question on intrinsic and extrinsic geometry

Consider hypersurfaces embedded in the Euclidean $\mathbb{R}^3$ manifold.

Compute the intrinsic and extrinsic curvature of:
(**i**) an embedded 2^{d} plane
(**ii**) an (infinite) cylinder

solution

We recall some definitions and make a comment:

A **hypersurface** of $(\mathcal{M},\, n+1)$ is the image Σ of an n-dimensional manifold $\hat{\Sigma}$ by an embedding $\Phi : \hat{\Sigma} \to \mathcal{M}$: $\Sigma = \Phi(\hat{\Sigma})$.

An **embedding** means that the map $\Phi : \hat{\Sigma} \to \Sigma$ is a homeomorphism. Geometrically this avoids self-intersection.

A hypersurface can be defined **locally** as the set of points for which a scalar field on $\mathcal{M}$ is constant. Denote this field by σ and set the constant to zero, we get:

$$\forall p \in \mathcal{M}, \qquad p \in \Sigma \Longleftrightarrow \sigma(p) = 0$$

If $\Phi : \hat{\Sigma} \to \mathcal{M}$ is the embedding which induces the submanifold structure of $\mathcal{M}$, the pullback map Φ^* **induces** the natural metric $\gamma = \Phi^* g$ on Σ.
In components we have:

$$\gamma_{\mathrm{IJ}}(x) = g_{kl}(\Phi(x)) \frac{\partial \Phi^k}{\partial x^{\mathrm{I}}} \frac{\partial \Phi^l}{\partial x^{\mathrm{J}}}, \qquad \text{(eq:pullbackmetric)}$$

where (Φ^k) denote the coordinates of $\Phi(x)$.

Given a Manifold, metric and associated Levi-Civita connection $(\mathcal{M},\, g,\, \nabla)$, the intrinsic curvature is usually introduced as the anti-commutator: $[\nabla_a,\, \nabla_b]\omega_c = \mathrm{Riem}_{abc}{}^d \omega_d$. We may expand the definition of the covariant derivative to provide a method for computation:

$$\mathrm{Riem}_{abc}{}^d[g] = -2\partial_{[a}\Gamma^d{}_{b]c} + 2\Gamma^e{}_{c[a}\Gamma^d{}_{b]e}; \qquad \text{(eq:Riem)}$$

with Christoffel symbols given by:

$$\Gamma^c{}_{ab}[g] = \frac{1}{2} g^{cd}(\partial_a[g_{bd}] + \partial_b[g_{ad}] - \partial_d[g_{ab}]). \qquad \text{(eq:Christoffel)}$$

The extrinsic curvature of Σ may be computed with the projector and unit normal n^a to Σ:

$$K_{ab} := -\gamma_a^c \gamma_b^d \nabla_{(c}[n_{d)}] = -\gamma_a^c \gamma_b^d \nabla_c[n_d], \qquad \text{(eq:Extrinsic)}$$

this may also be written via direct evaluation over appropriate basis vectors:

$$K_{\mathrm{IJ}} = \mathbf{K}(\boldsymbol{\partial}_{\mathrm{I}},\, \boldsymbol{\partial}_{\mathrm{J}}) = -\nabla_i[n_j](\partial_{\mathrm{I}})^i(\partial_{\mathrm{J}})^j. \qquad \text{(eq:Extrinsic2)}$$

We need to construct: Embedding map, induced metric, unit normal, intrinsic curvature Riem (Wald convention), extrinsic curvature K (negative convention).

(i)

Let $\hat{\Pi}$ be a connected submanifold of $\mathcal{M} := \mathbb{R}^3$ with topology $\mathbb{R}^2$, i.e., the 2-d plane. Introduce locally (In this case they also hold globally) a set of coordinates for $\mathcal{M}$, $(x^i) = (x, y, z)$, such that σ spans $\mathbb{R}$ and (x, y) are Cartesian coordinates spanning $\mathbb{R}^2$. The hypersurface Π is then defined by the coordinate condition $\sigma = 0$. An explicit form of the mapping Φ can be obtained by considering $(x^{\mathrm{I}}) = (x, y)$ as coordinates on $\hat{\Pi}$:

$$\begin{aligned} \Phi :=& \hat{\Pi} \to \mathcal{M}, \\ (x, y) \mapsto& (x, y, 0). \end{aligned}$$

In short, we view the hypersurface Π as the $z = 0$ plane. The scalar function σ defining Π is thus $\sigma = z$.

In order to find the induced metric we make use of Eq.(eq:pullbackmetric) and note that the Euclidean metric on $\mathcal{M}$ is simply δ_{ij}. Clearly we must have $\gamma_{\mathrm{IJ}} = \mathrm{diag}(1,\, 1)$.

Furthermore, it follows trivially from Eq.(eq:Riem) and Eq.(eq:Christoffel) that Riem vanishes (note $\Gamma[\delta]=0$).

In order to compute the extrinsic curvature we select a (global) unit normal to Π, namely, $n^i=(0,\,0,\,1)$. For the present case we have $n_i\delta^{ij}=n^j$. Due to this and $\Gamma[\delta]=0$ we have $\nabla_a[n_b]=\partial_a[n_b]=(0,\,0,\,0)$. Based on Eq.(eq:Extrinsic) we immediately conclude that the extrinsic curvature vanishes as well.

