ADM in spherical symmetry

In spherical symmetry we may take the general form of a spatial metric to be:

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \gamma_1(t,\,r)\mathrm{d}r^2 + r^2\gamma_2(t,\,r)\mathrm{d}\Omega^2, \qquad \mathrm{d}\Omega^2 := \mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\varphi^2.$$

More generally, analogous forms may be adopted for any symmetric tensor field $S_{(ij)} = S_{ij} \in \mathcal{T}_2(\Sigma_t)$.

Define the variables:

$$\mathrm{D}_\alpha := \partial_r[\log(\alpha)], \qquad \Gamma_1 := \partial_r[\log(\gamma_1)], \qquad \Gamma_2 := \partial_r[\log(\gamma_2)],$$

it can be shown that when working in vacuum and in the absence of spatial shift the standard ADM evolution equations may be re-written as:

$$\begin{aligned}
\partial_t[\gamma_1] &= -2\alpha\gamma_1\overline{\kappa}_1,\\
\partial_t[\gamma_2] &= -2\alpha\gamma_2\overline{\kappa}_2,\\
\partial_t[\Gamma_1] &= -2\alpha(\overline{\kappa}_1\mathrm{D}_\alpha + \partial_r[\overline{\kappa}_1]),\\
\partial_t[\Gamma_2] &= -2\alpha(\overline{\kappa}_2\mathrm{D}_\alpha + \partial_r[\overline{\kappa}_2]),\\
\partial_t[\overline{\kappa}_1] &= -\frac{\alpha}{\gamma_1}\Bigg[\partial_r[\mathrm{D}_\alpha + \Gamma_2] + \mathrm{D}_\alpha^2 - \frac{\mathrm{D}_\alpha\Gamma_1}{2} + \frac{\Gamma_2^2}{2} - \frac{\Gamma_1\Gamma_2}{2}\\
&\qquad - \gamma_1\overline{\kappa}_1(\overline{\kappa}_1 + 2\overline{\kappa}_2) - \frac{1}{r}(\Gamma_1 - 2\Gamma_2)\Bigg],\\
\partial_t[\overline{\kappa}_2] &= -\frac{\alpha}{2\gamma_1}\Bigg[\partial_r[\Gamma_2] + \mathrm{D}_\alpha\Gamma_2 + \Gamma_2^2 - \frac{\Gamma_1\Gamma_2}{2} - \frac{1}{r}(\Gamma_1 - 2\mathrm{D}_\alpha - 4\Gamma_2)\\
&\qquad - \frac{2}{\gamma_2}\left\{\frac{(\gamma_1-\gamma_2)}{r^2}\right\}\Bigg] + \alpha\overline{\kappa}_2(\overline{\kappa}_1 + 2\overline{\kappa}_2);
\end{aligned}$$

whereas the constraints may be put in the form:

$$\begin{aligned}
\mathcal{H} &:= -\partial_r[\Gamma_2] + \left\{\frac{\gamma_1-\gamma_2}{r^2\gamma_2}\right\} + \gamma_1\overline{\kappa}_2(2\overline{\kappa}_1 + \overline{\kappa}_2)\\
&\qquad + \frac{1}{r}(\Gamma_1 - 3\Gamma_2) + \frac{\Gamma_1\Gamma_2}{2} - \frac{3\Gamma_2^2}{4} = 0,\\
\mathcal{M}_r &:= -\partial_r[\overline{\kappa}_2] + \left\{\frac{\overline{\kappa}_1 - \overline{\kappa}_2}{r}\right\} + \frac{1}{2}(\overline{\kappa}_1 - \overline{\kappa}_2)\Gamma_2 = 0;
\end{aligned}$$

where $\overline{\kappa}_I := \kappa_I/\gamma_I$ (no sum, $I = 1,\,2$).

(I): Derive the above system.
(II): Recall that adapted coordinates can carry a downside of having to treat apparent coordinate singularities by regularizing. In the present case we have formal behaviour in the vicinity of $r=0$:

$$\gamma_I \sim \gamma_I^0 + \mathcal{O}(r^2), \qquad \overline{\kappa}_I \sim \overline{\kappa}_I^0 + \mathcal{O}(r^2);$$

what are the analogous conditions for Γ_I? These yield us parity conditions that we looked at imposing in the last tutorial through a staggered grid.

(III): If we take into account local flatness (at $r=0$) what additional relations must be simultaneously satisfied? Why does this complicate matters?

