Numerical relativity — Exercise sheet # 3

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15.05.2021

Maximal slicing

Exercise 1.1: Maximal Slicing of Schwarzschild

The 3+1 decomposition can be performed with different choices for the slicing of spacetime. The choice of time slicing amounts to making a choice for the lapse α by either prescribing a function or an equation. Since α can vary depending on the position on the spatial slice, the proper time is allowed to advance at different rates at different points on a given slice. A common choice is the *maximal slicing* condition:

$$K = 0. \tag{1}$$

If we want maximal slicing to hold at all times, then we should impose $\partial_t K = 0$ as well. In this case, the evolution equation for the extrinsic curvature scalar reduces to an elliptic equation for the lapse function

$$D_i D^i \alpha = \alpha \left(K_{ij} K^{ij} \right). \tag{2}$$

Step 1: In Schwarzschild coordinates, consider $\bar{t} = t + h(r)$ and enforce $K = 0$. Find the family of time independent slicings.

Step 2: From the general result in *Step 1*, recover the specific solutions for Schwarzschild and isotropic metric.

Step 3: Draw the Kruskal-Szekeres diagrams for the Schwarzschild and the isotropic slicings.

Step 4: Draw the embedding diagram.

Exercise 1.2: Extremum of K

In the lecture notes it was demonstrated (see §6.6) that imposing the maximal slicing condition of Eq.(1) extremises volume. Argue that in the case of a Lorentzian manifold the extremum constitutes a maximum.

Approximations

Exercise 1.1: Newtonian limit

Recall that if the gravitational field is weak and static then it is always possible to find a coordinate system $(x^\alpha) = (x^0 = t,\, x^i)$ such that the metric components take the form:

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)f_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{3}$$

where Φ is the gravitational potential and $|\Phi| \ll 1$.

In performing $3+1$ and conformal decompositions, we have found that the mean curvature evolves according to:

$$\mathcal{L}_m[K] = -D_iD^i[\alpha] + \alpha(4\pi(E+S) + K_{ij}K^{ij}), \tag{4}$$

Working within the regime of validity for Eq.(3) show that Eq.(4) reduces to the Poisson equation:

$$\mathcal{D}_i\mathcal{D}^i[\Phi] = 4\pi\rho, \tag{5}$$

where $\mathcal{D}$ is the derivative operator associated with the flat metric f_{ij}.