NR

22.06.2021

GAUGE CONDITIONS

Maximal slicing: $0 = K = -\nabla_a n^a = -\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\, n^a\right)$

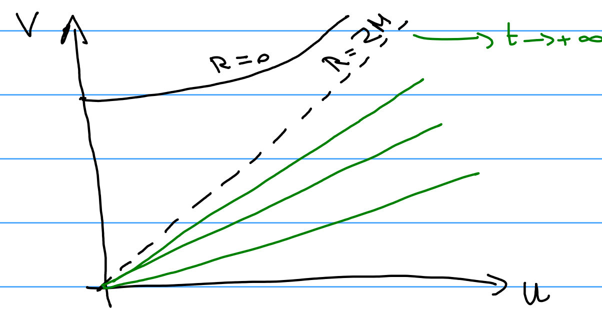
$\hookrightarrow \quad D^i D_i \alpha - \alpha\left[4\pi(E+S) + K_{ij}K^{ij}\right] = 0$ Eq. for α (MSα)

Schwarzschild foliations

(t, R) Schw. coords

$R > 2M \quad t = \text{const.}$

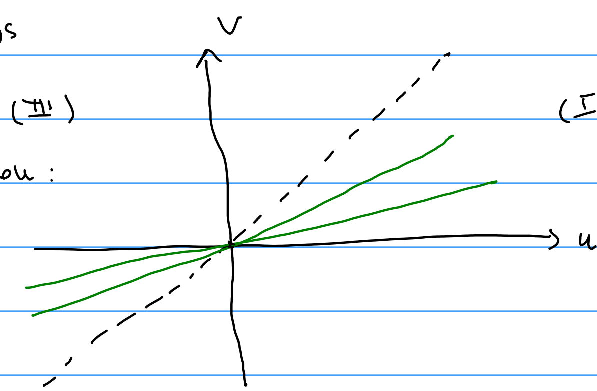
$K_{ab} \equiv 0 \Rightarrow K = 0$



- Maximal
- Not horizon penetrating

r : isotropic radius

Extended foliation:



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$$\alpha = \left(1 - \frac{M}{2r}\right)\left(1 + \frac{M}{2r}\right)^{-1}$$

- α antisymmetric
- $\alpha < 0$ in III

$R<2M$ $\quad$ $R=\text{const}$ $\quad$ (R is timelike)

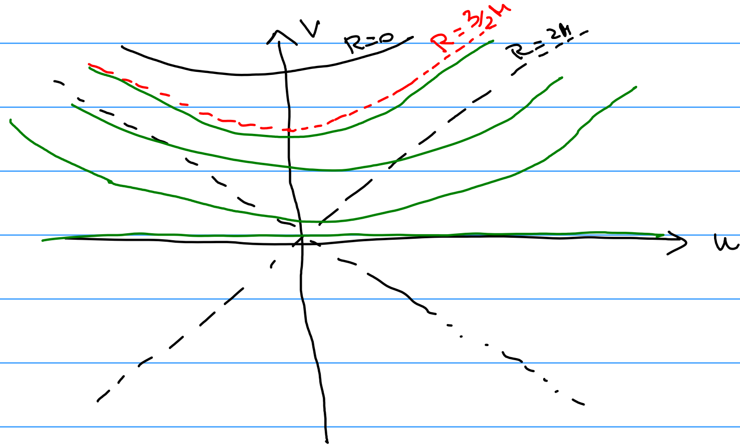
$$\alpha = \left(\frac{2M}{R}-1\right)^{-1} \qquad \gamma = R^4 \sin^2\theta \left(\frac{2M}{R}-1\right)$$

Use: $K = -\frac{1}{\alpha}\mathcal{L}_m \ln\sqrt{\gamma}$:

$$K = \frac{3M-2R}{R^2\left(\frac{2M}{R}-1\right)^{1/2}}$$

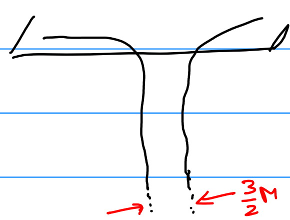
$\hookrightarrow$ $R=\frac{3M}{2}$ is a maximal ($K=0$) slice

