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CONFORMAL DECOMPOSITION OF 3+1 GR

HISTORICAL REMARKS

- 1944 LICHNEROWICZ : C.D. to deal with the initial data problem

- 1971 YORK : Dynamical d.o.f. of GR (GWs) are encoded in the conformal equivalence class of 3-metric

$$\boxed{\gamma_{ij} = \psi^4 \, \tilde{\gamma}_{ij}} \qquad (1)$$

↳ Shown using ADM + Cotton-York tensor

Consequence (example) : $\tilde{\gamma}_{ij} = f_{ij} \Rightarrow$ no GWs

- 90s – . C.D. of ADMY Eqs to establish well-posed (hyperbolic) scheme for free-evolution (Cauchy IBVP) of GR

Examples : BSSN, Z4c — used for black hole evolutions / astrophys. applications

C.D. OF THE METRIC

Common choice of conformal metrics in (1) are UNI-DETERMINANT :

$$\det \tilde{\gamma}_{ij} = \tilde{\gamma} = 1$$

Property :

A is nxn matrix, c is a number $\Big\}$ $\det(cA) = c^n \det A$

↳ $\det \gamma_{ij} = \gamma = \psi^{12} \, \tilde{\gamma}$

$$\tilde{\gamma} = 1 \quad \Rightarrow \quad \psi = \gamma^{1/12}$$

<u>Problem</u> : ψ is <u>not</u> a scalar!

$\tilde{\gamma}_{ij}$ is <u>not</u> a tensor (is a tensor density)

$\hookrightarrow$ it has <u>no</u> Levi-Civita connection associated

<u>Solution</u> : BACKGROUND METRIC on Σ_t (unphysical) such that:

(i) f_{ij} signature $+,+,+$ and $\det f_{ij} = f$ (Riemannian)

(ii) Time independent

$$\mathcal{L}_{\partial_t} f_{ij} = \frac{\partial f_{ij}}{\partial t} = 0$$

(iii) Inverse metric $f^{ij} f_{jk} = \delta^i{}_k$

(iv) Levi-Civita connection

$$\mathcal{D}_k f_{ij} = 0$$

$$\mathcal{D}^i = f^{ij} \mathcal{D}_j$$

$$\bar{\Gamma}^k_{ij} = \frac{1}{2} f^{kl} \left(\partial_i f_{lj} + \partial_j f_{il} - \partial_l f_{ij} \right)$$

$\hookrightarrow$

$$\boxed{\begin{array}{ll} \psi := \left(\dfrac{\gamma}{f}\right)^{1/12} & \text{SCALAR !} \\ \tilde{\gamma}_{ij} = \psi^{-4} \gamma_{ij} , & \tilde{\gamma} = f \\ \tilde{\gamma}^{ij} = \psi^{4} \gamma^{ij} & \end{array}}$$

Now $\tilde{\gamma}_{ij}$ is a "good" metric

$\hookrightarrow$ $\tilde{D}_k \tilde{\gamma}_{ij} = 0$ Conformal connection

$\tilde{\Gamma}^k_{ij}$ christoffel symbols.

Q: What to use as f_{ij} ?

A: Simple choice for Cartesian coordinates:

$$f_{ij} = \mathrm{diag}(1,1,1) \qquad f = 1$$

$$\Gamma^k_{ij} = 0 \qquad \mathcal{D}_i = \partial_i$$

Other possibility for spherical coordinates:

$$f_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta) \qquad f = r^4 \sin^2\theta$$

$$\Gamma^k_{ij} \neq 0 \qquad \mathcal{D}_i \sim J_{ij}\partial_j$$

Example: Weak-field metric

$$ds^2 = -(1+2\phi)\,dt^2 + \underbrace{(1-2\phi)\, f_{ij}\, dx^i dx^j}_{\gamma_{ij}}$$

ϕ: Newton-potential

$$\gamma_{ij} = \psi^4 \hat{\gamma}_{ij} \qquad \text{with} \qquad \psi^4 = (1-2\phi)$$

$$\hat{\gamma}_{ij} = f_{ij}$$

↳ Conf. metric = background metric = flat Euclidean 3-metric

↳ $\psi = (1-2\phi)^{1/4} \underset{\phi \ll 1}{\approx} 1 - \frac{1}{2}\phi$

Conformal factor ⇝ Newtonian potential

Example: Schw. metric in isotropic coords.

