HYPERBOLIC BAND THEORY THROUGH HIGGS BUNDLES

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Abstract. Hyperbolic lattices underlie a relatively new form of quantum matter with potential applications to quantum computing and simulation and which, to date, have been engineered artificially. A corresponding hyperbolic band theory has emerged, on the one hand as a response to this development and on the other hand as a natural extension of 2-dimensional Euclidean band theory to higher-genus configuration spaces. Attempts to develop the hyperbolic analogue of Bloch's theorem have revealed the role of complex algebro-geometric objects — notably, moduli spaces of stable bundles on a curve — when periodic condensed matter systems are considered on hyperbolic geometries. In this article, we generalize this portion of the theory by considering Higgs bundles instead of vector bundles. In the process, we discover that Higgs bundles and their moduli enjoy more than one interpretation in the context of band theory.

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1. Introduction

There are a number of established and emerging connections between high energy and condensed matter physics, some of which are mediated by physical and mathematical dualities such as AdS/CFT. This article aims to describe one such emerging connection, specifically between the moduli space of Higgs bundles, a feature of Yang-Mills gauge theory in 2 dimensions, and the band theory of physical systems with periodic potentials. The former originates in [22], where the self-dual Yang-Mills equations are dimensionally reduced so that they are written on a smooth complex bundle on a compact Riemann surface. For the latter, we take the point of view not only of the familiar 2-dimensional Euclidean band theory given by the symmetry of a lattice tiling the plane, but also expand our attention to include its hyperbolic generalization that has recently appeared in [34, 33]. The hyperbolic version of the theory considers the Schrödinger equation with a Hamiltonian that is invariant under the noncommutative translations of a tessellation of the hyperbolic plane $\mathbb{H}$. One of the great successes of conventional band theory is the explanation of electronic and optical properties of solids, which allows for materials to be realized with desired semiconducting or insulating properties through band-gap engineering and which provides the theoretical foundation for solid-state devices such as transistors. The hyperbolic realization of band-gap phenomena anticipates new forms of quantum matter and topoelectric circuits with applications to quantum computing and simulation. To date, such circuits, albeit photonic rather than electronic in nature, have been artificially engineered [32].

One of the reasons why the hyperbolic generalization of band theory is attractive is because it continues to admit Bloch wave solutions, as established in [34, 33]. The Bloch decomposition here is automorphic in nature, with phase factors given by unitary representations (in all ranks) of the Fuchsian group Γ of the tessellation. The appearance of representations of surface groups of higher-genus curves $\mathbb{H}/\Gamma$ is a natural entry point for Higgs bundles into the band theory.

After reviewing the hyperbolic band theory and the definition and basic features of Higgs bundles and their moduli, we connect the two subjects in two ways. First, we package the crystal lattice and abelian crystal momenta of the band theory into the spectral data of Higgs bundles, which thereby injects the base of the Hitchin map on the moduli space of Higgs bundles into the space of available crystals. Alternatively, we interpret the Higgs field of a Higgs bundle as an imaginary crystal momentum state, which leverages the similarity between spectral curves and band structures within a tight-binding model. Even in the Euclidean case, these interpretations elicit new aspects of the band theory, which we will illustrate with some suggestive calculations for parabolic Higgs bundles on the projective line with elliptic spectral curves.

In the final section of the paper, we speculate on connections to topological materials (which are in some sense the major impetus for the recent injection of algebraic techniques into condensed matter theory, to fractional quantum Hall states, to supersymmetric field theories, to the geometric Langlands correspondence, and more. We end the article with a rather provocative picture that aims to put many ideas in high-energy physics, algebraic geometry, and condensed matter into the same orbit.

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2. Hyperbolic band theory

We begin by building the framework of hyperbolic band theory. The foundation is a hyperbolic version of the classical Bloch theorem, which constructs the eigenstates of the crystal Hamiltonian from crystal momenta. Hyperbolic Bloch states were introduced in [34], and a corresponding version of Bloch's theorem was formulated in [33].

2.1. **Euclidean crystallography.** Hyperbolic crystallography is modeled on Euclidean crystallography, so we review the usual setup for inspiration. This material is standard (see for instance [2]), but presented here with the hyperbolic generalization in mind. A crystal is defined by a periodic potential on $\mathbb{R}^n$. The crystal structure describes the periodicity data, often represented with a lattice Γ of points on $\mathbb{R}^n$. The potential has crystal structure Γ if $V(x) = V(x+\gamma)$ for all γ in Γ. Alternatively, translation by Γ defines a subgroup of the isometry group of $\mathbb{R}^n$. Alternatively, the lattice is described by its unit cell, a polyhedral set whose translates under Γ tile all of $\mathbb{R}^n$. The shape of the unit cell uniquely determines the lattice. Moreover, periodicity identifies opposite faces of the cell, which glue together to form the torus $\mathbb{R}^n/\Gamma$. Periodic functions on $\mathbb{R}^n$ are uniquely described by ordinary functions on $\mathbb{R}^n/\Gamma$. We are interested in hyperbolic crystals in 2 dimensions, which are analogous to Euclidean crystals in $\mathbb{R}^2 \cong \mathbb{C}$.

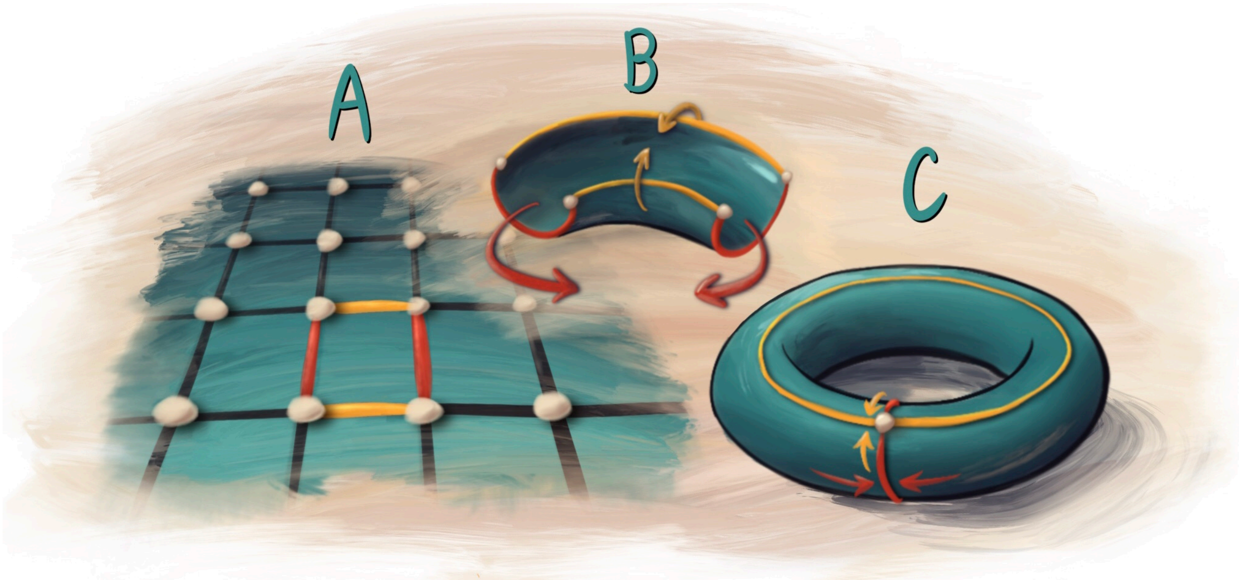


FIGURE 1. A) A Euclidean crystal, given by a lattice Γ in $\mathbb{C}$, with a unit cell highlighted. B) to construct $\mathbb{C}/\Gamma$, the edges of the unit cell of like color are identified. C) The resulting surface is a torus.

2.1.1. *Bloch theorem.* Consider a quantum particle on this periodic potential. Its wavefunction is a complex valued function $\psi : \mathbb{C} \to \mathbb{C}$, whose evolution is governed by the Hamiltonian $H = \Delta + V$. Here Δ is the Laplacian and $V : \mathbb{C} \to \mathbb{R}$ is the potential energy. Band theory is founded on *Bloch's theorem*, which says the Hamiltonian has an eigenbasis ψ_k satisfying $\psi_k(z+\gamma) = e^{2\pi i(k_x\gamma_x + k_y\gamma_y)}\psi_k(z)$, which are called Bloch waves with crystal momentum $k = k_x + ik_y \in \mathbb{C}$.

There are many derivations of Bloch's theorem, but the one lending itself to fruitful generalization comes from group theory. To summarize, the Hilbert space $\mathcal{H} = L^2(\mathbb{C})$ splits into irreducible representations of Γ parameterized by k, each representation containing Bloch waves with crystal momentum k. The Hamiltonian is symmetric under lattice translations so preserves these subspaces. Each has their own energy eigenbasis, which together form a complete eigenbasis for H. More carefully, we have an action of Γ on the Hilbert space $\mathcal{H} = L^2(\mathbb{C})$ via translation operators $T_\gamma\psi(z) = \psi(z+\gamma)$. These form a representation of Γ because $T_{\gamma_1+\gamma_2} = T_{\gamma_1} + T_{\gamma_2}$. The Hilbert space $\mathcal{H}$ splits into a direct product of irreducible representations of Γ, all of which are one dimensional since Γ is abelian. Denoting the space of these representations by $\Gamma^\vee$,

$$\mathcal{H} = \prod_{\chi_k \in \Gamma^\vee} \mathcal{H}_k, \qquad \mathcal{H}_k = \{\psi \in \mathcal{H} | T_\gamma\psi = \chi_k(\gamma)\psi\} \tag{1}$$

The representations are defined by $\chi_k(\gamma) = e^{2\pi i k^*(\gamma)}$, parameterized by a linear function $k^* : \mathbb{C} \to \mathbb{C}$, the space of which is isomorphic to $\mathbb{C}$. This is trivial whenever $k^*(\gamma)$ is an integer for each $\gamma \in \Gamma$, which by definition means k^* lies in the dual lattice $\Gamma^* \subset \mathbb{C}$. We are left with the well-known isomorphism $\Gamma^\vee \cong \mathbb{C}/\Gamma^*$, showing the space of irreducible representations is a 1-complex dimensional torus. We can expand out the action of k^* via $k^*(\gamma) = k \cdot \gamma = k_x\gamma_x + k_y\gamma_y$. So, $\mathcal{H}_k$ exactly consists of Bloch waves with crystal momentum k. The identification $\Gamma^\vee \cong \mathbb{C}^\times/\Gamma^*$ reflects the periodicity of crystal momentum space.

