Extrinsic curvature

Exercise 1.1:

Consider a sphere of radius a under the embedding in $\mathbb{R}^3$ specified through $t := r - a = 0$ with $r = \sqrt{x^2 + y^2 + z^2}$. Compute the intrinsic and extrinsic curvature.

(Soln) 1.1

For the general idea: see example from last tutorial.

We can introduce spherical coordinates $x^\alpha \doteq (r,\, \vartheta,\, \varphi)$ where:

$$
\begin{aligned}
x(r,\, \vartheta,\, \varphi) &= r \sin(\vartheta) \cos(\varphi), \\
y(r,\, \vartheta,\, \varphi) &= r \sin(\vartheta) \sin(\varphi), \\
z(r,\, \vartheta,\, \varphi) &= r \cos(\vartheta).
\end{aligned}
$$

This provides us (components of) the embedding map $\Phi : \hat{\mathbb{S}}^2_a \to \mathbb{R}^3$. In particular we think of $\Phi^k \doteq (x(a,\, \vartheta,\, \varphi),\, y(a,\, \vartheta,\, \varphi),\, z(a,\, \vartheta,\, \varphi))$. Consequently via pullback we have the induced metric:

$$
\begin{aligned}
\gamma_{IJ} \mathrm{d}x^I \mathrm{d}x^J = {} & \underbrace{g_{kl}}_{=\delta_{kl}} \frac{\partial \Phi^k}{\partial x^{\mathrm{I}}} \frac{\partial \Phi^l}{\partial x^{\mathrm{J}}} \mathrm{d}x^I \mathrm{d}x^J, \\
= {} & \underbrace{\left[a^2 \cos^2(\varphi) \cos^2(\vartheta) + a^2 \cos^2(\vartheta) \sin^2(\varphi) + a^2 \sin^2(\vartheta)\right]}_{a^2} \mathrm{d}\vartheta^2 \\
& + \underbrace{\left[-a^2 \cos(\varphi) \cos(\vartheta) \sin(\varphi) \sin(\vartheta) + a^2 \cos(\varphi) \cos(\vartheta) \sin(\varphi) \sin(\vartheta) + 0\right]}_{=0} \mathrm{d}\vartheta \mathrm{d}\varphi \\
& + \underbrace{\left[-a^2 \cos(\varphi) \cos(\vartheta) \sin(\varphi) \sin(\vartheta) + a^2 \cos(\varphi) \cos(\vartheta) \sin(\varphi) \sin(\vartheta) + 0\right]}_{=0} \mathrm{d}\varphi \mathrm{d}\vartheta \\
& + \underbrace{\left[a^2 \sin^2(\varphi) \sin^2(\vartheta) + a^2 \cos^2(\varphi) \sin^2(\vartheta) + 0\right]}_{a^2 \sin^2(\vartheta)} \mathrm{d}\varphi^2 \\
= {} & a^2 \Big(\underbrace{\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\, \mathrm{d}\varphi^2}_{=:\mathrm{d}\Omega^2} \Big),
\end{aligned}
$$

where we think of $x^I \doteq (\vartheta,\, \varphi)$.

So we have the *induced metric* - what about Riemann, Ricci and scalar curvature?

First recall that these are related by (Wald convention) $\mathrm{Riem}_{ikjl} g^{kl} = \mathrm{Ric}_{ij}$ and $g^{ij} \mathrm{Ric}_{ij} = \mathcal{R}$.

Dimensionality:

In general for $(\mathcal{M},\, g)$ with $\dim(\mathcal{M}) = n$ one has that the number of independent components of $\mathrm{Riem}[g]_{abcd}$ goes as $\frac{1}{12} n^2 (n^2 - 1)$ (try to justify this).

For $\dim(\Sigma) = 2$ the tensor Riem should therefore only have one independent component.

