NR

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ASYMPTOTIC FLATNESS & GLOBAL QUANTITIES

Global "charges" in GR:

(i) Asymptotically Flat (AF) spacetimes

$\hookrightarrow$ ADM energy & momentum

(ii) In presence of killing vectors (symmetries)

$\hookrightarrow$ Komar charges

ASYMPTOTICAL FLATNESS

DEF. A globally hyp. spacetime is A.F. iff

$\forall\ \Sigma_t \quad \exists\ f_{ij}$ "background metric" :

(i) f_{ij} is FLAT except in a compact domain ("strong-field region)

(ii) $\exists\ \{x^i\}$: $f_{ij} = \mathrm{diag}(1,1,1)$ for $r = \sqrt{x^i x_i} \to +\infty$

(iii) For $r \to +\infty$:

$$
(1) \begin{cases}
\gamma_{ij} = f_{ij} + \mathcal{O}(r^{-1}) & (1a) \\
\partial_k \gamma_{ij} = \mathcal{O}(r^{-2}) & (1b) \\
K_{ij} = \mathcal{O}(r^{-2}) & (1c) \\
\partial_k K_{ij} = \mathcal{O}(r^{-3}) & (1d)
\end{cases}
$$

"decay conditions"

The region $r \to +\infty$ is SPATIAL INFINITY (i_0)

Example: Schw. is A.F. ✓

Example: Flat + GW spacetime

$$\gamma_{ij} = f_{ij} + \frac{1}{r} h_{ij}(\underbrace{t-r}_{u}) + \mathcal{O}(r^{-2}) \quad ✓ \quad (1a)$$

check the drvts:

$$\partial_k \gamma_{ij} = -\frac{h'_{ij}(u)}{r}\frac{x^k}{r} - \frac{h_{ij}(u)}{r^2}\frac{x^k}{r} + O(r^{-3}) \qquad \times \quad (1b)$$

Not A.F.

Note:

- AF def. depends on Σ_t and on x^i

Coordinate transf. that preserves A.F.

$$x^\mu \longrightarrow \Lambda^\mu{}_\alpha x^\alpha + c^\mu(\theta,\varphi) + O(r^{-1})$$

↳ Rotations, Translations, Boosts ("Spi" group)

ADM ENERGY/MASS

A.F. spacetime

GR action:

$$S_{GR} = \underbrace{\int_{\mathcal{U}\subset\mathcal{M}} {}^4R\,\sqrt{g}\,d^4x}_{S_{\text{Hilbert}}} + \underbrace{\oint_{\partial\mathcal{U}} (Y - Y_0)\sqrt{h}\,d^3y}_{\text{BOUNDARY TERM}} \qquad (2)$$

Necessary to obtain EFE by varying S_{GR}

$$\frac{\delta S_{GR}}{\delta g^{ab}}$$

$\partial\mathcal{U}$: boundary of $\mathcal{U}$ (TIMELIKE)

h_{ab} the induced metric on $\partial\mathcal{U}$ as embedded in (M, g)

Y_{ab} the extrinsic curvature of $\partial\mathcal{U}$ in (M, g)

Y in (2) is the $Tr(Y_{ab})$

h_{0ab} the induced metric on $\partial\mathcal{U}$ as embedded in (M, η)

Y_{0ab} the extrinsic curvature of $\partial\mathcal{U}$ — " —

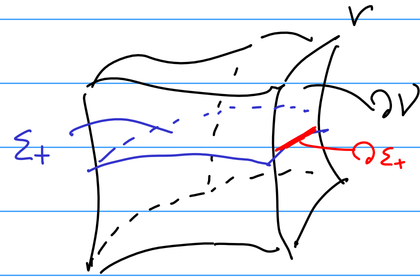
Y_0 in (2) is the $Tr(Y_{0ab})$

$\hookrightarrow \dfrac{\delta S}{\delta g_{ab}} = 0$ w/ $\delta g_{ab}\big|_{\text{boundary}} = 0$ $\rightarrow$ EFE.

Hamiltonian:

(3) $$H = -\overbrace{\int_{\Sigma_t} \sqrt{\gamma}\,\left(\alpha C_0 + 2\beta^i C_i\right)}^{(3a)} - 2\oint_{\partial\Sigma_t} \sqrt{q}\,\left[\alpha\,(k - k_0) + \beta^i s^j \left(K_{ij} - K\gamma_{ij}\right)\right]$$

lapse Ham. constr. shift Mom. constr.