To connect this splitting with the Hamiltonian, note that translation by γ is an isometry, so T_γ is unitary and commutes with the Laplacian. Since the potential is periodic, it also commutes with lattice translations, meaning the whole Hamiltonian satisfies $T_\gamma H = HT_\gamma$. Hence, H preserves every subspace $\mathcal{H}_k$. Denoting the restriction of H to $\mathcal{H}_k$ by H_k, we have:

Theorem 1 (Bloch's theorem)**.** *A periodic Hamiltonian H with lattice $\Gamma \subset \mathbb{C}$ has an eigenbasis of Bloch waves. This basis consists of eigenfunctions of H_k for k varying in crystal momentum space $\mathbb{C}^\times/\Gamma^*$.*

The central object of Band theory is the spectrum of H_k as k varies across the crystal momentum space.

Remark 1*.* We should heed that the given derivation is not rigorous. The Hilbert space is not really $L^2(\mathbb{C})$, as any nonzero quasiperiodic function is not square integrable. Instead,this is a motivating physical argument.

2.2. **Hyperbolic crystallography.** In this paper we study hyperbolic crystals, which are periodic structures on the hyperbolic plane $\mathbb{H}$. This is a two-dimensional open disk endowed with a metric of constant negative curvature. The symmetries of $\mathbb{H}$ are isometries, forming the group $PSL(2,\mathbb{R})$. A crystal lattice is described by a discrete subgroup of isometries $\Gamma \subset PSL(2,\mathbb{R})$ (A Fushcian group), and each element $\gamma \in \Gamma$ acts on $\mathbb{H}$. A potential $V : \mathbb{H} \to \mathbb{R}$ is invariant under Γ, meaning $V(x) = V(\gamma(x))$.

(**NOTE**: Explain hyperbolic crystals, the analoug of lattices and unit cells, their associated riemann surfaces)

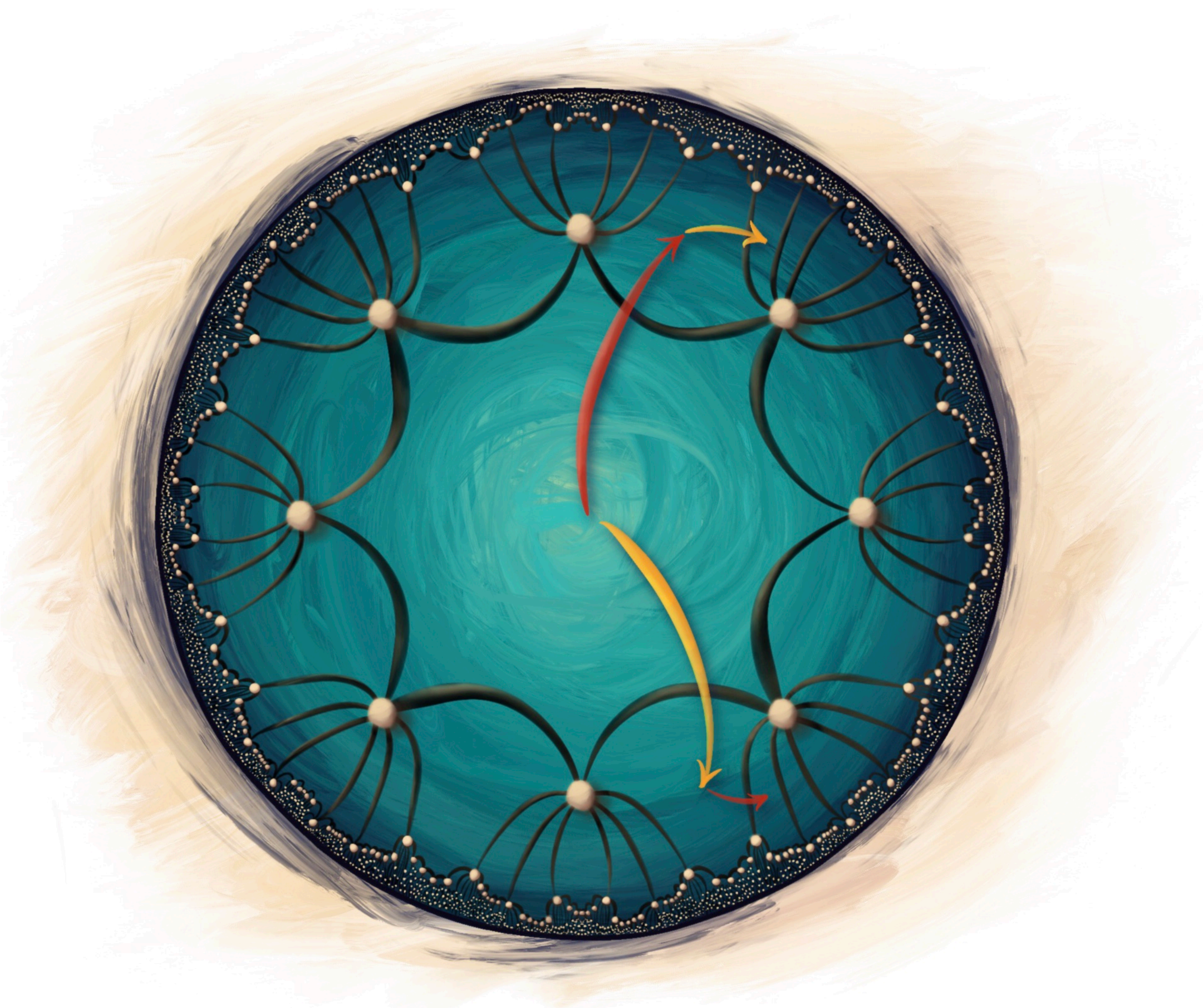


FIGURE 2. The translations of a hyperbolic lattice form a Nonabelian group. The red and yellow arrows indicate translations for a basis of cycles in Σ

2.2.1. *Hyperbolic Bloch theorem.* Just as conventional band theory rests upon Bloch's theorem, Hyperbolic band theory must be built on a hyperbolic analog of Bloch's theorem. The difficulty in extending the technique of section 2.1.1 comes from the noncommutativity lattice, meaning not every irreducible representation is one dimensional.

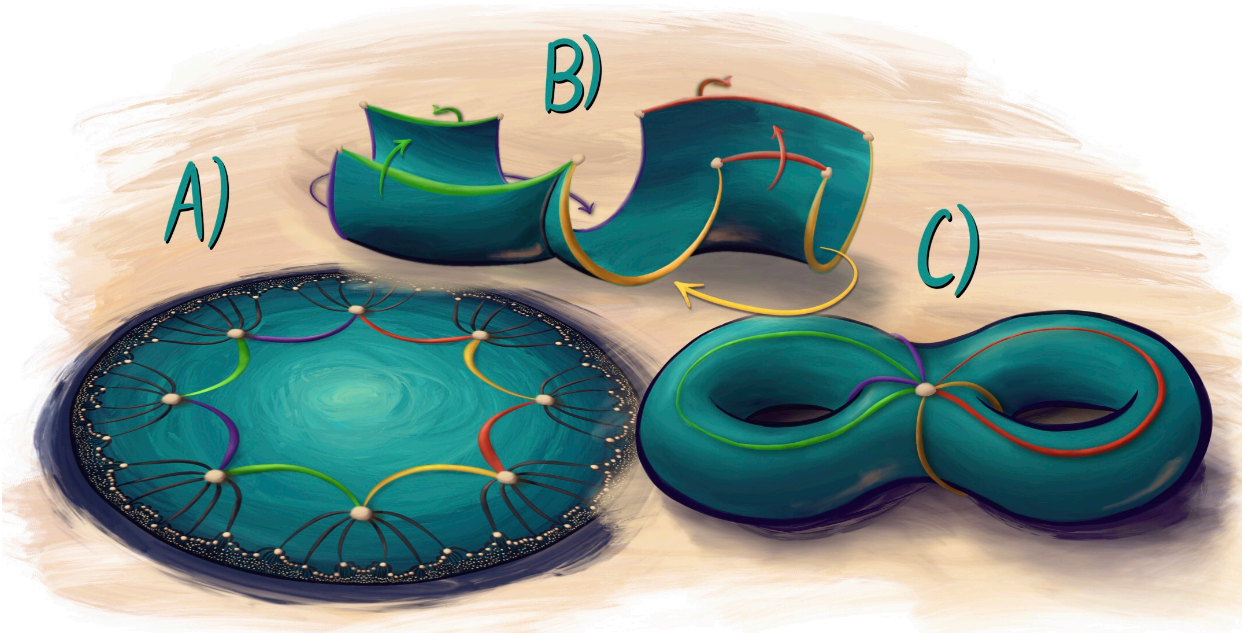


FIGURE 3. A) A hyperbolic crystal, given by the a hyperbolic lattice Γ in $\mathbb{H}$ with unit cell highlighted. This is a polygon with $4g$ sides B) To construct $\mathbb{H}/\Gamma$, the edges of the unit cell of like color are identified. C) The resulting surface is a genus g surface (**NOTE**: I messed up, and identified the edges according to the usual parameterization of the fundamental polygon, instead of by lattice translations. This is not something I can fix easily. Not sure what to do about this.)

We denote the symmetry group of a hyperbolic crystal by $\Gamma \subset PSL(2,\mathbb{R})$, meaning the potential is invariant under the action of Γ. Just like before, this acts on Hilbert space $\mathcal{H} = L^2(\mathbb{H})$ by precomposition $T_\gamma\psi(z) = \psi(\gamma(z))$, furnishing a representation of Γ. Since this once again acts by isometries of $\mathbb{H}$, T_γ is unitary and commutes with both the potential and the Laplacian. We again split $\mathcal{H}$ into irreducible representations of Γ, but since Γ is noncommutative we must now brave the troubles of higher dimensional irreducible representations.

Consider a unitary representation $\rho : \Gamma \to U(n)$. We say a vector in $\mathcal{H}$ transforms by ρ if it belongs to an n-dimensional subspace $\langle \psi_1, \ldots, \psi_n \rangle$ where each vector $v_1\psi_1 + \cdots + v_n\psi_n$ satisfies

$$T_\gamma \begin{pmatrix} \psi_1 & \ldots & \psi_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \psi_1 & \ldots & \psi_n \end{pmatrix} \rho(\gamma) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \tag{2}$$

In physics language, we say the vector belongs to a multiplet mixed by translations. Collect all the vectors transforming by ρ into a subspace $\mathcal{H}_\rho \subset \mathcal{H}$. Since ρ is irreducible, each vector belongs to at most one subspace $\mathcal{H}_\rho$. Since H commutes with each T_γ, it can preserve the subspaces $\mathcal{H}_\rho$. We check

$$T_\gamma \begin{pmatrix} H\psi_1 & \ldots & H\psi_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = HT_\gamma \begin{pmatrix} \psi_1 & \ldots & \psi_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} H\psi_1 & \ldots & H\psi_n \end{pmatrix} \rho(\gamma) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \tag{3}$$

When the vectors $\{H\psi_i\}$ are a linearly independent, they form a representation of ρ, so each vector $H\psi_i$ lives in $\mathcal{H}_\rho$. We are not so lucky when $\{H\psi_i\}$ is linearly dependent, as we instead get a lower dimensional representation, meaning $H\psi_i$ does not live in $\mathcal{H}_\rho$. This can only happen when H has nontrivial kernel. When the kernel is trivial, H preserves $\mathcal{H}_\rho$ for all ρ, so it restricts to a self adjoint operator H_ρ with its own spectrum, giving us:

Theorem 2 (Hyperbolic Bloch theorem)**.** *When the Hamiltonian H has trivial kernel, eigenfunctions of H are all eigenfunctions of H_ρ for some irreducible representation $\rho : \Gamma \to U(n)$. These eigenfunctions belong to a n-dimensional subspace of $\mathcal{H}$, which transforms under T_γ by $\rho(\gamma)$.*

This argument is essentially given in [33], though framed slightly differently. That paper dealt with a finite subset of the full hyperbolic lattice in the tight binding model, and consequently a finite dimensional Hilbert space. This avoids a number of subtle issues with the above argument, like the lack of a proper Hilbert space and the assertion that all representations are finite dimensional. Once again, we forego a rigorous treatment of the preceding argument and take it as a physical heuristic describing what hyperbolic crystal momentum space ought to be.