$$ds^2 = -\frac{\left(1-\frac{M}{2r}\right)^2}{\left(1+\frac{M}{2r}\right)^2}\,dt^2 + \overbrace{\underbrace{\left(1+\frac{M}{2r}\right)^4}_{\psi} \underbrace{\left(dr^2 + r^2 d\Omega^2\right)}_{f_{ij}}}^{\gamma_{ij}}$$

$$\psi = \left(1 + \frac{M}{2r}\right) \quad \rightarrow \quad \phi = -\frac{M}{r} \quad \text{weak-field limit !}$$

$$\tilde{\gamma}_{ij} = f_{ij} \quad \text{flat metric in sph. coords} \quad = \quad \text{background metric.}$$

CONF. CONNECTION & RICCI TENSOR

Relation between D_i and $\tilde{D}_i$ (both live on Σ_t) is the standard relation between connections on the same manifold:

$$\text{"} \quad D_j T^{\cdots}_{\cdots} = \tilde{D}_j T^{\cdots}_{\cdots} + \Sigma\, C\, T^{\cdots}_{\cdots} - \Sigma\, C\, T^{\cdots}_{\cdots} \quad \text{"}$$

$$C^k{}_{ij} = \Gamma^k{}_{ij} - \tilde{\Gamma}^k{}_{ij}$$

Using this the Ricci are related by

$$R_{ij} = \tilde{R}_{ij} + \text{"}\tilde{D} C - \tilde{D} C + C C - C C\text{"} =$$

$$= \tilde{R}_{ij} + R^{\psi}{}_{ij}$$

second cov. derivs. of conf. factor

$$\sim \tilde{D}\tilde{D}\psi + \tilde{D}\psi\,\tilde{D}\psi$$

Ricci scalar:

$$R = \gamma^{ij} R_{ij} = \ldots = \psi^{-4}\tilde{R} - 8\psi^{-5}\tilde{D}_i\tilde{D}^i\psi \qquad (H)$$

C.D. OF EXTRINSIC CURVATURE

First, Traceless + trace part:

$$K_{ij} = \underbrace{A_{ij}}_{\text{Traceless part}} + \frac{1}{3}\overbrace{K}^{\text{trace}}\gamma_{ij} \qquad (1')$$

Second, C.D. of A^{ij}: $\quad A^{ij} = \psi^{p}\,\underbrace{\bar{A}^{ij}}_{\text{CONFORMAL TRACELESS PART OF } K^{ij}}$

Two choices for p:

(i) $p=-4$: based on $\mathcal{L}_m \gamma_{ij} = \dots$

$\llcorner$ evolution schemes, hyperbolic part of ADMY eqs.

(ii) $p=-10$: based on $C^i = 0$

$\llcorner$ constraint eqs.

Choice (i): Take the "kinematical" eq for K_{ij}

$$\mathcal{L}_m(\psi^4 \hat{\gamma}_{ij}) = -2\alpha A_{ij} - \frac{2}{3}\alpha K \psi^4 \hat{\gamma}_{ij}$$

$$\cancel{\psi^{-4}\psi^4} \mathcal{L}_m \hat{\gamma}_{ij} = \underbrace{-4\psi^{-4}\psi^3 \mathcal{L}_m \psi}_{\mathcal{L}_m \ln\psi} \cdot \hat{\gamma}_{ij} - 2\alpha\psi^{-4} A_{ij} - \frac{2}{3}\alpha K \cancel{\psi^{-4}\psi^4} \hat{\gamma}_{ij} \qquad (\times)$$

Trace $(\times)$ with $\hat{\gamma}^{ij}$:

$$\tilde{\gamma}^{ij}\mathcal{L}_m \hat{\gamma}_{ij} = -4\underbrace{\tilde{\gamma}^{ij}\hat{\gamma}_{ij}}_{=3} \mathcal{L}_m \ln\psi - 2\alpha\psi^{-4}\underbrace{\tilde{\gamma}^{ij}A_{ij}}_{=0} - \frac{2}{3}\alpha K \underbrace{\tilde{\gamma}^{ij}\hat{\gamma}_{ij}}_{=3}$$

$$= -12\,\mathcal{L}_m \ln\psi - 2\alpha K$$

$$\tilde{\gamma}^{ij}\mathcal{L}_m \tilde{\gamma}_{ij} \sim \mathrm{Tr}(M^{-1}\delta M) = \delta(\ln\det M)$$

$$= \mathcal{L}_m \ln\tilde{\gamma} = (\partial_t - \mathcal{L}_\beta)\ln f = -\mathcal{L}_\beta \ln f =$$

$$= -\tilde{\gamma}^{ij}\mathcal{L}_\beta \tilde{\gamma}_{ij} = -\tilde{\gamma}^{ij}\left(\beta^k \underbrace{\hat{D}_k \tilde{\gamma}_{ij}}_{=0} + 2\tilde{\gamma}_{k(i}\hat{D}_{j)}\beta^k\right) =$$

$$= -2\hat{D}_i\beta^i$$

$$\hookrightarrow \boxed{6\,\mathcal{L}_m \ln\psi + \alpha K = \hat{D}_i\beta^i}$$

