Chiral Green's functions and Ward identities

Matthias Heinz

TECHNISCHE UNIVERSITÄT DARMSTADT

Outline:

1. Ward identities in a $\mathrm{U}(1)$ example
2. Chiral Ward identities via the algebra of currents
3. The chiral generating functional

Scalar Φ^4 theory with a global $\mathrm{U}(1)$ symmetry

$$\mathcal{L}^0 = \frac{1}{2}((\partial_\mu \Phi^\dagger)(\partial^\mu \Phi)) - \frac{m^2}{2}\Phi^\dagger \Phi - \frac{\lambda}{4}(\Phi^\dagger \Phi)^2$$

Global $\mathrm{U}(1)$ symmetry:

$$\Phi \to (1 + i\epsilon)\Phi, \quad \Phi^\dagger \to (1 - i\epsilon)\Phi^\dagger,$$

Conserved Noether current:

$$J^\mu = i((\partial^\mu \Phi^\dagger)\Phi - \Phi^\dagger(\partial^\mu \Phi))$$

Scalar Φ^4 theory with a global $\mathrm{U}(1)$ symmetry

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Example Green's function:

$$G^{\mu}(x,y,z) = \langle 0 | T[\Phi(x) J^{\mu}(y) \Phi^{\dagger}(z)] | 0 \rangle ,$$

Symmetry constraint:

$$J^{\mu} \to J^{\mu}, \quad G^{\mu} \to G^{\mu},$$

Example Ward identity:

$$\begin{aligned} \partial_{\mu}^{y} G^{\mu}(x,y,z) = & (\delta^4(y-x) - \delta^4(y-z)) \langle 0 | T[\Phi(x) \Phi^{\dagger}(z)] | 0 \rangle \\ & + \langle 0 | T[\Phi(x) (\partial_{\mu}^{y} J^{\mu}(y)) \Phi^{\dagger}(z)] | 0 \rangle , \end{aligned}$$

Generating functional for Φ^4

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Generating functional:

$$W[j, j^*, j_\mu] = \langle 0 | T[\exp\left\{ i \int d^4x [j(x)\Phi^\dagger(x) + j^*(x)\Phi(x) + j_\mu(x)J^\mu(x)] \right\}] | 0 \rangle ,$$

Our example Green's function:

$$G^\mu(x, y, z) = (-i)^3 \frac{\delta^3 W[j, j^*, j_\mu]}{\delta j^*(x) \delta j_\mu(y) \delta j(z)} \Bigg|_{j=0, j^*=0, j_\mu=0} ,$$

As path integral:

$$W[j, j^*, j_\mu] = \int \mathcal{D}\Phi^* \mathcal{D}\Phi \exp\left(i \int d^4x [\mathcal{L}^0(x) + \mathcal{L}_{\mathrm{ext}}(x)] \right),$$

$$\mathcal{L}_{\mathrm{ext}}(x) = j(x)\Phi^*(x) + j^*(x)\Phi(x) + j_\mu(x)J^\mu(x),$$

Generating functional for Φ^4

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As path integral:

$$W[j, j^*, j_\mu] = \int \mathcal{D}\Phi^* \mathcal{D}\Phi \exp\big(iS[\Phi, \Phi^*, j, j^*, j_\mu] \big),$$

Note: Only in the presence of external fields can we demand $\mathcal{L}$ remain invariant under *local* transformations.

The master equation for Φ^4

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Demanding $S[\Phi, \Phi^\dagger, j, j^*, j_\mu] = S[\Phi', \Phi'^\dagger, j', j'^*, j'_\mu]$ gives:

$$\begin{aligned} j(x) &\to (1 + i\epsilon(x))j(x), \\ j^*(x) &\to (1 - i\epsilon(x))j^*(x), \\ j_\mu(x) &\to j_\mu - (\partial_\mu \epsilon(x)), \end{aligned}$$

We observe that this also means:

$$W[j, j^*, j_\mu] = W[j', j'^*, j'_\mu],$$

Master equation:

$$0 = \int d^4x \epsilon(x) \left[ij(x) \frac{\delta}{\delta j(x)} - ij^*(x) \frac{\delta}{\delta j^*(x)} + \partial_\mu^x \frac{\delta}{\delta j_\mu(x)} \right] W[j, j^*, j_\mu],$$

The master equation for Φ^4

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Demanding $S[\Phi, \Phi^\dagger, j, j^*, j_\mu] = S[\Phi', \Phi'^\dagger, j', j'^*, j'_\mu]$ gives:

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$$W[j, j^*, j_\mu] = W[j', j'^*, j'_\mu],$$

Master equation:

$$0 = \left[j(x)\frac{\delta}{\delta j(x)} - j^*(x)\frac{\delta}{\delta j^*(x)} - i\partial_\mu^x \frac{\delta}{\delta j_\mu(x)} \right] W[j, j^*, j_\mu],$$

QCD in the chiral limit

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$$\mathcal{L}^0_{\mathrm{QCD}} = \sum_{l=u,d,s} (\bar{q}_{R,l} i \not{D} q_{R,l} + \bar{q}_{L,l} i \not{D} q_{L,l}) - \frac{1}{4} \mathcal{G}_{a\mu\nu} \mathcal{G}_a^{\mu\nu},$$

Symmetry group:

$$\mathrm{U}(3)_L \times \mathrm{U}(3)_R \xrightarrow{\text{Quantization}} \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$$

Conserved currents:

- $V_a^\mu = R_a^\mu + L_a^\mu = \bar{q}\gamma^\mu \frac{\lambda_a}{2} q,$
- $A_a^\mu = R_a^\mu - L_a^\mu = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda_a}{2} q,$
- $V^\mu = R^\mu + L^\mu = \bar{q}\gamma^\mu q,$

Color-neutral quadratic forms:

- $S_a(x) = \bar{q}(x)\lambda_a q(x),$
- $P_a(x) = i\bar{q}(x)\gamma_5\lambda_a q(x),$

Note: $a = 0, ..., 8$

Chiral Green's functions and Ward identities

An example

Green's function:

$$G^{\mu}_{APab}(x,y) = \langle 0| T[A^{\mu}_a(x) P_b(y)] |0\rangle ,$$

Ward identity:

$$\partial^x_\mu G^{\mu}_{APab}(x,y) = \delta(x_0 - y_0) \langle 0| \left[A^0_a(x), P_b(y)\right] |0\rangle + \langle 0| T[(\partial^x_\mu A^{\mu}_a(x)) P_b(y)] |0\rangle ,$$

Generalization to any $(n+1)$-point functions:

$$\begin{aligned}
\partial^x_\mu \langle 0| T[J^\mu(x) A_1(x_1) \dots A_n(x_n)] |0\rangle &= \langle 0| T[(\partial^x_\mu J^\mu(x)) A_1(x_1) \dots A_n(x_n)] |0\rangle \\
&+ \delta(x^0 - x^0_1) \langle 0| T[[J_0(x), A_1(x_1)] A_2(x_2) \dots A_n(x_n)] |0\rangle \\
&+ \dots \\
&+ \delta(x^0 - x^0_n) \langle 0| T[A_1(x_1) A_2(x_2) \dots [J_0(x), A_n(x_n)]] |0\rangle ,
\end{aligned}$$

Algebra of currents

- We could now evaluate $[J_0(x), A_n(x_n)]$ commutators
- *But we have to be careful*
- QED current example:
  - $[J_0(t, \vec{x}), J_i(t, \vec{y})] = 0$
  - from which one can show $\langle 0 | J_0(t, \vec{x}) | n \rangle = 0$
- Fix: Schwinger term in original charge-current commutator
- In general, charge-current commutation relations only determined up to a derivative of a delta function
- Another problem: used naive time-ordered product rather than *covariant* time-ordered product
- Seagull terms from covariant time-ordering cancel with Schwinger terms (Feynman)

Chiral generating functional

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Extend chiral Lagrangian to include external fields (sources):

$$\mathcal{L} = \mathcal{L}^0_{\mathrm{QCD}} + \mathcal{L}_{\mathrm{ext}}$$

with

$$\mathcal{L}_{\mathrm{ext}} = \sum_{a=1}^{8} v_a^\mu V_a^\mu + \frac{1}{3} v_{(s)}^\mu V^\mu + \sum_{a=1}^{8} a_a^\mu A_a^\mu - \sum_{a=0}^{8} s_a S_a + \sum_{a=0}^{8} p_a P_a,$$

