Appendix B

Conformal Killing operator and conformal vector Laplacian

Contents

In this Appendix, we investigate the main properties of two important vectorial operators on Riemannian manifolds: the *conformal Killing operator* and the associated *conformal vector Laplacian*. The framework is that of a single three-dimensional manifold Σ, endowed with a positive definite metric (i.e. a Riemannian metric). In practice, Σ is embedded in some spacetime $(\mathcal{M}, \boldsymbol{g})$, as being part of a 3+1 foliation $(\Sigma_t)_{t\in\mathbb{R}}$, but we shall not make such an assumption here. For concreteness, we shall denote Σ's Riemannian metric by $\tilde{\boldsymbol{\gamma}}$, because in most applications of the 3+1 formalism, the conformal Killing operator appears for the metric $\tilde{\boldsymbol{\gamma}}$ conformally related to the physical metric $\boldsymbol{\gamma}$ and introduced in Chap. 6. But again, we shall not use the hypothesis that $\tilde{\boldsymbol{\gamma}}$ is derived from some "physical" metric $\boldsymbol{\gamma}$. So in all what follows, $\tilde{\boldsymbol{\gamma}}$ can be replaced by the physical metric $\boldsymbol{\gamma}$ or any other Riemannian metric, as for instance the background metric $\boldsymbol{f}$ introduced in Chap. 6 and 7.

B.1 Conformal Killing operator

B.1.1 Definition

The ***conformal Killing operator*** $\tilde{\boldsymbol{L}}$ associated with the metric $\tilde{\boldsymbol{\gamma}}$ is the linear mapping from the space $\mathcal{T}(\Sigma)$ of vector fields on Σ to the space of symmetric tensor fields of type $\binom{2}{0}$ defined by

$$\boxed{\forall \boldsymbol{v} \in \mathcal{T}(\Sigma), \quad (\tilde{L}v)^{ij} := \tilde{D}^i v^j + \tilde{D}^j v^i - \frac{2}{3}\tilde{D}_k v^k \, \tilde{\gamma}^{ij}}, \tag{B.1}$$

where $\tilde{\boldsymbol{D}}$ is the Levi-Civita connection associated with $\tilde{\boldsymbol{\gamma}}$ and $\tilde{D}^i := \tilde{\gamma}^{ij}\tilde{D}_j$. An immediate property of $\tilde{\boldsymbol{L}}$ is to be traceless with respect to $\tilde{\boldsymbol{\gamma}}$, thanks to the $-2/3$ factor: for any vector $\boldsymbol{v}$,

$$\tilde{\gamma}_{ij}(\tilde{L}v)^{ij} = 0. \tag{B.2}$$

B.1.2 Behavior under conformal transformations

An important property of $\tilde{\boldsymbol{L}}$ is to be invariant, except for some scale factor, with respect to conformal transformations. Indeed let us consider a metric $\boldsymbol{\gamma}$ conformally related to $\tilde{\boldsymbol{\gamma}}$:

$$\boldsymbol{\gamma} = \Psi^4 \tilde{\boldsymbol{\gamma}}. \tag{B.3}$$

In practice $\boldsymbol{\gamma}$ will be the metric induced on Σ by the spacetime metric $\boldsymbol{g}$ and Ψ the conformal factor defined in Chap. 6, but we shall not employ this here. So $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ are any two Riemannian metrics on Σ that are conformally related (we could have called them $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$) and Ψ is simply the conformal factor between them. We can employ the formulæ derived in Chap. 6 to relate the conformal Killing operator of $\tilde{\boldsymbol{\gamma}}$, $\tilde{\boldsymbol{L}}$, with that of $\boldsymbol{\gamma}$, $\boldsymbol{L}$ say. Formula (6.35) gives

$$\begin{aligned} D^j v^i &= \gamma^{jk} D_k v^i = \Psi^{-4}\tilde{\gamma}^{jk}\left[\tilde{D}_k v^i + 2\left(v^l \tilde{D}_l \ln\Psi\, \delta^i{}_k + v^i \tilde{D}_k \ln\Psi - \tilde{D}^i \ln\Psi\, \tilde{\gamma}_{kl} v^l\right)\right] \\ &= \Psi^{-4}\left[\tilde{D}^j v^i + 2\left(v^k \tilde{D}_k \ln\Psi\, \tilde{\gamma}^{ij} + v^i \tilde{D}^j \ln\Psi - v^j \tilde{D}^i \ln\Psi\right)\right]. \end{aligned} \tag{B.4}$$

