NR

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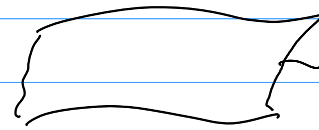
INITIAL DATA PROBLEM

$$\begin{cases} C_0 := R + K^2 - K_{ij}K^{ij} - 16\pi E = 0 \\ C_i := D_j K^j_i - D_i K - 8\pi P_i = 0 \end{cases}$$

Want: γ_{ij}, K_{ij} on Σ_0



$\Sigma_{t=0}$

(i) Constraints are satisfied

(ii) Physically meaningful

↳ BH spacetime, N^{th} BH spacetime, NS spacetime ...

Here (today): Matter fields are zero or given.

4 equations for 12 unknowns!

↳ Need to prescribe 8 quantities, and solve for the other 4.

- CONSTRAINED DATA (4)
- FREE DATA (8)

Q: How to?

A: "Principles": 1) Physical (astro) expectation for Σ_0

2) Euristic / intuition on the fields we prescribe/solve fo

3) Mathematical necessity

↳ $C_a = 0$: Solvable!

- Linear equations, decouple
- Well-posed BVP

Conformal decomposition allows us to implement 1–3)

Two main formalisms: • Conformal Transverse Traceless formalism (CTT, York 1973)

• Conformal Thin Sandwich formalism (CTS, York 1999)

LICHNEROWICZ EQ (HAM. CONSTRAINT)

(L) $$\boxed{C_0 := \underbrace{\tilde{D}_i \tilde{D}^i}_{\sim \Delta\psi} \psi - \frac{1}{8} \hat{A}_{ij} \hat{A}^{ij} \psi^{-7} + \left(-\frac{K^2}{12} + 2\pi E \right) \psi^5 = 0}$$ $(p = -10)$

$\sim \Delta \psi \rightsquigarrow$ Nonlinear eq for ψ !?

- An eq for the conf. factor ψ (Elliptic type ?)
- $K^2 \equiv 0$ the equation simplifies ...

↳ the BVP can be studied : several known results about well-posedness of (L) under the hypothesis $\boxed{K \equiv \text{const}}$

Constant Mean Curvature (CMC) spacetime

Example :

$$\left.\begin{array}{l} \text{A.F. spacetimes} \\ \text{CMC} \quad \text{-"-} \quad \text{w/ } K \equiv 0 \\ E = 0 \end{array}\right\} \Rightarrow \text{BVP w/ (L) is } \underline{\text{solvable}} \text{ for a "large" class of } \tilde{\gamma}_{ij}$$

Prototype equation for (L) w/ $K \equiv 0$:

$$\Delta u = + f^2 u^p \qquad \text{in flat spacetime}$$

Theorems : (i) $p = 1$ (linear) : BVP w/ data on $\partial\Omega$ boundary of domain and $u|_{\partial\Omega} \equiv 0 \Rightarrow u \equiv 0$ unique

(ii) (nonlinear) unique solution iff $p > 0$

(same sign in front of the positive coefficient of u^p)

How about (L) ? Linearize (L) and use (i) [set $K \equiv 0$] :

$$(L) : \quad \tilde{\Delta} \psi + H(\psi) = 0$$

$$\psi = \psi_0 + \epsilon \qquad \psi_0 : \text{solution}$$

$$H(\psi) = H(\psi_0) + \frac{\partial H}{\partial \epsilon}\Big|_0 \epsilon + O(\epsilon^2)$$

$\hookrightarrow$ $\tilde{\Delta}\epsilon = f\epsilon$ with :

$$\boxed{f = \frac{1}{8}\tilde{R} + \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\psi_0^{-8} - 10\pi E\psi_0^4}$$

≥ 0 ≥ 0

Note : $K \equiv 0 \Rightarrow \tilde{R} > 0$

f is not positive because of the "matter term"

$\hookrightarrow$ not solvable !

We can still solve (L) w/ $K \equiv 0$ if we rescale the matter term :

$$\tilde{E} := \psi^s E$$

$$\hookrightarrow -5 \cdot 2\pi E\psi_0^4 \longrightarrow \underbrace{+(s-5)\,2\pi\,\tilde{E}\,\psi_0^{4-s}}_{>0 \quad \forall s>5}$$

This is not a trick :

(!) $\Rightarrow$ In practice nonlinear elliptic eqs are solved by iteration (linearization)

$\hookrightarrow$ Specifying the $\tilde{E}$ r.h.s. is key to find a numerical solution

Q: What to use for s ?

A : $s = 8$: this way the dominant energy condition can be expressed in conf. variables :

$$\tilde{E} := \psi^8 E \quad \tilde{P}^i := \psi^{10} P_i$$

$$\tilde{E}^2 \geq \tilde{P}^2 \Rightarrow E^2 \geq P^2$$

$$\left[\psi^{2s} E^2 \geq \psi^{-4}\psi^{10}\psi^{10} P^2\right]$$

MAXIMAL SLICING

Q: What is the meaning of $K = 0$?