$\partial\Sigma_t = \partial V \cap \Sigma_t$

s^i : normal vector of $\partial\Sigma_t$

q_{ij} : induced metric on $\partial\Sigma_t$ as embedded in (Σ, γ)

k_{ij} : extrinsic curvature -"- -"-

k in (3) is the trace

k_{0ij} : extrinsic curvature of $\partial\Sigma_t$ as embedded in (Σ, f)

k_0 in (3) is the trace

<u>Key observation</u>: on a <u>solution</u> of GR (3a) = 0, the boundary term gives us an "energy" for Σ_t ...

DEF. ADM MASS is defined iff for $\alpha \equiv 1$ $\beta^i \equiv 0$:

(4) $$M_{ADM} := -\frac{1}{8\pi}\lim_{r\to+\infty}\oint_{\partial\Sigma_t}\sqrt{q}\,(k - k_0) \qquad (4a)$$

$$= +\frac{1}{16\pi}\lim_{r\to+\infty}\oint_{\partial\Sigma_t}\sqrt{q}\,s^i\left[D^j\gamma_{ij} - D_i\left(f^{kl}\gamma_{kl}\right)\right] \qquad (4b)$$

$$= +\frac{1}{16\pi}\lim_{r\to+\infty}\oint\sqrt{q}\,s^i\left(\partial_j\gamma_{ij} - \partial_i\gamma^k_k\right) \qquad (4c)$$

$$= -\frac{1}{2\pi} \lim_{r\to+\infty} \oint_{\partial\Sigma_t} \sqrt{q}\, s^i \left(\mathcal{D}_i \Psi - \frac{1}{8} \mathcal{D}^j \tilde{\gamma}_{ij} \right) \qquad (4d)$$

(4b) Specifies to the background metric

(4c) — " — to Cartesian coords

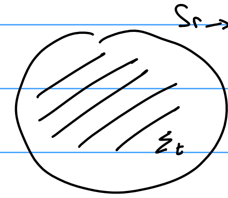
(4d) Written in terms of conf. variables ($\to$ proof in the notes)

Note: $\boxed{(4c) \Rightarrow \text{A.F. guarantees the integral exists!}}$

Example: Weak-field

$$\tilde{\gamma}_{ij} = f_{ij} \qquad \mathcal{D}^i \tilde{\gamma}_{ij} = 0 \qquad \Psi = 1 - \frac{1}{2}\phi \qquad \mathcal{D}_i \Psi = -\mathcal{D}_i \phi/2$$

$$(4d) \Rightarrow M_{ADM} = +\frac{1}{4\pi} \lim_{r\to+\infty} \oint_{S_r} \sqrt{f}\, s^i \mathcal{D}_i \phi =$$



$S_{r\to+\infty}$ Σ_t

$$= +\frac{1}{4\pi} \int_{\Sigma_t} \sqrt{f}\, d^3x\, \underbrace{\mathcal{D}^i \mathcal{D}_i \phi}_{\Delta\phi = 4\pi\rho} =$$

$$= \int_{\Sigma_t} \sqrt{f}\, d^3x\, \rho$$

Example: Schwarzschild (isotropic coords)

$$\Psi = 1 + \frac{M}{2r} \qquad \tilde{\gamma}_{ij} = f_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$$

$$\sqrt{q}\, d^2y = r^2 \sin\theta\, d\theta\, d\varphi \quad \text{at } i_0 \quad \text{w/ } \partial\Sigma_t \sim S_r$$

$$s^i \mathcal{D}_i = \partial_r \quad \text{at } i_0$$

$$M_{ADM} = (4d) = -\frac{1}{2\pi} \lim_{r\to+\infty} \oint_{S_r} \overbrace{\frac{\partial\Psi}{\partial r} r^2}^{-\frac{M}{2r^2}\cdot r^2} \sin\theta\, d\theta\, d\varphi = M$$

TH. (Schoen & Yau '79, '81 ; Witten '81) :

if "T_{ab}" obeys the dominant energy condition $(E^2 \geq p^2)$,

then (i) $M_{ADM} \geq 0$

(ii) $M_{ADM} = 0 \iff \Sigma_t$ is Minkowski

(iii) $\frac{dM_{ADM}}{dt} = 0$ (H is not time dependent)

EURISTIC

Start from Newton :

$$M = \int_V \rho = \frac{1}{4\pi}\int_V \Delta\phi = \oint_S s^i \partial_i \phi \overset{*}{=}$$