(IV): In order to implement the full set of conditions found in the previous part it is useful to introduce a new auxiliary variable:

$$\lambda := \frac{1}{r}\left(1 - \frac{\gamma_1}{\gamma_2}\right).$$

- What is the parity condition on λ?
- Use this variable to remove the curled brace term in $\partial_t[\overline{\kappa}_2]$ together with $\mathcal{H}$.
- Derive a *regular* evolution equation for λ. To do this differentiate and make use of the momentum constraint component.

(V): In principle we would like to be able to perform an evolution for some test problem. We could pick (e.g.) Bona-Masso slicing:

$$\partial_t[\alpha] = -\alpha^2 f(\alpha)\mathcal{K} = -\alpha^2 f(\alpha)[\overline{\kappa}_1 + 2\overline{\kappa}_2].$$

Derive an equation for $\partial_t[\mathrm{D}_\alpha]$.

(VI): An immediate question arises as to whether there are any obvious restrictions on $f(\alpha)$. To answer this consider putting $u := (\alpha,\, \gamma_1,\, \gamma_1,\, \lambda)$ together with $v = (\mathrm{D}_\alpha,\, \Gamma_1,\, \Gamma_2,\, \overline{\kappa}_1,\, \overline{\kappa}_2)$ such that we may generically write our system as:

$$\begin{aligned}\partial_t[u_i] &= q_i(u,\, v),\\ \partial_t[v_i] &= M_i^j(u)\partial_r[v_j] + p_i(u,\, v);\end{aligned}$$

where q and p are source terms. Investigate the characteristic structure of M_i^j writing down the eigenfields and eigenvalues.

(VII): (Optional) verify that the above steps of regularization significantly mitigate stability issues encountered in the example code of the last tutorial.

(VIII): (Optional) recall that a spatial slice of Schwarzschild may be written in isotropic form as:

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \psi^4(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2), \qquad \psi = 1 + M/(2r).$$

In the static puncture evolution technique one extracts analytically the conformal factor through field re-definitions:

$$\begin{aligned}\tilde{\gamma}_1 &:= \gamma_1/\psi^4, & \tilde{\gamma}_2 &:= \gamma_2/\psi^4;\\ \tilde{\Gamma}_1 &:= \Gamma_1 - 4\partial_r[\log(\psi)], & \tilde{\Gamma}_2 &:= \Gamma_2 - 4\partial_r[\log(\psi)];\end{aligned}$$

with other variables left as previously regularized. Modify your regularized example code to perform an evolution for the case that $f = 2/\alpha$ and comment on what you observe.

(soln)

(I):

See notebook.

(II):

Collectively, we have behaviour for $r \to 0$ as:

$$\gamma_I \sim \gamma_I^0 + \mathcal{O}(r^2), \qquad \Gamma_I \sim \mathcal{O}(r), \qquad \overline{\kappa}_I \sim \overline{\kappa}_I^0 + \mathcal{O}(r^2);$$

Thus γ_I and $\overline{\kappa}_I$ are *even* about the origin whereas Γ_I are *odd*.

(III):

The main reason matters are complicated is that we want to simultaneously impose more conditions than we have easy access to do.

As the fields $\mathrm{D}_\alpha \sim \mathcal{O}(r)$ and $\Gamma_I \sim \mathcal{O}(r)$ near the origin terms such as D_α/r and Γ_I/r should not cause problems.

On the other hand curled braces terms such as $(\gamma_1 - \gamma_2)/r^2$ and $(\overline{\kappa}_1 - \overline{\kappa}_2)/r$ require differences of terms to precisely follow:

$$\gamma_1 - \gamma_2 \sim \mathcal{O}(r^2), \qquad \overline{\kappa}_1 - \overline{\kappa}_2 \sim \mathcal{O}(r^2).$$

In other words we must simultaneously also have that (compare behaviour in prior part):

$$\gamma_1^0 = \gamma_2^0,$$

and this must hold for all t, thus $\overline{\kappa}_1^0 = \overline{\kappa}_2^0$ is also required.

Recall that for local flatness at $r = 0$ we noticed that it must be possible to locally write (with $R := R(r)$):

$$
\begin{aligned}
\mathrm{d}s^2\big|_{R\sim 0} &= \mathrm{d}R^2 + R^2\mathrm{d}\Omega^2,\\
\implies \mathrm{d}s^2\big|_{r\sim 0} &= \left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^2\Bigg|_{r=0}\left[\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right].
\end{aligned}
$$

or relating this to our initial assumption on γ we have (at least analytically) that $\gamma_1^0 = \gamma_2^0$ and similarily $\overline{\kappa}_1^0 = \overline{\kappa}_2^0$.