A: Gauge condition that extremize the volume of Σ.

$$K = \gamma^{ab} K_{ab} = -\frac{1}{2\alpha} \gamma^{ij} \mathcal{L}_m \gamma_{ij} = -\frac{1}{2\alpha} \mathcal{L}_m \ln \gamma = -\frac{1}{\alpha} \mathcal{L}_m \ln \sqrt{\gamma}$$

identity (see last lecture)

determinant

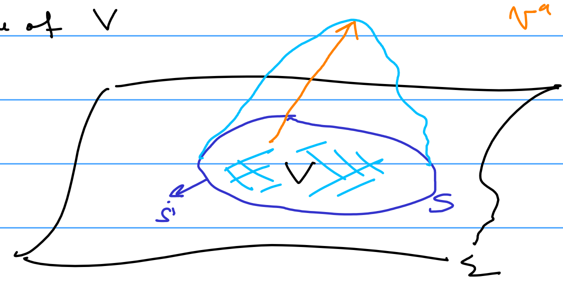
volume element! on Σ

Volume: $V = \int \sqrt{\gamma}\, d^3x$

Perform a variation of V

$N^a = \delta t (\alpha n^a + \beta^a)$:

$N^a |_S = 0$



$$\frac{\delta V}{\delta t} = \int \partial_t \sqrt{\gamma}\, d^3x = \int \left(-\alpha K + D_i \beta^i\right) \sqrt{\gamma}\, d^3x = -\int \alpha K \sqrt{\gamma}\, d^3x + \underbrace{\oint_S \beta^i s_i}_{=0} = -\int \alpha K \sqrt{\gamma}\, d^3x$$

$K = 0 \Rightarrow \delta V = 0$

Example: Same diff. geom. problem of film of soap on a ring S

↳ the soap film MINIMIZE the V (Euclidean geom).

But we are in a Lorentzian geometry $\Rightarrow$ MAXIMUM.

CTT ($p=-10$ scaling of K_{ij})

- Use (L) for $C_0 = 0$
- Derive an eq from $C_i = 0$ for a quantity X^i (vector) obtained by a further decomposition of

$$\hat{A}^{ij} := \hat{A}^{ij}_L + \hat{A}^{ij}_{TT} \qquad (1)$$

where:

$$TT : \quad \tilde{\gamma}_{ij}\hat{A}^{ij} = 0 \quad \text{and} \quad \tilde{D}_j \hat{A}^{ij}_{TT} = 0$$

$$L : \quad (\tilde{L}X)^{ij} = \tilde{D}^i X^j + \tilde{D}^j X^i - \frac{2}{3}\tilde{D}_k X^k \tilde{\gamma}^{ij}$$

Conformal Killing operator on X^i

determined by taking the divergence of (1):

$$\tilde{D}_j \hat{A}^{ij} = \tilde{D}_j (\tilde{L}X)^{ij} = \tilde{D}_j \tilde{D}^j X^i + \frac{1}{3}\tilde{D}^i \tilde{D}_j X^j + \tilde{R}^i{}_j X^j =: \tilde{\Delta}_L X^i$$

Conformal vector Laplacian operator

$\exists!$ L+TT decomposition (1) iff $\exists! X^i$ of the conf. Laplacian eq:

$$\boxed{\tilde{\Delta}_L X^i = \tilde{D}_j \hat{A}^{ij}}$$

Th (Cantor '79):

$$\left.\begin{array}{l} \Sigma \text{ is A.F.} \\ \partial_k \partial_l \tilde{\gamma}_{ij} = O(r^{-3}) \end{array}\right\} \Rightarrow \exists! \text{ is guaranteed.}$$

$\boxed{\text{CTT}}$

$$\begin{cases} (L) & (C_0 = 0) \\ \tilde{\Delta}_L X^i - \frac{2}{3}\tilde{D}^i K \psi^6 - 8\pi \tilde{p}^i = 0 & (C_i = 0) \end{cases}$$

Constrained data: ψ, X^i

Free data : $\tilde{\gamma}_{ij}$, $\hat{A}^{ij}_{TT}$, K

Note : (i) under max. slicing $K \equiv 0$
the 2 equations decouple !

(ii) for CMC slices they partially decouple
(can solve one after the other)

(iii) $\tilde{\gamma}_{ij}, \hat{A}^{ij}_{TT} \rightsquigarrow$ GW content of Σ

CTT : CONF. FLAT, ASYMPTOTICALLY FLAT, MAX. SLICING DATA, VACUUM

2) 1) 3) 4)

Simplest choice of free data:

$\tilde{\gamma}_{ij} = f_{ij}$ 2)

$K \equiv 0$ 3)

$E = 0 = p^i$ 4)

$\hat{A}^{ij}_{TT} \equiv 0$ 5)

$\hookrightarrow$ $\tilde{D}_i = D_i$ $\tilde{D}_i\tilde{D}^i = D_iD^i = \Delta$

$\tilde{R} = 0$ $\tilde{L} = L$

$\hookrightarrow$ CTT :
$$\begin{cases} \Delta\psi + \frac{1}{8}(LX)_{ij}(LX)^{ij}\psi^{-7} = 0 & \text{(CTT1)} \\ \Delta_L X^i = \Delta X^i + \frac{1}{3}D_jD^iX^j = 0 & \text{(CTT2)} \end{cases}$$

These are "easy" : flat (Euclidean) operators & decoupled.

BVP :

$\begin{cases} \text{OUTER BCs : A.F.} \quad \psi = 1 \ , \ X^i = 0 \quad \text{at } i_0 \ (r \to +\infty) \ 1) \\ \text{INNER BCs (strong field) : fix the topology} \quad (*) \end{cases}$

CASE 1: ⊛ $\Sigma = \mathbb{R}^3$ (No inner BC)

$$(CTT2) \Rightarrow X^i \equiv 0$$

$$(CTT1) \Rightarrow \begin{cases} \Delta\psi = 0 \\ \psi = 1 \quad r \to +\infty \end{cases} \Rightarrow \psi \equiv 1$$

$\hookrightarrow$ Flat spacetime.

If we want a nontrivial solution we need an inner BC.

... Next time ...