$$

and will also make use of the *vector Laplacian*:

$$\tilde{D}_j[\hat{A}^{ij}] = \tilde{D}_j[\mathbb{L}_{\tilde{\gamma}}[X]^{ij}] = \tilde{D}_j[\tilde{D}^j[X^i]] + \frac{1}{3}\tilde{D}^i[\tilde{D}_j[X^j]] + \tilde{R}^i_j X^j =: \Delta_{\tilde{\gamma}}[X^i].$$

Under this decomposition the constraints become:

$$\Delta_{\tilde{\gamma}}[X^i] - \frac{2}{3}\tilde{D}^i[K]\Psi^6 - 8\pi\tilde{P}^i = 0, \tag{1}$$

$$\tilde{D}_i[\tilde{D}^i[\Psi]] - \frac{1}{8}\tilde{R}\Psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\Psi^{-7} - \frac{1}{12}K^2\Psi^5 + 2\pi\tilde{E}\Psi^{-3} = 0; \tag{2}$$

where ($\tilde{\gamma}$, $\hat{A}_{\mathrm{TT}}{}^{ij}$, K, E, P^i) may be freely prescribed and (Ψ, X^i) are to be solved for.

Our goal here is to solve Eq.(1) and approximate a solution to Eq.(2) from scratch.

1. Consider working in vacuum and assume the maximal slicing condition. Verify the constraints decouple, and that if we further impose conformal flatness while working with Cartesian coordinates the momentum constraint becomes:

$$\partial^j[\partial_j[X^i]] + \frac{1}{3}\partial^i[\partial_j X^j] = 0. \tag{3}$$

2. Solutions to Eq.(3) are known as "Bowen-York" solutions. By further decomposing the vector potential as $X_i = Y_i + \partial_i[Z]$ and making a suitable choice for Z show that Eq.(3) may be rewritten as the coupled scalar Poisson equations:

$$\partial^j[\partial_j[Z]] = -\frac{1}{4}\partial_j[Y^j], \quad \partial^j[\partial_j[Y_i]] = 0. \tag{4}$$

Impose spherical symmetry and find a simple solution for Z, Y_i and hence X^i.

3. Define the normal vector $n^i := x^i/r$ where $r^2 = x^2 + y^2 + z^2$. Verify that another solution to the momentum constraint, assuming conformal flatness is given by:

$$X^i = \tilde{\varepsilon}^{ijk} n_j J_k / r^2, \tag{5}$$

where J^i satisfies $\tilde{D}_i[J_j] = 0$ and $\tilde{\varepsilon}^{ijk}$ is the Levi-Civita tensor associated with $\tilde{\gamma}_{ij}$ such that $\tilde{D}_i[\tilde{\varepsilon}^{jkl}] = 0$. Verify that substitution of Eq.(5) into the conformal Killing operator allows us to write:

$$\mathbb{L}_{\tilde{\gamma}}[X]^{ij} = \frac{6}{r^3} n^{(i} \tilde{\varepsilon}^{j)kl} J_k n_l = \hat{A}_{\mathrm{L}}{}^{ij}. \tag{6}$$

4. Show that in spherical polar coordinates the non-vanishing components (up to symmetries) of X^i and $\hat{A}_{\mathrm{L}}{}^{ij}$ are given by:

$$X^\phi = -\frac{J}{r^3}, \quad \hat{A}^{\mathrm{L}}{}_{r\phi} = \frac{3J}{r^2}\sin^2\theta, \tag{7}$$

where J is the magnitude of the vector J^i aligned with the polar axis.

5. Consider the rotating black hole solution of Eq.(7) and assume $\hat{A}^{ij}{}_{\mathrm{TT}} = 0$. This furnishes us a solution for the traceless extrinsic curvature. To complete an initial data specification we also need to provide Ψ. Thus, the Hamiltonian constraint must be solved which is non-linear in general and typically requires a numerical approach. Here we construct an approximate expansion. Given our $\hat{A}^{ij}$ and prior assumptions notice that the only non-vanishing source term in Eq.(2) arises from:

$$\hat{A}_{ij}\hat{A}^{ij} = \frac{18J^2}{r^6}\sin^2\theta. \tag{8}$$

Furthermore as $J \to 0$ the Hamiltonian becomes the Laplace equation and hence, formally, $\Psi \to 1 + \mathcal{M}/(2r) =: {}^{(0)}\Psi$. This suggests we write:

$$\Psi = {}^{(0)}\Psi + \frac{J^2}{\mathcal{M}^4}{}^{(2)}\Psi + \mathcal{O}(J^4) = 1 + \frac{\mathcal{M}}{2r} + \frac{J^2}{\mathcal{M}^4}{}^{(2)}\Psi + \mathcal{O}(J^4). \tag{9}$$

Show that subtitution of Eq.(9) into Eq.(2) leads to:

$$\tilde{D}^2\left[{}^{(2)}\Psi\right] = -\frac{9}{4}\frac{\mathcal{M}^4 r}{(r+\mathcal{M}/2)^7}\sin^2\theta. \tag{10}$$

6. To find the first correction offered by Eq.(10) split the angular dependence of ${}^{(2)}\Psi$ using the Legendre polynomials:

$$P_0(\cos\theta) = 1, \quad P_2(\cos\theta) = (3\cos^2\theta - 1)/2; \tag{11}$$

such that:

$${}^{(2)}\Psi(r,\,\theta) = {}^{(2)}\Psi_0(r)P_0(\cos\theta) + {}^{(2)}\Psi_2(r)P_2(\cos\theta). \tag{12}$$

Show that the resulting equations are solved by:

$${}^{(2)}\Psi_0(r) = -\left(1+\frac{\mathcal{M}}{2r}\right)^{-5}\frac{\mathcal{M}}{5r}\left(5\left(\frac{\mathcal{M}}{2r}\right)^3 + 4\left(\frac{\mathcal{M}}{2r}\right)^4 + \left(\frac{\mathcal{M}}{2r}\right)^5\right), \tag{13}$$

and

$${}^{(2)}\Psi_2(r) = -\frac{1}{10}\left(1+\frac{\mathcal{M}}{2r}\right)^{-5}\left(\frac{\mathcal{M}}{r}\right)^3. \tag{14}$$

Remarks: The interpretation of the the data $(\hat{A}^{ij}, \Psi)$ is that of a rotating black hole. A similar approach can also be used to construct a boosted black hole. A natural question to ask is whether following the Bowen-York solution scheme can lead us to initial data describing a spatial slice of Kerr. Unfortunately this turns out not to be the case (see also Remark 8.2.2 of the lecture notes).