Working the relations between geometric objects:

$$
\mathrm{Ric}_{IJ} = \mathrm{Riem}_{IKJ}{}^K = \mathrm{Riem}_{I1J}{}^1 + \mathrm{Riem}_{I2J}{}^2,
$$

Therefore (using antisymmetry of Riem):

$$
\mathrm{Ric}_{11} = \mathrm{Riem}_{121}{}^2, \qquad \mathrm{Ric}_{12} = \mathrm{Riem}_{122}{}^2, \qquad \mathrm{Ric}_{21} = \mathrm{Riem}_{211}{}^1, \qquad \mathrm{Ric}_{22} = \mathrm{Riem}_{212}{}^1;
$$

which permits us to write:

$$
\mathcal{R} = \gamma^{IJ} \mathrm{Ric}_{IJ} = \gamma^{11} \mathrm{Riem}_{121}{}^2 + \gamma^{12} \mathrm{Riem}_{122}{}^2 + \gamma^{21} \mathrm{Riem}_{211}{}^1 + \gamma^{22} \mathrm{Riem}_{212}{}^1.
$$

Using:

$$
\gamma^{IJ} \doteq \frac{1}{\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}} \begin{bmatrix} \gamma_{22} & -\gamma_{12} \\ -\gamma_{21} & \gamma_{11} \end{bmatrix},
$$

leads to:

$$\begin{aligned}\mathcal{R} &= \frac{1}{\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}}\left[\gamma_{22}\mathrm{Riem}_{121}{}^{2} + -\gamma_{12}\mathrm{Riem}_{122}{}^{2} + -\gamma_{21}\mathrm{Riem}_{211}{}^{1} + \gamma_{11}\mathrm{Riem}_{212}{}^{1}\right],\\ &= \frac{2}{\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}}\mathrm{Riem}_{1212};\end{aligned}$$

where we also used:

$$\begin{aligned}\gamma_{22}\mathrm{Riem}_{121}{}^{2} - \gamma_{21}\mathrm{Riem}_{211}{}^{1} = \gamma_{22}\mathrm{Riem}_{121}{}^{2} + \gamma_{21}\mathrm{Riem}_{121}{}^{1} = \gamma_{2I}\mathrm{Riem}_{121}{}^{I} = \mathrm{Riem}_{1212},\\ -\gamma_{12}\mathrm{Riem}_{122}{}^{2} + \gamma_{11}\mathrm{Riem}_{212}{}^{1} = \gamma_{12}\mathrm{Riem}_{212}{}^{2} + \gamma_{11}\mathrm{Riem}_{212}{}^{1} = \mathrm{Riem}_{2121}.\end{aligned}$$

In principle, we therefore only need to compute Riem_{1212}. Non-trivial Christoffel symbols are:

$$\Gamma[\gamma]^{2}{}_{12} = \Gamma[\gamma]^{2}{}_{21} = \cot(\vartheta), \qquad \Gamma[\gamma]^{1}{}_{22} = -\cos(\vartheta)\sin(\vartheta);$$

which leads to $\mathrm{Riem}_{1212} = a^2\sin^2(\vartheta)$. When combined with the expression above for $\mathcal{R}$ we find:

$$\mathcal{R}[\gamma] = \frac{2}{a^2}.$$

Extrinsic quantities :

For this we identify an outward pointing normal to $\mathbb{S}^2_a$ in Cartesian coordinates as:

$$n^\alpha = \frac{1}{a}(x,\, y,\, z),$$

thus, working in these coordinates we immediately notice $\nabla_\beta[n^\alpha] = \partial_\beta[n^\alpha] = a^{-1}\delta^\alpha_\beta$ and therefore:

$$\nabla_\beta[n_\alpha] = a^{-1}g_{\alpha\beta}.$$

Recall that for the extrinsic curvature we have $K_{IJ} = \mathbf{K}(\boldsymbol{\partial}_I,\, \boldsymbol{\partial}_J) = -\nabla_\beta[n_\alpha](\partial_I)^\alpha(\partial_J)^\beta$, thus if we take the basis vectors $\boldsymbol{\partial}_I \doteq (\boldsymbol{\partial}_\vartheta,\, \boldsymbol{\partial}_\varphi)$ we find $\mathbf{K} = -a^{-1}\boldsymbol{\gamma}$. In components:

$$K_{IJ} = \begin{bmatrix} K_{\vartheta\vartheta} & K_{\vartheta\varphi}\\ K_{\varphi\vartheta} & K_{\varphi\varphi}\end{bmatrix} = \begin{bmatrix} -a & 0\\ 0 & -a\sin^2(\vartheta)\end{bmatrix}.$$