This momentum space is the moduli space of irreducible unitary representations of Γ, $\mathrm{Hom}_{\mathrm{irr}}(\Gamma \to U(n))/U(n)$. We mod out by the conjugation action of $U(n)$, because two conjugate crystal momenta are equivalent. The most familiar components is the rank one part, described by $\Gamma^\vee \equiv \mathrm{Hom}(\Gamma, U(1))$. Since $U(1)$ is abelian, any homomorphism $\Gamma \to U(1)$ factors through the abelianization of Γ, which is isomorphic to $\mathbb{Z}^{2g}$ for an integer $g \geq 0$. Therefore, we have

$$\Gamma^\vee \cong \mathrm{Hom}(\mathbb{Z}^{2g}, U(1)) \cong U(1)^{2g}$$

This component is a g complex dimensional complex torus. (c.f the Euclidean case, which has $g = 1$ and yields a complex dimension 1 torus as its momentum space.) We call this the *Abelian Brillouin zone*, which was first introduced in [**maciejko_hyperbolic_2020**]. The other connected components of momentum space are the character varieties $\mathrm{Hom}_{\mathrm{irr}}(\Gamma, U(n))/U(n)$, which we call the *Non-abelian Brillouin zones*. We can express these geometrically using...

2.3. **Vector bundles on Riemann surfaces.** The algebraic description of crystal momentum space arose by thinking of a crystal through its symmetry group. Geometry comes when we switch to the unit cell, namely the Riemann surface $\Sigma = \mathbb{H}/\Gamma$. The fundamental group $\pi_1(\Sigma)$ must equal Γ, because $\mathbb{H}$ is simply connected and Γ acts freely. So, the crystal momenta are representations of the fundamental group. These are geometrically realized by a flat connection with monodromy reproducing the representation. These perspectives are equivalent by the Riemann-Hilbert correspondence.

We can put this in action for abelian Bloch states, parameterized by $U(1)$ representations of the fundamental group [34]. We start with a trivial complex line bundle over $\mathbb{H}$. A Bloch state ψ a is a section of this bundle with factor of automorphy χ. We wish to represent ψ as a section of a line bundle L over the unit cell Σ. We form a topologically trivial line bundle L from $\mathbb{H} \times_\chi \mathbb{C} \to \Sigma$ by quotienting by Γ, acting by $\gamma : (x, v) \to (\gamma(x), \chi(\gamma)(v)$ (**NOTE**: add picture). In effect, the fibers rotate by $\chi(\gamma)$ over any loop γ, which we encode as the parallel transport of a flat $U(1)$ connection D. Up to gauge transforms, the monodromy $\pi_1(\Sigma) \to U(1)$ uniquely determines D. Note that L has a Hermitian metric, pulled back from the standard Hermitian metric on the trivial bundle $\mathbb{H} \times \mathbb{C}$, $h(x, y) = x\bar{y}$. We can regard D as a $\mathbb{C}^*$ connection on the line bundle, which is Hermitian with respect to this h, reducing the structure group from $\mathbb{C}^*$ to $U(1)$.

Moving to the realm of algebraic geometry, we can interpret the $(0, 1)$ part of the D as a Dolbeault operator $\bar{\partial}_L$ on L, endowing L with a holomorphic structure. Through this map, the abelian Brillouin zone $\mathrm{Hom}(\Gamma, U(1))/U(1)$ is equivalent to the moduli of degree 0 [1] holomorphic line bundles on Σ, better known as the Jacobian variety $\mathrm{Jac}(\Sigma)$. This is a g dimensional complex torus with g the genus of Σ, in accordance with last section. For $g = 1$, the Jacobian is the dual abelian variety to the curve, reproducing the duality between momentum and position space for Euclidean crystals.

(**NOTE**: $E = \Sigma \times_\rho \mathbb{C}^n$)

A similar scene plays out for higher rank representations, but the details are more involved [33]. Starting from a trivial vector bundle $\mathbb{H} \times_\rho \mathbb{C}^n \to \mathbb{H}$, we obtain a topologically trivial [2] vector bundle $E \to \Sigma$ by quotienting out Γ which acts by $\gamma : (x, v) \to (\gamma(x), \rho(\gamma)(v)$. Pulling back the standard Hermitian structure on $\mathbb{C}^n$ gives a Hermitian vector bundle (E, h) over Σ. We can associate the crystal momentum representation

[1] the degree is zero because the bundle is topologically trivial

[2] The topological type of a vector bundle is determined by its Chern classes, and on a Riemann surface the only nonzero Chern class is c_1. So, a vector bundle is topologically trivial if and only if it has degree zero.

$\pi_1(\Sigma) \to U(n)$ to a flat $U(n)$ connection D. In Algebraic geometry realm, this gives a holomorphic structure $\bar{\partial}_E = D^{0,1}$ on E. The Narasimhan–Seshadri theorem says that a holomorphic bundles comes from an irreducible representations are if and only if it is stable. Stability is a constraint on the degrees of holomorphic subbundles. Specifically, we require all subbundles $V \subset E$ to satisfy

$$\frac{\deg(V)}{\mathrm{rank}(V)} < \frac{\deg(E)}{\mathrm{rank}(E)};$$

that is, E has the maximal normalized degree of all its subbundles. Stability ensures that the automorphism group of a vector bundle is as small as possible, which is necessary for a well behaved (i.e. Hausdorff) moduli space. The nonabelian Brillouin zone $\mathrm{Hom}_{\mathrm{irr}}(\Gamma, U(n))$ is diffeomorphic to the moduli space of stable vector bundles $\mathcal{N}(\Sigma_g, U(n))$. This is a smooth manifold with real dimension $2n^2(g-1)+2$. Notably, it is no longer a torus when $n > 1$, and has a complicated but well studied topology. It is noncompact, but at the expense of adding singular points we can compactify by including semistable bundles. These are bundles where $<$ in the definition of stability is replaced by $\leq$. The moduli space of semistable bundles $\mathcal{N}^{ss}(\Sigma_g, U(n))$ is diffeomorphic to that of *reducible* representations of $U(n)$. The application of these moduli spaces to hyperbolic band theory is discussed in [33]

2.4. **Hamiltonian for abelian Bloch states.** With Bloch's theorem in hand, we can proceed to band theory. We begin with abelian Bloch states, identified with a representation $\chi : \Gamma \to U(1)$. We wish to know how the spectrum $H_\chi = H|_{\mathcal{H}_\chi}$ varies with the representation. It amounts to solving an eigenvalue problem for H on the unit cell with twisted periodic boundary conditions, $\psi(\gamma(x)) = \chi(\gamma)\psi(x)$ on the boundary of the cell. This was studied numerically using the finite element method in reference [34]. However, these periodic boundary conditions are unwieldy for analytic manipulation, so we will use a closed form of H_k.

Theorem 3. *Let A be a closed one form satisfying $\exp(\int_\gamma A) = \chi(\gamma)$ for every loop γ. Then, every eigenfunction of H_χ corresponds uniquely to an eigenfunciton of*

$$H_A = (d + A)^*(d + A) + V \tag{4}$$

acting on $L^2(\Sigma)$, with the same eigenvalue. In particular, these operators have the same spectrum.

When working with this, we will pass between this geometric form and its expression in local coordinates. Choose a conformal coordinate z on a patch of Σ, in which the metric has the form $g(z, \bar{z})\mathrm{d}z^2$ and $A = A'(z)\mathrm{d}z + A''(z)\mathrm{d}\bar{z}$. The Hamiltonian is then

$$H_A = \frac{1}{g}(\partial_z + A')(\partial_{\bar{z}} + A'') + V \tag{5}$$

Proof: Theorem 3. Once again, we follow the Euclidean derivation for guidance. In this case, we can express any Bloch wave as $\psi_k(z) = e^{ik\cdot z}u(z)$, where k, z are points in $\mathbb{C}$ thought of as vectors in $\mathbb{R}^2$, and $u(z)$ is periodic with the lattice Γ. In effect, we split into a predetermined phase factor $e^{ik\cdot z}$ which satisfies the proper boundary conditions, and a periodic remaining part. Denoting $k_z = k_x + ik_y, k_{\bar{z}} = k_x - ik_y$, and noting that $g = 1$ on the torus, we have

$$H\psi_k = (\partial_z\partial_{\bar{z}} + V)e^{ik\cdot z}u = e^{ik\cdot z}((\partial_z + ik_z)(\partial_{\bar{z}} + ik_{\bar{z}}) + V)u \tag{6}$$

So, if ψ_k is an eigenfunction of H with eigenvalue λ, then u is an eigenfunction of

$$(\partial_z + ik_z)(\partial_{\bar{z}} + ik_{\bar{z}}) + V$$

with the same eigenvalue. Or, twisted eigenfunctions of the untwisted Hamiltonian equal periodic eigenfunctions of the 'twisted' Hamiltonian. In coordinate invariant language, this 'twisted' Hamiltonian is

$$H_A = (\mathrm{d} + k_z\mathrm{d}z + k_{\bar{z}}\mathrm{d}\bar{z})^*(\mathrm{d} + k_z\mathrm{d}z + k_{\bar{z}}\mathrm{d}\bar{z}) + V \tag{7}$$

