Sources

These notes exist as a form of preparation for a seminar I have to give on this topic. The primary source that these notes are essentially completely based off of is "A Primer for Chiral Perturbation Theory" by Scherer and Schindler. I also looked some additional things up in the papers by Gasser and Leutwyler published from 1982 to 1985. None of the content in these notes are my own developments. This is purely an attempt at reassembling the content of Scherer and Schindler's notes in a manner that gets at the essential points relevant for my seminar.

Introduction

In quantum field theories, scattering amplitudes can be calculated via the evaluation of appropriate Green's functions and the Lehmann-Symanzik-Zimmermann reduction formalism. This means symmetries of the relevant theory that constrain scattering amplitudes also constrain Green's functions. Additionally, these symmetries also provide constraints *among* Green's functions, by considering, for example, the divergence of some Green's function. These relations are known as Ward-Fradkin-Takahashi identities, which we will just refer to in short as Ward identities.

Our goal in these notes is to understand how these Ward identities can be derived in general (focusing on an approach within the path integral formalism) and to see what Ward identities look like in QCD near the chiral limit.

Ward identities in a simple $\mathrm{U}(1)$ example

For our simple example, we consider a simple scalar field theory with two real scalar fields (Φ_1 and Φ_2) with a global U(1) symmetry:

$$\mathcal{L} = \frac{1}{2}((\partial_\mu \Phi^\dagger)(\partial^\mu \Phi)) - \frac{m^2}{2}\Phi^\dagger\Phi - \frac{\lambda}{4}(\Phi^\dagger\Phi)^2, \tag{1}$$

with

$$\begin{aligned}\Phi(x) &= \frac{1}{\sqrt{2}}(\Phi_1(x) + i\Phi_2(x)),\\ \Phi^\dagger(x) &= \frac{1}{\sqrt{2}}(\Phi_1(x) - i\Phi_2(x)).\end{aligned}$$

Our Lagrangian is invariant under a global U(1) transformation:

$$\Phi \to (1 + i\epsilon)\Phi,$$
$$\Phi^\dagger \to (1 - i\epsilon)\Phi^\dagger,$$

giving us the conserved current:

$$J^\mu = i((\partial^\mu \Phi^\dagger)\Phi - \Phi^\dagger(\partial^\mu \Phi)). \tag{2}$$

After canonical quantization, one can show that

$$J^\mu \to J^\mu$$

under the previously mentioned infinitesimal global transformation. When considering the Green's function

$$G^\mu(x, y, z) = \langle 0| T[\Phi(x) J^\mu(y) \Phi^\dagger(z)] |0\rangle\,, \tag{3}$$

we see that it is also invariant under such a transformation, a constraint imposed upon it by the symmetry of the theory. Note that Green's functions will not always be invariant, but will have specified transformation behavior based on the transformation behavior of the fields. Furthermore, when considering the divergence of G^μ, one finds:

$$\partial^y_\mu G^\mu(x, y, z) = (\delta^4(y - x) - \delta^4(y - z)) \langle 0| T[\Phi(x)\Phi^\dagger(z)] |0\rangle\,. \tag{4}$$

This Ward identity relates a 3-point Green's function to a 2-point Green's function. Next, we will go through the steps necessary to arrive at this result in the path integral formalism, which will allow us to learn some general properties of the approach. These will help us when we move on to the more complicated $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ and will ultimately make the path integral formalism the approach of choice.

To briefly recap the path integral formalism, we can calculate correlation functions (in this case a 2-point correlation function) via the following functional integral:

$$\langle 0| T[\Phi^\dagger(x)\Phi(y)] |0\rangle = \frac{\int \mathcal{D}\Phi^* \mathcal{D}\Phi \Phi^*(x)\Phi(y) \exp(iS[\Phi, \Phi^*])}{\int \mathcal{D}\Phi^* \mathcal{D}\Phi \exp(iS[\Phi, \Phi^*])}, \tag{5}$$

where $S[\Phi, \Phi^*]$ is the action of our theory:

$$S[\Phi,\Phi^*]=\int d^4x\mathcal{L}(x).$$

To unite all correlation functions containing these operators in one object, we define a generating functional:

$$W[j,j^*]=\ \langle 0|T[\exp\Big\{i\int d^4x[j(x)\Phi^\dagger(x)+j^*(x)\Phi(x)]\Big\}]|0\rangle\,, \tag{6}$$

where we have introduced external sources $j(x)$ and $j^*(x)$. With this functional, we can get any Green's function by taking functional derivatives with respect to j and j^* at the coordinates we are interested in. For example, the previous 2-point correlation function can be written as:

$$\langle 0|T[\Phi^\dagger(x)\Phi(y)]|0\rangle=\left[\left(-i\frac{\delta}{\delta j(x)}\right)\left(-i\frac{\delta}{\delta j^*(y)}\right)W[j,j^*]\right]_{j=0,j^*=0}.$$