Finally taking the trace with γ:

$$K = -\frac{2}{a}.$$

Exercise 1.2:

Consider Schwarzschild in isotropic coordinates:

$$\mathrm{d}s^2 = -\alpha^2\mathrm{d}t^2 + \psi^4(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2),$$

where $\psi := (1 + M/2r)$, $\alpha := (1 - M/2r)/\psi$ and $\mathrm{d}\Omega^2$ is the standard 2-sphere metric. Identify spatial slices Σ with hypersurfaces of constant coordinate time t.

- What is the 3-metric?
- Compute the intrinsic and extrinsic curvature.

(Soln) 1.2

As a preliminary note that in $3+1$ decomposition the ambient metric can be written in the form:

$$\mathrm{d}s^2 = -\alpha^2\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + \beta^i\mathrm{d}t)(\mathrm{d}x^j + \beta^j\mathrm{d}t); \qquad \text{(eq:metrDec)}$$

Furthermore:

$$n^a = (\alpha^{-1},\, -\alpha^{-1}\beta^i), \qquad n_a = (-\alpha,\, 0,\, 0,\, 0).$$

It will also be useful for us to recall the evolution equation for the spatial metric:

$$\partial_t[\gamma_{ij}] = -2\alpha K_{ij} + D_i[\beta_j] + D_i[\beta_j], \qquad \text{(eq:evogam)}$$

where D_i is the Levi-Civita connection associated with the induced metric γ.

Clearly it is the case that $\boldsymbol{\beta} = 0$ and that the 3-metric is given by:

$$\gamma_{ij} = \psi^4\mathrm{diag}\big(1,\, r^2,\, r^2\sin^2\vartheta\big).$$

As γ_{ij} is time-independent and $\boldsymbol{\beta} = 0$ we immediately see that due to Eq.(eq:evogam) the extrinsic curvature and its trace are zero.

Notice that γ_{ij} is diagonal. In this case Christoffel symbols may be computed with the assistance of:

$$\begin{aligned} \Gamma^a{}_{bc}[\gamma] &= 0, & \Gamma^a{}_{bb}[\gamma] &= -\frac{1}{2\gamma_{aa}}\partial_a[\gamma_{bb}], \\ \Gamma^a{}_{ab}[\gamma] &= \partial_b\left[\log(\gamma_{aa})^{1/2}\right], & \Gamma^a{}_{aa}[\gamma] &= \partial_a\left[\log(\gamma_{aa})^{1/2}\right]; \end{aligned}$$

where $a \neq b \neq c$ and there is no summation over repeated indices.

As a check one should find that the non-zero Ricci 3-tensor components are:

$$\begin{aligned} \mathrm{Ric}_{rr} &= -\frac{8rM}{(2r^2+Mr)^2)}, \\ \mathrm{Ric}_{\vartheta\vartheta} &= \frac{4r^3M}{(2r^2+Mr)^2}, \\ \mathrm{Ric}_{\varphi\varphi} &= \sin^2\vartheta\,\mathrm{Ric}_{\vartheta\vartheta}; \end{aligned}$$

with scalar curvature $\mathcal{R}[\gamma] = 0$.

Gauss-Codazzi-Ricci

The Gauss-Codazzi-Ricci equations are identities relating the $3+1$ projections of the 4^{d} Riemann and Ricci tensors.

