## Sources

These notes exist as a form of preparation for a seminar I have to give on this topic. The primary source that these notes are essentially completely based off of is "A Primer for Chiral Perturbation Theory" by Scherer and Schindler. I also looked some additional things up in the papers by Gasser and Leutwyler published from 1982 to 1985. None of the content in these notes are my own developments. This is purely an attempt at reassembling the content of Scherer and Schindler's notes in a manner that gets at the essential points relevant for my seminar.

## Introduction

In quantum field theories, scattering amplitudes can be calculated via the evaluation of appropriate Green's functions and the Lehmann-Symanzik-Zimmermann reduction formalism. This means symmetries of the relevant theory that constrain scattering amplitudes also constrain Green's functions. Additionally, these symmetries also provide constraints *among* Green's functions, by considering, for example, the divergence of some Green's function. These relations are known as Ward-Fradkin-Takahashi identities, which we will just refer to in short as Ward identities.

**Our goal in these notes is to understand how these Ward identities can be derived in general (focusing on an approach within the path integral formalism) and to see what Ward identities look like in QCD near the chiral limit.**

## Ward identities in a simple $\mathrm{U}(1)$ example

For our simple example, we consider a simple scalar field theory with two real scalar fields ($\Phi_1$ and $\Phi_2$) with a global U(1) symmetry:

$$\mathcal{L} = \frac{1}{2}((\partial_\mu \Phi^\dagger)(\partial^\mu \Phi)) - \frac{m^2}{2}\Phi^\dagger\Phi - \frac{\lambda}{4}(\Phi^\dagger\Phi)^2, \tag{1}$$

with

$$\begin{aligned}\Phi(x) &= \frac{1}{\sqrt{2}}(\Phi_1(x) + i\Phi_2(x)),\\ \Phi^\dagger(x) &= \frac{1}{\sqrt{2}}(\Phi_1(x) - i\Phi_2(x)).\end{aligned}$$

Our Lagrangian is invariant under a global U(1) transformation:

$$\Phi \to (1 + i\epsilon)\Phi,$$
$$\Phi^\dagger \to (1 - i\epsilon)\Phi^\dagger,$$

giving us the conserved current:

$$J^\mu = i((\partial^\mu \Phi^\dagger)\Phi - \Phi^\dagger(\partial^\mu \Phi)). \tag{2}$$

After canonical quantization, one can show that

$$J^\mu \to J^\mu$$

under the previously mentioned infinitesimal global transformation. When considering the Green's function

$$G^\mu(x, y, z) = \langle 0| T[\Phi(x) J^\mu(y) \Phi^\dagger(z)] |0\rangle\,, \tag{3}$$

we see that it is also invariant under such a transformation, a constraint imposed upon it by the symmetry of the theory. Note that Green's functions will not always be invariant, but will have specified transformation behavior based on the transformation behavior of the fields. Furthermore, when considering the divergence of $G^\mu$, one finds:

$$\partial^y_\mu G^\mu(x, y, z) = (\delta^4(y - x) - \delta^4(y - z)) \langle 0| T[\Phi(x)\Phi^\dagger(z)] |0\rangle\,. \tag{4}$$

This Ward identity relates a 3-point Green's function to a 2-point Green's function. Next, we will go through the steps necessary to arrive at this result in the path integral formalism, which will allow us to learn some general properties of the approach. These will help us when we move on to the more complicated $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ and will ultimately make the path integral formalism the approach of choice.

To briefly recap the path integral formalism, we can calculate correlation functions (in this case a 2-point correlation function) via the following functional integral:

$$\langle 0| T[\Phi^\dagger(x)\Phi(y)] |0\rangle = \frac{\int \mathcal{D}\Phi^* \mathcal{D}\Phi\, \Phi^*(x)\Phi(y) \exp(iS[\Phi, \Phi^*])}{\int \mathcal{D}\Phi^* \mathcal{D}\Phi \exp(iS[\Phi, \Phi^*])}, \tag{5}$$

where $S[\Phi, \Phi^*]$ is the action of our theory:

$$S[\Phi, \Phi^*] = \int d^4x \mathcal{L}(x).$$

To unite all correlation functions containing these operators in one object, we define a generating functional:

$$W[j, j^*] = \langle 0|T[\exp\left\{ i \int d^4x [j(x)\Phi^\dagger(x) + j^*(x)\Phi(x)] \right\}]|0\rangle \,, \tag{6}$$

where we have introduced external sources $j(x)$ and $j^*(x)$. With this functional, we can get any Green's function by taking functional derivatives with respect to $j$ and $j^*$ at the coordinates we are interested in. For example, the previous 2-point correlation function can be written as:

$$\langle 0|T[\Phi^\dagger(x)\Phi(y)]|0\rangle = \left[ \left( -i\frac{\delta}{\delta j(x)} \right) \left( -i\frac{\delta}{\delta j^*(y)} \right) W[j, j^*] \right]_{j=0, j^*=0} .$$

These functional derivatives bring down factors of $\Phi^\dagger$ and $\Phi$ from our functional. To discuss Green's functions containing our Noether current $J^\mu$, like $G^\mu$, we need to add a source term for our current as well, giving us

$$W[j, j^*, j_\mu] = \langle 0|T[\exp\left\{ i \int d^4x [j(x)\Phi^\dagger(x) + j^*(x)\Phi(x) + j_\mu(x)J^\mu(x)] \right\}]|0\rangle \,. \tag{7}$$

