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3+1 DECOMPOSITION OF GR (cont.)

RECAP: ADMY Eqs in Geo. gauge $(\alpha = 1, \beta^i = 0)$

(ADMY, GG)

$$\left\{\begin{array}{l} \boxed{\begin{array}{l} \partial_t \gamma_{ij} = -2K_{ij} \\ \partial_t K_j^i = R_j^i + K K_j^i - 2 K_{ik} K^k_j + (\text{matter terms}) \end{array}} \\ \boxed{\begin{array}{l} 0 = R + K^2 - K_{ij}K^{ij} - 16\pi E =: C_0 \\ 0 = D_j K^j_i - D_i K - 8\pi P_i =: C_i \end{array}} \end{array}\right.$$

EV. EQS:

PRINCIPAL PART, HIGHEST DRVTS

$$-\ddot{\gamma}_{ij} + \underbrace{\gamma^{kl}(\gamma_{ij})}_{\text{Rational poly in } \gamma_{ij}} \left[\partial_k \partial_l \gamma_{ij} + \partial_i\partial_j \gamma_{kl} - 2\partial_{(i|}\partial_k \gamma_{l|j)k}\right] \simeq 0$$

- 2^{nd} time & spatial derivatives ; "WAVE-LIKE" **HYPERBOLIC**
- QUASILINEAR (= linear in highest derivatives)
- Quadratic in the 1^{ST} spatial derivatives (not shown above)

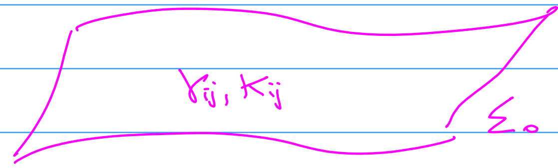
TODAY:
- What kind of EQS are the constraints ?
- Are constraints "transported along the evolution" ? (cf Maxwell/EM)
- Hamiltonian formulation of GR.

CONSTRAINT EQS

$$\left\{\begin{array}{ll} 0 \simeq \gamma^{ik}\gamma^{jl}\partial_k\partial_l \gamma_{ij} - \gamma^{ij}\gamma^{kl}\partial_k\partial_l\gamma_{ij} & (C_0) \\ 0 \simeq \gamma^{jk}\partial_j \dot{\gamma}_{ki} - \gamma^{kl}\partial_i \dot{\gamma}_{kl} & (C_i) \end{array}\right.$$

- Contain only 2^{nd} spatial derivatives of γ_{ij} or $\dot{\gamma}_{ij}$
  $\simeq$ Elliptic-like (No known type !)

- Solve w/ BOUNDARY DATA $\longrightarrow$ INITIAL DATA PROBLEM IN GR



- $\boxed{\begin{array}{ll} 6+6=12 & \text{unknowns} \\ 4 & \text{eqs} \end{array}}$

IF we want γ_{ij}, K_{ij} on Σ_0 we must:

(i) PRESCRIBE 8 quantities "FREE DATA"

(ii) Solve $C_\mu = 0$ for 4 quantities

$\hookrightarrow$ Need to understand how to prescribe the free data

$\hookrightarrow$ CONF. DEC. (later...)

- Assuming we have a solution on Σ_0, we could then evolve (Σ_0, γ, K) to $t>0$ ...

INITIAL DATA PROBLEM $\longrightarrow$ EVOLUTION ...

Q: Do we need to solve $C_\mu = 0$ during the evolution? i.e. $\forall\, t>0$?

A: No.

CONSTRAINTS ARE TRANSPORTED ALONG THE DYNAMICS

Consequence of $\nabla^b\, {}^4G_{ab} = 0$

Simple approach based on Z4 system:

Consider the following extended theory :

$$\boxed{{}^4G_{ab} + \underline{2\nabla_{(a}Z_{b)}} - \underline{g_{ab}\nabla_c Z^c} = 8\pi\, T_{ab}} \qquad (Z4)$$

New terms, and new field Z^a

→ Z4 reduces to GR if $Z^a \equiv 0$ (or Z^a is a KV).

3+1 split of (Z4) :

$$(Z4, 3+1) \quad \begin{cases} \mathcal{L}_m \gamma_{ij} = -2\alpha K_{ij} \\ \mathcal{L}_m K_{ij} = (\text{as in ADMY eqs}) + \alpha D_{(i}Z_{j)} \\ \mathcal{L}_m \theta = \frac{1}{2} C_0 - \theta K + D_k Z^k - Z^k D_k \ln\alpha \\ \mathcal{L}_m Z_i = C_i + D_i\theta - \theta D_i \ln\alpha - 2K^k{}_i Z_k \end{cases}$$

w/ $-\theta := n^a Z_a$

Obs :

- Constraints all become ev. eqs !

  ∂_t " Z vector " ~ " constraints "

- System of hyperbolic eqs
  (No constraints)

- GR is obtained under the algebraic constraint $Z^a \equiv 0$

Bianchi ids :

$$\nabla^a\, {}^4G_{ab} = 0 \implies \boxed{0 = \Box Z_a + R_{ab} Z^b}$$

wave eqs. for Z^a (2^{nd} order in time/space)

if $Z^a \equiv 0 \equiv \dot{Z}^a = C^a$, then $Z^a \equiv 0 \quad \forall\, t > 0$.