These functional derivatives bring down factors of $\Phi^\dagger$ and Φ from our functional. To discuss Green's functions containing our Noether current J^μ, like G^μ, we need to add a source term for our current as well, giving us

$$W[j,j^*,j_\mu]=\ \langle 0|T[\exp\Big\{i\int d^4x[j(x)\Phi^\dagger(x)+j^*(x)\Phi(x)+j_\mu(x)J^\mu(x)]\Big\}]|0\rangle\,. \tag{7}$$

From this, we can write G^μ as

$$G^\mu(x,y,z)=\left[(-i)^3\frac{\delta^3W[j,j^*,j_\mu]}{\delta j^*(x)\delta j_\mu(y)\delta j(z)}\right]_{j=0,j^*=0,j_\mu=0}. \tag{8}$$

To see the effects of symmetries of our theory on our generating functional, let us consider its path integral representation

$$W[j,j^*,j_\mu]=\frac{\int\mathcal{D}\Phi^*\mathcal{D}\Phi\Phi^*(x)\Phi(y)\exp(iS[\Phi,\Phi^*])}{\int\mathcal{D}\Phi^*\mathcal{D}\Phi\exp(iS[\Phi,\Phi^*,j,j^*,j_\mu])}, \tag{9}$$

with

$$S[\Phi,\Phi^*,j,j^*,j_\mu]=S[\Phi,\Phi^*]+\int d^4x[j(x)\Phi^*(x)+j^*(x)\Phi(x)+j_\mu(x)J^\mu(x)].$$

Demanding that this action remain invariant under a *local* infinitesimal U(1) transformation gives us the necessary simultaneous transformation behavior of the external sources:

$$\begin{aligned} j(x) &\to (1 + i\epsilon(x))j(x), \\ j^*(x) &\to (1 - i\epsilon(x))j^*(x), \\ j_\mu(x) &\to j_\mu - (\partial_\mu \epsilon(x)). \end{aligned}$$

One then also sees that as a result the generating functional is invariant under the same transformation:

$$W[j, j^*, j_\mu] = W[j', j'^*, j'_\mu]. \tag{10}$$

An expansion of the difference between the original and transformed generating functionals in powers of $\epsilon(x)$ gives the condition

$$0 = \int d^x \epsilon(x) \left[ij(x)\frac{\delta}{\delta j(x)} - ij^*(x)\frac{\delta}{\delta j^*(x)} + \partial^x_\mu \frac{\delta}{\delta j_\mu(x)} \right] W[j, j^*, j_\mu],$$

which due to being valid for all $\epsilon(x)$ simplifies to

$$\left[j(x)\frac{\delta}{\delta j(x)} - j^*(x)\frac{\delta}{\delta j^*(x)} - i\partial^x_\mu \frac{\delta}{\delta j_\mu(x)} \right] W[j, j^*, j_\mu] = 0. \tag{11}$$

We can now return to the divergence of G^μ and derive the Ward identity from before:

$$\begin{aligned} \partial^y_\mu G^\mu(x, y, z) &= \left[(-i)^3 \partial^y_\mu \frac{\delta^3 W[j, j^*, j_\mu]}{\delta j^*(x)\delta j_\mu(y)\delta j(z)} \right]_{j=0, j^*=0, j_\mu=0}, \\ &= (-i^2) \left\{ \frac{\delta^2}{\delta j^*(x)\delta j(z)} \left[(-i)\partial^y_\mu \frac{\delta W}{\delta j_\mu(y)} \right] \right\}_{j=0, j^*=0, j_\mu=0}, \\ &= (-i^2) \left\{ \frac{\delta^2}{\delta j^*(x)\delta j(z)} \left[j^*(y)\frac{\delta W}{\delta j^*(y)} - j(y)\frac{\delta W}{\delta j(y)} \right] \right\}_{j=0, j^*=0, j_\mu=0}, \\ &= (-i^2) \left\{ \delta^4(y - x)\frac{\delta^2 W}{\delta j^*(y)\delta j(z)} - \delta^4(y - z)\frac{\delta W}{\delta j^*(x)\delta j(y)} \right\}_{j=0, j^*=0, j_\mu=0}. \end{aligned}$$

