NR

04.05.2021

2

3+1 DECOMPOSITION OF GR

RECAP: 3+1 "SPLIT" $(M, g, \nabla) \longrightarrow (\Sigma, \gamma, D, K)$

^{4}Riemann Riemann

GAUSS - CODAZZI - RICCI IDENTITIES

Projections of ^{4}Riemann (and projections' contractions)

(GCR)

(G) · GAUSS EQS: $\gamma^p{}_a \gamma^q{}_b \gamma^r{}_c \gamma^s{}_d \, {}^4R_{pqrs} = R_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc}$

(C) · CODAZZI EQS: $\gamma^p{}_a \gamma^q{}_b \gamma^r{}_c n^s \, {}^4R_{pqrs} = D_a K_{bc} - D_b K_{ac}$

(R) · RICCI EQS: $\gamma_{ap} n^r \gamma^q{}_b n^s \, {}^4R^p{}_{rps} =$

$= \alpha^{-1} \mathcal{L}_m K_{ab} + \alpha^{-1} D_a D_b \alpha + K_{as} K^s{}_b$

SEE: CHAP. 4 Lecture Notes Eq. (35, 36, 37)

3+1 GR = Project EFE and use (GCR) to write

$$\boxed{{}^4G_{ab} = 8\pi \, T_{ab}} \qquad \text{(EFE)}$$

in terms of: (γ, K) with D and $\mathcal{L}_m$

$\hookrightarrow$ CONSTRAINTS $(G_{ab} - 8\pi T_{ab}) n^b = 0$

EVOLUTION EQS

$\mathcal{L}_m \gamma_{ab} = -2\alpha K_{ab}$ "KINEMATICAL EQ"

$\mathcal{L}_m K_{ab} = \dots$ "DYNAMICAL EQ"

HAMILTONIAN CONSTRAINT

$$ {}^4G_{ab}\, n^a n^b = {}^4R_{ab}\, n^a n^b - \frac{1}{2}\, {}^4R\, g_{ab} n^a n^b = \underbrace{{}^4R_{ab} n^a n^b + \frac{1}{2}\, {}^4R}_{\frac{1}{2} \times \text{L.H.S. contracted (G)}} = $$

$$ -g_{ab} n^a n^b = -\underbrace{n^a n^b \gamma_{ab}}_{=0} \ldots = -n^a n^b (\gamma_{ab} - n_a n_b) = +1 $$

$$ = \frac{1}{2} \left(R + K^2 - K_{ab} K^{ab} \right) $$

$T_{ab} n^a n^b =: E$ energy density measured by Eulerian obs.

$$ \boxed{C_0 := R + K^2 - K_{ab} K^{ab} - 16\pi E = 0} \qquad \text{(ADMY-H)} $$

- Scalar eq
- Defined on Σ
- Contains only spatial quantities & spatial derivatives.

MOMENTUM CONSTRAINT

$$ {}^4G_{pq}\, n^q \gamma^p{}_a = {}^4R_{qp}\, \gamma^p{}_a n^q - \frac{1}{2}\, {}^4R\, g_{qp} \gamma^p{}_a n^q = \underbrace{{}^4R_{qp} \gamma^p{}_a n^q}_{\text{L.H.S. OF (C) CONTRACTED EQ}} = $$

$$ g_{qp} \gamma^p{}_a n^q = (\gamma_{qp} - \overbrace{n_q n_p)\gamma^p{}_a}^{=0} n^q = 0, \qquad \underbrace{\gamma_{qp}\gamma^p{}_a n^q}_{=0} $$

$$ = D_a K - D_b K^b{}_a $$

X corrected minus sign

$T_{pq} \gamma^p{}_a n^q =: -P_a$ momentum density measured by Eulerian obs.

$$ \boxed{C_a := D_b K^b{}_a - D_a K - 8\pi P_a = 0} \qquad \text{(ADH-M)} $$

- 1 tensor eq (rank 1) ("3 components") defined on Σ

DYNAMICAL EV. EQ

Trace reverse EFE : $\quad {}^4R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2} T g_{ab}\right)$

${}^4R_{pq}\, \gamma^p{}_a \gamma^q{}_b \quad = \quad$ L.H.S. OF CONTRACTED (R) EQ. COMBINED WITH CONTRACTE (G)

× corrected minus sign

$T_{pq}\, \gamma^p{}_a \gamma^q{}_b = + S_{ab}$ stress tensor measured by Eulerian obs
(purely spatial tensor)