Note that:
$\gamma^a_b = g^{ac}\gamma_{cb} = \delta^a_b + n^a n_b$ is the $3+1$ spatial projection operator and n^a the unit normal vector to hypersurfaces Σ_t

exercise 2.1: derivation of Gauss equations

The Gauss equation is the spatial projection of the 4^{d} Riemann tensor ${}^4R_{abcd}$ that can be expressed in terms of the spatial (3^{d}) Riemann tensor R_{abcd} and extrinsic curvature K_{ab} as

$$\gamma^p_a\gamma^q_b\gamma^r_c\gamma^s_d{}^4R_{pqrs} = R_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc}.$$

Compute the above relation and the contractions

$$\begin{aligned} \gamma^p_a\gamma^q_b{}^4R_{pq} + \gamma_{ap}n^q\gamma^r_b n^s{}^4R^p_{qrs} &= R_{ab} + KK_{ab} - K_{ap}K^p_b, \\ {}^4R + 2n^a n^b{}^4R_{ab} &= R + K^2 - K_{ab}K^{ab}. \end{aligned}$$

(Soln) 2.1

We show:

$$\gamma^p_a\gamma^q_b\gamma^r_c\gamma^s_d{}^{[4]}\mathrm{Riem}_{pqrs} = {}^{[3]}\mathrm{Riem}_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc}.$$

As a preliminary recall the projector convention $\gamma^p_a = \delta^p_a + n^p n_a$.

Idea: we relate ${}^{[d]}\mathrm{Riem}$ and ${}^{[d-1]}\mathrm{Riem}$ definitions directly. To fix conventions suppose we have the ambient manifold, metric and Levi-Civita connection in the triplet ($\mathcal{M}$, g, ∇) together with analogous quantities for the submanifold (Σ, γ, D). Our first goal is to relate the two covariant derivatives.

Suppose $V \in \mathcal{T}^1(\Sigma)$. We have:

$$\begin{aligned} D_a[V^b] &= \gamma^c_a\gamma^b_d\nabla_c[V^d] = \gamma^c_a\Big(\underbrace{\delta^b_d}_{=g^b_d} + n_d n^b\Big)\nabla_c[V^d], \\ &= \gamma^c_a\nabla_c[V^b] + \gamma^c_a n_d n^b\nabla_c[V^d] =: (\star). \end{aligned}$$

Note:

$$\begin{aligned} n_dV^d = 0 \Longrightarrow 0 = \nabla_c[n_dV^d] &= \big(\nabla_c[n_d]\big)V^d + n_d\nabla_c[V^d], \\ \Longleftrightarrow -\nabla_c[n_d]V^d &= n_d\nabla_c[V^d]. \end{aligned}$$

Thus from $(\star)$:

$$
\begin{aligned}
D_a[V^b] &= \gamma^c_a \nabla_c[V^b] - \gamma^c_a V^d n^b \nabla_c[n_d], \\
&= \gamma^c_a \nabla_c[V^b] - \gamma^c_a V^e \gamma^d_e n^b \nabla_c[n_d], \\
&= \gamma^c_a \nabla_c[V^b] + K_{ae} V^e n^b;
\end{aligned}
$$

where we use $K_{ab} = -\gamma^c_a \gamma^d_b \nabla_c[n_d]$.

We need a second derivative (Riemann requires commutator). Thus:

$$
\begin{aligned}
D_a[D_b[V^c]] = D_a\left[\gamma^p_b {\gamma^c}_q \nabla_p[V^q]\right] &= \gamma^r_a \gamma^s_b \gamma^c_t \nabla_r\left[\gamma^p_s \gamma^t_q \nabla_p[V^q]\right], \\
&= \gamma^r_a \gamma^s_b \gamma^c_t \left[\gamma^p_s \gamma^t_q \nabla_r[\nabla_p[V^q]] + \nabla_p[V^q] \nabla_r[\gamma^p_s \gamma^t_q]\right], \\
&= \gamma^r_a \gamma^s_b \gamma^c_t \gamma^p_s \gamma^t_q \nabla_r[\nabla_p[V^q]] + \gamma^r_a \gamma^s_b \gamma^c_t \nabla_p[V^q] \nabla_r[\gamma^p_s \gamma^t_q], \\
&= \gamma^r_a \gamma^c_q \gamma^p_b \nabla_r[\nabla_p[V^q]] + \underbrace{\gamma^r_a \gamma^s_b \gamma^c_t \nabla_p[V^q]\left[\gamma^p_s \nabla_r[\gamma^t_q] + \gamma^t_q \nabla_r[\gamma^p_s]\right]}_{(\star\star)};
\end{aligned}
$$

