NR

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INITIAL DATA PROBLEM (cont.)

RECAP: Formalisms based on conformal decomposition of 3+1 EFE

- CTT
- CTS

} Lichnerowicz eq + (different choices) Momentum constr.

FREE DATA (8) + CONSTRAINED DATA (4)

CTT:
$$\begin{cases} \tilde{D}_i \tilde{D}^i \psi - \frac{1}{8}\tilde{R}\psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-7} + 2\pi \hat{E}\psi^{-3} - \frac{K}{12}\psi^5 = 0 \quad (L) \\ \tilde{\Delta}_L X^i - \frac{2}{3}\psi^6 \tilde{D}^i K - 8\pi \hat{p}^i = 0 \end{cases}$$

Conformal vector Laplacian
$$= \tilde{D}_j \tilde{D}^j X^i + \frac{1}{3}\tilde{D}^i \tilde{D}_j X^j + \tilde{R}^i_j X^j$$

with $\hat{A}^{ij} = (\tilde{L}X)^{ij} + \hat{A}^{ij}_{TT} = \underbrace{2\tilde{D}^{(i}X^{j)} - \frac{2}{3}\tilde{D}_k X^k \tilde{\gamma}^{ij}}$ $+ \hat{A}^{ij}_{TT}$

FREE DATA: $\tilde{\gamma}_{ij}$, $\hat{A}^{ij}_{TT}$, K, E, p^i

CONSTRAINED DATA: ψ, X^i

CTT: Asymp. Flat, Conf. Flat, Time Symmetric ID

Vacuum: $E = 0 = p^i$

$$\left.\begin{array}{ll} \tilde{\gamma}_{ij} = f_{ij} & (C.F.) \\ K = 0 & (\text{Max. Slicing}) \\ \hat{A}_{TT} = 0 & \end{array}\right\} \Rightarrow \tilde{D}_i = D_i \quad \tilde{D}_i\tilde{D}^i = \Delta \quad \tilde{R} = 0$$

$K_{ij} = 0$ (see later)

$$\begin{cases} \Delta\psi + \frac{1}{8}(LX)_{ij}(LX)^{ij}\psi^{-7} = 0 \\ \Delta_L X^i = 0 \end{cases}$$

with outer BCs

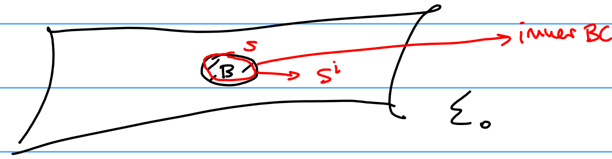
$$\psi \to 1, \quad X^i \to 0 \qquad r \to +\infty \qquad (A.F.)$$

$\hookrightarrow$ Inner BCs $\longrightarrow$ Topology of Σ_0

$\boxed{\Sigma_0 \sim \mathbb{R}^3}$ (No inner BCs)

$$\dot{x}^i \equiv 0 \implies \psi \equiv 1$$: Flat spacetime on Σ_0 (Not stationary, see later)

$\boxed{\Sigma_0 \sim \mathbb{R}^3 \setminus B}$



Simplest choice : $\dot{x}^i|_S = 0$ $\qquad \psi|_S = 1$

$$\dot{x}^i \equiv 0 \implies \psi \equiv 1 \implies \text{Flat } \Sigma_0$$

A more interesting choice :

S is a closed <u>minimal</u> surface :

$$0 = D_i s^i|_S = \psi^{-6} \tilde{D}_i(\psi^6 s^i) = \psi^{-6} \tilde{D}_i(\psi^4 \tilde{s}^i) =$$

$\tilde{s}^i := \psi^{+2} s^i$ unit normal vector w.r.t. $\tilde{\gamma}$:

$$\tilde{\gamma}(\tilde{s},\tilde{s}) = \psi^{-4} \gamma(\tilde{s},\tilde{s}) = \gamma(s,s) = 1$$

$$= \psi^{-6} \frac{1}{\sqrt{f}} \partial_i \left(\sqrt{f}\, \psi^4 \tilde{s}^i\right) \qquad (1)$$

Take B sphere radius $r = a$ $\quad (x^i = (r,\theta,\varphi))$ $\quad f_{ij} = \mathrm{diag}(1, r^2, r^2 \sin\theta)$

$$\tilde{s}^i = (1,0,0)$$

(1) : $$\frac{1}{r^2}\partial_r(r^2\psi^4)\Big|_{r=a} = 0 \quad \longleftrightarrow \quad \underbrace{\left(\partial_r \psi + \frac{\psi}{2r}\right)\Big|_{r=a} = 0}_{\text{Neumann-Dirichlet BCs for } \Delta\psi + \ldots = 0}$$

Solution of (L) + BCs :

$$\psi = 1 + \frac{a}{r}$$

Q: What is a ?