Hence

$$D^i v^j + D^j v^i = \Psi^{-4}\left(\tilde{D}^i v^j + \tilde{D}^j v^i + 4 v^k \tilde{D}_k \ln\Psi\, \tilde{\gamma}^{ij}\right) \tag{B.5}$$

Besides, from Eq. (6.36),

$$-\frac{2}{3} D_k v^k\, \gamma^{ij} = -\frac{2}{3}\left(\tilde{D}_k v^k + 6 v^k \tilde{D}_k \ln\Psi\right)\Psi^{-4}\tilde{\gamma}^{ij}. \tag{B.6}$$

Adding the above two equations, we get the simple relation

$$\boxed{(Lv)^{ij} = \Psi^{-4}(\tilde{L}v)^{ij}}. \tag{B.7}$$

Hence the conformal Killing operator is invariant, up to the scale factor Ψ^{-4}, under a conformal transformation.

B.1.3 Conformal Killing vectors

Let us examine the kernel of the conformal Killing operator, i.e. the subspace $\ker\tilde{\boldsymbol{L}}$ of $\mathcal{T}(\Sigma)$ constituted by vectors $\boldsymbol{v}$ satisfying

$$(\tilde{L}v)^{ij} = 0. \tag{B.8}$$

A vector field which obeys Eq. (B.8) is called a ***conformal Killing vector***. It is the generator of some conformal isometry of $(\Sigma, \tilde{\boldsymbol{\gamma}})$. A ***conformal isometry*** is a diffeomorphism $\Phi : \Sigma \to \Sigma$ for which there exists some scalar field Ω such that $\Phi_* \tilde{\boldsymbol{\gamma}} = \Omega^2 \tilde{\boldsymbol{\gamma}}$. Notice that any isometry is a conformal isometry (corresponding to $\Omega = 1$), which means that every Killing vector is a conformal Killing vector. The latter property is obvious from the definition (B.1) of the conformal Killing operator. Notice also that any conformal isometry of $(\Sigma, \tilde{\boldsymbol{\gamma}})$ is a conformal isometry of $(\Sigma, \boldsymbol{\gamma})$, where $\boldsymbol{\gamma}$ is a metric conformally related to $\tilde{\boldsymbol{\gamma}}$ [cf. Eq. (B.3)]. Of course, $(\Sigma, \tilde{\boldsymbol{\gamma}})$ may not admit any conformal isometry at all, yielding $\ker \tilde{\boldsymbol{L}} = \{0\}$. The maximum dimension of $\ker \tilde{\boldsymbol{L}}$ is 10 (taking into account that Σ has dimension 3). If $(\Sigma, \tilde{\boldsymbol{\gamma}})$ is the Euclidean space $(\mathbb{R}^3, \boldsymbol{f})$, the conformal isometries are constituted by the isometries (translations, rotations) augmented by the homotheties.

B.2 Conformal vector Laplacian

B.2.1 Definition

The ***conformal vector Laplacian*** associated with the metric $\tilde{\boldsymbol{\gamma}}$ is the endomorphism $\tilde{\boldsymbol{\Delta}}_L$ of the space $\mathcal{T}(\Sigma)$ of vector fields on Σ defined by taking the divergence of the conformal Killing operator:

$$\boxed{\forall \boldsymbol{v} \in \mathcal{T}(\Sigma), \quad \tilde{\Delta}_L \, v^i := \tilde{D}_j (\tilde{L} v)^{ij}}. \tag{B.9}$$