$$
\begin{aligned}
(\star\star) &= \gamma^s_a \gamma^s_b \gamma^c_t \nabla_p[V^q]\left[\gamma^p_s \nabla_r[n^t n_q] + \gamma^t_q \nabla_r[n^p n_s]\right], \\
&= \gamma^s_a \gamma^s_b \gamma^c_t \nabla_p[V^q]\left[\gamma^p_s \left[\nabla_r[n^t] n_q + \underbrace{\nabla_r[n_q] n^t}_{=0}\right] + \gamma^t_q \left[n^p \nabla_r[n_s] + \underbrace{n_s \nabla_r[n^p]}_{=0}\right]\right], \\
&= \gamma^r_a \gamma^s_b \gamma^c_t \nabla_p[V^q]\left[\gamma^p_s n_q \nabla_r[n^t] + \gamma^t_q n^p \nabla_r[n_s]\right], \\
&= -\gamma^r_a \gamma^s_b \gamma^c_t \nabla_p[n_q] V^q \gamma^p_s \nabla_r[n^t] + \gamma^r_a \gamma^s_b \gamma^c_t \nabla_p[V^q] \gamma^t_q n^p \nabla_r[n_s], \\
&= \gamma^r_a \gamma^s_b \gamma^c_t K_{sq} V^q \nabla_r[n^t], \\
&= -{K_a}^c K_{bq} V^q - K_{ab} \gamma^c_q n^p \nabla_p[V^q];
\end{aligned}
$$

Thus, we have:

$$
D_a[D_b[V^c]] = \gamma^r_a \gamma^c_q \gamma^p_b \nabla_r[\nabla_p[V^q]] - {K_a}^c K_{bq} V^q - K_{ab} \gamma^c_q n^p \nabla_p[V^q].
$$

Recall that for $\omega \in \Omega(\Sigma)$ we have:

$$
{}^{[3]}\mathrm{Riem}_{abc}{}^d \omega_d = [D_a,\, D_b]\omega_c.
$$

Let $\omega_d = \gamma_{ad} V^d$ then ${}^{[3]}\mathrm{Riem}_{abcd} V^d = [D_a,\, D_b]\omega_c$ and ${}^{[3]}\mathrm{Riem}_{ab}{}^{cd} \omega_d = [D_a,\, D_b]\omega^c$.

We have:

$$
\begin{aligned}
D_a[D_b[V^c]] - D_b[D_a[V^c]] &= \gamma^r_a \gamma^c_q \gamma^p_b \nabla_r[\nabla_p[V^q]] - {K_a}^c K_{bq} V^q - K_{ab} \gamma^c_q n^p \nabla_p[V^q] \\
&\quad - \left[\gamma^r_b \gamma^c_q \gamma^p_a \nabla_r[\nabla_p[V^q]] - {K_b}^c K_{aq} V^q - K_{ba} \gamma^c_q n^p \nabla_p[V^q]\right], \\
&= \gamma^r_a \gamma^c_q \gamma^p_b [\nabla_r,\, \nabla_p] V^q - {K_a}^c K_{bq} V^q + {K_b}^c K_{aq} V^q, \\
&= {}^{[3]}{R_{ab}{}^c}_q V^q, \\
&= \gamma^r_a \gamma^c_q \gamma^p_b {}^{[4]}\mathrm{Riem}_{rp}{}^q{}_s V^s - {K_a}^c K_{bq} V^q + {K_b}^c K_{aq} V^q, \\
&= \gamma^r_a \gamma^c_s \gamma^p_b {}^{[4]}\mathrm{Riem}_{rp}{}^s{}_q V^q - {K_a}^c K_{bq} V^q + {K_b}^c K_{aq} V^q,
\end{aligned}
$$

where we relabel dummy indices $s \leftrightarrow q$ on the first term in the last expression - thus:

$$
\Longrightarrow \gamma^r_a \gamma^s_c \gamma^p_b \gamma^e_q {}^{[4]}\mathrm{Riem}_{rpse} = {}^{[3]}\mathrm{Riem}_{abcq} + K_{ac} K_{bq} - K_{bc} K_{aq}.
$$

