

question on intrinsic and extrinsic geometry

Consider hypersurfaces embedded in the Euclidean \mathbb{R}^3 manifold.

Compute the intrinsic and extrinsic curvature of:

- (i) an embedded 2^d plane
- (ii) an (infinite) cylinder

solution

We recall some definitions and make a comment:

A **hypersurface** of $(\mathcal{M}, n+1)$ is the image Σ of an n -dimensional manifold $\hat{\Sigma}$ by an embedding $\Phi : \hat{\Sigma} \rightarrow \mathcal{M}$: $\Sigma = \Phi(\hat{\Sigma})$.

An **embedding** means that the map $\Phi : \hat{\Sigma} \rightarrow \Sigma$ is a homeomorphism. Geometrically this avoids self-intersection.

A hypersurface can be defined **locally** as the set of points for which a scalar field on \mathcal{M} is constant. Denote this field by σ and set the constant to zero, we get:

$$\forall p \in \mathcal{M}, \quad p \in \Sigma \iff \sigma(p) = 0$$

If $\Phi : \hat{\Sigma} \rightarrow \mathcal{M}$ is the embedding which induces the submanifold structure of \mathcal{M} , the pullback map Φ^* **induces** the natural metric $\gamma = \Phi^*g$ on Σ .

In components we have:

$$\gamma_{IJ}(x) = g_{kl}(\Phi(x)) \frac{\partial \Phi^k}{\partial x^I} \frac{\partial \Phi^l}{\partial x^J}, \quad (\text{eq:pullbackmetric})$$

where (Φ^k) denote the coordinates of $\Phi(x)$.

Given a Manifold, metric and associated Levi-Civita connection (\mathcal{M}, g, ∇) , the intrinsic curvature is usually introduced as the anti-commutator: $[\nabla_a, \nabla_b]\omega_c = \text{Riem}_{abc}{}^d \omega_d$. We may expand the definition of the covariant derivative to provide a method for computation:

$$\text{Riem}_{abc}{}^d[g] = -2\partial_{[a}\Gamma^d{}_{b]c} + 2\Gamma^e{}_{c[a}\Gamma^d{}_{b]e}; \quad (\text{eq:Riem})$$

with Christoffel symbols given by:

$$\Gamma^c{}_{ab}[g] = \frac{1}{2}g^{cd}(\partial_a[g_{bd}] + \partial_b[g_{ad}] - \partial_d[g_{ab}]). \quad (\text{eq:Christoffel})$$

The extrinsic curvature of Σ may be computed with the projector and unit normal n^a to Σ :

$$K_{ab} := -\gamma_a^c \gamma_b^d \nabla_c[n_d] = -\gamma_a^c \gamma_b^d \nabla_c[n_d], \quad (\text{eq:Extrinsic})$$

this may also be written via direct evaluation over appropriate basis vectors:

$$K_{IJ} = \mathbf{K}(\partial_I, \partial_J) = -\nabla_i[n_j](\partial_I)^i(\partial_J)^j. \quad (\text{eq:Extrinsic2})$$

We need to construct: Embedding map, induced metric, unit normal, intrinsic curvature Riem (Wald convention), extrinsic curvature K (negative convention).

(i)

Let $\hat{\Pi}$ be a connected submanifold of $\mathcal{M} := \mathbb{R}^3$ with topology \mathbb{R}^2 , i.e., the 2-d plane. Introduce locally (In this case they also hold globally) a set of coordinates for \mathcal{M} , $(x^i) = (x, y, z)$, such that σ spans \mathbb{R} and (x, y) are Cartesian coordinates spanning \mathbb{R}^2 . The hypersurface Π is then defined by the coordinate condition $\sigma = 0$. An explicit form of the mapping Φ can be obtained by considering $(x^I) = (x, y)$ as coordinates on $\hat{\Pi}$:

$$\begin{aligned} \Phi &:= \hat{\Pi} \rightarrow \mathcal{M}, \\ (x, y) &\mapsto (x, y, 0). \end{aligned}$$

In short, we view the hypersurface Π as the $z = 0$ plane. The scalar function σ defining Π is thus $\sigma = z$.

In order to find the induced metric we make use of Eq.(eq:pullbackmetric) and note that the Euclidean metric on \mathcal{M} is simply δ_{ij} . Clearly we must have $\gamma_{IJ} = \text{diag}(1, 1)$.

Furthermore, it follows trivially from Eq.(eq:Riem) and Eq.(eq:Christoffel) that R_{iem} vanishes (note $\Gamma[\delta] = 0$).