C.D. of the "kinematic" eq $\mathcal{L}_m \gamma = -2\alpha K$:

(2)
$$\boxed{\begin{aligned} \mathcal{L}_m \ln\psi &= \frac{1}{6}\left(\tilde{D}_i\beta^i - \alpha K\right) \\ \mathcal{L}_m \tilde{\gamma}_{ij} &= -2\alpha \underbrace{\psi^{-4} A_{ij}}_{} - \frac{2}{3}\tilde{D}_k\beta^k \tilde{\gamma}_{ij} \end{aligned}}$$

Natural choice is $p = -4$

$=: \tilde{A}_{ij} \; (= \bar{A}_{ij})$

Choice (ii) : Take $0 = C^i \sim D_j K^{ij} \sim \underline{D_j A^{ij}} + \frac{1}{3} D^i K$

$D_j A^{ij} = \tilde{D}_j A^{ij} + C^i_{jk} A^{kj} + C^j_{jk} A^{ik} = \ldots$

$= \psi^{-10} \tilde{D}_j (\underbrace{\psi^{10}} A^{ij})$

Natural choice is $p = -10$

$=: \hat{A}^{ij} \quad (= \bar{A}^{ij})$

(3)
$$\boxed{0 = C^i = \tilde{D}_j \hat{A}^{ij} - \frac{2}{3}\psi^6 \tilde{D}^i K - 8\pi \psi^{10} P^i}$$
MOM. CONSTRAINT.

We're deriving the C.D. of ADM...

HAMILTONIAN CONSTRAINT $\quad 0 = C^0 = R + K^2 - \underline{K_{ij}K^{ij}} - 16\pi E$

expressed before in terms of $\tilde{R}$ and ψ already a conf. var!

$K_{ij}K^{ij} = (A_{ij} + \frac{1}{3}K\gamma_{ij})(A^{ij} + \frac{1}{3}K\gamma^{ij}) = A_{ij}A^{ij} + \frac{2}{3}K \overbrace{\gamma_{ij}A^{ij}}^{=0} + \frac{1}{\cancel{3}}\frac{1}{3}K^2 \overbrace{\gamma_{ij}\gamma^{ij}}^{=\cancel{3}} = A_{ij}A^{ij} + \frac{1}{3}K^2$

Put things together:

(p = −4) $$\boxed{0 = C^0 = \hat{D}_i \hat{D}^i \psi - \frac{1}{8}\hat{R}\psi + \left(\frac{1}{8}\tilde{A}_{ij}\hat{A}^{ij} - \frac{1}{12}K^2 + 2\pi E\right)\psi^5}$$

(p = −10) $$\boxed{0 = C^0 = \hat{D}_i \hat{D}^i \psi - \frac{1}{8}\hat{R}\psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-7} + \left(2\pi E - \frac{1}{12}K^2\right)\psi^5} \quad (4)$$

— LICHNEROWICZ EQ —

OTHER C.D. ADMY : DYNAMICAL EQS

$$\mathcal{L}_m K_{ij} = \dots$$

$$\longrightarrow \boxed{\begin{cases} \mathcal{L}_m K = \dots \\ \mathcal{L}_m \tilde{A}_{ij} = \dots \end{cases}} \quad (5)$$

$$\begin{cases} \boxed{\begin{matrix} (5) \\ (2) \end{matrix}} & \text{Ev. equations for } \psi, \hat{\gamma}_{ij}, K, \hat{A}_{ij} \\ \boxed{\begin{matrix} (4) \\ (5) \end{matrix}} & \text{Constraints} \end{cases}$$

Conf. dec. of ADMY

Conf. variables defined by (1) $\gamma_{ij} = \psi^4 \hat{\gamma}_{ij}$

(1′) Traceless and conf. dec. of K_{ij}

ISEMBERG-WILSON-MATHEWS APPROXIMATION* TO GR

* Exact for Schw. / 1PN

Hypothesis (i) $\hat{\gamma}_{ij} = f_{ij}$ Conf. flat metric

(ii) $K \equiv 0$ Max. slicing condition

$\longrightarrow$ C.D. 3+1 ADMY $\longrightarrow$ $\Delta\psi = \dots$ $\Delta\beta^i = \dots$ $\Delta\alpha = \dots$ $\boxed{\text{Elliptic system}}$