From Eq. (B.1),

$$\begin{aligned} \tilde{\Delta}_L \, v^i &= \tilde{D}_j \tilde{D}^i v^j + \tilde{D}_j \tilde{D}^j v^i - \frac{2}{3} \tilde{D}^i \tilde{D}_k v^k \\ &= \tilde{D}^i \tilde{D}_j v^j + \tilde{R}^i{}_j v^j + \tilde{D}_j \tilde{D}^j v^i - \frac{2}{3} \tilde{D}^i \tilde{D}_j v^j \\ &= \tilde{D}_j \tilde{D}^j v^i + \frac{1}{3} \tilde{D}^i \tilde{D}_j v^j + \tilde{R}^i{}_j v^j, \end{aligned} \tag{B.10}$$

where we have used the contracted Ricci identity (6.42) to get the second line. Hence $\tilde{\Delta}_L \, v^i$ is a second order operator acting on the vector $\boldsymbol{v}$, which is the sum of the vector Laplacian $\tilde{D}_j \tilde{D}^j v^i$, one third of the gradient of divergence $\tilde{D}^i \tilde{D}_j v^j$ and the curvature term $\tilde{R}^i{}_j v^j$:

$$\boxed{\tilde{\Delta}_L \, v^i = \tilde{D}_j \tilde{D}^j v^i + \frac{1}{3} \tilde{D}^i \tilde{D}_j v^j + \tilde{R}^i{}_j v^j} \tag{B.11}$$

The conformal vector Laplacian plays an important role in 3+1 general relativity, for solving the constraint equations (Chap. 8), but also for the time evolution problem (Sec. 9.3.2). The main properties of $\tilde{\boldsymbol{\Delta}}_L$ have been first investigated by York [274, 275].

B.2.2 Elliptic character

Given $p \in \Sigma$ and a linear form $\boldsymbol{\xi} \in \mathcal{T}_p^*(\Sigma)$, the ***principal symbol*** of $\tilde{\boldsymbol{\Delta}}_L$ with respect to p and $\boldsymbol{\xi}$ is the linear map $\boldsymbol{P}_{(p,\boldsymbol{\xi})} : \mathcal{T}_p(\Sigma) \to \mathcal{T}_p(\Sigma)$ defined as follows (see e.g. [101]). Keep only the terms involving the highest derivatives in $\tilde{\boldsymbol{\Delta}}_L$ (i.e. the second order ones): in terms of components, the operator is then reduced to

$$v^i \longmapsto \tilde{\gamma}^{jk} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k} v^i + \frac{1}{3} \tilde{\gamma}^{ik} \frac{\partial}{\partial x^k} \frac{\partial}{\partial x^j} v^j \tag{B.12}$$

Replace each occurrence of $\partial/\partial x^j$ by the component ξ_j of the linear form $\boldsymbol{\xi}$, thereby obtaining a mapping which is no longer differential, i.e. that involves only values of the fields at the point p; this is the principal symbol of $\tilde{\boldsymbol{\Delta}}_L$ at p with respect to $\boldsymbol{\xi}$:

$$\begin{array}{rccl} \boldsymbol{P}_{(p,\boldsymbol{\xi})}: & \mathcal{T}_p(\Sigma) & \longrightarrow & \mathcal{T}_p(\Sigma) \\ & \boldsymbol{v}=(v^i) & \longmapsto & \boldsymbol{P}_{(p,\boldsymbol{\xi})}(\boldsymbol{v}) = \left(\tilde{\gamma}^{jk}(p)\xi_j\xi_k\, v^i + \frac{1}{3}\tilde{\gamma}^{ik}(p)\xi_k\xi_j v^j\right), \end{array} \tag{B.13}$$