$S := g^{ab} S_{ab}$: trace

Can verify : $\boxed{T_{ab} = S_{ab} + 2\, n_{(a} P_{b)} + n_a n_b E}$

Trace : $T = g^{ab} T_{ab} = S - E$

(ADM-K) $$\boxed{\mathcal{L}_m K_{ab} = -D_a D_b \alpha + \alpha \left\{ R_{ab} + K K_{ab} - 2 K_{ac} K^c{}_b + 4\pi\left[(S-E)\gamma_{ab} - 2 S_{ab} \right]\right\}}$$

- Contains $\mathcal{L}_m$
- 1 tensor eq (rank 2, symmetric) ("6 component")

ADMY

Constraints (4)

Σ

$$\begin{cases} C_0 = 0 & \text{(ADM-H)} \\ C_a = 0 & \text{(ADM-M)} \\ \mathcal{L}_m \gamma_{ab} = -2\alpha K_{ab} & \text{(ADM-}\gamma\text{)} \quad \text{"kinematical eq"} \\ \mathcal{L}_m K_{ab} = \dots & \text{(ADM-K)} \quad \text{"Dynamical eq"} \end{cases}$$

Dynamical eq (6+6)

- Involve (γ, K) as fundamental vars
- —〃— $\mathcal{L}_m$, D, DD
- —〃— α, β^a but no Eqs are available for lapse, shift !

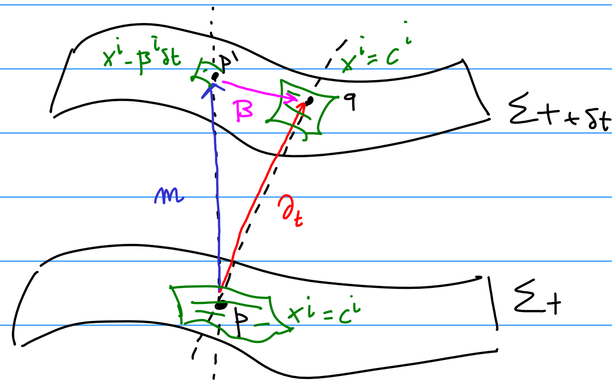
↳ α, β^a reflects "gauge freedom" for

- foliation choice
- 3 spatial coordinates on Σ

↳ MUST BE PRESCRIBED.

ADAPTED COORDINATES

$$x^\mu = (t, x^i) \Rightarrow \text{"natural basis" } T_p(M) : e_\mu = \partial_\mu = (\partial_t, \partial_i)$$



In general:

$$(\partial_t)^a = n^a + \beta^a$$

↳ timelike vector

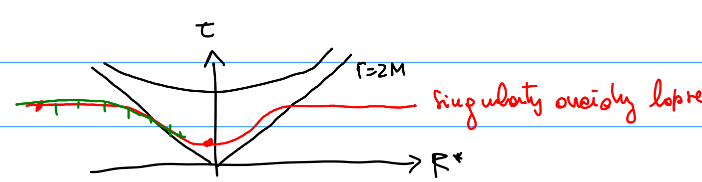
↳ spatial vector : $\beta^a n_a = 0$

$$(\partial_t)^2 = \underset{\substack{\parallel \\ (\alpha n)^2}}{n^2} + \beta^2 = -\alpha^2 + \beta^2$$

∂_t	if :
timelike	$\beta^2 < \alpha^2$
null	$\beta^2 = \alpha^2$
spacelike	$\beta^2 > \alpha^2$

Possibility of superluminal shift.

↳ Example: BH evolution



Specify to adapted coords:

$$(\partial_t)^a = (1,0,0,0)$$
$$\beta^a = (0,\beta^i)$$

$$n^a = \alpha^{-1}\left[(\partial_t)^a - \beta^a\right] = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right)$$

$$n_a = (-\alpha,0,0,0)$$

Metric:

$$g_{00} = g(\partial_t,\partial_t) = -\alpha^2 - \beta^2$$

[× tensor notation $= g_{ab}(\partial_t)^a(\partial_t)^b$]

$$g_{0i} = g(\partial_t,\partial_i) = \beta_i$$

$$g_{ij} = g(\partial_i,\partial_j) = \gamma_{ij}$$

$$\boxed{ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)}$$

Determinants:

$$g := \det g_{\mu\nu} \qquad \gamma := \det \gamma_{ij}$$

Trace: $\left(K = g^{ab}K_{ab} = K^a{}_a\right)$

$$\left.\begin{aligned} g^{00} &= -\alpha^{-2} \\ &\underset{\substack{\uparrow \\ \text{Cramer rule}}}{=} \frac{1}{\det g_{\mu\nu}}\det g_{ij} = \frac{1}{\det g_{\mu\nu}}\det \gamma_{ij}\end{aligned}\right\} \Rightarrow \boxed{\sqrt{-g} = \alpha\sqrt{\gamma}}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{0i} & g_{ij}\end{pmatrix} = \begin{pmatrix} -\alpha^2+\beta^2 & \beta_i \\ \beta_i & \gamma_{ij}\end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{0i} & g^{ij}\end{pmatrix} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^i \\ \alpha^{-2}\beta^i & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2}\end{pmatrix}$$

Mind: $g_{ij} = \gamma_{ij}$ BUT $g^{ij} \neq \gamma^{ij}$

Lie derivatives:

$$\mathcal{L}_m = \mathcal{L}_{\partial_t} - \mathcal{L}_\beta = \partial_t - \mathcal{L}_\beta$$

$$\mathcal{L}_\beta \gamma_{ij} = \beta^k \underbrace{D_k\gamma_{ij}}_{=0} + 2D_{(i}\beta_{j)} = 2\,\partial_{(i}\beta_{j)} - 2\Gamma^k_{ij}\beta_k$$

$$\mathcal{L}_\beta K_{ij} = \beta^k \partial_k K_{ij} + K_{ik}\partial_j \beta^k + K_{jk}\partial_i \beta^k$$

Now we can write (ADMY) as PDEs !

ADMY in GEODESIC GAUGE

Geod. gauge $\begin{cases} \alpha \equiv 1 & t = \tau \text{ proper time of Eulerian obs} \\ \beta^i \equiv 0 & (\partial_t = n) \end{cases}$

Recall: acceleration of Eulerian obs : $a_a = n^b \nabla_b n_a = D_a \ln \alpha \overset{\text{geod. gauge}}{=} 0$

↳ WORLDLINES OF EULERIAN OBS ARE GEODESICS.

ADMY in Geo. Gauge:

$$(\text{ADMY, GG}) \begin{cases} \boxed{\begin{aligned} \partial_t \gamma_{ij} &= -2K_{ij} \\ \partial_t K_{ij} &= R_{ij} + KK_{ij} - 2K_{ik}K_j{}^k + (\text{matter terms}) \end{aligned}} \\ C_0 = R + K^2 - K_{ij}K^{ij} - 16\pi E = 0 \\ C_i = D_j K^j{}_i - D_i K - 8\pi P_i = 0 \end{cases}$$

Q: What type of PDEs ?

A: EVOLUTION EQ $\sim$ first-order in time

second-order in space wave-like eq.

Linearize : $\gamma_{ij} \approx f_{ij} + h_{ij}$ (flat metric + perturbation)

$$\longrightarrow \begin{cases} \partial_t h_{ij} = -2K_{ij} \\ \partial_t K_{ij} \simeq R_{ij} \simeq -\frac{1}{2}\partial_k \partial^k \gamma_{ij} + \dots \end{cases}$$

symmetric 2-tensor version of scalar wave eq :

$$\Box \phi = -\partial_{tt}\phi + \partial_k \partial^k \phi = 0 \quad \longrightarrow \quad \begin{cases} \partial_t \phi = \Pi \\ \partial_t \Pi = \partial_k \partial^k \phi \end{cases}$$

Can re-write (ADMY, G4) as "wave eq" for γ_{ij}
(substitute back $K_{ij} = -\partial_t \gamma_{ij} = -\dot{\gamma}_{ij}$) :

$$\boxed{-\ddot{\gamma}_{ij} + \underbrace{\gamma^{kl}\left(\partial_k \partial_l \gamma_{ij} + \partial_i \partial_j \gamma_{kl} - \partial_i \partial_k \gamma_{jl} - \partial_j \partial_k \gamma_{il}\right)}_{\text{PRINCIPAL PART (highest derivatives)}} \simeq 0}$$

Must consider : $\gamma^{kl} = \gamma^{kl}(\gamma_{ij})$ ~ rational polynomial of γ_{ij}

- 2^{nd} time derivative, 2^{nd} spatial derivatives
- QUASILINEAR = linear in the highest derivative (P.P.)
- quadratic in the first spatial metric derivatives (not written above)

$\Big\}$ as PDEs

NOTE : TWO WRONG MINUS SIGN CORRECTED !