

Chiral Green's functions and Ward identities

Matthias Heinz



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Scalar ϕ^4 theory with a global $U(1)$ symmetry



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$$\mathcal{L} = \frac{1}{2}((\partial_\mu \Phi^\dagger)(\partial^\mu \Phi)) - \frac{m^2}{2}\Phi^\dagger \Phi - \frac{\lambda}{4}(\Phi^\dagger \Phi)^2$$

Global $U(1)$ symmetry:

$$\Phi \rightarrow (1 + i\epsilon)\Phi, \quad \Phi^\dagger \rightarrow (1 - i\epsilon)\Phi^\dagger,$$

Conserved Noether current:

$$J^\mu = i((\partial^\mu \Phi^\dagger)\Phi - \Phi^\dagger(\partial^\mu \Phi))$$

Scalar ϕ^4 theory with a global $U(1)$ symmetry



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Example Green's function:

$$G^\mu(x, y, z) = \langle 0 | T[\Phi(x) J^\mu(y) \Phi^\dagger(z)] | 0 \rangle,$$

Symmetry constraint:

$$J^\mu \rightarrow J^\mu, \quad G^\mu \rightarrow G^\mu,$$

Example Ward identity:

$$\begin{aligned} \partial_\mu^y G^\mu(x, y, z) = & (\delta^4(y - x) - \delta^4(y - z)) \langle 0 | T[\Phi(x) \Phi^\dagger(z)] | 0 \rangle \\ & + \langle 0 | T[\Phi(x) (\partial_\mu^y J^\mu(y)) \Phi^\dagger(z)] | 0 \rangle, \end{aligned}$$



Generating functional:

$$W[j, j^*, j_\mu] = \langle 0 | T[\exp \left\{ i \int d^4x [j(x)\Phi^\dagger(x) + j^*(x)\Phi(x) + j_\mu(x)J^\mu(x)] \right\}] | 0 \rangle ,$$

Our example Green's function:

$$G^\mu(x, y, z) = (-i)^3 \frac{\delta^3 W[j, j^*, j_\mu]}{\delta j^*(x) \delta j_\mu(y) \delta j(z)} \Big|_{j=0, j^*=0, j_\mu=0} ,$$

As path integral:

$$W[j, j^*, j_\mu] = \int \mathcal{D}\Phi^* \mathcal{D}\Phi \exp \left(i \int d^4x [\mathcal{L}(x) + j(x)\Phi^*(x) + j^*(x)\Phi(x) + j_\mu(x)J^\mu(x)] \right) ,$$



Generating functional:

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As path integral:

$$W[j, j^*, j_\mu] = \int \mathcal{D}\Phi^* \mathcal{D}\Phi \exp(iS[\Phi, \Phi^*, j, j^*, j_\mu]) ,$$

Note: Only in the presence of external fields can we demand \mathcal{L} remain invariant under *local* transformations.

The master equation for Φ^4



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Demanding $S[\Phi, \Phi^\dagger, j, j^*, j_\mu] = S[\Phi', \Phi'^\dagger, j', j'^*, j'_\mu]$ gives:

$$\begin{aligned}j(x) &\rightarrow (1 + i\epsilon(x))j(x), \\j^*(x) &\rightarrow (1 - i\epsilon(x))j^*(x), \\j_\mu(x) &\rightarrow j_\mu - (\partial_\mu \epsilon(x)),\end{aligned}$$

We observe that this also means:

$$W[j, j^*, j_\mu] = W[j', j'^*, j'_\mu],$$

Master equation:

$$0 = \int d^4x \epsilon(x) \left[ij(x) \frac{\delta}{\delta j(x)} - ij^*(x) \frac{\delta}{\delta j^*(x)} + \partial_\mu^x \frac{\delta}{\delta j_\mu(x)} \right] W[j, j^*, j_\mu],$$

The master equation for Φ^4



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We observe that this also means:

$$W[j, j^*, j_\mu] = W[j', j'^*, j'_\mu],$$

Master equation:

$$0 = \left[j(x) \frac{\delta}{\delta j(x)} - j^*(x) \frac{\delta}{\delta j^*(x)} - i \partial_\mu^x \frac{\delta}{\delta j_\mu(x)} \right] W[j, j^*, j_\mu],$$



$$\mathcal{L}_{\text{QCD}}^0 = \sum_{l=u,d,s} (\bar{q}_{R,l} i \not{D} q_{R,l} + \bar{q}_{L,l} i \not{D} q_{L,l}) - \frac{1}{4} \mathcal{G}_{a\mu\nu} \mathcal{G}_a^{\mu\nu},$$