Geometric intuition: The unit vector stays constant when translated along Π and hence the extrinsic curvature (measuring how the hypersurface bends in the ambient space) is zero. Furthermore, on a plane, the internal angles of a triangle must sum to π - the intrinsic curvature vanishes verifies this fact.

(ii)

For the hypersurface describing our (infinite) cylinder $\mathcal{C}$ we take the local definition $\sigma:=\rho-a=0$, where $\rho:=\sqrt{x^2+y^2}$ and $a\in\mathbb{R}^{>0}$ is the radius of the cylinder. We introduce the cylindrical coordinates $(x^i)=(\rho,\,\varphi,\,z)$, such that $\varphi\in[0,\,2\pi)$, $x=\rho\cos\varphi$ and $y=\rho\sin\varphi$. On $\mathcal{C}$, $(x^{\mathrm{I}})=(\varphi,\,z)$ constitute a coordinate system.

Thus, we take:

$$\Phi:(\varphi,\,z)\mapsto(\rho\cos\varphi,\,\rho\sin\varphi,\,z).$$

From Eq.(eq:pullbackmetric) we can compute the induced metric:

$$\begin{aligned}\boldsymbol{\gamma}=\gamma_{\mathrm{IJ}}\mathrm{d}x^{\mathrm{I}}\otimes\mathrm{d}x^{\mathrm{J}}=&\delta_{ij}\frac{\partial\Phi^i}{\partial x^{\mathrm{I}}}\frac{\partial\Phi^j}{\partial x^{\mathrm{J}}}\mathrm{d}x^{\mathrm{I}}\otimes\mathrm{d}x^{\mathrm{J}}\\ =&a^2\mathrm{d}\varphi\otimes\mathrm{d}\varphi+\mathrm{d}z\otimes\mathrm{d}z;\end{aligned}$$

Thus $\gamma_{\mathrm{IJ}}=\mathrm{diag}(a^2,\,1)$.

Notice that a is constant. We may therefore introduce another coordinate $\eta:=a\varphi$ and further map the induced metric via coordinate transformation to:

$$\boldsymbol{\gamma}=\mathrm{d}\eta\otimes\mathrm{d}\eta+\mathrm{d}z\otimes\mathrm{d}z.$$

This is in the form of the Euclidean metric and therefore we once again conclude that the intrinsic curvature Riem is zero.

In order to compute extrinsic curvature we need a unit normal. Notice that on $\mathcal{C}$ for fixed φ, translation along the z-direction should leave the unit normal unaffected. We make the out-ward pointing choice:

$$n^i=\left(\frac{x}{a},\,\frac{y}{a},\,0\right)\Longrightarrow n_i=\delta_{ij}n^j=\left(\frac{x}{a},\,\frac{y}{a},\,0\right),$$

with a constant. It follows then that:

$$\nabla_i[n_j]=\partial_i[n_j]=\mathrm{diag}(1/a,\,1/a,\,0);$$

as $\Gamma[\gamma]=0$.

Consider Eq.(eq:Extrinsic2). We have the natural basis $(\boldsymbol{\partial}_{\mathrm{I}})=(\boldsymbol{\partial}_\varphi,\,\boldsymbol{\partial}_z)$ associated with the coordinates $(\varphi,\,z)$. The components $(\partial_{\mathrm{I}})^i$ are of the vector $\boldsymbol{\partial}_{\mathrm{I}}$ we take with respect to the natural basis $(\boldsymbol{\partial})_i=(\boldsymbol{\partial}_x,\,\boldsymbol{\partial}_y,\,\boldsymbol{\partial}_z)$ in Cartesian coordinates. As $\boldsymbol{\partial}_\varphi=-y\boldsymbol{\partial}_x+x\boldsymbol{\partial}_y$, one has $(\partial_\varphi)^i=(-y,\,x,\,0)$ and $(\partial_z)^i=(0,\,0,\,1)$.

We find:

$$\begin{aligned}\mathbf{K}(\boldsymbol{\partial}_{\mathrm{I}},\,\boldsymbol{\partial}_{\mathrm{J}})\mathrm{d}x^{\mathrm{I}}\otimes\mathrm{d}x^{\mathrm{J}}=&-\left(\frac{1}{a}(\boldsymbol{\partial}_I)^1(\boldsymbol{\partial}_J)^1+\frac{1}{a}(\boldsymbol{\partial}_I)^2(\boldsymbol{\partial}_J)^2\right)\mathrm{d}x^{\mathrm{I}}\otimes\mathrm{d}x^{\mathrm{J}},\\ =&-\frac{1}{a}\left[(-y)(-y)+x^2\right]\partial_\varphi\otimes\partial_\varphi,\\ =&-a\partial_\varphi\otimes\partial_\varphi.\end{aligned}$$

The trace of the extrinsic curvature can be found using the relation $\gamma^{\mathrm{IJ}}=\mathrm{diag}(a^{-2},1)$ which results in $K=-a^{-1}$.

other things to try:

- Intrinsic / extrinsic quantities associated with sphere embedded in $\mathbb{R}^3$.
- Given Schwarzschild in isotropic coordinates compute (spatial) 3-metric, extrinsic curvature and its trace. (see Eq.6.11 of lecture notes)