$\hookrightarrow$ (can be proven) this is a limit slice of



②

" Trumpet slices "



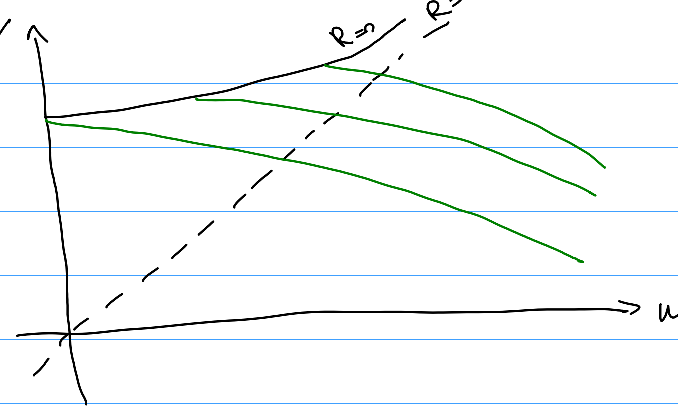
- Maximal
- Symmetric w.r.t. $R=2M$ ($r=\frac{M}{2}$)
- Horizon penetrating

① and ② correspond to solutions of (MS α) w/ BCs :

$$\begin{cases} \alpha \to 1 \text{ at } i_0 \\ \alpha(2M)=0 \ \text{(Dirichlet)} \to ① \\ \quad \text{OR} \\ \partial_R \alpha\left(\frac{3}{2}M\right)=0 \ \text{(Newmann)} \to ② \end{cases} \Bigg\} \ \underline{\text{SINGULARITY AVOIDING}}$$

Ingoing Eddington-Finkelstein (Kerr-Schild) :

$$t_{KS} = t + 2M \ln\left(\frac{r}{2M} - 1\right)$$



Example of NON-SINGULARITY AVOIDING foliation !

Q: How to construct in general max. foliations of Schw ?

A: Consider the transformation :

$$t \to \tilde{t} = t + h(R)$$

Height function

Then determine the $h(R)$:

$$0 = K \quad \Rightarrow \quad h'(R) = \frac{C^2}{A^2(R)\left[A(R)R^4 + C^2\right]}$$

w/ $A(R) := \left(1 - \frac{2M}{R}\right)$ and C is a constant of integration.

Metric :

$$\alpha = f(R) \qquad\qquad f(R) := 1 + \frac{2M}{R} + \frac{C^2}{R^4}$$

$$\beta^r = \frac{C\sqrt{f(R)}}{R^2}, \quad \gamma_{ij}dx^i dx^j = f^{-1}(R)\,dR^2 + R^2 d\Omega$$

Family of foliation for different choices of C :

- $C = 0$: standard Schw. time (1)
- other choices possible :

$$C = \frac{3}{4}\sqrt{3}\,M^2$$ (2)

Main property of max. slicing (generic feat.)

$\boxed{\alpha \to 0}$ in highest curvature regions

"lapse freezing" or "Collapse of the lapse"

$\hookrightarrow$ singularity avoiding gauge.

Harmonic slicing

$\Box x^\mu = 0$ Harmonic gauge

$\mu = 0$ $\Box t = 0$ Harmonic slicing

$$0 = \Box t = \sqrt{-g}\, \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \underbrace{\partial_\nu t}_{\delta_\nu{}^0} \right) = \sqrt{-g}\, \partial_\mu \left(\sqrt{-g}\, g^{\mu 0} \right)$$

$$\hookrightarrow \quad 0 = \partial_t \left(\alpha \sqrt{\gamma} \underbrace{g^{00}}_{\alpha^{-2}} \right) + \partial_i \left(\alpha \sqrt{\gamma} \underbrace{g^{0i}}_{\beta^i/\alpha^2} \right)$$

$$\hookrightarrow \quad 0 = \partial_t \alpha - \beta^i \partial_i \alpha - \alpha \underbrace{\left[\frac{1}{\sqrt{\gamma}} \partial_t \sqrt{\gamma} - \underbrace{\frac{1}{\sqrt{\gamma}} \partial_i \left(\sqrt{\gamma} \beta^i \right)}_{= D_i \beta^i} \right]}_{= -\alpha K}$$

$$\boxed{\mathcal{L}_m \alpha = -\alpha^2 K}$$ Harm. slicing (ev. equation for α)