A: Try to compute <u>global quantities</u> :

$$M_{ADM} = -\frac{1}{2\pi} \lim_{r \to +\infty} \int \frac{\partial \psi}{\partial r} r^2 \sin\theta \, d^2\Omega = \ldots = 2a$$

$$\hookrightarrow \quad a = \frac{M_{ADM}}{2}$$

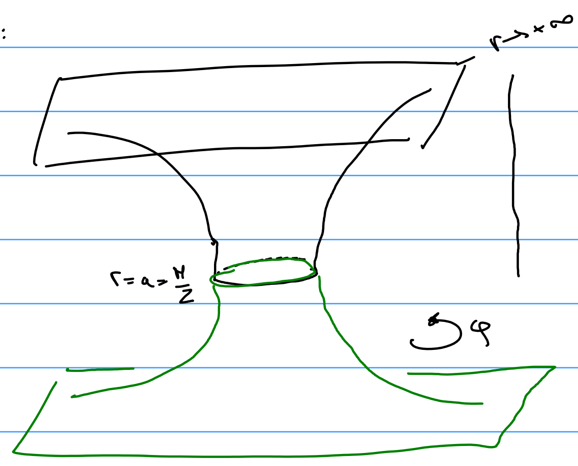
Putting together things :

$$\gamma_{ij} = \psi \, \tilde{\gamma}_{ij} = \psi f_{ij} = \left(1 + \frac{M_{ADM}}{2r}\right) f_{ij}$$

$\hookrightarrow$ Schwarzschild in isotropic coords !

[The Einstein-Rosen bridge is minimal surface]

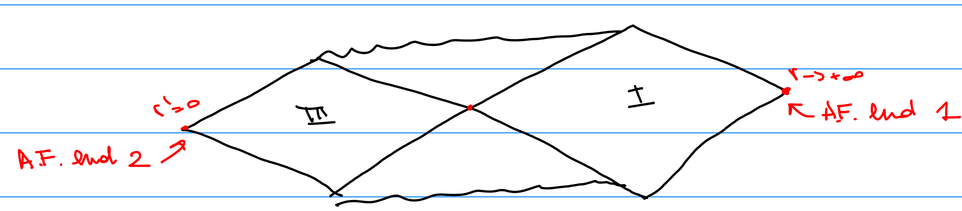
Embedding diagram :



$$r \to r' = \frac{M^2}{4r}$$

$$\left[\frac{M}{2}, \infty\right) \to \left(0, \frac{M}{2}\right)$$

$$\underbrace{\gamma(r,\theta,\varphi) = \gamma(r',\theta,\varphi)}_{\text{ISOMETRY}}$$



<u>Time Symmetry</u> : spatial slice of Schw. in iso. coords

$$\left.\begin{array}{l} \hat{A}^{ij}_{TT} \equiv 0 \text{ and } X^i \equiv 0 \Rightarrow \hat{A}^{ij} \equiv 0 \\ K \equiv 0 \end{array}\right\} \Rightarrow K_{ij} \equiv 0$$

$\hookrightarrow \mathcal{L}_m \gamma_{ij} = 0$ at $t=0$

$\hookrightarrow \Sigma_0$ is MOMENTARILY static

$t \to -t$ line element is invariant

TIME SYMMETRY

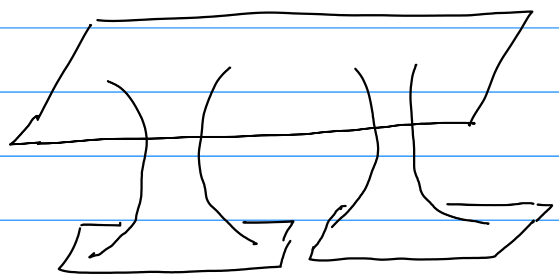
$\to$ There could be a nontrivial evolution of Σ_0 !

Example : geodesic gauge.

Multiple BH ID by superposition :

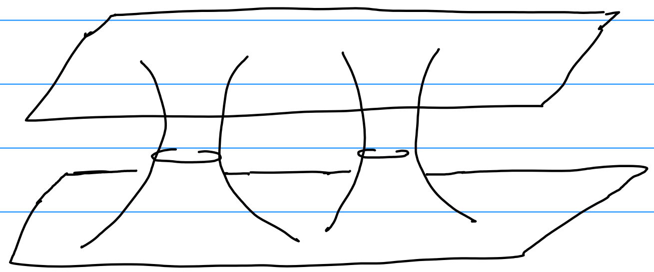
(BL) $$\psi = 1 + \underbrace{\sum_{p=1}^{N} \frac{M_p}{|x^i - C^i_p|}}_{\psi_{BL}}$$ Brill & Lindquist data

N=2



More complicated data : Misner data

N=2



Can be constructed by suitable inner BCs ...