Going from the second-to-last line to the last line, we used that $\delta j^*(y)/\delta j^*(x) = \delta^4(y - x)$ and $\delta j(y)/\delta j(z) = \delta^4(y - z)$ and the chain rule terms leaving $j^*(y)$ and $j(y)$ untouched disappear when our sources get set to 0. Due to the delta functions, we can replace the y coordinates in the functional derivatives with x and z and evaluate the functional derivatives to obtain Green's functions, giving us

$$\partial_\mu^y G^\mu(x,y,z) = \left(\delta^4(y-x) - \delta^4(y-z)\right) \langle 0|T[\Phi(x)\Phi^\dagger(z)]|0\rangle \,.$$

The key points to take from this example are:

- the generating functional is invariant under *local* symmetry transformations,
- the invariance of the functional can be used to derive a master equation that can be used to derive all Ward identities.

Chiral Green's functions

Recall that in the chiral limit (quarks are massless) the light quark QCD Lagrangian can be written as

$$\mathcal{L}^0_{\mathrm{QCD}} = \sum_{l=u,d,s} (\bar{q}_{R,l} i \not{D} q_{R,l} + \bar{q}_{L,l} i \not{D} q_{L,l}) - \frac{1}{4}\mathcal{G}_{a\mu\nu}\mathcal{G}^{\mu\nu}_a. \tag{12}$$

This Lagrangian is invariant under a global $\mathrm{U}(3) \times \mathrm{U}(3)$ symmetry:

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \to \exp\left\{-i\sum_{a=1}^{8} \Theta_{La}\frac{\lambda_a}{2}\right\} e^{-i\Theta_L} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}, \tag{13}$$

$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \to \exp\left\{-i\sum_{a=1}^{8} \Theta_{Ra}\frac{\lambda_a}{2}\right\} e^{-i\Theta_R} \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}, \tag{14}$$

which has been explicitly decomposed into an $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_L \times \mathrm{U}(1)_R$ representation. Under a corresponding local transformation, we find conserved left-handed and right-handed SU(3) currents, L^μ_a and R^μ_a, and conserved left-handed and right-handed singlet currents, L^μ and R^μ. After quantizing our theory, we find that all the SU(3) currents are still conserved, defining useful vector and axial-vector linear combinations like:

$$V^\mu_a = R^\mu_a + L^\mu_a = \bar{q}\gamma^\mu \frac{\lambda_a}{2} q, \tag{15}$$
$$A^\mu_a = R^\mu_a - L^\mu_a = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda_a}{2} q. \tag{16}$$

We also find that the corresponding singlet vector current

$$V^\mu = R^\mu + L^\mu = \bar{q}\gamma^\mu q \tag{17}$$

is conserved, but the singlet axial-vector current

$$A^\mu = R^\mu - L^\mu = \bar{q}\gamma^\mu \gamma_5 q$$

is no longer conserved due to quantum corrections, referred to as the anomaly.

For chiral Green's functions, we are interested in vacuum expectation values of time-ordered products of color-neutral, Hermitian quadratic forms (which can be related to processes involving mesons) along with our conserved currents. We introduce the scalar and pseudoscalar densities

$$S_a(x) = \bar{q}(x)\lambda_a q(x), \tag{18}$$
$$P_a(x) = \bar{q}(x)\gamma_5\lambda_a q(x), \tag{19}$$

where $a = 0, ..., 8$.