Carrying this to the hyperbolic case, we can once again decompose any Bloch wave $\psi_k(z) = s(z)u(z)$ where $s(z)$ is a nowhere vanishing 'phase factor' and $u(z)$ is periodic. For fixed χ, pick such an s. **TODO**: show this always exist . Treating it as a "multiplication by s" operator, Leibniz's rule for a derivation δ manifests as

$$\delta s = s(\delta + \frac{\delta s}{s}) \tag{8}$$

If ψ_k is an eigenfunction of the Hamiltonian $\mathrm{d}^*\mathrm{d} + V = \frac{1}{g}\partial_z\partial_{\bar{z}} + V$ with eigenvalue λ, then this rule says

$$\lambda su = (\frac{1}{g}\partial_z\partial_{\bar{z}} + V)su = s(\frac{1}{g}(\partial_z + \frac{\partial_z s}{s})(\partial_{\bar{z}} + \frac{\partial_{\bar{z}} s}{s}) + V)u = sH_A u \tag{9}$$

The one form A is

$$A = \frac{\partial_z s}{s}\mathrm{d}z + \frac{\partial_{\bar{z}} s}{s}\mathrm{d}\bar{z} = \frac{\mathrm{d}s}{s} = \mathrm{d}\log s$$

We must next check that $\mathrm{d}s/s$ obeys all the properties listed in the theorem. First, both $\mathrm{d}s$ and s are factors of automorphy for the representation χ_k, so their ratio is a well defined one form on Σ. It is also closed. Its monodromy is best studied on $\mathbb{H}$, where for a loop γ we get

$$\int_\gamma \mathrm{d}\log(s) = \log(s) + 2\pi i\, n|_p^{\gamma(p)} \tag{10}$$

$$= \log\left(\frac{s(\gamma(p))}{s(p)}\right) + 2\pi i\, n \tag{11}$$

$$= \log\left(\chi(\gamma)\right) + 2\pi i\, n \tag{12}$$

for an ambiguous factor $n \in \mathbb{Z}$. The exponential takes care of the ambiguity, and we're left with the desired $\chi(\gamma)$.

Finally, we need to show that every one form A satisfying the given conditions is equal to $\mathrm{d}s'/s'$ for some nowhere vanishing factor of automorphy s'. The form $A - \mathrm{d}s/s$ is closed, and has its integral around any loop is $2\pi n$ for some integer n. Therefore, it equals $\mathrm{d}\log p(z)$ for some periodic non-vanishing function p **TODO**: justify . Rearranging, we get

$$A = \mathrm{d}\log(s) + \mathrm{d}\log(p) = \mathrm{d}\log(sp) \tag{13}$$

so, sp is the desired factor of automorphy. □

Any physical Hamiltonian must be self-adjoint, which only holds for imaginary A. Indeed, the Hermitian conjugate of H_A is:

$$H_A^\dagger = \left(\frac{1}{g}(\partial_z + A')(\partial_{\bar{z}} + A'')\right)^\dagger + V^\dagger \tag{14}$$

$$= \frac{1}{g}(\partial_{\bar{z}}^\dagger + A''^\dagger)(\partial_z^\dagger + A'^\dagger) + V \tag{15}$$

$$= \frac{1}{g}(-\partial_z + \bar{A}'')(-\partial_{\bar{z}} + \bar{A}') + V \tag{16}$$

Where we used $\partial_z^\dagger = \bar{\partial}_z^* = -\partial_{\bar{z}}$ and likewise for $\partial_{\bar{z}}$. Comparing to equation 5, this equals H_A when $\bar{A}'' = -A'$. Without coordinates, this says $\bar{A} = -A$ This also arises if we only allow phase factors $s(z)$ with magnitude 1 everywhere. If we constrain $s\bar{s} = 1$ and take derivatives, we see

$$A' = \frac{\partial_z s}{s} = -\frac{\partial_z \bar{s}}{\bar{s}} \qquad A'' = \frac{\partial_{\bar{z}} s}{s} = -\frac{\partial_{\bar{z}} \bar{s}}{\bar{s}}$$

We again find $-\bar{A}' = A''$.

2.4.1. *Cohomological interpretation.* This proof suggests that $A = \mathrm{d}s/s$ should only be defined up to a choice of s. So, the space of A is the space of closed one forms, modulo exact ones and those with trivial monodromy. In terms of cohomology, A lives in $H^1(\Sigma, \mathbb{C})/H_1(\Sigma, \mathbb{Z})^*$. Restricting to self-adjoint Hamiltonians means A truly lives in $H^1(\Sigma, \mathbb{C})/H_1(\Sigma, \mathbb{Z})^*$ Poincare duality gives the natural identification $H_1(\Sigma, \mathbb{Z})^* \cong H^1(\Sigma, \mathbb{Z})$. The space $H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$ carries a natural complex structure which is isogenic to $H^{0,1}(\Sigma, \mathbb{C})/H^1(\Sigma, \mathbb{Z})$, the Jacobian. (**CITE** T)hat is, the proper space parameterizing possible A is exactly the crystal momentum space!

2.4.2. *Physical interpretation.* We could have guessed the form of equation (4) from physics. A nonzero crystal momentum means the particle picks up a phase when traveling along a cycle in Σ, which is like a geometric phase (or Aharanov-Bohm phase). In the familiar Aharanov-Bohm effect, A charged particle moving in a vector potential picks up the exponential of the monodromy of the vector potential when traveling in a loop.Vector potentials enter the Hamiltonian through minimal coupling, replacing the derivative d with $d + A$. This suggests exactly the Hamiltonian from theorem (3). In our case, the magnetic field $B = \mathrm{d}A$ on the surface is zero, meaning A must be closed. Physically, this corresponds to magnetic fluxes through just the holes and not the surface of Σ [[Picture]]. The flux quantity determines the monodromy and thus the crystal momentum.

But which vector potential should we use? The Hamiltonian should be invariant under gauge transforms, but the only gauge invariant quantities in electromagnetism are the magnetic field and the monodromy. Both of these are fixed, so any and all closed one-forms with the prescribed monodromy should be equivalent. This agrees with the analysis in last subsection. In fact, the gauge freedom is exactly the freedom in choice of phase factor s (with norm 1 everywhere).

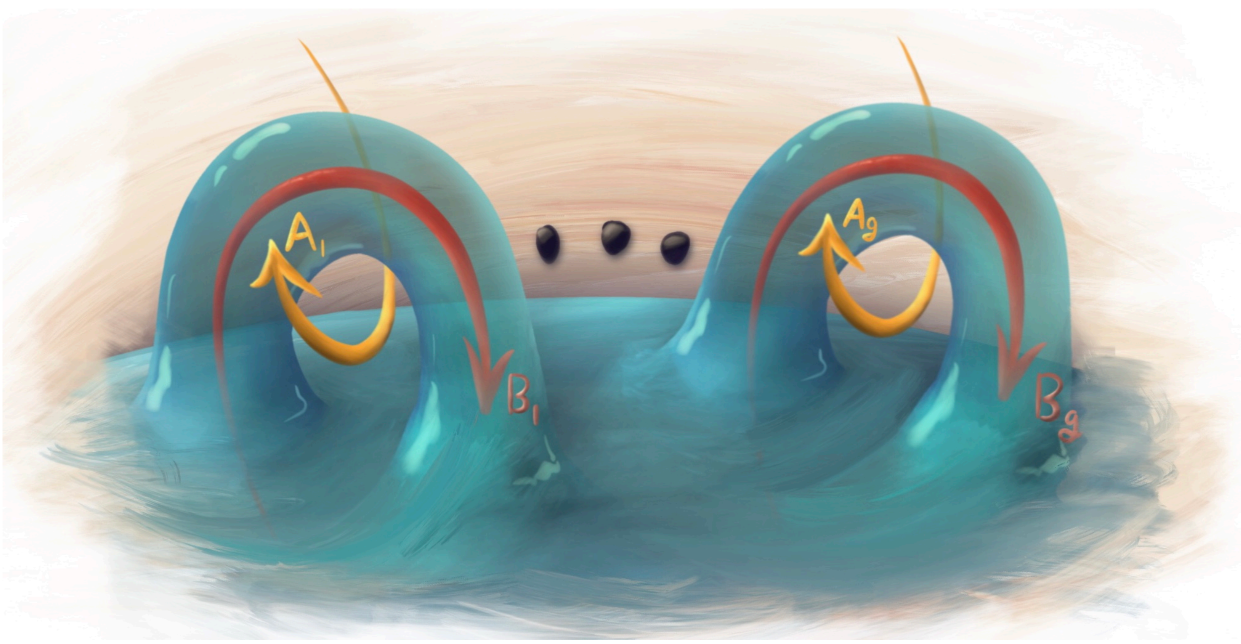


FIGURE 4. Magnetic fluxes through the handles of a Riemann surface cause nontrivial vector potentials. These arrows give a basis of harmonic one-forms.

2.4.3. *Geometric interpretation.* The connection with electromagnetism suggests a gauge theoretic approach, taking us back to the realm of geometry. Electromagnetism is a $U(1)$ gauge field, and the vector potential is a $U(1)$ principle connection whose curvature is the magnetic field. In this situation, the magnetic field is zero so the connection is flat. This is exactly the setup from section 2.3, meaning we can interpret χ as the monodromy of the flat $U(1)$ structure on a line bundle L. This gives the Hamiltonian a nice geometric interpretation: The differential operator $\mathrm{d} + A = \nabla_L$ is the flat connection, so the Hamiltonian is

$$H_L = \nabla_L^* \nabla_L + V \tag{17}$$

Here the adjoint is with respect to the standard Hermitian metric pulled back from $\mathbb{H} \times \mathbb{C}$. The Hamiltonian acts on sections of L, so its Hilbert space is $L^2(L)$. The eigenfunctions in $\mathcal{H}_\chi$ are the eigensections ψ_L of H_L. Trivializing the line bundle gives an isomorphism $L^2(L) \cong L^2(\Sigma)$, and this transforms between the Hamiltonians of equation 17 and 4.

