NR

15.06.2021

INITIAL DATA PROBLEM (cont.)

CONFORMAL THIN SANDWICH (CTS)

Different decomposition of $\hat{A}_{ij}$:

$$\left\{ \begin{aligned} \mathcal{L}_m \tilde{\gamma}_{ij} &= \partial_t \tilde{\gamma}_{ij} - \mathcal{L}_\beta \tilde{\gamma}_{ij} = \partial_t \tilde{\gamma}_{ij} + (\tilde{L}\beta)^{ij} + \frac{2}{3} \tilde{D}_k \beta^k \tilde{\gamma}_{ij} \\ &= 2\alpha \hat{A}_{ij} + \frac{2}{3} \tilde{D}_k \beta^k \tilde{\gamma}_{ij} \end{aligned} \right.$$

$$\boxed{\hat{A}_{ij} = (2\alpha)^{-1} \left[\dot{\tilde{\gamma}}_{ij} + (\tilde{L}\beta)_{ij} \right]}$$

(and similarly for $\hat{A}^{ij}$)

" Replaces of the L + TT decomposition of $\hat{A}_{ij}$ in CTS "

plug-in in $C_i = 0$

CTS :

$$\left\{ \begin{aligned} &\tilde{D}_j \left[\tilde{\alpha}^{-1} (\tilde{L}\beta)^{ij} \right] + \tilde{D}_j \left[\tilde{\alpha}^{-1} \dot{\tilde{\gamma}}^{ij} \right] - \frac{4}{3} \psi^6 \tilde{D}^i K - 16\pi \tilde{P}^i = 0 \qquad (1) \\ &\tilde{D}^i \tilde{D}_i \psi + \ldots = 0 \qquad (L) \end{aligned} \right.$$

FREE DATA : $\tilde{\gamma}_{ij}$, K, $\dot{\tilde{\gamma}}_{ij}$, E, $\tilde{P}^i$, $\tilde{\alpha}$

CONSTRAINED DATA : ψ, β^i

- $\dot{\tilde{\gamma}}_{ij}$ ~ can help in specifying free data, e.g. stationary data ...
- $\tilde{\alpha} := \underbrace{\psi^{-6}}_{\uparrow \text{ we want to solve for ...}} \alpha$: " conf. rescaled lapse " ⇝ ?
- $K \equiv 0$ ⇒ CTS eqs decouple
- Momentum constraint for β^i is linear

York – Pfeiffer (2003) : XCTS

↳ specify an eq. for $\tilde{\alpha}$

$$\mathcal{L}_m K = \dot{K} - \beta^i \tilde{D}_i K$$

$$= -\psi^{-4} \underbrace{\left(\tilde{D}_i \tilde{D}^i \alpha + 2\tilde{D}_i \ln\psi \, \tilde{D}^i \alpha \right)}_{\underset{\text{identity}}{=} \psi^{-1}\left[\tilde{D}_i \tilde{D}^i (\alpha\psi) + \alpha \tilde{D}_i \tilde{D}^i \psi\right]} + \alpha [\ldots]$$

$C_0 = 0$ (L)

$$\hookrightarrow \tilde{D}_i \tilde{D}^i (\hat{\alpha}\psi^7) - (\hat{\alpha}\psi^7)\left[\frac{\hat{R}}{8} + \frac{5}{12}K^2\psi^4 + \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-8} + 2\pi(\hat{E} + 2S\psi^8)\psi^{-4}\right] + (\dot{K} - \beta^i \hat{D}_i K)\psi^5 = 0 \qquad (2)$$

$$\text{XCTS} : \begin{cases} (1) \\ (L) \\ (2) \end{cases}$$

- $K \equiv 0$ do not decouple ⌢̈
- FREE DATA : $\boxed{\hat{\gamma}_{ij}, \dot{\hat{\gamma}}_{ij}, K, \dot{K}}$ → CAN GIVE NICE HANDLE ON FREE DATA

  CONSTR. DATA : $\psi, \beta^i, \tilde{\alpha}$
- There exist examples in which XCTS gives <u>non unique solutions</u>

<u>Example</u> : Conformally flat, asymptotically flat & "zero time-derivatives" in vacuum