Two more observations

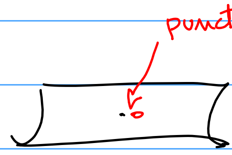
(i) This is only for nonspinning BH

No boost

(ii) The extended Σ_0 has topology is

$\Sigma_0 : \mathbb{R}^3 \setminus \{0\}$

puncture



Bowen-York '80

Ansatz solution for the $\Delta_L X^i + \ldots = 0$ eq:

$$\text{(BY)} \qquad X^i = -\frac{1}{4r}\left(7 f^{ij} P_j + \frac{1}{r^2} P_j x^i x^j\right) - \frac{1}{r^3} \epsilon^{ij}{}_k S_j x^k$$

w/ 6 parameters : P^i, S^i

$$\text{(BYA)} \qquad \hookrightarrow \hat{A}^{ij} = (LX)^{ij} = \underbrace{\frac{3}{2}\frac{1}{r^3}\left[x^i P^j + x^j P^i + \left(f^{ij} - \frac{x^i x^j}{r^2}\right) P^k x_k\right]}_{O(1/r^2)} +$$

$$+ \underbrace{\frac{3}{r^5}\left(\epsilon^{ik}{}_\ell S_k x^\ell x^j + \epsilon^{jk}{}_\ell S_k x^\ell x^i\right)}_{O(1/r^3)}$$

- Meaning of P^i : ADM momentum components
- S^i : angular mom. components (quasi-isotropic gauge)

Sketch for P_{ADM} :

$$P^i_{ADM} = \frac{1}{8\pi} \lim_{r \to +\infty} \int \hat{A}_{ik} x^k r \, d\Omega \overset{\text{(BYA)}}{=}$$

$$\sim \lim \int \frac{3}{2}\frac{1}{r^2}\left[x_i P_k x^k + \underbrace{x_k x^k}_{r^2} P^i - \underbrace{\left(\underbrace{f_{ik} x^k}_{x^i} - \frac{x^i \overbrace{x^k x_k}^{\cancel{r^2}}}{\cancel{r^2}}\right)}_{=0} P^j x_j + O(1/r^3)\right] d\Omega$$

$$\sim \ldots = P^i$$

$$\left(\text{Trick} : \int \frac{x^i x^k}{r^2} d\Omega = \delta^{ik} \int \frac{x^k x^k}{r^2} d\Omega = \frac{4\pi}{3} \delta^{ik}\right)$$

Summary : with the BY ansatz

- non rotating BH with M_{ADM} $\quad (S^i = 0 = P^i)$
- $P^i \neq 0$ $(S^i = 0)$: boosted BH
- $S^i \neq 0$ $(P^i = 0)$: spinning BH

Q : Did we find the Kerr solution? [$S^2 \neq 0$]

A : No!

(Garat & Price 2000) There exists NO Kerr foliation :

(i) axisymmetry
(ii) conformally flat
(iii) reduces to Schw. in nonrotating limit [$S^2 \equiv 0$]

↳ "There is NO conformally flat slice of Kerr"

↳ Moment of time symmetry ⟹ The BY data w/ $S^2 \neq 0$ are NONSTATIONARY!

↳ The evolution must be a nontrivial one.

Puncture solutions* : BY ansatz + solution of (L)

(Black hole solutions)

↑ require numerical approach (nonlinear eq for ψ)

Approaches :

(i) Generalized Misner data (Cook '90s)

↳ solve (L) w/ Newman-Dirichlet inner BCs

(ii) Generalized Brill-Lindquist data (*) (Brandt & Brügmann '97)

↳ Solve (L) on $\mathbb{R}^3$ by analytically separating the singular behaviour :

Ansatz : $\psi = \psi_{BL} + u = \sum_p \frac{M_p}{|x^i_p - c^i_p|} + u$

"correction" (to solve for)

$\Delta \psi_{BL} = 0 \implies$ Plug-in ansatz in (L) : eq for $\Delta u + \ldots = 0$

$$\boxed{\Delta u + \frac{\hat{A}_{ij}\hat{A}^{ij}}{8\,\psi_{BL}^{7}}\left(1+\frac{u}{\psi_{BL}}\right)^{-7} = 0} \qquad (u)$$

w/ outer BC : $u \sim 1 + \mathcal{O}(1/r)$

Key observation : NO INNER BC are needed !

by evaluating the fields near the punctures, one sees that

at $x^i = C_P^i$ (u) is $\boxed{\Delta u = 0}$ regular!

↳ Can be solved on $\mathbb{R}^3$.

Next : CTS , XCTS ⇝ stationary slices ✓

"more complex" (well-posedness/smooth) ✗