Symmetry group:

$$\text{U}(3)_L \times \text{U}(3)_R \xrightarrow{\text{Quantization}} \text{SU}(3)_L \times \text{SU}(3)_R \times \text{U}(1)_V$$

Conserved currents:

- ▶ $V_a^\mu = R_a^\mu + L_a^\mu = \bar{q} \gamma^\mu \frac{\lambda_a}{2} q,$
- ▶ $A_a^\mu = R_a^\mu - L_a^\mu = \bar{q} \gamma^\mu \gamma_5 \frac{\lambda_a}{2} q,$
- ▶ $V^\mu = R^\mu + L^\mu = \bar{q} \gamma^\mu q,$

Color-neutral quadratic forms:

- ▶ $S_a(x) = \bar{q}(x) \lambda_a q(x),$
- ▶ $P_a(x) = i \bar{q}(x) \gamma_5 \lambda_a q(x),$

Note: $a = 0, \dots, 8$

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An example



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Green's function:

$$G_{APab}^{\mu}(x, y) = \langle 0 | T[A_a^{\mu}(x) P_b(y)] | 0 \rangle ,$$

Ward identity:

$$\partial_{\mu}^x G_{APab}^{\mu}(x, y) = \delta(x_0 - y_0) \langle 0 | [A_a^0(x), P_b(y)] | 0 \rangle + \langle 0 | T[(\partial_{\mu}^x A_a^{\mu}(x)) P_b(y)] | 0 \rangle ,$$

Generalization to any $(n + 1)$ -point functions:

$$\begin{aligned} \partial_{\mu}^x \langle 0 | T[J^{\mu}(x) A_1(x_1) \dots A_n(x_n)] | 0 \rangle &= \langle 0 | T[(\partial_{\mu}^x J^{\mu}(x)) A_1(x_1) \dots A_n(x_n)] | 0 \rangle \\ &+ \delta(x^0 - x_1^0) \langle 0 | T[[J_0(x), A_1(x_1)] A_2(x_2) \dots A_n(x_n)] | 0 \rangle \\ &+ \dots \\ &+ \delta(x^0 - x_n^0) \langle 0 | T[A_1(x_1) A_2(x_2) \dots [J_0(x), A_n(x_n)]] | 0 \rangle , \end{aligned}$$



- ▶ We could now evaluate $[J_0(x), A_n(x_n)]$ commutators
- ▶ *But we have to be careful*
- ▶ QED current example:
 - ▶ $[J_0(t, \vec{x}), J_i(t, \vec{y})] = 0$
 - ▶ from which one can show $\langle 0 | J_0(t, \vec{x}) | n \rangle = 0$
- ▶ Fix: Schwinger term in original charge-current commutator
- ▶ In general, charge-current commutation relations only determined up to a derivative of a delta function
- ▶ Another problem: used naive time-ordered product rather than *covariant* time-ordered product
- ▶ Seagull terms from covariant time-ordering cancel with Schwinger terms (Feynman)



Extend chiral Lagrangian to include external fields (sources):

$$\mathcal{L} = \mathcal{L}_{\text{QCD}}^0 + \mathcal{L}_{\text{ext}}$$

with

$$\mathcal{L}_{\text{ext}} = \sum_{a=1}^8 v_a^\mu V_a^\mu + \frac{1}{3} v_{(s)}^\mu V^\mu + \sum_{a=1}^8 a_a^\mu A_a^\mu - \sum_{a=0}^8 s_a S_a + \sum_{a=0}^8 p_a P_a,$$