$\beta^i \equiv 0$: $\alpha = C(x^i) \sqrt{\gamma}$ $\rightsquigarrow$ Suggest some singularity avoiding property

Example : Schw.

$$\partial_t \alpha = 0 \quad \beta^i \equiv 0 \quad K \equiv 0$$

$\hookrightarrow$ $t = $ const slices of Schw. are harmonic slices.

Cf. 1

Bona-Masso family (1+log slicing)

$$\boxed{\mathcal{L}_m \alpha = -\alpha^2 f(\alpha) K} \qquad (BM) \quad (\text{ev. eq. for } \alpha)$$

$f(\alpha)$ arbitrary fun:

(i) $f(\alpha) \equiv 0$: Geodesic slicing
(ii) $f(\alpha) = 1$: Harmonic slicing
(iii) $f(\alpha) = \frac{2}{\alpha}$: "1+log" slicing

$\beta^i \equiv 0$: $\partial_t \alpha = \partial_t \ln \gamma$

$$\boxed{\alpha = 1 + \ln \gamma}$$

Example: Schwarzschild 1+log foliation (Height function method):

$\beta^i \equiv 0$

$$\alpha^2 = \underbrace{1 - \frac{2M}{R} + \frac{C^2}{R^4}}_{\text{max. slicing}} e^{\alpha} \qquad (\text{implicit eq. for } \alpha)$$

exp. term

cf. "Trumpet slices" $R_T \approx 1.31241\, M$

Spatial gauge (β^i)

- "Minimal distorsion"

$$\Theta_{ij} := \partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij} \qquad \text{"distorsion tensor"}$$

$$\Sigma_{ij} := \Theta_{ij} - \frac{1}{3}\Theta \gamma_{ij} = \ldots = -2\alpha A_{ij} + (L\beta)_{ij} = \Psi^4 \partial_t \tilde{\gamma}_{ij}$$

Functional:

$$I[\beta^i] := \int_{\Sigma_t} \Sigma_{ij} \Sigma^{ij} \sqrt{\gamma}\, d^3x = \int_{\Sigma_t} \left[4\alpha^2 A_{ij} A^{ij} - 4\alpha A_{ij} (L\beta)^{ij} + (L\beta)_{ij} (L\beta)^{ij} \right] \sqrt{\gamma}\, d^3x$$

$$0 = \delta I[\beta^i] = \int_{\Sigma_t} 2\delta[\Sigma_{ij}(L\beta)^{ij}]\sqrt{\gamma}\,d^3x =$$

$$= 2\int_{\Sigma_t} \Sigma_{ij}\left(D^i\delta\beta^j + D^j\delta\beta^i - \cancel{\tfrac{2}{3}D_k\delta\beta^k\gamma^{ij}}\right)\sqrt{\gamma}\,d^3x =$$

symm

$\Sigma = 0$

$$= 4\int_{\Sigma_t} \Sigma_{ij}D^i\delta\beta^j\sqrt{\gamma}\,d^3x =$$

$$= 4\int_{\Sigma_t}\left[D^i(\Sigma_{ij}\delta\beta^j) - D^i\Sigma_{ij}\cdot\delta\beta^j\right]\sqrt{\gamma}\,d^3x =$$

$$= 4\underbrace{\int_{\partial\Sigma_t}\cancel{\Sigma_{ij}\delta\beta^j\sqrt{\gamma}\,d^3x}}_{\delta\beta^j|_\partial = 0} - 4\int_{\Sigma_t} D^i\Sigma_{ij}\cdot\delta\beta^j\sqrt{\gamma}\,d^3x \quad \forall\,\delta\beta^i$$

