Numerical relativity — Exercise sheet # 4

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Exercise 1.1: ADM spherical symmetry

Note: In this exercise we investigate further some of the material touched upon in the last tutorial. Feel free to make use of the results / notebooks there.

In spherical symmetry we may take the general form of a spatial metric to be:

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \gamma_1(t,\, r)\mathrm{d}r^2 + r^2\gamma_2(t,\, r)\mathrm{d}\Omega^2, \quad \mathrm{d}\Omega^2 := \mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\varphi^2.$$

An analogous form may be chosen for any symmetric tensor field $S_{(ij)} = S_{ij} \in \mathcal{T}_2(\Sigma_t)$. Define the variables:

$$\mathrm{D}_\alpha := \partial_r[\log(\alpha)], \quad \Gamma_1 := \partial_r[\log(\gamma_1)], \quad \Gamma_2 := \partial_r[\log(\gamma_2)],$$

it can be shown that when working in vacuum and in the absence of spatial shift the standard ADM evolution equations may be particularized to spherical symmetry as:

$$\begin{aligned}
\partial_t[\gamma_1] &= -2\alpha\gamma_1\overline{\kappa}_1, & \partial_t[\gamma_2] &= -2\alpha\gamma_2\overline{\kappa}_2;\\
\partial_t[\Gamma_1] &= -2\alpha(\overline{\kappa}_1\mathrm{D}_\alpha + \partial_r[\overline{\kappa}_1]), & \partial_t[\Gamma_2] &= -2\alpha(\overline{\kappa}_2\mathrm{D}_\alpha + \partial_r[\overline{\kappa}_2]);
\end{aligned}$$

$$\partial_t[\overline{\kappa}_1] = -\frac{\alpha}{\gamma_1}\left[\partial_r[\mathrm{D}_\alpha + \Gamma_2] + \mathrm{D}_\alpha^2 - \frac{\mathrm{D}_\alpha\Gamma_1}{2} + \frac{\Gamma_2^2}{2} - \frac{\Gamma_1\Gamma_2}{2} - \gamma_1\overline{\kappa}_1(\overline{\kappa}_1 + 2\overline{\kappa}_2) - \frac{1}{r}(\Gamma_1 - 2\Gamma_2)\right];$$

$$\begin{aligned}
\partial_t[\overline{\kappa}_2] = &-\frac{\alpha}{2\gamma_1}\left[\partial_r[\Gamma_2] + \mathrm{D}_\alpha\Gamma_2 + \Gamma_2^2 - \frac{\Gamma_1\Gamma_2}{2} - \frac{1}{r}(\Gamma_1 - 2\mathrm{D}_\alpha - 4\Gamma_2) - \frac{2}{\gamma_2}\left\{\frac{(\gamma_1 - \gamma_2)}{r^2}\right\}\right]\\
&+ \alpha\overline{\kappa}_2(\overline{\kappa}_1 + 2\overline{\kappa}_2);
\end{aligned}$$

whereas the constraints may be put in the form:

$$\mathcal{H} := -\partial_r[\Gamma_2] + \left\{\frac{\gamma_1 - \gamma_2}{r^2\gamma_2}\right\} + \gamma_1\overline{\kappa}_2(2\overline{\kappa}_1 + \overline{\kappa}_2) + \frac{1}{r}(\Gamma_1 - 3\Gamma_2) + \frac{\Gamma_1\Gamma_2}{2} - \frac{3\Gamma_2^2}{4} = 0;$$

$$\mathcal{M}_r := -\partial_r[\overline{\kappa}_2] + \left\{\frac{\overline{\kappa}_1 - \overline{\kappa}_2}{r}\right\} + \frac{1}{2}(\overline{\kappa}_1 - \overline{\kappa}_2)\Gamma_2 = 0;$$

where $\overline{\kappa}_I := \kappa_I/\gamma_I$ (no sum, $I = 1,\, 2$) and we assume that the extrinsic curvature may be written as:

$$K_{ij}\mathrm{d}x^i\mathrm{d}x^j = \kappa_1(t,\, r)\mathrm{d}r^2 + r^2\kappa_2(t,\, r)\mathrm{d}\Omega^2.$$

Questions:

1. Derive the above system.
2. Recall that adapted coordinates can carry a downside of having to treat apparent coordinate singularities by regularizing. In the present case we have formal behaviour in the vicinity of $r = 0$:

   $$\gamma_I \sim \gamma_I^0 + \mathcal{O}(r^2), \quad \overline{\kappa}_I \sim \overline{\kappa}_I^0 + \mathcal{O}(r^2);$$

   what are the analogous conditions for Γ_I? These yield us parity conditions that we looked at imposing in the last tutorial through a staggered grid.
3. If we take into account local flatness (at $r = 0$) what additional relations must be simultaneously satisfied? Why does this complicate matters?
4. In order to implement the full set of conditions found in the previous part it is useful to introduce a new auxiliary variable:

   $$\lambda := \frac{1}{r}\left(1 - \frac{\gamma_1}{\gamma_2}\right).$$

   - What is the parity condition on λ?
   - Use λ to regularize the curled brace term in $\partial_t[\overline{\kappa}_2]$ together with $\mathcal{H}$.
   - Derive a *regular* evolution equation for λ. Hint: this requires the evolution equations and the constraints.
5. In principle we would like to be able to perform an evolution for some test problem. We could pick (e.g.) Bona-Masso slicing:

   $$\partial_t[\alpha] = -\alpha^2 f(\alpha)\mathcal{K} = -\alpha^2 f(\alpha)[\overline{\kappa}_1 + 2\overline{\kappa}_2].$$

   Derive an equation for $\partial_t[\mathrm{D}_\alpha]$.
6. An immediate question arises as to whether there are any obvious restrictions on $f(\alpha)$. To answer this consider putting $\mathbf{u} := (\alpha,\, \gamma_1,\, \gamma_1,\, \lambda)$ together with $\mathbf{v} = (\mathrm{D}_\alpha,\, \Gamma_1,\, \Gamma_2,\, \overline{\kappa}_1,\, \overline{\kappa}_2)$ such that we may (schematically) write our system as:

   $$\begin{aligned}\partial_t[u_i] &= q_i(u,\, v),\\ \partial_t[v_i] &= M_i^j(u)\partial_r[v_j] + p_i(u,\, v);\end{aligned}$$

   where q and p are source terms. Investigate the characteristic structure by writing down the eigenfields and eigenvalues as based on M_i^j thus providing a restriction on the possible choices of $f(\alpha)$.

7. (Optional): suitably modify the example code of last tutorial using the equations derived here in order to mitigate the stability issues we encountered.

8. (Optional): recall that a spatial slice of Schwarzschild may be written in isotropic form as:
$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \psi^4(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2), \quad \psi = 1 + M/(2r). \tag{1}$$
In the static puncture evolution technique one extracts analytically the conformal factor through field re-definitions:
$$\begin{aligned} \tilde{\gamma}_1 &:= \gamma_1/\psi^4, & \tilde{\gamma}_2 &:= \gamma_2/\psi^4; \\ \tilde{\Gamma}_1 &:= \Gamma_1 - 4\partial_r[\log(\psi)], & \tilde{\Gamma}_2 &:= \Gamma_2 - 4\partial_r[\log(\psi)]; \end{aligned} \tag{2}$$
with other variables left as previously regularized. Modify your regularized code to perform an evolution for the case that $f = 2/\alpha$ and comment on what you observe.