In order to compute the extrinsic curvature we select a (global) unit normal to Π , namely, $n^i = (0, 0, 1)$. For the present case we have $n_i \delta^{ij} = n^j$. Due to this and $\Gamma[\delta] = 0$ we have $\nabla_a[n_b] = \partial_a[n_b] = (0, 0, 0)$. Based on Eq.(eq:Extrinsic) we immediately conclude that the extrinsic curvature vanishes as well.

Geometric intuition: The unit vector stays constant when translated along Π and hence the extrinsic curvature (measuring how the hypersurface bends in the ambient space) is zero. Furthermore, on a plane, the internal angles of a triangle must sum to π - the intrinsic curvature vanishes verifies this fact.

(ii)

For the hypersurface describing our (infinite) cylinder \mathcal{C} we take the local definition $\sigma := \rho - a = 0$, where $\rho := \sqrt{x^2 + y^2}$ and $a \in \mathbb{R}^{>0}$ is the radius of the cylinder. We introduce the cylindrical coordinates $(x^i) = (\rho, \varphi, z)$, such that $\varphi \in [0, 2\pi)$, $x = \rho \cos \varphi$ and $y = \rho \sin \varphi$. On \mathcal{C} , $(x^I) = (\varphi, z)$ constitute a coordinate system.

Thus, we take:

$$\Phi : (\varphi, z) \mapsto (\rho \cos \varphi, \rho \sin \varphi, z).$$

From Eq.(eq:pullbackmetric) we can compute the induced metric:

$$\begin{aligned} \gamma &= \gamma_{IJ} dx^I \otimes dx^J = \delta_{ij} \frac{\partial \Phi^i}{\partial x^I} \frac{\partial \Phi^j}{\partial x^J} dx^I \otimes dx^J \\ &= a^2 d\varphi \otimes d\varphi + dz \otimes dz; \end{aligned}$$

Thus $\gamma_{IJ} = \text{diag}(a^2, 1)$.

Notice that a is constant. We may therefore introduce another coordinate $\eta := a\varphi$ and further map the induced metric via coordinate transformation to:

$$\gamma = d\eta \otimes d\eta + dz \otimes dz.$$

This is in the form of the Euclidean metric and therefore we once again conclude that the intrinsic curvature R_{iem} is zero.

In order to compute extrinsic curvature we need a unit normal. Notice that on \mathcal{C} for fixed φ , translation along the z -direction should leave the unit normal unaffected. We make the out-ward pointing choice:

$$n^i = \left(\frac{x}{a}, \frac{y}{a}, 0 \right) \implies n_i = \delta_{ij} n^j = \left(\frac{x}{a}, \frac{y}{a}, 0 \right),$$

with a constant. It follows then that:

$$\nabla_i[n_j] = \partial_i[n_j] = \text{diag}(1/a, 1/a, 0);$$

as $\Gamma[\gamma] = 0$.

Consider Eq.(eq:Extrinsic2). We have the natural basis $(\partial_I) = (\partial_\varphi, \partial_z)$ associated with the coordinates (φ, z) . The components $(\partial_I)^i$ are of the vector ∂_I we take with respect to the natural basis $(\partial)_i = (\partial_x, \partial_y, \partial_z)$ in Cartesian coordinates. As $\partial_\varphi = -y\partial_x + x\partial_y$, one has $(\partial_\varphi)^i = (-y, x, 0)$ and $(\partial_z)^i = (0, 0, 1)$.

We find:

$$\begin{aligned} \mathbf{K}(\partial_I, \partial_J) dx^I \otimes dx^J &= - \left(\frac{1}{a} (\partial_I)^1 (\partial_J)^1 + \frac{1}{a} (\partial_I)^2 (\partial_J)^2 \right) dx^I \otimes dx^J, \\ &= - \frac{1}{a} [(-y)(-y) + x^2] \partial_\varphi \otimes \partial_\varphi, \\ &= - a \partial_\varphi \otimes \partial_\varphi. \end{aligned}$$

The trace of the extrinsic curvature can be found using the relation $\gamma^{IJ} = \text{diag}(a^{-2}, 1)$ which results in $K = -a^{-1}$.

other things to try:

- Intrinsic / extrinsic quantities associated with sphere embedded in \mathbb{R}^3 .
- Given Schwarzschild in isotropic coordinates compute (spatial) 3-metric, extrinsic curvature and its trace. (see Eq.6.11 of lecture notes)