2.5. **Hamiltonian for nonabelian Bloch states.** We can find similar expressions for the Hamiltonian in the nonabelian case. Consider the representation $\rho : \pi_1(\Sigma) \to U(n)$, associated to the vector bundle $E \to \Sigma$. The Hamiltonian H_ρ is H restricted to the subspace of nonabelian Bloch states which transform under ρ

Theorem 4. *Let A be an* $\mathrm{End}(E)$ *valued one form with monodromy $\rho(\gamma)$ about every loop γ. Then, every eigenfunction of H_ρ uniquely corresponds to an eigensection of $H_E = (d+A)^*(d+A) + V \cdot Id$ acting on sections of E.*

Proof. We follow the proof of theorem 3, and start by separating out a phase factor and a periodic part. We accomplish this by picking a frame. Let $\tilde{s}_1, \ldots, \tilde{s}_n$ be a unitary frame with respect to the Hermitian metric on E, which transform according to ρ. Such a frame always exists, because E is topologically trivial $(* \quad *)$. Every section ψ of E can be written as

$$\psi = s_1(z)u_1(z) + \cdots + s_n(z)u_n(z)$$

for $u_i(z)$ are functions on Σ. In this frame, ψ is represented by a vector valued function $\vec{u}$. Now we act on ψ by the Hamiltonian, by acting over $\mathbb{H}$ then pushing down to a section of E:

$$\vec{\psi} = \begin{pmatrix} \vec{s}_1 & \ldots & \vec{s}_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = S\vec{u} \tag{18}$$

$$\mathrm{d}\vec{\psi} = \mathrm{d}S\,\vec{u} + S\mathrm{d}\vec{u} = S(\mathrm{d} + S^{-1}\mathrm{d}S)\vec{u} \tag{19}$$

$$(\mathrm{d}^*\mathrm{d} + V)\vec{\psi} = S\left((\mathrm{d} + S^{-1}\mathrm{d}S)^*(\mathrm{d} + S^{-1}\mathrm{d}S) + V\right)\vec{u} \tag{20}$$

In the last line, the adjoint is with respect to the standard Hermitian inner product on $\mathbb{C}^n \times \mathbb{H}$. Note that $\vec{\psi}$ is an eigenfunction of H_ρ exactly when $\vec{u}$ is an eigenfunction of $H_{S^{-1}\mathrm{d}S}$. Note that $S(\gamma(z)) = \rho_k S(z)$, meaning $S^{-1}(\gamma(z))\mathrm{d}S(\gamma(z)) = S^{-1}(z)\mathrm{d}S(z)$, so $S^{-1}\mathrm{d}S$ descends to an $\mathrm{End}(E)$ valued one form on Σ.

TODO: Show that $S^{-1}\mathrm{d}S$ satisfies the monodromy property

Finally, we must contend with gauge transformations, which are realized by choosing a different frame. We call the change of frame matrix D, which is a section of $\mathrm{End}E$. Outside of overall conjugation, this gauge transform replaces d with $\mathrm{d} + D^{-1}\mathrm{d}D$. However, because D is a true section, the monodromy of $D^{-1}\mathrm{d}D$ must be trivial. Since the only gauge invariant quantity is monodromy, every A with given monodromy is representable as $S^{-1}\mathrm{d}S + D^{-1}\mathrm{d}D$ for some endomorphism D. ◻

Remark 2. For abelian crystal momenta, this Hamiltonian also arose from minimal coupling with a $U(1)$ gauge field, see section 2.4.2. In Higher rank, it should couple to a $U(n)$ gauge theory. The flat connections are solutions to the Yang-Mills equations on Σ, which we can describe with $U(n)$ fluxes threading the holes of Σ like in Figure 4. In greater generality, the Kobayashi–Hitchin correspondence says every stable bundle admits a unique Hermitian Yang-Mills connection.

2.5.1. *Geometric interpretation.* Much like the Abelian case, we can think of $(\mathrm{d}+A)$ as a flat $U(n)$ connection ∇_E on E. The Hamiltonian has geometric form $H_E = \nabla_E^*\nabla_E + V$. This lets us mathematically state the central problem of Hyperbolic band theory in full generality:

Problem 1 (The band theory problem)**.** *Consider a Riemann surface Σ with a degree zero, Hermitian vector bundle (E, h) with $U(N)$ flat connection ∇_E and a real potential $V : \Sigma \to \mathbb{R}$. How does the spectrum of the operator*

$$H_E = \nabla_E^*\nabla_E + V \tag{21}$$

vary along the moduli space of flat irreducible unitary connections?

We can bring this closer to algebraic geometry by trading flat structures with holomorphic structures. The flat connection splits into holomorphic and antiholomorphic components, $\nabla_E = \nabla'_E + \nabla''_E$. The $(0,1)$ part defines a Dolbeault operator $\bar{\partial}_E = \nabla''_E$, making E into a holomorphic vector bundle. This map is a diffeomorphism between the moduli space of flat irreducible unitary connections to the moduli space of stable holomorphic vector bundles. We can also express the Laplacian through $\bar{\partial}_E$. The Laplacian decomposes as

$$\nabla_E^*\nabla_E = (\nabla'_E + \nabla''_E)^*(\nabla'_E + \nabla''_E) = \nabla'^*_E\nabla'_E + \nabla''^*_E\nabla''_E$$

The cross terms vanish due to type considerations. A Weitzenböck identity states (∗ ∗)[3]

$$\nabla_E''^{*}\nabla_E'' = \nabla_E'^{*}\nabla_E' + i\star F_{\nabla_E} \tag{22}$$

The curvature F_{∇_E} vanishes, as ∇_E is flat. Hence, $\nabla_E^*\nabla_E = 2\nabla_E''^{*}\nabla_E'' = 2\bar{\partial}_E^*\bar{\partial}_E$. (∗ factor of 2? ∗)This gives an alternate version of the hyperbolic band theory problem:

Problem 2 (The band theory problem, holomorphic version)**.** *Consider a Riemann surface Σ with a degree zero topological vector bundle E and a real potential $V : \Sigma \to \mathbb{R}$. How does the spectrum of the operator*

$$H_E = 2\bar{\partial}_E^*\bar{\partial}_E + V \tag{23}$$

vary along the moduli space of stable bundles?

The hyperbolic Brillouin zone is the moduli space of all stable bundles. The connected components are stable bundles of a specified rank, starting with the abelian Brillouin zone (Jacobian of the curve), but increasing in dimension and complexity as the rank goes to infinity (See Figure 5. We wish to find the band structure: A graph of the spectrum of H_E, expressed as a many-sheeted cover of hyperbolic Brillouin zone.

We note one more geometric perspective of this Hamiltonian. As the Hermitian metric retrieved from the standard one on $\mathbb{H} \times \mathbb{C}^n$ is flat, It's Chern connection relative to the holomorphic structure $\bar{\partial}_E$ equals the canonical flat connection ∇_E (∗ ∗). Looking at equation 22, we can control a scalar part of the Laplacian by changing the curvature. More precisely, equation 22 applied to the Chern connection ∇_h for a Hermitian metric h says

$$\nabla_h^*\nabla_h = \bar{\partial}_E^*\bar{\partial}_E + i\star\Theta(h)$$

where $\Theta(h)$ is the curvature of the metric. Choose h such that $i\star\Theta = \frac{1}{2}V\cdot\mathrm{Id}$, so the curvature is central and scaled by the potential. Then, we get a third variant of the hyperbolic band theory problem:

Problem 3 (The band theory problem, Chern version)**.** *Consider a Riemann surface Σ with a degree zero topological vector bundle E and fixed Hermitian metric h with curvature $\frac{1}{2}\star V$, and denote the Chern connection by $_h$. How does the spectrum of the operator*

$$H_E = \nabla_h^*\nabla_h \tag{24}$$

vary with the holomorphic structure of E? [4]

This formulation is at home with the Higgs bundle approach described in the rest of this paper, and is perhaps the optimal framework for discussing more general questions (∗ ∗). However, the potential will suffice for this paper, and it simplifies some of the exposition.

3. Higgs bundles

Now we begin the primary intent of this paper, studying the solutions to the eigenvalue problem described in the last section using Higgs bundles. We will review some basic ideas about Higgs bundles, taking a route optimized for the problem at hand.

3.1. Riemann surfaces as spectral curves.

As motivation, recall that a hyperbolic crystal with rank 1 crystal momentum is described by a Riemann surface Σ with line bundle L. Suppose the crystal had a symmetry, represented by a finite group G acting by holomorphic maps on Σ. To remove redundancy, We should represent the crystal data as a geometric structure over a fundamental domain of G, with proper periodicity. This is represented by the quotient $C = \Sigma/\Gamma$, which inherits the Riemann surface structure of Σ. The projection $p : \Sigma \to C$ is a N-to-1 branched cover. So, we want to encode a branched covering $p : \Sigma \to C$ and line bundle $L \to \Sigma$ as a geometric structure over C. Over a regular point $b \in C$, this data consist of a set of points $p_i \in p^{-1}(b)$ that we can represent as 'heights' $\lambda_i \in \mathbb{C}$, each with a one dimensional subspace $L|_{p_i}$. We package this into a $N \times N$ matrix acting on the vector space $\oplus_i L|_{p_i}$, with eigenvalues λ_i and eigenvectors $L|_{p_i}$.

[3] Specifically, this is the Nakano-Akizuki formula. The usual expression on the right is $[F_{\nabla_E}, \Lambda]$, which simply equals $\star F_{\nabla_E}$ on a Riemann surface.

[4] Note that the adjoint is with respect to the curved h, not the flat metric $h_\flat$ induced from $\mathbb{H} \times \mathbb{C}^n$. Since $\Theta(h)$ is central, we can take h as a multiple of the identity with respect to a unitary frame of the flat basis. This means the adjoint with respect to h and $h_\flat$ agree.

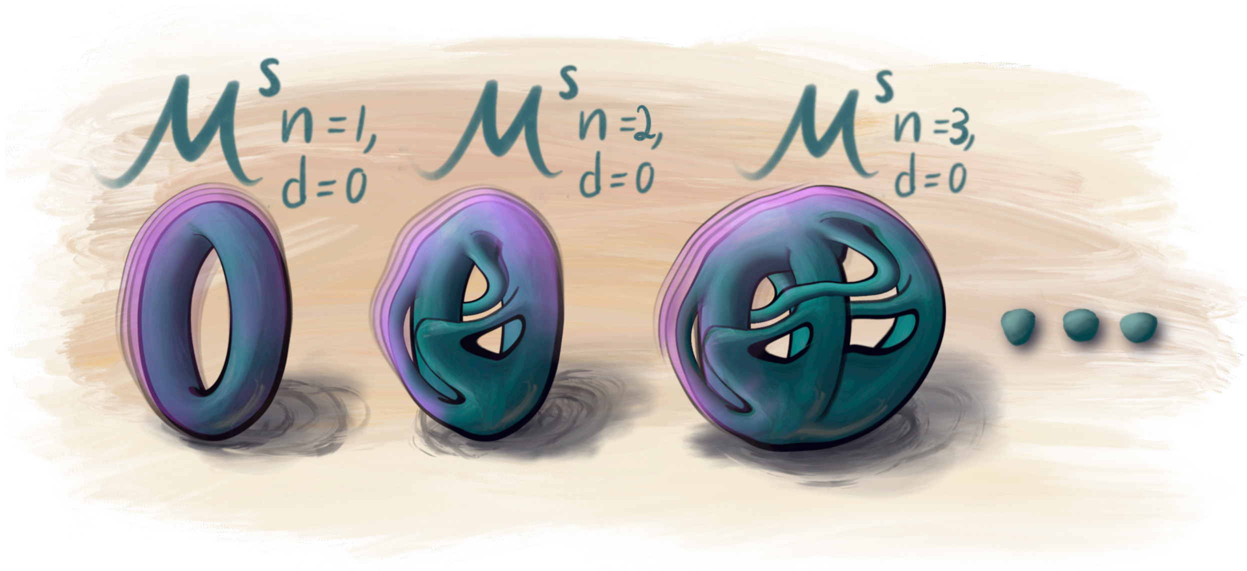


FIGURE 5. Band structure over stable bundles (**NOTE**: Not sure whether to include this ... it doesn't really explain anything mathematical in particular)

Roughly speaking, A Higgs bundle is a *global, holomorphic* version of this construction. It consists of a holomorphic vector bundle $E \to C$, and a Higgs field: a holomorphic section ϕ of End$E \otimes K$, where $K = T^{*(1,0)}(C)$ is the holomorphic cotangent bundle of C, also called the canonical bundle. After choosing a frame for E, *phi* is a matrix of holomorphic one-forms, and its eigenvalues are valued in K. These generalize to twisted Higgs bundles, whose eigenvalues can live in any line bundle $K(D) = K \otimes \mathcal{O}(D)$, where $\mathcal{O}(D)$ is the line bundle associated to some divisor on C. The graph of the eigenvalues defines a *spectral curve* Σ as a subset of $\mathrm{Tot}(K(D))$, the total space of $K(D)$.