NOTE: this approach is rather general, and can be used for other theories / set of equations that have constraints / "mixed character" w/ hyperbolic and elliptic eqs.

Example: MHD

Hydrodynamics with perfectly conducting fluid

$$0 = \partial_\mu T^{\mu\alpha} = \partial_\mu \left(T^{\mu\alpha}_{P.F.} + T^{\mu\alpha}_{EM} \right)$$

INDUCTION EQ.

$$\hookrightarrow \begin{cases} \boxed{\partial_t B^i + \partial_k \left(B^k v^i - B^i v^k \right) = 0} \\ \boxed{C := \partial_i B^i = 0} \end{cases}$$

EV. eq for Magnetic field

$\operatorname{div} \vec{B} = 0$ constraint

Extended system (w/ new scalar field ψ):

$$\begin{cases} \partial_t B^i + \underline{\partial_k \left(B^k v^i - B^i v^k \right)} + \partial^i \underline{\psi} = 0 \\ \partial_t \psi + \underbrace{\partial_i B^i}_{C} \boxed{+ \delta \psi} = 0 \end{cases}$$

DAMPING TERM (*)

- Only ev. eqs!
- Time drvt of the second eq:

$$0 = \partial_{tt} \psi + \partial_t \partial_i B^i =$$

sub. 1st eq

$$= \partial_{tt} \psi - \partial_i \underline{\partial^i} \psi + 0$$

$\partial_k \partial_i$ sym, $(B^k v^i - B^i v^k)$ antisym

$$\hookrightarrow \quad 0 = \Box \psi + \delta \dot{\psi}$$

Compare [exercise]: $0 = \Box C + \delta \dot{C}$

constraints are transported

Imagine you solve numerically MHD:

a violation of $C=0$ can "appear" from algorithmic errors...

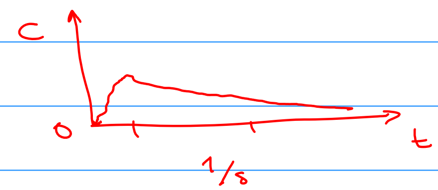
If you are solving the extended system:

then the violation will propagate and be transported to the domain boundary because of:

$$\Box C + \delta \dot{C} = 0$$

IDEA: I could ADD a DAMPING TERM ($\delta > 0$)

↳ the violation is also damped



↳ DIVERGENCE CLEANING METHOD IN MHD

[Dedner + 2012]

NOTE:

- Free-evolution schemes: hyperbolic eqs. are maximized; constraints are not solved during evolution but monitored
- Fully-constrained formulation: elliptic eqs (constraints + gauge + ...) are maximized and solved $\forall t$

  2 dynamical (time-dependent) D.O.F. → GWs

  [2 hyp. eqs]

ADM HAMILTONIAN FORMULATION OF GR

$$S_{GR} = \int_M {}^4R \sqrt{-g} + (\text{boundary terms})$$

$$= \int dt \int_{\Sigma_t} \underbrace{(R + K_{ij}K^{ij} - K^2)}_{\text{LAGRANGIAN DENSITY, } L} \alpha \sqrt{\gamma}$$

Conj. momenta : $$\pi^{ij} := \frac{\partial L}{\partial \dot{\gamma}_{ij}} = \sqrt{\gamma}\left(K \gamma^{ij} - K^{ij}\right)$$

<u>Hamiltonian density</u> :

$$\mathcal{H} = \pi^{ij} \dot{\gamma}_{ij} - L = \ldots =$$

$$= \sqrt{\gamma} \left[\alpha C_0 + 2\beta^i C_i + 2 D_j (K\beta^j - K^j{}_i \beta^i)\right]$$

understood as fcn of (γ_{ij}, π^{ij})

<u>Hamiltonian</u> :

$$\boxed{H = \int_{\Sigma_t} \mathcal{H} = \int_{\Sigma_t} \sqrt{\gamma} \left(\alpha C_0 + 2\beta^i C_i\right)}$$

<u>EoM</u>:

$$\dot{\gamma}_{ij} = \frac{\partial H}{\partial \pi^{ij}} = -2 \frac{\alpha}{\sqrt{\gamma}} \overbrace{\left(\pi_{ij} - \frac{1}{2}\pi \gamma_{ij}\right)}^{\sim K_{ij}} + 2 D_{(i}\beta_{j)}$$

$$-\dot{\pi}^{ij} = \frac{\partial H}{\partial \gamma_{ij}} = \ldots \left(\sim \mathcal{L}_m K_{ij} \text{ in ADM} + C_0\right) \ldots$$

$$0 = \frac{\partial H}{\partial \alpha} = C_0$$

$$0 = \frac{\partial H}{\partial \beta^i} = C_i$$

Equivalent to ADM ⟵
modulo constraints additions

<u>Obs</u> :

ADM	York
• (γ_{ij}, π^{ij})	(γ_{ij}, K_{ij}) w/ $K_{ij} = -\frac{1}{\sqrt{\gamma}}\left(\pi_{ij} - \frac{1}{2}\pi\gamma_{ij}\right)$

• α, β^i : Lagrangian multipliers that enforce the constraints.

Next :

- Conformal decomposition of 3+1 GR
  - Approach the initial data problem (How to solve it?)
  - Free-ev. schemes like BSSNOK / Z4c
- Cauchy (IVP) problem (Hyperbolicity, well-posedness)
- Gauge