(IV):

To impose the additional conditions it is suggested to introduce the auxiliary field λ.

It can be seen that λ is of *odd* parity. To see this one can write:

$$
\lambda \sim \frac{1}{r}\left(1 - \frac{\gamma_1^0 + \gamma_1^2 r^2 + \mathcal{O}(r^4)}{\gamma_1^0 + \gamma_2^2 r^2 + \mathcal{O}(r^4)}\right),
$$

where notice we have taken $\gamma_1^0 = \gamma_2^0$ and peform Taylor expansion in r about $r=0$:

$$
\lambda \sim \frac{r}{\gamma_1^0}(\gamma_2^2 - \gamma_1^2) + \mathcal{O}(r^3),
$$

and so this and terms of the form λ/r should be safe to evaluate.

Indeed we may rewrite the equation of $\partial_t[\overline{\kappa}_2]$ and the Hamiltonian constraint immediately in terms of this variable as:

$$
\begin{aligned}
\partial_t[\overline{\kappa}_2] = -\frac{\alpha}{2\gamma_1}\Bigg[&\partial_r[\Gamma_2] + \mathrm{D}_\alpha\Gamma_2 + \Gamma_2^2 - \frac{\Gamma_1\Gamma_2}{2} - \frac{1}{r}(\Gamma_1 - 2\mathrm{D}_\alpha - 4\Gamma_2)\\
&+ \frac{2\lambda}{r}\Bigg] + \alpha\overline{\kappa}_2(\overline{\kappa}_1 + 2\overline{\kappa}_2),
\end{aligned}
$$

and:

$$
\mathcal{H} = -\partial_r[\Gamma_2] - \frac{\lambda}{r} + \gamma_1\overline{\kappa}_2(2\overline{\kappa}_1 + \overline{\kappa}_2) + \frac{1}{r}(\Gamma_1 - 3\Gamma_2) + \frac{\Gamma_1\Gamma_2}{2} - \frac{3\Gamma_2^2}{4}.
$$

In order to evolve λ itself we can construct a simple expression through direct differentiation and use of the first two ADM evolution equations:

$$
\partial_t[\lambda] = \frac{2\alpha\gamma_1}{\gamma_2}\left\{\frac{\overline{\kappa}_1 - \overline{\kappa}_2}{r}\right\},
$$

where the curled brace term may be removed with $\mathcal{M}_r$ to yield:

$$
\partial_t[\lambda] = \frac{2\alpha\gamma_1}{\gamma_2}\left[\partial_r[\overline{\kappa}_2] - \frac{\Gamma_2}{2}(\overline{\kappa}_1 - \overline{\kappa}_2)\right].
$$

(V):

Here we just notice that $\partial_t[\mathrm{D}_\alpha] = \partial_t[\log(\alpha)] = \partial_t[\alpha]/\alpha$ and it follows that:

$$
\partial_t[\mathrm{D}_\alpha] = -\partial_r[\alpha f(\alpha)(\overline{\kappa}_1 + 2\overline{\kappa}_2)].
$$

(VI):

note: for a guide to the concept of hyperbolicity see Alcubierre's book §5.3. Important for use here is that the *principal symbol* of the system of equations has a complete set of eigenvectors (eigenfields), if the eigenvalues are also real and distinct then we have so-called strongly hyperbolic system.

Consider $\boldsymbol{v} := (\mathrm{D}_\alpha, \Gamma_1, \Gamma_2, \overline{\kappa}_1, \overline{\kappa}_2)$ then we can investigate the characteristic matrix (principal part of system):

$$
M := \alpha\begin{bmatrix}
0 & 0 & 0 & -f(\alpha) & -2f(\alpha)\\
0 & 0 & 0 & -2 & 0\\
0 & 0 & 0 & 0 & -2\\
-1/\gamma_1 & 0 & -1/\gamma_1 & 0 & 0\\
0 & 0 & -1/(2\gamma_1) & 0 & 0
\end{bmatrix},
$$

which leads to the collection of eigenvalues:

$$\boldsymbol{\xi} = (\xi^0,\, \xi^l_-,\, \xi^l_+,\, \xi^f_-,\, \xi^f_+) = \left(0,\; -\frac{\alpha}{\sqrt{\gamma_1}},\; +\frac{\alpha}{\sqrt{\gamma_1}},\; -\alpha\sqrt{\frac{f}{\gamma_1}},\; +\alpha\sqrt{\frac{f}{\gamma_1}}\right),$$

where $\xi^l_\pm$ is the coordinate speed of light whereas $\xi^f_\pm$ are the gauge speeds.