We can pass between twisted Higgs bundles $(E, \phi : E \to E \otimes K(D))$ and their spectral data $(\Sigma \subset \mathrm{Tot}(K(D)), L \to \Sigma)$. Starting from the spectral data, E is related to L at a regular point b by $E|_b = \oplus_i L|_{p_i}$. We extend this over the branch points using the language of sheaves. Denoting the sheaf of holomorphic sections of L by $\mathcal{O}(L)$, we define the pushforward sheaf [5] $p_*\mathcal{O}(L)$: On an open set $U \subset C$, the space of holomorphic sections $H^0(U, p_*\mathcal{O}(L))$ equals $H^0(p^{-1}(U), calO(L))$, the holomorphic sections of L on $p^{-1}(U)$. For a branched cover p the pushforward sheaf is locally free, so defines the pushforward vector bundle $E = p_*L$. The pushforward option uniquely maps holomorphic sections of L to those of p_*L, allowing one to define pushforwards of operators. For a good introduction, see [23, chapter 2]. We can also obtain the Higgs field. the total space of $K(D)$ has projection $\pi : \mathrm{Tot}(K(D)) \to C$, and the pullback $\pi^*K(D)$ defines the tautological line bundle over $\mathrm{Tot}(K(D))$. This also has a tautological section λ, which fiberwise is the identity function $\lambda(z) = z$. Multiplying by λ acts fiberwise on L by scaling $L|_{p_i}$ by λ_i. The pushforward of this operation gives the Higgs field $\phi : p_*L \to p_*LK(D)$.

From the Higgs bundle, we get the spectral curve Σ by graphing the eigenvalues of ϕ in $\mathrm{Tot}(K(D))$. More invariantly, Σ is the locus of zeros of the characteristic polynomial $\det(\phi - \lambda I) = \lambda^d + a_1\lambda^{d-1} + \cdots + a_d$, with coefficients $a_i = \mathrm{Tr}(\phi^i) \in H^0(\Sigma, K(D)^i)$. In fact these coefficients characterize the spectral curve, which we will exploit to define a fibration on the space of Higgs bundles in section 3.3. The projection $\mathrm{Tot}(K(D)) \to C$ restricts to the ramified covering $p : \Sigma \to C$. Finally, the line bundle on Σ is the unique one with pushforward p_*L.

3.2. **Hyperelliptic curves.** Let us see how this applies Hyperelliptic curves, or double branched covers of the Riemann sphere $\mathbb{P}^1$. This is an important special case, which we will use as an example throughout this paper.

[5] This is often called the "direct image sheaf"

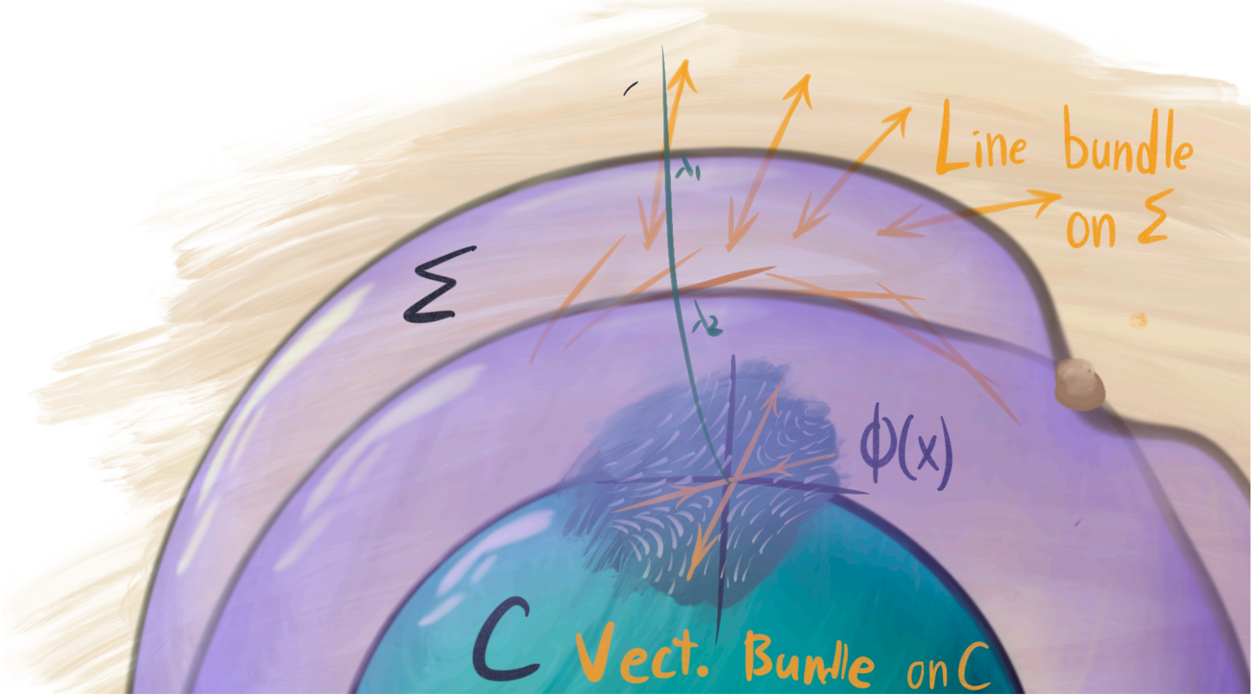


FIGURE 6. Spectral curve (**NOTE**: I was rushed when I made this, so it doesn't hold up to quality standards.)

Consider a crystal with spatial inversion symmetry. About any point in the hyperbolic plane, there is an isometry σ locally acting as $z \to -z$ called inversion. Drawing the Poincare disk model centered about this point, inversion is just euclidean inversion of the disk. For a crystal with spatial inversion symmetry, the fundamental polygon is symmetric about the origin, as depicted in figure 7A. This also shows the fundamental domain for inversion is half the polygon. The quotient Σ/σ is formed formed by cutting out the fundamental domain, and gluing together the boundaries according to σ. Each side of the half polygon is identified with with its flipped self up to lattice translation, as shown in figure 7B. Folding this up gives a topological sphere, in figure 7C. More formally, the branch points of $p : \Sigma \to \Sigma/\sigma$ are the fixed points of σ, consisting of the center, vertex, and midpoint of half the edges of the polygon. If g is the genus of Σ, then the polygon has $4g$ edges, and σ has $2g + 2$ fixed points. The Riemann-Hurwitz formula guarantees that if $p : \Sigma \to C$ is a branched cover with $2g + 2$ branch points, then C must be $\mathbb{P}^1$. So, a spatially symmetric crystal is a double branched cover $p : \Sigma \to \Sigma/\sigma = \mathbb{P}^1$, depicted by figure 7D. Σ is called a hyperelliptic curve, and the map σ is the hyperelliptic involution. Conversely, any hyperelliptic curve can be represented by a spatially symmetric fundamental polygon. Explicitly, Σ is the locus of solutions to the equation $\lambda^2 + P(z) = 0$ for a degree $2g+2$ polynomial $P(z)$ with roots at the branch points of p.

We can represent the crystal data (Σ, L) as a twisted Higgs bundle (p_*L, ϕ) over $\mathbb{P}^1$, with ϕ valued in $K(D)$. The characteristic polynomial of ϕ is $\lambda^2 + P(z)$, where $P(z)$ is a section of $K(D)^2$. The degree of the polynomial $P(z)$ is $2g+2$, so the degree of $K(D)$ is $g+1$. On $\mathbb{P}^1$, this uniquely characterizes $K(D) = \mathcal{O}(g+1)$. Relating the Euler characteristics of the bundles L and p_*L shows that the degree of p_*L is $-(g + 1)$. Since p_*L lives over $\mathbb{P}^1$, the Birkhoff–Grothendieck theorem says it splits as

$$p_*L = \mathcal{O}(-k) \oplus \mathcal{O}(k - (g + 1)) \tag{25}$$

for some integer k. Using this decomposition, we can write ϕ as a matrix, each entry a holomorphic section of a different line bundle on $\mathbb{P}^1$:

$$\phi \in H^0\left(\Sigma, \begin{pmatrix} \mathcal{O}(-k) \otimes \mathcal{O}(-k)^* & \mathcal{O}(-k) \otimes \mathcal{O}(k-(g+1))^* \\ \mathcal{O}(-k)^* \otimes \mathcal{O}(k-(g+1)) & \mathcal{O}(k-(g+1)) \otimes \mathcal{O}(k-(g+1))^* \end{pmatrix} \otimes \mathcal{O}(g+1)\right)$$

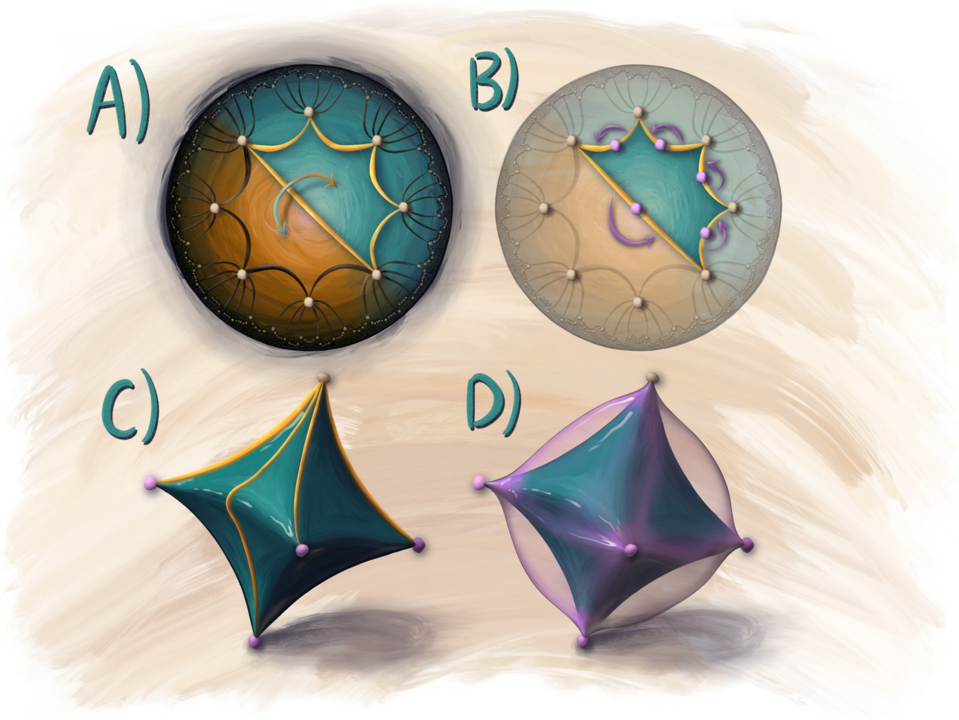


FIGURE 7. Hyeprelliptic and orbifold

$$= \begin{pmatrix} \mathcal{O}(g+1) & \mathcal{O}(2(-k+g+1)) \\ \mathcal{O}(2k) & \mathcal{O}(g+1) \end{pmatrix}$$

So, the Higgs field can be represented by a 2-by2 matrix of polynomials, of degrees given above. Note that a Higgs field only exists if all entries have nonnegative degree, implying $0 \leq k \leq (g+1)$.