$\hookrightarrow$ $\boxed{D^i\Sigma_{ij} = 0}$ $\Rightarrow$ Eq. for β^i : Minimal distorsion shift

$$\boxed{\Delta_L\beta^i = 2D_j(\alpha A^{ij}) - 16\pi\alpha P^i + \frac{4}{3}\alpha D^i K + 2A^{ij}D_j\alpha}$$

(Elliptic eq. for β^i)

<u>Observation</u> : $Q_{ij}(\Sigma_{ij}) \equiv 0$ if ∂_t is a K.V.

$\hookrightarrow$ Minimal distortion satisfied for <u>stationary</u> spacetimes (in adapted coords).

- "Approximate" minimal distortion equations :

$$0 = D^i\Sigma_{ij} = D^i(\Psi^4\partial_t\tilde{\gamma}_{ij}) \approx \tilde{D}^i(\partial_t\tilde{\gamma}_{ij}) \approx \mathcal{D}_j\dot{\tilde{\gamma}}_{ij}$$

$\hookrightarrow$ Elliptic eq. for β^i

- $\underline{\Gamma\text{-freezing}}$:

(I.32)

$$0 = \mathcal{D}_j \dot{\tilde{\gamma}}_{ij} = \partial_t \mathcal{D}_j \tilde{\gamma}_{ij} = \dots = -\partial_t \tilde{\Gamma}^i$$

with: $$\tilde{\Gamma}^i := -\mathcal{D}_j \tilde{\gamma}^{ij} = \left(\tilde{\Gamma}^i_{jk} - F^i_{jk}\right)\tilde{\gamma}^{jk}$$

↳ Elliptic eq for β^i :

$$\boxed{\tilde{\gamma}^{jk} \mathcal{D}_j \mathcal{D}_k \beta^i + \dots = 0} \qquad \underline{\Gamma\text{-freezing}}$$

Parabolic Γ-driver

↳ idea: write evolution equations for β^i : the solution "asymptote" to the "equilibrium" solution of the Γ-freezing.

$$\boxed{\partial_t \beta^i = k\, \partial_t \tilde{\Gamma}^i \cong k\left(\tilde{\gamma}^{jk} \mathcal{D}_j \mathcal{D}_k \beta^i + \dots\right)} \qquad k \in \mathbb{R}^+$$

" for $t \to +\infty$ ~ Γ-freezing solution "

Hyperbolic Γ-driver

$$\partial_{tt} \beta^i = k\, \partial_t \tilde{\Gamma}^i - (\eta - \partial_t \ln k)\, \partial_t \beta^i =$$

$$= k\left(\tilde{\gamma}^{jk} \partial_j \partial_k \beta^i + \cancel{\cdot}\right) - (\eta - \partial_t \ln k)\, \partial_t \beta^i$$

$$\boxed{\partial_{tt} \beta^i = k\, \tilde{\gamma}^{ij} \partial_j \partial_k \beta^i \underbrace{- \eta\, \partial_t \beta^i}_{\text{damping term, with parameter } \eta > 0}}$$

Simpler (first-order) version:

$$\boxed{\partial_t \beta^i = \mu_S \tilde{\Gamma}^i - \eta \beta^i + \beta^j \partial_j \beta^i}$$

(hyperbolic) Γ-driver

advection-like equation (eq.) for β^i

with speed μ_S

with damping term with coef. $\eta > 0$

Cf. (hyp.) Γ-driver w/ harmonic shift eq. $\partial_t \beta^i = \ldots$

— Summary —

- Geodesic gauge: $\alpha = 1$ $\beta^i = 0$

$\alpha \leftarrow$ slicing
- Maximal slicing: $0 = K$
- Harmonic slicing: $\Box t = 0$
- Bona-Masso family (Harm. slicing, 1+log slicing)

$\beta^i \leftarrow$ spatial
- Minimal distortion
  - Γ-drivers

1+log + Γ-driver evolution of punctured BH