NR

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3+1 GEOMETRY

Spacetime 4D

(M, g, ∇)

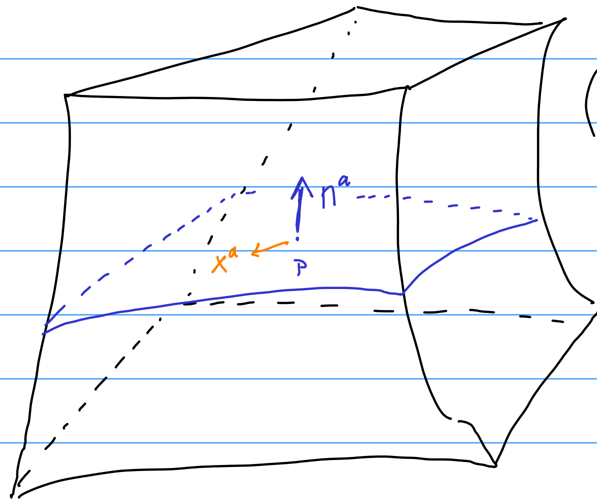
↳ ^{4}Riemann

${}^4R_{\alpha\beta\gamma\delta}$

Hypersurfaces 3D

(Σ_t, γ, D)

X^a : spacelike $T_p(\Sigma_t)$



SCALAR FIELD $t : M \to \mathbb{R}$: $(dt)^a = g^{ab}(dt)_b$ is TIMELIKE

↳ SPATIAL HYPERSURFACES (3D) : NORMAL VECTOR

$$n^a := -\alpha\,(dt)^a$$

$$\alpha := \left(-(dt)^a(dt)_a\right)^{-1/2} \qquad \text{LAPSE FUNCTION}$$

MATH BOX

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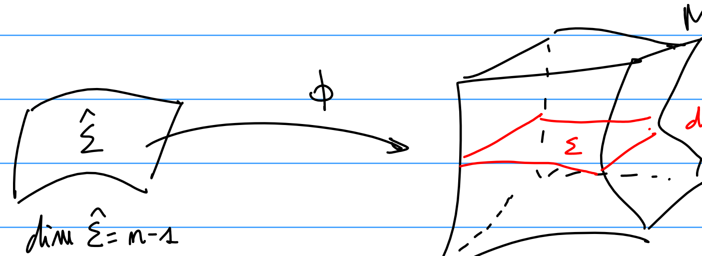
Embedding

M "AMBIENT MANIFOLD"



dim Σ = n-1

dim $\hat{\Sigma}$ = n-1

dim M = n

1-to-1 map

"Move tensor fields" from/to $\hat{\Sigma}$ to/from M : pullback / push forward

Identify Σ with $\hat{\Sigma}$.

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↳ A metric γ on Σ is given by the pullback of g on M

OR

The ambient manifold induces a metric on Σ.

$\hookrightarrow$ INDUCED METRIC $\quad \gamma := \phi^* g$

$$\gamma_{ij} = \frac{\partial x^\alpha}{\partial x^i}\frac{\partial x^\beta}{\partial x^j} g_{\alpha\beta} \quad \text{in some coords.}$$

$\hookrightarrow$ CONNECTION $D \longrightarrow$ Riemann tensor R

components: R_{ijkh}

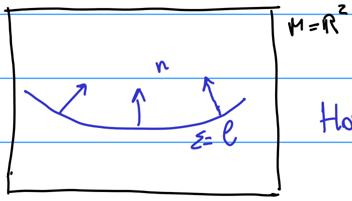
$R_{ijkh} \sim$ "internal" / intrinsic curvature

There is " another curvature " ...

Q: How Σ bends / deforms into M ?

A: The EXTRINSIC CURVATURE measures this.

Example:



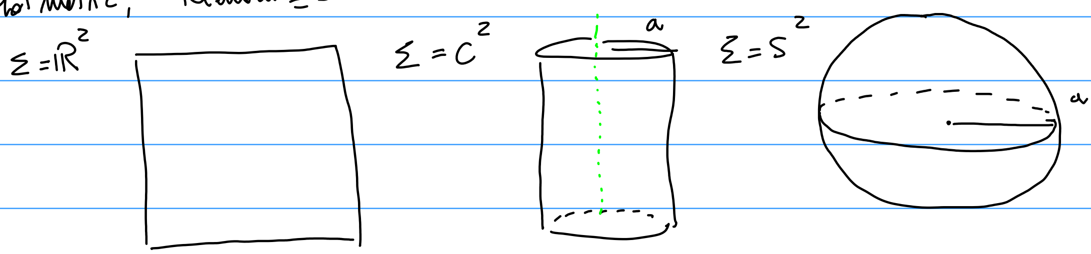
How does n change if transported along ℓ ?

