Numerical relativity — Exercise sheet # 2

Boris Daszuta
boris.daszuta@uni-jena.de

03.05.2021

Extrinsic curvature

Exercise 1.1: Sphere embedded in $\mathbb{R}^3$

Consider a sphere of radius a under the embedding in $\mathbb{R}^3$ specified through $t := r - a = 0$ with $r = \sqrt{x^2 + y^2 + z^2}$. Compute the intrinsic and extrinsic curvature.

Exercise 1.2: Extrinsic curvature of Schwarzschild

Consider Schwarzschild in isotropic coordinates

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + \psi^4 (\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2)$$

where $\psi := (1 + M/2r)$, $\alpha := (1 - M/2r)/\psi$ and $\mathrm{d}\Omega^2$ is the standard 2-sphere metric. Identify spatial slices Σ with hypersurfaces of constant coordinate time t.

- What is the 3-metric?
- Compute the intrinsic and extrinsic curvature.

Gauss-Codazzi-Ricci equations

Choose one of the following exercises.

Exercise 2.1: Derivation of Gauss equations (Gauss-Codazzi-Ricci equations)

The Gauss-Codazzi-Ricci equations are identities relating the 3+1 projections of the 4D Riemann and Ricci tensors. The Gauss equation is the spatial projection of the 4D Riemann tensor ${}^4R_{abcd}$ that can be expressed in terms of the spatial (3D) Riemann tensor R_{abcd} and extrisinc curvature K_{ab} as

$$\gamma_a^p \gamma_b^q \gamma_c^r \gamma_d^s {}^4R_{pqrs} = R_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc} \ , \tag{1}$$

where $\gamma^a_b = g^{ac}\gamma_{cb} = \delta^a_b + n^a n_b$ is the 3+1 spatial projection operator and n^a the unit normal vector to hypersurfaces Σ_t. Compute the above relation and the contractions

$$\gamma^p_a \gamma^q_b {}^4R_{pq} + \gamma_{ap} n^q \gamma^r_b n^{s4} R^p_{qrs} = R_{ab} + K K_{ab} - K_{ap} K^p_b \ , \tag{2}$$

$$^4R + 2n^a n^{b4} R_{ab} = R + K^2 - K_{ab}K^{ab} \ . \tag{3}$$

Exercise 2.2: Derivation of Codazzi equations (Gauss-Codazzi-Ricci equations)

The Gauss-Codazzi-Ricci equations are identities relating the 3+1 projections of the 4D Riemann and Ricci tensors. The Codazzi equation is the projection of the 4D Riemann tensor $^4R_{abcd}$

$$\gamma^p_a \gamma^q_b \gamma^r_c n^{s4} R_{pqrs} = D_b K_{ac} - D_a K_{bc} \ , \tag{4}$$

expressed in terms of the spatial (3D) Riemann tensor R_{abcd} and extrisinc curvature K_{ab}. Above $\gamma^a_b = g^{ac}\gamma_{cb} = \delta^a_b + n^a n_b$ is the 3+1 spatial projection operator, n^a the unit normal vector to hypersurfaces Σ_t, and D_a the covariant derivative of (Σ_t, γ_{ab}). Compute the above relation and the contraction

$$\gamma^p_a n^{q4} R_{pq} = D_a K - D_s K^s_a \ . \tag{5}$$

Exercise 2.3: Derivation of Ricci equations (Gauss-Codazzi-Ricci equations)

The Gauss-Codazzi-Ricci equations are identities relating the 3+1 projections of the 4D Riemann and Ricci tensors. The Ricci equation is the spatial projection of the 4D Riemann tensor $^4R_{abcd}$

$$\gamma^p_a \gamma^q_b n^r n^{s4} R_{prqs} = \mathcal{L}_m K_{ab} + \alpha^{-1} D_a D_b \alpha + K^d_b K_{ad} \ , \tag{6}$$

expressed in terms of the spatial (3D) Riemann tensor R_{abcd} and extrisinc curvature K_{ab}. Above $\gamma^a_b = g^{ac}\gamma_{cb} = \delta^a_b + n^a n_b$ is the 3+1 spatial projection operator, n^a the unit normal vector to hypersurfaces Σ_t, $m^a = \alpha n^a$ is the normal evolution vector, D_a the covariant derivative of (Σ_t, γ_{ab}), and $\mathcal{L}_n$ is the Lie drivative along n^a. Derive this equations.

The term "$\gamma\gamma n n^4 R$" appears also in the contracted Ricci equation,

$$\gamma^p_a n^{q4} R_{pq} = D_a K - D_s K^s_a \ . \tag{7}$$

Combine the two equations to obtain

$$\gamma^p_a \gamma^{q4}_b R_{pq} = -\alpha^{-1}\mathcal{L}_m K_{ab} - \alpha^{-1} D_a D_b \alpha + R_{ab} + K K_{ab} - 2K_{ar} K^r_b \ . \tag{8}$$