One example Green's function is

$$G^\mu_{APab}(x, y) = \langle 0| T[A^\mu_a(x) P_b(y)] |0\rangle \,, \tag{20}$$

which is related to the pion decay process. Chiral Ward identities are refer to Ward identities which relate the divergence of a Green's function containing V^μ_a or A^μ_a to some other Green's functions. In this context, *chiral* refers to the underlying $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ group to which these currents correspond. We can compute the divergence of our Green's function above, for example, and find

$$\partial^x_\mu G^\mu_{APab}(x, y) = \delta(x_0 - y_0) \langle 0| [A^0_a(x), P_b(y)] |0\rangle + \langle 0| T[(\partial^x_\mu A^\mu_a(x)) P_b(y)] |0\rangle \,. \tag{21}$$

Let us discuss briefly the general properties of this Ward identity. First, we get an equal-time commutator of the charge density and the remaining operator. Second, we get a term which replaces the Noether current in the original Green's function with its divergence. In the exact chiral limit, this term vanishes. If the breaking of chiral symmetry is small, we can incorporate this term perturbatively. In general, for the divergence of a Green's function with a Noether current and multiple other operators, we get a sum of terms with equal-time commutators between the Noether current and the other operators and a last term with the divergence of the Noether current.

Chiral Ward identities via the current algebra

To get a simplified form for these chiral Ward identities, we need to evaluate commutators of the charge density of our Noether current of interest with other Noether currents and our scalar and pseudoscalar densities. The original algebra for the charge densities of our Neother currents is

$$[Q_{La}, Q_{Lb}] = if_{abc}Q_{Lc}, \tag{22}$$

$$[Q_{Ra}, Q_{Rb}] = if_{abc}Q_{Rc}, \tag{23}$$

$$[Q_{La}, Q_{Rb}] = 0, \tag{24}$$

$$[Q_{La}, Q_V] = [Q_{Ra}, Q_V] = 0. \tag{25}$$

This can easily be naively extended to our vector and axial-vector charge densities:

$$[Q_{Va}, Q_{Vb}] = if_{abc}Q_{Vc}, \tag{26}$$

$$[Q_{Aa}, Q_{Ab}] = if_{abc}Q_{Vc}, \tag{27}$$

$$[Q_{Va}, Q_{Ab}] = if_{abc}Q_{Ac}, \tag{28}$$

$$[Q_{Aa}, Q_{Vb}] = if_{abc}Q_{Ac}, \tag{29}$$

$$[Q_{Aa}, Q_V] = [Q_{Va}, Q_V] = 0. \tag{30}$$

However, a general derivation of the algebra obeyed by our Noether currents and scalar and pseudoscalar densities is potentially error-prone. The reason lies in a conceptual inconsistency first noticed by Schwinger, which we will illustrate via an example from QED. The electromagnetic current space and time components can be, through the use of canonical commutation relations, shown to commute:

$$[J_0(t,\vec{x}, J_i(t,\vec{y}] = 0.$$

This implies that the following commutator vanishes:

$$[J_0(t,\vec{x}, \vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}_y \cdot \vec{J}(t,\vec{y})] = -[J_0(t,\vec{x}), \partial_t J_0(t,\vec{y})] = 0.$$

Evaluating this commutator between the ground state for $\vec{x} = \vec{y}$ yields the result

$$\langle 0|J_0(t,\vec{x})|n\rangle = 0$$

for any intermediate state $|n\rangle$, which is unphysical because it would imply that the charge density operator could not generate e^+e^- pairs when operating on the vacuum. A correction to the original commutator containing a derivative of the delta function called the Schwinger term alleviates this issue. This is because the equal-time commutators can only be determined up to a factor of the derivative of the delta function.

So what does this mean for the chiral Ward identity we saw before? Well, it is correct, which one can verify via the path integral formalism that we will discuss soon. This comes from the fact that we are considering the naive time-ordered product rather than the covariant time-ordered product, which has different behavior at the equal-time points. Feynman postulated that the Schwinger terms cancel with the seagull terms from the covariant time-ordered product so just using the naive time-ordered product and omitting Schwinger terms gives the right results. However, one must tread carefully, and overall this “solution” is fairly unsatisfactory.

Chiral Ward identities from the generating functional

The previously mentioned problems related to Schwinger terms are absent in the path integral formalism, making it more convenient and allowing us to proceed with confidence. We will proceed by constructing a generating functional for our Green’s functions of interest and demanding its invariance under a local transformation of the external fields. This invariance is equivalent to the set of all chiral Ward identities and can be used to look at divergences of Green’s functions.

Things left to do:

- Introduce Lagrangian with external fields
- Motivate external fields as conserved currents of global symmetry

- Determine form of external fields by demanding invariances of Lagrangian
  - Hermitian
  - Lorentz scalar
  - Even under P, C, and T
  - Invariant under local chiral transformations
- List how external fields must transform under chiral transformations
- Give one concrete example

Key takeaways