Chiral generating functional

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$$\mathcal{L} = \mathcal{L}^0_{\mathrm{QCD}} + \mathcal{L}_{\mathrm{ext}}$$

with

$$\mathcal{L}_{\mathrm{ext}} = \bar{q}\gamma_\mu \left(v^\mu + \frac{1}{3} v^\mu_{(s)} + \gamma_5 a^\mu \right) q - \bar{q}(s - i\gamma_5 p)q,$$

using definitions

$$v^\mu = \sum_{a=1}^{8} v^\mu_a \frac{\lambda_a}{2}, \quad a^\mu = \sum_{a=1}^{8} a^\mu_a \frac{\lambda_a}{2},$$

$$s = \sum_{a=0}^{8} s_a \lambda_a, \quad p = \sum_{a=0}^{8} p_a \lambda_a,$$

Chiral generating functional

Some examples

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Generating functional:

$$W[v, a, s, p] = \langle 0| T[\exp\left\{ i \int d^4 x \mathcal{L}_{\mathrm{ext}}(x) \right\}]|0\rangle_0 \,,$$

Chiral limit example:

$$\bar{u}u = \frac{1}{2}\bar{q}\left(\sqrt{\frac{2}{3}}\lambda_0 + \lambda_3 + \frac{1}{\sqrt{3}}\lambda_8\right) q,$$

$$\langle 0|\bar{u}(x)u(x)|0\rangle_0 = \frac{i}{2}\left[\sqrt{\frac{2}{3}}\frac{\delta}{\delta s_0(x)} + \frac{\delta}{\delta s_3(x)} + \frac{1}{\sqrt{3}}\frac{\delta}{\delta s_8(x)}\right] W[v, a, s, p]\Bigg|_{v=a=s=p=0} \,,$$

Chiral generating functional

Some examples

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Generating functional:

$$W[v,a,s,p] = \langle 0| T[\exp\left\{ i \int d^4x \mathcal{L}_{\mathrm{ext}}(x) \right\}]|0\rangle_0 \,,$$

Physical example:

$$\langle 0| T[A_a^\mu(x) P_b(y)]|0\rangle = (-i)^2 \, \frac{\delta^2}{\delta a_{a\mu}(x)\delta p_b(y)} W[v,a,s,p]\Big|_{v=a=p=0,s=\mathrm{diag}(m_u,m_d,m_s)} \,,$$

Constraining external fields

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We demand of $\mathcal{L}$ that it is:

- Hermitian Lorentz scalar
- Even under P and C
- Invariant under local chiral transformations

Constraining external fields

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Parity:

$$v^\mu \xrightarrow{P} v_\mu,$$
$$v^\mu_{(s)} \xrightarrow{P} v^{(s)}_\mu,$$
$$a^\mu \xrightarrow{P} -a_\mu,$$
$$s \xrightarrow{P} s,$$
$$p \xrightarrow{P} -p,$$

Constraining external fields

We demand of $\mathcal{L}$ that it is:

- Hermitian Lorentz scalar
- Even under P and C
- Invariant under local chiral transformations

Charge conjugation:

$$
\begin{aligned}
v_\mu &\xrightarrow{C} -v_\mu^T, \\
v_\mu^{(s)} &\xrightarrow{C} -v_\mu^{(s)\,T}, \\
a_\mu &\xrightarrow{C} a_\mu^T, \\
s &\xrightarrow{C} s^T, \\
p &\xrightarrow{C} p^T,
\end{aligned}
$$

Constraining external fields

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Local chiral transformation:

$$q_R \to \exp\left\{-i\frac{\Theta(x)}{3}\right\} V_R(x) q_R, \quad q_L \to \exp\left\{-i\frac{\Theta(x)}{3}\right\} V_L(x) q_L,$$

After splitting our external fields into $r_\mu = v_\mu + a_\mu$ and $l_\mu = v_\mu - a_\mu$:

$$\begin{aligned}
r_\mu &\to V_R r_\mu V_R^\dagger + i V_R(\partial_\mu V_R^\dagger), \\
l_\mu &\to V_L l_\mu V_L^\dagger + i V_L(\partial_\mu V_L^\dagger), \\
v_\mu^{(s)} &\to v_\mu^{(s)} - (\partial_\mu \Theta), \\
s + ip &\to V_R(s + ip) V_L^\dagger, \\
s - ip &\to V_L(s - ip) V_R^\dagger,
\end{aligned}$$

Key takeaways

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$\mathrm{U}(1)$ example:

- Local invariance of generating functional contains all Ward identities of theory

Chiral Ward identities from algebra of currents:

- Using the algebra of currents, one must tread with caution (Schwinger and seagull terms)

Generating functional for chiral Green's functions:

- Allows one to compute Green's functions for chiral limit and “real" world
- Can constrain transformation behavior of external fields by invariance of generating functional under local transformations

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