For the contracted result use the metric and argue using ${}^{[3]}\mathrm{Riem}$ intrinsic to Σ etc.

exercise 2.2: derivation of Codazzi equations

The Codazzi equation is the projection of the 4^{d} Riemann tensor ${}^4R_{abcd}$

$$
\gamma^p_a \gamma^q_b \gamma^r_c n^{s\,4} R_{pqrs} = D_b K_{ac} - D_a K_{bc},
$$

expressed in terms of the spatial (3^{d}) Riemann tensor R_{abcd} and extrinsic curvature K_{ab}. Compute the above relation and the contraction

$$
\gamma^p_a n^{q\,4} R_{pq} = D_a K - D_s K^s_a.
$$

(Soln) 2.2

We show:

$$
\gamma^p_a \gamma^q_b \gamma^r_c n^{s\,[4]}\mathrm{Riem}_{pqrs} = D_b[K_{ac}] - D_a[K_{bc}].
$$

As a preliminary define $\mathcal{T}_1(\Sigma) \ni a_b := n^a \nabla_a[n_b]$ and notice that we can write:

$$K_{ab} = -\gamma_a^p \gamma_b^q \nabla_p[n_q] = -\left(\delta_a^p + n^p n_a\right)\left(\delta_b^q + n^q n_b\right)\nabla_p[n_q] = -\nabla_a[n_b] - n_a a_b.$$

Now:

$$\begin{aligned}
D_a[K_{bc}] &= \gamma_a^p \gamma_b^q \gamma_c^r \nabla_p[K_{qr}], \\
&= \gamma_a^p \gamma_b^q \gamma_c^r \nabla_p\Big[-\gamma_q^s \gamma_r^t \nabla_s[n_t]\Big], \\
&= \gamma_a^p \gamma_b^q \gamma_c^r \nabla_p\Big[-\nabla_q[n_r] - n_q a_r\Big], \\
&= -\gamma_a^p \gamma_b^q \gamma_c^r (\nabla_p[\nabla_q[n_r]] + \nabla_p(n_q a_r)), \\
&= -\gamma_a^p \gamma_b^q \gamma_c^r (\nabla_p[\nabla_q[n_r]] + a_r \nabla_p[n_q]) \qquad (n \perp \gamma), \\
&= -\gamma_a^p \gamma_b^q \gamma_c^r \nabla_p[\nabla_q[n_r]] + a_c K_{ac},
\end{aligned}$$

Notice:

$$\begin{aligned}
D_a[K_{bc}] - D_b[K_{ac}] = D_{[a}[K_{b]c}] &= -\gamma_a^p \gamma_b^q \gamma_c^r \nabla_{[p}[\nabla_{q]}[n_r]], \\
&= \gamma_a^p \gamma_b^q \gamma_c^r n^s\, {}^{[4]}\mathrm{Riem}_{pqrs}.
\end{aligned}$$

exercise 2.3: derivation of Ricci equations

The Ricci equation is the spatial projection of the 4^{d} Riemann tensor ${}^4R_{abcd}$

$$\gamma_a^p \gamma_b^q n^r n^{s4} R_{prqs} = \mathcal{L}_m K_{ab} + \alpha^{-1} D_a D_b \alpha + K_b^d K_{ad},$$

expressed in terms of the spatial (3^{d}) Riemann tensor R_{abcd} and extrisinc curvature K_{ab}. Above D_a the covariant derivative of (Σ_t, γ_{ab}), and $\mathcal{L}_n$ is the Lie derivative along n^a. Derive this equations. The term "$\gamma\gamma n n^4 R$" appears also in the contracted Ricci equation,

$$\gamma_a^p n^{q4} R_{pq} = D_a K - D_s K_a^s.$$

Combine the two equations to obtain

$$\gamma_a^p \gamma_b^{q4} R_{pq} = -\alpha^{-1} \mathcal{L}_m K_{ab} - \alpha^{-1} D_a D_b \alpha + R_{ab} + K K_{ab} - 2 K_{ar} K_b^r.$$