The differential operator $\tilde{\boldsymbol{\Delta}}_L$ is said to be ***elliptic*** on Σ iff the principal symbol $\boldsymbol{P}_{(p,\boldsymbol{\xi})}$ is an isomorphism for every $p \in \Sigma$ and every non-vanishing linear form $\boldsymbol{\xi} \in \mathcal{T}_p^*(\Sigma)$. It is said to be ***strongly elliptic*** if all the eigenvalues of $\boldsymbol{P}_{(p,\boldsymbol{\xi})}$ are non-vanishing and have the same sign. To check whether it is the case, let us consider the bilinear form $\tilde{\boldsymbol{P}}_{(p,\boldsymbol{\xi})}$ associated to the endomorphism $\boldsymbol{P}_{(p,\boldsymbol{\xi})}$ by the conformal metric:

$$\forall (\boldsymbol{v},\boldsymbol{w}) \in \mathcal{T}_p(\Sigma)^2, \quad \tilde{\boldsymbol{P}}_{(p,\boldsymbol{\xi})}(\boldsymbol{v},\boldsymbol{w}) = \tilde{\boldsymbol{\gamma}}\left(\boldsymbol{v},\, \boldsymbol{P}_{(p,\boldsymbol{\xi})}(\boldsymbol{w})\right). \tag{B.14}$$

Its matrix $\tilde{P}_{ij}$ is deduced from the matrix $P^i_{\ j}$ of $\boldsymbol{P}_{(p,\boldsymbol{\xi})}$ by lowering the index i with $\tilde{\boldsymbol{\gamma}}(p)$. We get

$$\tilde{P}_{ij} = \tilde{\gamma}^{kl}(p)\xi_k\xi_l\,\tilde{\gamma}_{ij}(p) + \frac{1}{3}\xi_i\xi_j. \tag{B.15}$$

Hence $\tilde{\boldsymbol{P}}_{(p,\boldsymbol{\xi})}$ is clearly a symmetric bilinear form. Moreover it is positive definite for $\boldsymbol{\xi} \neq 0$: for any vector $\boldsymbol{v} \in \mathcal{T}_p(\Sigma)$ such that $\boldsymbol{v} \neq 0$, we have

$$\tilde{\boldsymbol{P}}_{(p,\boldsymbol{\xi})}(\boldsymbol{v},\boldsymbol{v}) = \tilde{\gamma}^{kl}(p)\xi_k\xi_l\,\tilde{\gamma}_{ij}(p)v^iv^j + \frac{1}{3}(\xi_i v^i)^2 > 0, \tag{B.16}$$

where the > 0 follows from the positive definite character of $\tilde{\boldsymbol{\gamma}}$. $\tilde{\boldsymbol{P}}_{(p,\boldsymbol{\xi})}$ being positive definite symmetric bilinear form, we conclude that $\boldsymbol{P}_{(p,\boldsymbol{\xi})}$ is an isomorphism and that all its eigenvalues are real and strictly positive. Therefore $\tilde{\boldsymbol{\Delta}}_L$ is a strongly elliptic operator.

B.2.3 Kernel

Let us now determine the kernel of $\tilde{\boldsymbol{\Delta}}_L$. Clearly this kernel contains the kernel of the conformal Killing operator $\tilde{\boldsymbol{L}}$. Actually it is not larger than that kernel:

$$\boxed{\ker \tilde{\boldsymbol{\Delta}}_L = \ker \tilde{\boldsymbol{L}}}. \tag{B.17}$$

Let us establish this property. For any vector field $\boldsymbol{v} \in \mathcal{T}(\Sigma)$, we have

$$\begin{aligned} \int_\Sigma \tilde{\gamma}_{ij} v^i \tilde{\Delta}_L\, v^j \sqrt{\tilde{\gamma}}\, d^3x &= \int_\Sigma \tilde{\gamma}_{ij} v^i \tilde{D}_l (\tilde{L}v)^{jl} \sqrt{\tilde{\gamma}}\, d^3x \\ &= \int_\Sigma \left\{ \tilde{D}_l \left[\tilde{\gamma}_{ij} v^i (\tilde{L}v)^{jl}\right] - \tilde{\gamma}_{ij}\tilde{D}_l v^i\, (\tilde{L}v)^{jl} \right\} \sqrt{\tilde{\gamma}}\, d^3x \\ &= \oint_{\partial\Sigma} \tilde{\gamma}_{ij} v^i (\tilde{L}v)^{jl} \tilde{s}_l \sqrt{\tilde{q}}\, d^2y - \int_\Sigma \tilde{\gamma}_{ij}\tilde{D}_l v^i\, (\tilde{L}v)^{jl} \sqrt{\tilde{\gamma}}\, d^3x, \end{aligned} \tag{B.18}$$

where the Gauss-Ostrogradsky theorem has been used to get the last line. We shall consider two situations for $(\Sigma, \boldsymbol{\gamma})$:

- Σ is a *closed manifold*, i.e. is compact without boundary;
- $(\Sigma, \tilde{\gamma})$ is an *asymptotically flat manifold*, in the sense made precise in Sec. 7.2.