Trouble arises when we look at the eigenfields $\mathbf{w} := M^{-1}\boldsymbol{v} = (w^0,\, w^l_-,\, w^l_+,\, w^f_-,\, w^f_+)$:

$$\begin{aligned} w_0 &= \frac{\mathrm{D}_\alpha}{f} - (\Gamma_1 + 2\Gamma_2)/2, \\ w^l_\pm &= \gamma_1^{1/2}\overline{\kappa}_2 \mp \Gamma_2/2, \\ w^f_\pm &= \gamma_1^{1/2}\left(\overline{\kappa}_1 + 2\frac{f+1}{f-1}\overline{\kappa}_2\right) \mp \left(\frac{\mathrm{D}_\alpha}{f^{1/2}} + 2\frac{\Gamma_2}{f-1}\right), \end{aligned}$$

where it is clear that as $f \to 1$ the eigenfields become ill-defined. Thus for a choice of $f = 1$ in the gauge our system does not have a (complete) set of eigenfields and can only be weakly-hyperbolic which in turn would mean any numerical evolution would be generally untenable.

(VII):

See code.

(VIII):

Under the rescaling:

$$\begin{aligned} \tilde{\gamma}_1 &:= \gamma_1/\psi^4, & \tilde{\gamma}_2 &:= \gamma_2/\psi^4; \\ \tilde{\Gamma}_1 &:= \Gamma_1 - 4\partial_r[\log(\psi)], & \tilde{\Gamma}_2 &:= \Gamma_2 - 4\partial_r[\log(\psi)]; \end{aligned}$$

the regularized equations become:

Equations for $\partial_t[\gamma_I]$ become:

$$\partial_t[\tilde{\gamma}_I] = -2\alpha\tilde{\gamma}_I\overline{\kappa}_I,$$

whereas $\partial_t[\Gamma_I]$ goes to:

$$\partial_t[\tilde{\Gamma}_I] = -2\alpha(\overline{\kappa}_I\mathrm{D}_\alpha + \partial_r[\overline{\kappa}_I]).$$

In the case of λ and $\partial_t[\lambda]$:

$$\lambda = \frac{1}{r}\left(1 - \frac{\tilde{\gamma}_1}{\tilde{\gamma}_2}\right),$$

and

$$\partial_t[\lambda] = \frac{2\alpha\tilde{\gamma}_1}{\tilde{\gamma}_2}\left[\partial_r[\overline{\kappa}_2] - \left(\tilde{\Gamma}_2 + 4\mathrm{D}_\psi\right)(\overline{\kappa}_1 - \overline{\kappa}_2)/2\right],$$

where we have defined $\mathrm{D}_\psi := \partial_r[\log(\psi)]$.

Equation for $\partial_t[\overline{\kappa}_1]$:

$$\begin{aligned} \partial_t[\overline{\kappa}_1] = -\frac{\alpha}{\gamma_1}\Bigg[&\partial_r[\mathrm{D}_\alpha + \Gamma_2] + \mathrm{D}_\alpha^2 - \frac{\mathrm{D}_\alpha\Gamma_1}{2} + \frac{\Gamma_2^2}{2} - \frac{\Gamma_1\Gamma_2}{2} \\ &- \gamma_1\overline{\kappa}_1(\overline{\kappa}_1 + 2\overline{\kappa}_2) - \frac{1}{r}(\Gamma_1 - 2\Gamma_2)\Bigg], \end{aligned}$$

becomes:

$$\begin{aligned} \partial_t[\overline{\kappa}_1] = -\frac{\alpha}{\tilde{\gamma}_1\psi^4}\Bigg[&\partial_r[\mathrm{D}_\alpha + \tilde{\Gamma}_2] + 4\partial_r[\mathrm{D}_\psi] + \mathrm{D}_\alpha^2 - \frac{\mathrm{D}_\alpha}{2}(\tilde{\Gamma}_1 + 4\mathrm{D}_\psi) + \frac{(\tilde{\Gamma}_2 + 4\mathrm{D}_\psi)^2}{2} \\ &- \frac{(\tilde{\Gamma}_1 + 4\mathrm{D}_\psi)(\tilde{\Gamma}_2 + 4\mathrm{D}_\psi)}{2} - \frac{1}{r}([\tilde{\Gamma}_1 + 4\mathrm{D}_\psi] - 2[\tilde{\Gamma}_2 + 4\mathrm{D}_\psi])\Bigg] + \alpha\overline{\kappa}_1(\overline{\kappa}_1 + 2\overline{\kappa}_2), \end{aligned}$$

where we have expanded out a $\propto \gamma_1$ term from within the bracket and cancelled.