Simplest choice :
$$\begin{cases} \hat{\gamma}_{ij} = f_{ij} \\ \dot{\hat{\gamma}}_{ij} = 0 = \dot{K} \\ K \equiv 0 \end{cases}$$

$$\hookrightarrow \quad (*) \begin{cases} \Delta\psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-7} = 0 \\ D_j\left[\tilde{\alpha}^{-1}(L\beta)^{ij}\right] = 0 \\ \Delta(\tilde{\alpha}\psi^7) - \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\tilde{\alpha}\psi^{-7} = 0 \end{cases}$$

<u>A.F. conditions</u> : $\psi = 1$ $\quad \beta^i = 0$ $\quad \alpha = 1$ $\quad r \to +\infty$

$$\begin{cases} \boxed{\beta^i \equiv 0} & \text{is a solution} \checkmark \\ \dot{\tilde{\gamma}}_{ij} = 0 \end{cases} \Rightarrow \hat{A}^{ij} \equiv 0$$

$$(*): \begin{cases} \Delta\psi = 0 & (1) \\ \Delta(\underbrace{\tilde{\alpha}\psi^7}_{\phi}) = 0 & (2') \end{cases}$$

with <u>same</u> BC as (1)

$$\begin{cases} \alpha = 1 \\ \psi = 1 \end{cases} \downarrow \quad \phi \to 1, \; r \to +\infty$$

<u>Inner BC</u> : $\Sigma_0 \approx \mathbb{R}^3 \setminus \{0\}$

$$(1) \hookrightarrow \boxed{\psi = 1 + \frac{M}{2r}}$$

$$(2') \hookrightarrow \boxed{\phi = \tilde{\alpha}\psi^7 = 1 + \frac{a}{r}}$$

Solve for lapse:

$$\alpha = \psi^6\tilde{\alpha} = (\underbrace{\tilde{\alpha}\psi^7}_{=\phi})\psi^{-1} = \left(1+\frac{a}{r}\right)\left(1+\frac{M}{2r}\right)^{-1} = \frac{r+a}{r+\frac{M}{2}}$$

"$\alpha(r)$"

Q: What is 'a' ?

A: a is fixed by the <u>choice</u> of $\alpha(r=0)$

$$\alpha_0 = \lim_{r\to 0} \alpha(r)$$

Simplest choice : $\underline{\alpha_0 = +1}$

$\hookrightarrow a = +\frac{M}{2} \Rightarrow \boxed{\alpha \equiv 1}$

$\hookrightarrow$ Schwarzschild's spatial slice in isotropic coords and geodesic gauge

$$g = -dt^2 + \underbrace{\psi^4(dr^2 + r^2 d\Omega^2)}_{\text{Spatial slice of Schw. iso r}}$$

Another choice : $\underline{\alpha_0 = -1}$

$\hookrightarrow a = -\frac{M}{2} \Rightarrow \boxed{\alpha = \left(1-\frac{M}{2r}\right)\left(1+\frac{M}{2r}\right)}$

$$g = g^{Schw}_{\text{isotropic coords}}(t, r, \theta, \varphi)$$

Observations :

(i) Moment of time symmetry

(ii) The time development of Σ_0 are different !

$$\begin{cases} \alpha_0 = +1 \longrightarrow \Sigma_0 \text{ evolves in a nontrivial way} \\ \alpha_0 = -1 \longrightarrow \Sigma_0 \text{ does not evolve : } \partial_t \text{ is a K.V.} \end{cases}$$

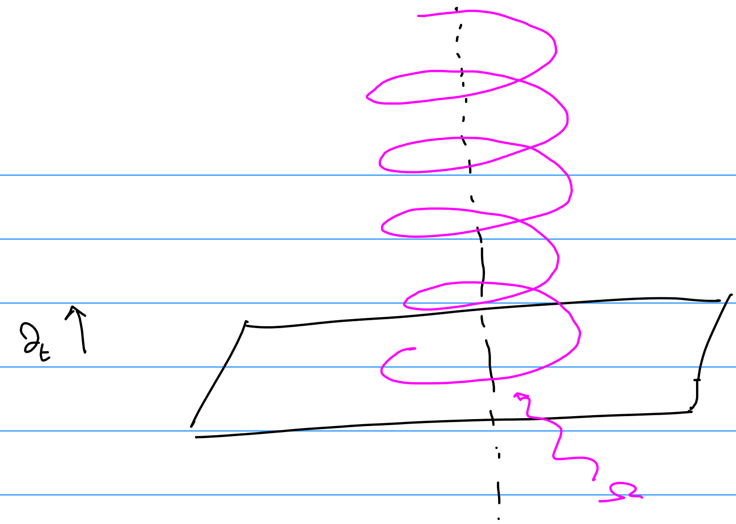
(iii) $\alpha < 0$? n future pointing $\Rightarrow$ "time will run backwards for $r \to 0$"