(Soln) 2.3

Preliminaries:

For this we need $K_{ab} = -\nabla_a[n_b] - n_a a_b$, $\mathcal{T}_1(\Sigma) \ni a_b := n^a \nabla_a[n_b]$, $a_b = D_b[\log(\alpha)]$, $D_a[a_b] = -a_a a_b + \frac{1}{\alpha} D_a[D_b[\alpha]]$.

Note: $a_b = D_b[\log(\alpha)]$ essentially follows from $\nabla_a[\nabla_b[t]] = \nabla_b[\nabla_a[t]]$.
Recall: If hypersurfaces Σ are level surfaces of coordinate time function t. Put $\Omega_a := \nabla_a[t]$ and introduce the usual normalisation such that $\Omega_a := \frac{1}{\alpha} n_a$ $(n^a n_a = -1)$.

$$\begin{aligned}
a_b = n^a \nabla_a[n_b] &= n^a \nabla_a[\alpha \Omega_b] = n^a \nabla_a[\alpha \nabla_b[t]], \\
&= n^a \Big[\nabla_a[\alpha] \nabla_b[t] + \alpha \nabla_a[\nabla_b[t]]\Big], \\
&= n^a \Big[\nabla_a[\alpha] \Omega_b + \alpha \nabla_b[\nabla_a[t]]\Big], \\
&= n^a \Big[\nabla_a[\alpha] n_b / \alpha + \alpha \nabla_b[\Omega_a]\Big], \\
&= n^a (n_b \nabla_a[\log(\alpha)] + \alpha \nabla_b[n_a/\alpha]), \\
&= n^a \Big(n_b \nabla_a[\log(\alpha)] - \frac{1}{\alpha} \nabla_b[\alpha] + \nabla_b[n_a]\Big)
\end{aligned}$$

Projecting yields: $a_b = \frac{1}{\alpha} D_a[\alpha]$.

Consider derivative:

$$\begin{aligned}
D_a[a_b] = D_a\Big[\frac{1}{\alpha} D_b[\alpha]\Big] &= D_a[1/\alpha] D_b[\alpha] + \frac{1}{\alpha} D_a[D_b[\alpha]], \\
&= -\frac{1}{\alpha^2} D_a[\alpha] D_b[\alpha] + \frac{1}{\alpha} D_a[D_b[\alpha]], \\
&= -\Big[\frac{1}{\alpha} D_a[\alpha]\Big]\Big[\frac{1}{\alpha} D_b[\alpha]\Big] + \frac{1}{\alpha} D_a[D_b[\alpha]], \\
&= -a_a a_b + \frac{1}{\alpha} D_a[D_b[\alpha]].
\end{aligned}$$

Derivation of result:
Consider Lie-derivative along normal direction:

$$
\begin{aligned}
\mathcal{L}_{\mathbf{n}}[K_{ab}] &= n^c\nabla_c[K_{ab}] + K_{ac}\nabla_b[n^c] + K_{bc}\nabla_a[n^c],\\
&= n^c\nabla_c[-\nabla_a[n_b] - n_a a_b] + K_{ac}\nabla_b[n^c] + K_{bc}\nabla_a[n^c],\\
&= n^c\nabla_c[-\nabla_a[n_b] - n_a a_b] + K_a{}^c \underbrace{\nabla_b[n_c]}_{=-K_{bc}-n_b a_c} + K_b{}^c \underbrace{\nabla_a[n_c]}_{=-K_{ac}-n_a a_c},\\
&= -n^c\nabla_c[\nabla_a[n_b]] - n^c\nabla_c[n_a a_b] - K_a{}^c(K_{bc} + n_b a_c) - K_b{}^c(K_{ac} + n_a a_c),
\end{aligned}
$$