In the former case the lack of boundary of Σ implies that the first integral in the right-hand side of Eq. (B.18) is zero. In the latter case, we will restrict our attention to vectors $\boldsymbol{v}$ which decay at spatial infinity according to (cf. Sec. 7.2)

$$v^i = O(r^{-1}) \qquad \text{(B.19)}$$

$$\frac{\partial v^i}{\partial x^j} = O(r^{-2}), \qquad \text{(B.20)}$$

where the components are to be taken with respect to the asymptotically Cartesian coordinate system (x^i) introduced in Sec. 7.2. The behavior (B.19)-(B.20) implies

$$v^i (\tilde{L}v)^{jl} = O(r^{-3}), \qquad \text{(B.21)}$$

so that the surface integral in Eq. (B.18) vanishes. So for both cases of Σ closed or asymptotically flat, Eq. (B.18) reduces to

$$\int_\Sigma \tilde{\gamma}_{ij} v^i \tilde{\Delta}_L \, v^j \, \sqrt{\tilde{\gamma}} \, d^3x = - \int_\Sigma \tilde{\gamma}_{ij} \tilde{D}_l v^i \, (\tilde{L}v)^{jl} \sqrt{\tilde{\gamma}} \, d^3x. \qquad \text{(B.22)}$$

In view of the right-hand side integrand, let us evaluate

$$\begin{aligned} \tilde{\gamma}_{ij}\tilde{\gamma}_{kl}(\tilde{L}v)^{ik}(\tilde{L}v)^{jl} &= \tilde{\gamma}_{ij}\tilde{\gamma}_{kl}(\tilde{D}^i v^k + \tilde{D}^k v^i)(\tilde{L}v)^{jl} - \frac{2}{3}\tilde{D}_m v^m \underbrace{\tilde{\gamma}^{ik}\tilde{\gamma}_{ij}}_{=\delta^k{}_j} \tilde{\gamma}_{kl}(\tilde{L}v)^{jl} \\ &= \left(\tilde{\gamma}_{kl}\tilde{D}_j v^k + \tilde{\gamma}_{ij}\tilde{D}_l v^i\right)(\tilde{L}v)^{jl} - \frac{2}{3}\tilde{D}_m v^m \underbrace{\tilde{\gamma}_{jl}(\tilde{L}v)^{jl}}_{=0} \\ &= 2\tilde{\gamma}_{ij}\tilde{D}_l v^i \, (\tilde{L}v)^{jl}, \end{aligned} \qquad \text{(B.23)}$$

where we have used the symmetry and the traceless property of $(\tilde{L}v)^{jl}$ to get the last line. Hence Eq. (B.22) becomes

$$\int_\Sigma \tilde{\gamma}_{ij} v^i \tilde{\Delta}_L \, v^j \, \sqrt{\tilde{\gamma}} \, d^3x = -\frac{1}{2} \int_\Sigma \tilde{\gamma}_{ij}\tilde{\gamma}_{kl}(\tilde{L}v)^{ik}(\tilde{L}v)^{jl} \sqrt{\tilde{\gamma}} \, d^3x. \qquad \text{(B.24)}$$

Let us assume now that $\boldsymbol{v} \in \ker \tilde{\boldsymbol{\Delta}}_L$: $\tilde{\Delta}_L \, v^j = 0$. Then the left-hand side of the above equation vanishes, leaving

$$\int_\Sigma \tilde{\gamma}_{ij}\tilde{\gamma}_{kl}(\tilde{L}v)^{ik}(\tilde{L}v)^{jl} \sqrt{\tilde{\gamma}} \, d^3x = 0. \qquad \text{(B.25)}$$

Since $\tilde{\boldsymbol{\gamma}}$ is a positive definite metric, we conclude that $(\tilde{L}v)^{ij} = 0$, i.e. that $\boldsymbol{v} \in \ker \tilde{\boldsymbol{L}}$. This demonstrates property (B.17). Hence the "harmonic functions" of the conformal vector Laplacian $\tilde{\boldsymbol{\Delta}}_L$ are nothing but the conformal Killing vectors (one should add "which vanish at spatial infinity as (B.19)-(B.20)" in the case of an asymptotically flat space).