3.2.1. *Parabolic Higgs bundles.* Parabolic Higgs bundles offer a dual perspective to twisted Higgs bundles. Instead of a holomorphic section of $\mathrm{End}(E) \otimes K(D)$, ϕ is a meromorphic section of $\mathrm{End}(E) \otimes K$, with poles allowed on the divisor D.

This only blows up at $z = 0$ along a specific subspace of the fiber of E at zero. The data for a meromorphic Higgs bundle contains both the location and the subspace along which it blows up, and more generally the rate. We thus define

Definition 1 (Parabolic bundle)**.** Let the *parabolic divisor* D be a set of points $d_1, \dots, d_n$ on Σ, For a rank 2 vector bundle E, a *Quasi-periodic* structure is a choice of line F in the fibers of E at each point in D. To de-quasify this, we include with each point a set of *Weights* $\vec{\alpha}(p) = (\alpha_1, \alpha_2)$ with $0 \leq \alpha_1 < \alpha_2 < 1$, associated to the vector spaces E and F, respectively.

A *Parabolic vector bundle* E_* is the tuple $(E, \{F(p)\}_{p \in D}, \vec{\alpha})$

Remark 3. For general vector bundles, the choice of weight and line is replaced with a full weighted flag. The type of parabolic structure we're using are

A parabolic Higgs field is one which respects the parabolic structure. Namely, the residue of ϕ (the z^{-1} part) at $p \in D$ must send $E|_p$ to F, and send F to zero. This means ϕ is nilpotent at D, which has a number of implications. First, we know $\mathrm{Tr}(\phi)$ is a holomorphic section of $K(D)$, but nilpotency implies this is zero at every point of D, so it descends to a holomorphic section of K. On $\mathbb{P}^1$, $K \cong \mathcal{O}(-2)$, and so the only holomorphic section is the one that is globally zero. That is, ϕ is trace free and the Higgs bundle is valued in $SL(2, \mathbb{C})$.

Secondly, it means $P(z) = \det(\phi^2)$ vanishes on D. So, D is contained in the set of branch points. The degree of $K(D)$ must be $g+1$, so D contains $g+3$ points, and is contained within the $2g+2$ branch points. (**NOTE**: Say that it doesnt matter which of the points are taken as parabolic points)

3.2.2. *Orbifolds.* Perhaps the best way to see the role of parabolic bundles is through the parabolic bundle-orbifold bundle correspondence [5].

An orbifold is formed from the quotient of a manifold by a finite group action, which is properly discontinuous but not necessarily fixed. For example, if we think of $\mathbb{C}$ as a (very) large sheet of paper, then quotienting $\mathbb{C}$ by the involution $z \to \bar{z}$ amounts to folding the paper across the real axis. The result is not longer a manifold – The real axis is the boundary! Instead, each local chart is modeled on $\mathbb{C}$ or $\mathbb{C}/\mathbb{Z}_2$, or in general $\mathbb{C}$ modulo some finite group action. These charts hold the orbifold structure.

This is exactly the situation for covers of Riemann surfaces. In particular, take a hyperelliptic curve Σ with hyperelliptic involution τ over $\mathbb{P}^1$. The projective line is Σ modulo the action of $\mathbb{Z}_2$ acting by the involution τ, and τ does indeed have fixed points. About each fixed point (branch point), the action is locally $z \to -z$, so we have an orbifold structure for $\mathbb{P}^1$. A general chart is a map to $\mathbb{C}$, and a chart about a branch point goes to $\mathbb{C}/\{z \to -z\}$. By a stroke of good fortune, $\mathbb{C}$ and $\mathbb{C}/\mathbb{Z}_2$ are isomorphic (**NOTE**: what type of morphic?)so we can forget that $\mathbb{P}^1$ has an orbifold structure and just remember the Riemann surface structure.

Such are the usual proceedings, but we should not throw out the orbifold structure so hastily, as other objects on Σ are not isomorphic after quotienting by $\mathbb{Z}^2$. For instance, the metric. Keeping with our papercraft analogy, we make $\mathbb{P}^1$ by cutting out 1 half of the fundamental polygon, then gluing the edges together according to the hyperelliptic involution. This requires 'folding' each edge about the branch point, resulting in a cone point with angle π. The resulting $\mathbb{P}^1$ is constant negative curvature everywhere, except for the cone points with angle π at every branch point.

The orbifold $\Sigma/\mathbb{Z}_2$ has a natural action of $\mathbb{Z}^2$ on the fibers of E, which gives E a natural parabolic structure. Take the parabolic divisor to be the set of orbifold points p, define the distinguished line of $E|_p$ to be the fixed line on the $\mathbb{Z}_2$ action. (**NOTE**: weights $(0, 1/2)$)This is a particularly simple example of a very general correspondence between orbifold bundles and parabolic bundles, see [5].

To see this does what we want, consider a Higgs bundle about a branch point. We give Σ the local coordinate z, and C the corresponding coordinate $w = z^2$. For any local section $f(z)$, holomorphicity implies it enjoys an everywhere convergent power series. We split this into the even and odd parts, which we can write as functions of z^2 via

$$f(z) = f^e(z^2) + zf^o(z^2)$$

In fact, this shows the pushforward sheaf of L is locally free about this point, giving a holomorphic frame of p_*L on this set. The action of the Higgs field is multiplication by the eigenvalue, which is $\pm\sqrt{P(w)} \sim z$ around a root, thus ϕ acts on this basis through multiplication by z. In this basis, we have

$$\phi = \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}$$

But this is its form as a $K(D)$ valued Higgs bundle, and to see this properly we need to move back to K. Simply multiply by any section of $K(D)$, which have the form $\frac{s(w)}{P(w)}\mathrm{d}w$. We are at a root of $P(w)$, which has no repeated roots, so $P(w) \sim w$ and we gave

$$\phi = \begin{pmatrix} 0 & 1 \\ \frac{1}{w} & 0 \end{pmatrix} \mathrm{d}w$$

In particular the reside sends E to subspace consisting of even functions, and sends that subspace to zero. So, the parabolic structure at this point consists of E with this line. However, the even functions are exactly the sections of p_*L preserved by $z \to -z$! So, when a branch point p is included in the parabolic locus, the proper quasi parabolic structure is given from the orbifold construction.

(**NOTE**: Consider a branched double cover $p : \Sigma \to C$ with total ramification d, and a line bundle $L \to \Sigma$. The parabolic structure on p_*L has parabolic points at the branch points of p, and parabolic weights $(0, 1/2)$. If the degree of L is zero, the degree of p_*L is $-d/2$, and the parabolic degree is $\deg(p_*L) + \sum(w_i)$, where w_i are the weights of the parabolic points. For the weights given, $\sum(w_i) = d/2$, so the parabolic degree of p_*L is zero.)

Remark 4. Notice that the same Higgs field can be either a twisted Higgs field who's determinant is a simple zero at $w = 0$, or one whose determinant is a one form with pole at $w = 0$. The difference comes from whether or not we consider $w = 0$ a parabolic point. We start with a Higgs field with determinant having no poles and $2g + 2$ simple zeros. We then chose some subset of size $g + 3$ to declare parabolic, and assign these the line given from the orbifold construction. Now, the determinant has simple poles at the $g+3$ parabolic points, and simple zeros at the other branch points. The choice of the $g+3$ parabolic points among the branch points is ultimately arbitrary.