Equation for $\partial_t[\overline{\kappa}_2]$:

$$
\partial_t[\overline{\kappa}_2] = -\frac{\alpha}{2\gamma_1}\left[\partial_r[\Gamma_2] + \mathrm{D}_\alpha\Gamma_2 + \Gamma_2^2 - \frac{\Gamma_1\Gamma_2}{2} - \frac{1}{r}(\Gamma_1 - 2\mathrm{D}_\alpha - 4\Gamma_2) + \frac{2\lambda}{r}\right] + \alpha\overline{\kappa}_2(\overline{\kappa}_1 + 2\overline{\kappa}_2).
$$

becomes:

$$
\partial_t[\overline{\kappa}_2] = -\frac{\alpha}{2\tilde{\gamma}_1\psi^4}\left[\partial_r[\tilde{\Gamma}_2 + 4\mathrm{D}_\psi] + \mathrm{D}_\alpha(\tilde{\Gamma}_2 + 4\mathrm{D}_\psi) + (\tilde{\Gamma}_2 + 4\mathrm{D}_\psi)^2 - \frac{(\tilde{\Gamma}_1 + 4\mathrm{D}_\psi)(\tilde{\Gamma}_2 + 4\mathrm{D}_\psi)}{2} - \frac{1}{r}([\tilde{\Gamma}_1 + 4\mathrm{D}_\psi] - 2\mathrm{D}_\alpha - 4[\tilde{\Gamma}_2 + 4\mathrm{D}_\psi]) + \frac{2\lambda}{r}\right] + \alpha\overline{\kappa}_2(\overline{\kappa}_1 + 2\overline{\kappa}_2).
$$

For the lapse (with Bona-Masso):

$$
\partial_t[\alpha] = -\alpha^2 f(\alpha)[\overline{\kappa}_1 + 2\overline{\kappa}_2], \qquad \partial_t[\mathrm{D}_\alpha] = -\partial_r[\alpha f(\alpha)[\overline{\kappa}_1 + 2\overline{\kappa}_2]];
$$

which with $f(\alpha) := 2/\alpha$ translates into:

$$
\partial_t[\alpha] = -2\alpha[\overline{\kappa}_1 + 2\overline{\kappa}_2], \qquad \partial_t[\mathrm{D}_\alpha] = -\partial_r[\overline{\kappa}_1 + 2\overline{\kappa}_2].
$$

Should terms be further regrouped in these expressions?

For Schwarzschild in our choice of isotropic coordinates we have:

$$
\mathrm{D}_\psi = -\frac{M}{r(M+2r)},
$$
$$
\partial_r[\mathrm{D}_\psi] = \frac{1}{r^2} - \frac{4}{(M+2r)^2}.
$$

Thus, as $r \to 0$ we have:

$$
\psi^{-4} = \frac{16r^4}{M^4} + \mathcal{O}(r^5),
$$

and so poses no issue. On the other hand:

$$
\mathrm{D}_\psi = -\frac{1}{r} + \frac{2}{M} - \frac{4r}{M^2} + \mathcal{O}(r^2),
$$
$$
\partial_r[\mathrm{D}_\psi] = \frac{1}{r^2} - \frac{4}{M^2} + \frac{16r}{M^3} + \mathcal{O}(r^2).
$$

It may therefore be helpful to further group terms of the form:

$$
\psi^{-4}\mathrm{D}_\psi = -\frac{16r^3}{M} + \mathcal{O}(r^4).
$$

The λ term (its evolution equation in particular) may need to be rescaled also - we have however sufficient powers of r to achieve this too.

note: During numerical implementation, one can first start with the adapted lapse:

$$
\alpha = \frac{1 - M/(2r)}{1 + M/(2r)},
$$

before investigating non-trivial gauge-dynamics.