Where is XCTS useful?

Binary system in circular orbit Ω : $\cancel{\partial_t}$ no timelike K.V.

$\cancel{\partial_\varphi}$ no rotational K.V.

$$h^a := (\partial_t)^a + \Omega(\partial_\varphi)^a$$



Vector h^a is a k.v. HELICAL k.v.

XCTS : comoving frame : $h^a = (\partial_t)^a$

$$\boxed{\dot{\gamma}_{ij} = 0 = \dot{K}}$$

CTT + Bowen-York puncture method : exactly in circular orbits ?

GAUGE CONDITIONS

$$\left(\begin{aligned}\mathcal{L}_m \gamma_{ij} &= \dots \\ \mathcal{L}_m K_{ij} &= \dots\end{aligned}\right)$$

TODO:

$\{$ Choice of the foliation
  " " " spatial coords

Requirements:
- singularity avoiding
- Symmetry seeking
- Minimize grid distorsion

SLICING (α)

Geodesic slicing: $\boxed{\alpha = 1 \qquad \beta^i \equiv 0}$ → free falling observers

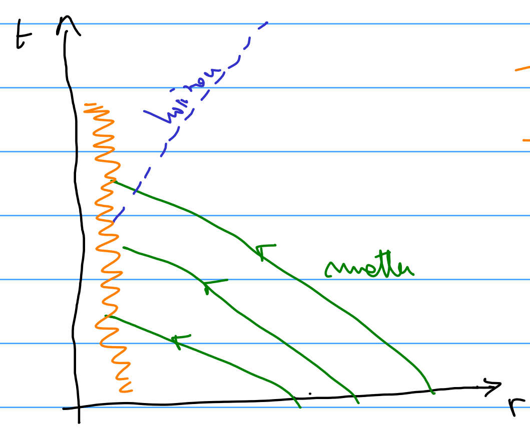
$$n^a = (\partial_t)^a \qquad (t = \tau)$$

$$a^a = D_a \ln \alpha \equiv 0$$

$$\begin{cases} \partial_t K = K_{ij}K^{ij} + 4\pi (E - S) \\ \partial_t \ln\sqrt{\gamma} = -K \end{cases}$$

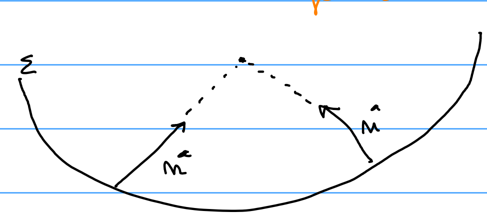
Geo. Slicing

Grav. collapse:



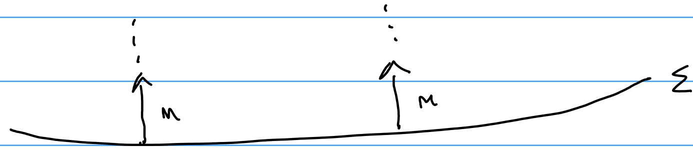
- curvature gets large
- coords. volume element decrease
  ↳ 0

Eulerian observers gets closer



(geo. slicing)

Maximal slicing: $\boxed{0 = K = -\nabla_a n^a}$ $\simeq$ incompressible fluid



$\hookrightarrow$ $\boxed{D_i D^i \alpha - \alpha\left[\,4\pi(E-S) + K_{ij}K^{ij}\right] = 0}$

Cf. to geo.

$a^a \neq 0$ Eulerian obs are accelerated

α lapse is solved for

$\hookrightarrow$ "counteracting effect" to the "singularity seeking" property of geo. slicing...

Next time...