The first term may be replaced with:

$$
\begin{aligned}
&{}^{[4]}\mathrm{Riem}_{dbac}n^d = 2\nabla_{[c}[\nabla_{a]}[n_b]] = (\nabla_c\nabla_a - \nabla_a\nabla_c)[n_b],\\
\iff &-{}^{[4]}\mathrm{Riem}_{dbac}n^d + \nabla_a[\nabla_c[n_b]] = \nabla_c[\nabla_a[n_b]].
\end{aligned}
$$

It follows then:

$$
\begin{aligned}
\mathcal{L}_{\mathbf{n}}[K_{ab}] &= n^c n^{d[4]}\mathrm{Riem}_{dbac} - n^c\nabla_a[\nabla_c[n_b]] - n^c\nabla_c[n_a]a_b - n^c n_a\nabla_c[a_b],\\
&\quad - K_a{}^c(K_{bc} + n_b a_c) - K_b{}^c(K_{ac} + n_a a_c),\\
&= n^c n^{d[4]}\mathrm{Riem}_{dbac} - n^c\nabla_a[\nabla_c[n_b]] - a_a a_b - n^c n_a\nabla_c[a_b],\\
&\quad - K_a{}^c(K_{bc} + n_b a_c) - K_b{}^c(K_{ac} + n_a a_c);
\end{aligned}
$$

Note the rewriting:

$$
\begin{aligned}
n^c\nabla_a[\nabla_c[n_b]] &= \nabla_a[\underbrace{n^c\nabla_c[n_b]}_{=a_b}] - \nabla_a[n^c]\nabla_c[n_b],\\
&= \nabla_a[a_b] - K_a{}^c K_{cb} - n_a a^c K_{cb}.
\end{aligned}
$$

We can therefore cancel terms:

$$
\implies \mathcal{L}_{\mathbf{n}}[K_{ab}] = -n^d n^{c[4]}R_{dbac} - \nabla_a[a_b] - n^c n_a\nabla_c[a_b] - a_a a_b - K^c{}_b K_{ac} - K_{ca}n_b a^c,
$$

Notice that $\mathcal{L}_{\mathbf{n}}[K_{ab}]$ is spatial, i.e.,

$$
\begin{aligned}
n^a\mathcal{L}_{\mathbf{n}}[K_{ab}] &= -\underbrace{n^a n^d n^{c[4]}\mathrm{Riem}_{dbac}}_{{}^{[4]}\mathrm{Riem}_{db(ac)}=0} - n^a\nabla_a[a_b] - n^c n^a n_a\nabla_c[a_b] - n^a a_a a_b\\
&\quad - \underbrace{n^a K_{ac}}_{=0}K^c{}_b - \underbrace{n^a K_{ca}}_{=0}n_b a^c,\\
&= -n^a\nabla_a[a_b] - n^c n^a n_a\nabla_c[a_b] = -n^a\nabla_a[a_b] + n^c\nabla_c[a_b] = 0.
\end{aligned}
$$

Therefore we lose no information taking a spatial projection on free indices:

$$
\begin{aligned}
\mathcal{L}_{\mathbf{n}}[K_{ab}] &= -n^d n^c\gamma_a^q\gamma_b^{r[4]}\mathrm{Riem}_{drqc} - \underbrace{\gamma_a^q\gamma_b^r\nabla_q[a_r]}_{D_a[a_b]} - a_a a_b - K^c{}_b K_{ac},\\
&= -n^d n^c\gamma_a^q\gamma_b^{r[4]}\mathrm{Riem}_{drqc} - \frac{1}{\alpha}D_a[D_b[\alpha]] - K^c{}_b K_{ac},
\end{aligned}
$$

where we used $D_a[a_b] = -a_a a_b + \frac{1}{\alpha}D_a[D_b[\alpha]]$.