B.2.4 Solutions to the conformal vector Poisson equation

Let now discuss the existence and uniqueness of solutions to the conformal vector Poisson equation

$$\boxed{\tilde{\Delta}_L\, v^i = S^i}, \tag{B.26}$$

where the vector field $\boldsymbol{S}$ is given (the source). Again, we shall distinguish two cases: the closed manifold case and the asymptotically flat one. When Σ is a closed manifold, we notice first that a necessary condition for the solution to exist is that the source must be orthogonal to any vector field in the kernel, in the sense that

$$\forall \boldsymbol{C} \in \ker \tilde{\boldsymbol{L}}, \qquad \int_\Sigma \tilde{\gamma}_{ij} C^i S^j \sqrt{\tilde{\gamma}}\, d^3x = 0. \tag{B.27}$$

This is easily established by replacing S^j by $\tilde{\Delta}_L\, v^i$ and performing the same integration by part as above to get

$$\int_\Sigma \tilde{\gamma}_{ij} C^i S^j \sqrt{\tilde{\gamma}}\, d^3x = -\frac{1}{2} \int_\Sigma \tilde{\gamma}_{ij} \tilde{\gamma}_{kl} (\tilde{L}C)^{ik} (\tilde{L}v)^{jl} \sqrt{\tilde{\gamma}}\, d^3x. \tag{B.28}$$

Since, by definition $(\tilde{L}C)^{ik} = 0$, Eq. (B.27) follows. If condition (B.27) is fulfilled (it may be trivial since the metric $\tilde{\boldsymbol{\gamma}}$ may not admit any conformal Killing vector at all), it can be shown that Eq. (B.26) admits a solution and that this solution is unique up to the addition of a conformal Killing vector.

In the asymptotically flat case, we assume that, in terms of the asymptotically Cartesian coordinates (x^i) introduced in Sec. 7.2

$$S^i = O(r^{-3}). \tag{B.29}$$

Moreover, because of the presence of the Ricci tensor in $\tilde{\boldsymbol{\Delta}}_L$, one must add the decay condition

$$\frac{\partial^2 \tilde{\gamma}_{ij}}{\partial x^k \partial x^l} = O(r^{-3}) \tag{B.30}$$

to the asymptotic flatness conditions introduced in Sec. 7.2 [Eqs. (7.1) to (7.4)]. Indeed Eq. (B.30) along with Eqs. (7.1)-(7.2) guarantees that

$$\tilde{R}_{ij} = O(r^{-3}). \tag{B.31}$$

Then a general theorem by Cantor (1979) [78] on elliptic operators on asymptotically flat manifolds can be invoked (see Appendix B of Ref. [246] as well as Ref. [91]) to conclude that the solution of Eq. (B.26) with the boundary condition

$$v^i = 0 \qquad \text{when } r \to 0 \tag{B.32}$$

exists and is unique. The possibility to add a conformal Killing vector to the solution, as in the compact case, does no longer exist because there is no conformal Killing vector which vanishes at spatial infinity on asymptotically flat Riemannian manifolds.

Regarding numerical techniques to solve the conformal vector Poisson equation (B.26), let us mention that a very accurate spectral method has been developed by Grandclément et al. (2001) [148] in the case of the Euclidean space: $(\Sigma, \tilde{\boldsymbol{\gamma}}) = (\mathbb{R}^3, \boldsymbol{f})$. It is based on the use of Cartesian components of vector fields altogether with spherical coordinates. An alternative technique, using both spherical components and spherical coordinates is presented in Ref. [63].