From this, we can write $G^\mu$ as

$$G^\mu(x, y, z) = \left[ (-i)^3 \frac{\delta^3 W[j, j^*, j_\mu]}{\delta j^*(x)\delta j_\mu(y)\delta j(z)} \right]_{j=0, j^*=0, j_\mu=0} . \tag{8}$$

To see the effects of symmetries of our theory on our generating functional, let us consider its path integral representation

$$W[j, j^*, j_\mu] = \frac{\int \mathcal{D}\Phi^*\mathcal{D}\Phi\Phi^*(x)\Phi(y)\exp(iS[\Phi, \Phi^*])}{\int \mathcal{D}\Phi^*\mathcal{D}\Phi \exp(iS[\Phi, \Phi^*, j, j^*, j_\mu])}, \tag{9}$$

with

$$S[\Phi, \Phi^*, j, j^*, j_\mu] = S[\Phi, \Phi^*] + \int d^4x [j(x)\Phi^*(x) + j^*(x)\Phi(x) + j_\mu(x)J^\mu(x)].$$

Demanding that this action remain invariant under a *local* infinitesimal U(1) transformation gives us the necessary simultaneous transformation behavior of the external sources:

$$\begin{aligned} j(x) &\to (1+i\epsilon(x))j(x),\\ j^*(x) &\to (1-i\epsilon(x))j^*(x),\\ j_\mu(x) &\to j_\mu - (\partial_\mu\epsilon(x)). \end{aligned}$$

One then also sees that as a result the generating functional is invariant under the same transformation:

$$W[j,j^*,j_\mu] = W[j',j'^*,j'_\mu]. \tag{10}$$

An expansion of the difference between the original and transformed generating functionals in powers of $\epsilon(x)$ gives the condition

$$0 = \int d^x\epsilon(x)\left[ij(x)\frac{\delta}{\delta j(x)} - ij^*(x)\frac{\delta}{\delta j^*(x)} + \partial^x_\mu\frac{\delta}{\delta j_\mu(x)}\right]W[j,j^*,j_\mu],$$

which due to being valid for all $\epsilon(x)$ simplifies to

$$\left[j(x)\frac{\delta}{\delta j(x)} - j^*(x)\frac{\delta}{\delta j^*(x)} - i\partial^x_\mu\frac{\delta}{\delta j_\mu(x)}\right]W[j,j^*,j_\mu] = 0. \tag{11}$$

We can now return to the divergence of $G^\mu$ and derive the Ward identity from before:

$$\begin{aligned} \partial^y_\mu G^\mu(x,y,z) &= \left[(-i)^3\partial^y_\mu\frac{\delta^3 W[j,j^*,j_\mu]}{\delta j^*(x)\delta j_\mu(y)\delta j(z)}\right]_{j=0,j^*=0,j_\mu=0},\\ &= (-i^2)\left\{\frac{\delta^2}{\delta j^*(x)\delta j(z)}\left[(-i)\partial^y_\mu\frac{\delta W}{\delta j_\mu(y)}\right]\right\}_{j=0,j^*=0,j_\mu=0},\\ &= (-i^2)\left\{\frac{\delta^2}{\delta j^*(x)\delta j(z)}\left[j^*(y)\frac{\delta W}{\delta j^*(y)} - j(y)\frac{\delta W}{\delta j(y)}\right]\right\}_{j=0,j^*=0,j_\mu=0},\\ &= (-i^2)\left\{\delta^4(y-x)\frac{\delta^2 W}{\delta j^*(y)\delta j(z)} - \delta^4(y-z)\frac{\delta W}{\delta j^*(x)\delta j(y)}\right\}_{j=0,j^*=0,j_\mu=0}. \end{aligned}$$

Going from the second-to-last line to the last line, we used that $\delta j^*(y)/\delta j^*(x) = \delta^4(y-x)$ and $\delta j(y)/\delta j(z) = \delta^4(y-z)$ and the chain rule terms leaving $j^*(y)$ and $j(y)$ untouched disappear when our sources get set to 0. Due to the delta functions, we can replace the $y$ coordinates in the functional derivatives with $x$ and $z$ and evaluate the functional derivatives to obtain Green's functions, giving us

$$\partial^y_\mu G^\mu(x,y,z) = \left(\delta^4(y-x) - \delta^4(y-z)\right) \langle 0|T[\Phi(x)\Phi^\dagger(z)]|0\rangle \, .$$

The key points to take from this example are:

- the generating functional is invariant under *local* symmetry transformations,
- the invariance of the functional can be used to derive a master equation that can be used to derive all Ward identities.

## Chiral Green's functions

### Things left to do:

- Introduce color neutral fields
- Write out example Green's functions
- Give intuitive interpretation for what these Green's functions are related to
- Example Green's function and result for divergence

## Chiral Ward identities via the current algebra

### Things left to do:

- Recap current algebra
- Show how naive application of current algebra can lead to wrong results
- Explain Schwinger term solution
- Feynman postulate that Schwinger terms and effects of covariant time time-ordering product cancel giving expected commutation relations

## Chiral Ward identities from the generating functional

### Things left to do:

- Introduce Lagrangian with external fields
- Motivate external fields as conserved currents of global symmetry
- Determine form of external fields by demanding invariances of Lagrangian

  - Hermitian
  - Lorentz scalar
  - Even under P, C, and T
  - Invariant under local chiral transformations

- List how external fields must transform under chiral transformations
- Give one concrete example

## Key takeaways