In GR : $\phi \sim h_{00}$

$\partial\phi \sim \partial h_{ij}$ "derivative of the metric"

Ham. constr. R + curvature = energy

$\partial\partial h_{ij} + 0 = E \approx \rho$

$$\overset{*}{=} \int_V R \sim \oint s^i(\partial_j h_{ij} + \partial_i h^k{}_k)$$

ADM MOMENTUM

Linear momentum $\rightsquigarrow$ spatial translations i_0 in direction ∂_k

DEF ADM mom. is defined from the boundary term at i_0 by taking

$\alpha = 0$ $\beta^i = (\partial_k)^i$

$$P_k^{ADM} := \frac{1}{8\pi} \lim_{r\to+\infty} \oint_{\partial\Sigma_t} \sqrt{q}\,(\partial_k)^i s^j (K_{ij} - K\gamma_{ij})$$

Note: (i) A.F. guarantees existance of these 3 quantities $(k=1,2,3)$

(ii) P_k^{ADM} transform like a 1-form !

↳ is a vector at i_0

↳ P_α^{ADM} $(-M_{ADM}, P_k^{ADM})$ transform "properly"

as 4-vector under Spi group

(ADM) ANG. MOMENTUM

Ang. momentum ⇝ Rotations at i_0 about the 3 axes:

$$\phi_x = -z\partial_y + y\partial_z$$
$$\phi_y = \dots$$
$$\phi_z = \dots$$

Note: $\phi_k \sim \mathcal{O}(r)$

DEF. Ang. mom. boundary term at i_0 w/ $\alpha \equiv 0$ $\beta^i = (\phi_k)^i$:

$$J_k^{ADM} := \frac{1}{8\pi} \lim_{r\to+\infty} \oint_{\partial\Sigma_t} \sqrt{q}\,(\phi_k)^i s^j (K_{ij} - K\gamma_{ij})$$

Note: (i) A.F. does <u>not</u> guarantee existance of these integrals !

$$(K_{ij} - K\gamma_{ij})(\phi_k)^j \approx \mathcal{O}(r^{-1})$$

but "sometimes" the contraction with ξ^i solves the problem (e.g. Kerr)

(ii) J_k^{ADM} do not transform like a 1-form at i_0 !!!

↳ A "working" def of $\vec{J}$ exists only in restricted class of metrics:

QUASI-ISOTROPIC GAUGE ← $\partial_k \hat{\gamma}_{ij} = \mathcal{O}(r^{-3})$

ASYMPTOTICALLY MAXIMAL GAUGE ← $K = \mathcal{O}(r^{-3})$

"stronger" decay conditions than A.F.

KOMAR MASS

In GR, Conserved charges appear as one has symmetries.

K^a K.V. $\qquad \mathcal{L}_K g_{ab} = 2\nabla_{(a}K_{b)} = 0$

TH: if $\exists\, K^a$ K.V., then

$$J^a := T^{ab} K_b$$

ANY! SYMMETRIC, DIVERGENCE-FREE TENSOR

is a conserved current:

$$\nabla_a J^a = 0$$

[Ex. prove TH]

A particular important current is constructed from the 4D Ricci tensor:

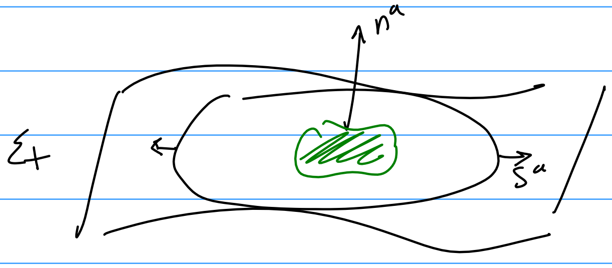
$$J_0^a := {}^4R^{ab} K_b = \nabla_b \nabla^a K^b = -\nabla_b (\underbrace{\nabla^b K^a}_{A^{ba} = A^{[ba]}})$$

Antisymmetric

↳ Stokes theorem ⇒

$$\boxed{\int_{\Sigma_t} d^3x \sqrt{\gamma}\, n_a \nabla_a A^{ab} = \oint_{\partial\Sigma_t} d^2y \sqrt{q}\,(s_a n_b - n_a s_b)\, A^{ab} = Q_K}$$

Komar charge (conserved)



Example: $K^a = (\partial_t)^a$ stationary spacetime

$Q_K =: M_K$ Komar mass

TH: if $K^a = n^a$ at i_0, then $M_K = M_{ADM}$.