Extend chiral Lagrangian to include external fields (sources):

$$\mathcal{L} = \mathcal{L}_{\text{QCD}}^0 + \mathcal{L}_{\text{ext}}$$

with

$$\mathcal{L}_{\text{ext}} = \bar{q} \gamma_\mu \left(v^\mu + \frac{1}{3} v_{(s)}^\mu + \gamma_5 a^\mu \right) q - \bar{q}(s - i\gamma_5 p)q,$$

using definitions

$$v^\mu = \sum_{a=1}^8 v_a^\mu \frac{\lambda_a}{2}, \quad a^\mu = \sum_{a=1}^8 a_a^\mu \frac{\lambda_a}{2},$$
$$s = \sum_{a=0}^8 s_a \lambda_a, \quad p = \sum_{a=0}^8 p_a \lambda_a,$$

Chiral generating functional

Some examples



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Generating functional:

$$W[v, a, s, p] = \langle 0 | T[\exp \left\{ i \int d^4x \mathcal{L}_{\text{ext}}(x) \right\}] | 0 \rangle_0,$$

Chiral limit example:

$$\bar{u}u = \frac{1}{2} \bar{q} \left(\sqrt{\frac{2}{3}} \lambda_0 + \lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 \right) q,$$

$$\langle 0 | \bar{u}(x) u(x) | 0 \rangle_0 = \frac{i}{2} \left[\sqrt{\frac{2}{3}} \frac{\delta}{\delta s_0(x)} + \frac{\delta}{\delta s_3(x)} + \frac{1}{\sqrt{3}} \frac{\delta}{\delta s_8(x)} \right] W[v, a, s, p] \Big|_{v=a=s=p=0},$$

Chiral generating functional

Some examples



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Generating functional:

$$W[v, a, s, p] = \langle 0 | T[\exp \left\{ i \int d^4x \mathcal{L}_{\text{ext}}(x) \right\}] | 0 \rangle_0 ,$$

Physical example:

$$\langle 0 | T[A_a^\mu(x) P_b(y)] | 0 \rangle = (-i)^2 \frac{\delta^2}{\delta a_{a\mu}(x) \delta p_b(y)} W[v, a, s, p] \Big|_{v=a=p=0, s=\text{diag}(m_u, m_d, m_s)} ,$$



We demand of \mathcal{L} that it is:

- ▶ Hermitian Lorentz scalar
- ▶ Even under P and C
- ▶ Invariant under local chiral transformations



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Parity:

$$V^\mu \xrightarrow{P} V_\mu,$$

$$V_{(s)}^\mu \xrightarrow{P} V_{\mu}^{(s)},$$

$$a^\mu \xrightarrow{P} -a_\mu,$$

$$s \xrightarrow{P} s,$$

$$p \xrightarrow{P} -p,$$

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Charge conjugation:

$$v_\mu \xrightarrow{C} -v_\mu^T,$$

$$v_\mu^{(s)} \xrightarrow{C} -v_\mu^{(s)T},$$

$$a_\mu \xrightarrow{C} a_\mu^T,$$

$$s \xrightarrow{C} s^T,$$

$$p \xrightarrow{C} p^T,$$

Local chiral transformation:

$$q_R \rightarrow \exp\left\{-i\frac{\Theta(x)}{3}\right\} V_R(x) q_R, \quad q_L \rightarrow \exp\left\{-i\frac{\Theta(x)}{3}\right\} V_L(x) q_L,$$

After splitting our external fields into $r_\mu = v_\mu + a_\mu$ and $l_\mu = v_\mu - a_\mu$:

$$r_\mu \rightarrow V_R r_\mu V_R^\dagger + iV_R(\partial_\mu V_R^\dagger),$$

$$l_\mu \rightarrow V_L l_\mu V_L^\dagger + iV_L(\partial_\mu V_L^\dagger),$$

$$v_\mu^{(s)} \rightarrow v_\mu^{(s)} - (\partial_\mu \Theta),$$

$$s + ip \rightarrow V_R(s + ip) V_L^\dagger,$$

$$s - ip \rightarrow V_L(s - ip) V_R^\dagger,$$



U(1) example:

- ▶ Local invariance of generating functional contains all Ward identities of theory

Chiral Ward identities from algebra of currents:

- ▶ Using the algebra of currents, one must tread with caution (Schwinger and seagull terms)

Generating functional for chiral Green's functions:

- ▶ Allows one to compute Green's functions for chiral limit and "real" world
- ▶ Can constrain transformation behavior of external fields by invariance of generating functional under local transformations



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