DEF:

$$\boxed{\begin{gathered} K : T_p(\Sigma) \times T_p(\Sigma) \longrightarrow \mathbb{R} \\ u \in T_p(\Sigma) \quad v \in T_p(\Sigma) \\ K(u,v) = K_{ab}\, u^a v^b := -u_a v^b \nabla_b n^a \end{gathered}} \quad (K1)$$

Examples

$M = \mathbb{R}^3$, flat metric, ${}^4\text{Riemann} \equiv 0$



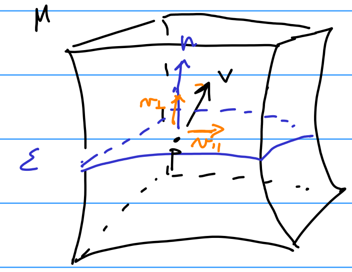
$R_{ijkh} \equiv 0$ $\qquad R_{ijkh} \equiv 0$ $\qquad R_{ijkh} \neq 0$

$K_{ij} \equiv 0$ $\qquad K_{\varphi\varphi} = -a$ $\qquad K_{ij} \neq 0$

$\qquad K = K^a{}_a = -\frac{1}{a}$ $\qquad K = -\frac{2}{a}$

PROJECTOR

→ To decompose tensors on M into tensors on Σ and "pieces" along n



$$T_p(M) = V_p(n) \oplus T_p(\Sigma)$$

$$v^a = v_\perp n^a + v_\parallel{}^a$$

$$P : T_p(M) \longrightarrow T_p(\Sigma)$$

$$v^a \longrightarrow v_\parallel{}^a$$

$$\boxed{P^a{}_b := \delta^a{}_b + n^a n_b} \qquad (P)$$

$$P^a{}_b v^b = \left(\delta^a{}_b + n^a n_b\right)\left(v_\perp n^b + v_\parallel^b\right) =$$

$$= v_\perp \cancel{n^a} - \cancel{n^a v_\perp} + v^a_\parallel - n^a \underbrace{n_b v^b_\parallel}_{=0} = v^a_\parallel$$

The induced metric is given by:

$$\boxed{\gamma_{ab} = P_a{}^c P_b{}^d g_{cd} = \dots = g_{ab} + n_a n_b}$$

→ $P^a{}_b = \gamma^a{}_b = g^{ac}\gamma_{cb}$ ⇝ (P)

From now on: $\gamma^a{}_b$ is the projector.

Covariant derivatives:

$$\text{"} \; DT = P \dots P \, \nabla T \; \text{"}$$

Extrinsic curvature:

$$\boxed{K_{ab} = -\gamma_a{}^c \gamma^d{}_b \nabla_{(c} n_{d)}} \qquad (K2)$$

→ Using P (or $\gamma_a{}^b$) every tensor can be expressed in its "3+1 form".

EULERIAN OBS : observers associated to n (worldlines defined by n)

$\hookrightarrow$ Σ_t is composed of all events simultaneous to the E.OBS.

DEF : Acceleration of E.OBS : $\boxed{a_a := n^b \nabla_b n_a}$ (a)

$(a_e n^a = 0)$

DEF : NORMAL EVOLUTION VECTOR : $m^a := \alpha n^a$

Properties :

- $m^2 = -\alpha^2$ $\quad (n^2 = -1)$
- $\nabla_m t = m^a (dt)_a = +1$

INTERPRETATION OF 3+1 GEOMETRY

" 3+1 geom ~ Kinematics of 3+1 GR "

(i) The vector m carries pts from Σ_t to $\Sigma_{t+\delta t}$;

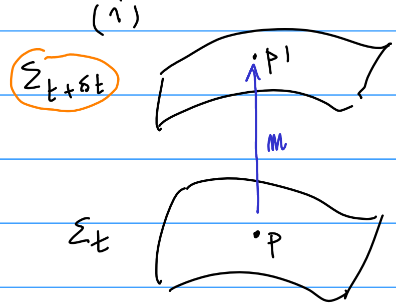
(ii) The lapse function α relates t to the proper time of E.OBS ;

(iii) The Lie derivative along m $\mathcal{L}_m$ transports tensors from Σ_t to $\Sigma_{t+\delta t}$;

(iv) $$\boxed{\mathcal{L}_m \gamma = -2\alpha K} \quad (K3)$$

Some hints :

(i) $t(p') = t(p + \delta t \cdot m) = t(p) + \delta t \underbrace{m^a (dt)_a}_{=+1} = t(p) + \delta t$



(ii) $d\tau^2 = -g(\delta t\, m, \delta t\, m) = -\underbrace{m^a}_{=\alpha n^a} m_a\, dt^2 = \alpha^2 dt^2$

$\hookrightarrow$ Lapse (wheeler)

(iii) ... follows from (i) and the def. of the Lie derivative ...

(iv) $\mathcal{L}_n \gamma_{ab} = \mathcal{L}_n (g_{ab} + n_a n_b) =$ ← Lie deriv in terms of cov. der.

$$= 2\nabla_{(a} n_{b)} + n_a \mathcal{L}_n n_b + n_b \mathcal{L}_n n_a =$$ ← $\mathcal{L}_n n_a = a_a$

$$= 2\left[\nabla_{(a} n_{b)} + n_{(a} a_{b)}\right] =$$

left for you! ← $\overset{?}{=} -2K_{ab}$

Hints:
- start from (K2), do the projections
- terms ~ $n n \nabla n$ ⇒ use (a)

Property: $\mathcal{L}_n \gamma_{ab} = \phi^{-1} \mathcal{L}_{\phi n} \gamma_{ab}$ $\forall\ \phi$ SCALAR

└ Follows from $\mathcal{L}$ def. and $\gamma_{ab} n^b = 0$

↳ (K3)

Note: 3 expressions for K_{ab} (K1), (K2), (K3)

Each can be taken as def. (and other derived)

Meaning of (i) - (iv):

- Σ_t and $\Sigma_{t+\delta t}$ are identified by diffeomorphism generated by m

↓

- (M, g) spacetime is the "time" development of (Σ, γ)

where "time evolution" is governed by $\mathcal{L}_m$

↓

- can identify: γ as the "main variable" of (3+1) GR

K as the "velocity" of γ

(K3) ⇒ "time derivative" ~ "velocity"

↳ KINEMATICAL RELATION ($\mathcal{L}_m \gamma \sim K$)