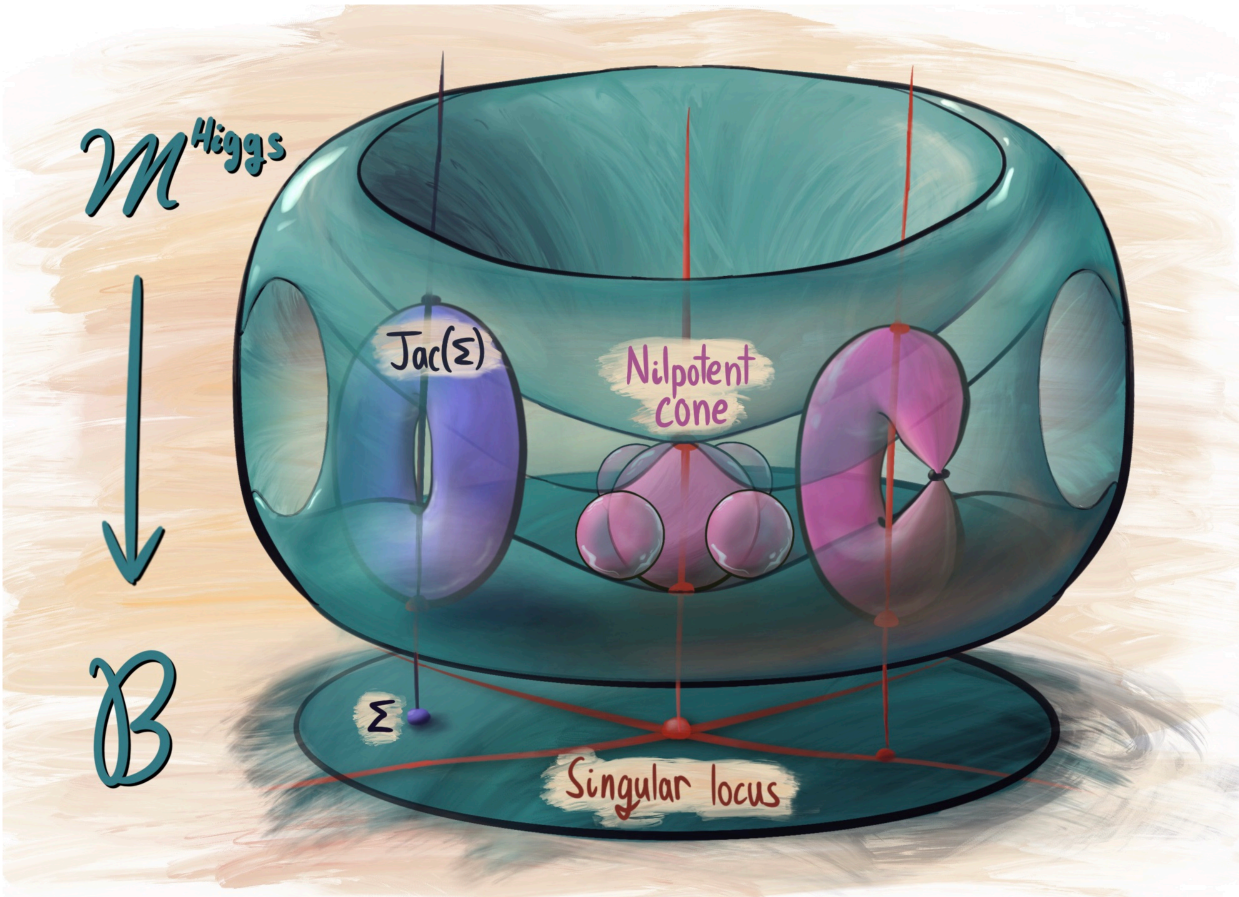


FIGURE 8. Illustration of the moduli space of Higgs bundles $\mathcal{M}^{\text{Higgs}}$. The Hitchin fibration maps $\mathcal{M}^{\text{Higgs}}$ to the Hitchin base $\mathcal{B}$, a vector space of half the dimension which parameterizes different spectral curves. A generic point in $\mathcal{B}$ gives a smooth spectral curve, with fiber its Jacobian (shown in purple). On the singular locus (shown in red), the spectral curve is singular and the fibers can degenerate (as indicated by the pink pinched torus). The most singular point is $0 \in B$, where the fiber is called the nilpotent cone (indicated here by the five pink spheres). This drawing is informed by the moduli space of Higgs bundles with 4 parabolic points over $\mathbb{P}^1$, described in detail in section 6

3.3. **Moduli of Higgs bundles.** (**NOTE**: Hitchin fibration, overdetermined system, stability?, cotangent bundle of moduli of stable bundles)

We now consider the moduli space of space of Higgs bundles, which is the space of pairs $(E, \bar{\partial}_E, \phi)$ modulo gauge equivalence, where $(E, \bar{\partial}_E)$ is a holomorphic vector bundle over a curve C and ϕ is a holomorphic section of $\text{End}(E) \otimes K(D)$. These pairs form an infinite dimensional affine space acted on by the group of gauge transforms, and the quotient (in the sense of geometric invariant theory (**CITE** ??)) gives a finite dimensional moduli space. To ensure a Hausdorff moduli space, we restrict to "stable" pairs (E, ϕ), where every ϕ-invariant holomorphic subbundle $V \subset E$ (That is, $\phi(V) \subseteq V \otimes N$) satisfies

$$\frac{\deg(V)}{\text{rank}(V)} < \frac{\deg(E)}{\text{rank}(E)}$$

In the rank 2 case, every subbundle of E is 1 dimensional, so has slope $\mu(L) = \deg(L)$. Slope stability implies $\deg(L) < \deg(E)/2$. We denote the moduli space of stable Higgs bundles by $\mathcal{M}$. The spectral data lends this space a fibration structure, by sending a Higgs bundle to its characteristic coefficients

$$h : (E, \bar{\partial}_E, \phi) \to (\mathrm{Tr}(\phi), \mathrm{Tr}(\wedge^2\phi), \ldots, \mathrm{Tr}(\wedge^n\phi)) \in \bigoplus_{i=0}^{n} H^0(C, K(D)^i) = \mathcal{B}$$

This maps $\mathcal{M}$ to an affine space $\mathcal{B}$ known as the Hitchin base. The point in $\mathcal{B}$ determines the spectral curve, so the remaining spectral data is the line bundle. Denote the spectral curve of a point $b \in \mathcal{B}$ by Σ_b. The regular points $b \in \mathcal{B}_{\mathrm{reg}} \subset \mathcal{B}$ have a characteristic polynomial has no repeated roots, so Σ_b is smooth and irreducible. The fiber of the Hitchin map $h^{-1}(b)$ is the moduli of line bundles $\mathrm{Jac}(\Sigma_b)$.

The regular points $\mathcal{B}_{\mathrm{reg}}$ are (Zariski) open and dense in $\mathcal{B}$, and so we arrive at a picture of the Higgs moduli space: it consists of equidimensional tori fibered over an affine base, which degenerate along a special locus $\mathcal{B}\backslash\mathcal{B}_{\mathrm{reg}}$. Moreover, $\mathcal{M}$ inherits a hyperkähler structure from the progenitor space of pairs (E, ϕ). One of the three commensurate symplecitc structures on $\mathcal{M}$ restricts to zero on the fibers of the Hitchin base, meaning the fibers are isotropic submanifolds. The coordinate functions on $\mathcal{B}$ Poisson commute with one another, making $\mathcal{M}$ a overdetermined integrable system.

(**NOTE**: Gauss-manin connection)

3.3.1. *Moduli of parabolic bundles.* A similar picture emerges when we treat these as parabolic Higgs bundles. Instead of just fixing the line bundle,

3.4. Canonical structures. (NOTE: Gauge fixing. Hitchins equations, harmonic metrics, etc)

(**NOTE**: nonabelian hodge theory for higgs bundbles)

Up until now, we have defined the Higgs moduli space through equivalence classes of Higgs bundles. We can make this more explicit if we pick a unique distinguished element from each equivalence class, essentially fixing the gauge. This is the role of Hitchin's equations.

$$F + [\phi, \phi^{*_H}] = \omega$$

4. Higgs bundles as crystal moduli

We may now embark on the primary exodus of this paper, and study the hyperbolic band theory of section 2 using the Higgs machinery exposited in section 3. In brief, the spectral data of a Higgs bundle encodes a Riemann surface with a line bundle, alternatively a crystal lattice with a rank 1 crystal momentum. The moduli space of Higgs bundles thus parameterizes a family of crystals. Abelianization map passes from crystal data on the base curve to that on the spectral curve. This sends a nonabelian structure (vector bundle on the base curve C) to an abelian structure (line bundle on the spectral curve Σ). It trades complexity in the structure group ($U(n)$ vs. $U(1)$) with complexity in the underlying curve (Σ necessarily has greater or equal genus to C). To use this in band theory, we need to relate the Hamiltonians on base and covering curve. Recalling the pushforward of a section of L is a section of p_*L, we define the "push forward operator" of the Hamiltonian H on L to be the operator on p_*H on p_*L that makes this diagram commute:

$$\begin{tikzcd} H^0(\Sigma, L) \arrow[r, "H"] \arrow[d, "p_*"'] & H^0(\Sigma, L) \arrow[d, "p_*"] \\ H^0(C, p_*L) \arrow[r, "p_*H"'] & H^0(C, p_*L) \end{tikzcd}$$

In this section we describe the pushforward of a crystal Hamiltonian, and relate that to Higgs bundles.

4.1. **Unbranched covers.** Suppose $p : \Sigma \to C$ is an unramified N-to-1 spectral cover. [6] A degree zero line bundle $L \to \Sigma$ pushes forward to a degree zero vector bundle $E \to C$. L carries a flat connection ∇_L, and this pushes forward to a flat connection ∇_E on E. Specifically, for a simply connected open set $U \subset C$, the preimage consists of N disconnected open sets $U_i \subset \Sigma$. The line bundle L is trivial on each U_i, so choose nowhere vanishing flat smooth sections f_i. Define a frame of p_*L on U by $e_i = p_*f_i$. The pushforward connection $\nabla_E = p_*\nabla_L$ is the unique connection such that this frame is flat. Since there are no branch points, this condition holds for an open cover of C, so extends to a connection on all of C.

[6] Note that the associated Higgs bundle on C is valued in a degree zero bundle.[35]

Consider a $\hat{V}$ on Σ with is symmetric under deck transforms. This is the lift of some potential V on the base C. The Hamiltonian on Σ is $H_L = \nabla_L^*\nabla_L + \hat{V}$. The pushforward is just what one would guess:

Proposition 1. *For an unramified cover $p : \Sigma \to C$ with $p_*L = E$, the pushforward of $H_L = \nabla_L^*\nabla_L + \hat{V}$ is $H_E = \nabla_E^*\nabla_E + V$*

Proof. Consider a section ψ of E, and denote $\hat{\psi}$ by the section of L satisfying $p_*\hat{\psi} = \psi$. We wish to show

$$H_E\psi = p_*H_L\hat{\psi}$$

First, we find the pushforward of the potential term. Since $\hat{V}$ is constant along the fibers,

$$p_*(\hat{\psi}\cdot\hat{V}) = p_*(\hat{\psi}\cdot V) = \psi\cdot V$$

Meaning the pushforward of $\hat{V}$ is V.

Next we want the pushforward of $\nabla_L^*\nabla_L$. The adjoint is with respect a Hermitian metric on $E \otimes T^*C$, where the metric on T^*C is induced by the Riemannian metric on C, and E carries the metric $h_\flat$ induced by the standard Hermitian metric on $\mathbb{H}\times\mathbb{C}^n$. This is defined by

$$< \varphi\otimes\nu, \psi\otimes\omega\rangle_E = h_\flat(\varphi,\psi)\cdot g_C(\nu,\omega)$$

Where $\varphi, \hat{\varphi}$ are sections of E and L respectively, related by $p_*\hat{\varphi} = \varphi$. The line bundle L also carries a Hermitian metric $\hat{h_\flat}$ induced from the standard one on $\mathbb{H}\times\mathbb{C}$.

(**NOTE**: Clean this up...)

$$< \varphi\,\nu, \psi\,\omega\rangle_E = p_*h_\flat(\varphi,\psi)\cdot g_C(\nu,\omega) = h_\flat(\hat{\varphi},\hat{\psi})\cdot g_C(\nu,\omega)$$

Where φ,ψ are sections of E, $p_*(\hat{\varphi}) = \varphi$ and likewise for ψ, and ν and ω represent the one-form parts. The last equality shows the definition of a pushforward metric. Now the result follows from juggling the definition of pushforward and adjoint:

$$\langle\varphi, \nabla_E^*\nabla_E\psi\rangle_E = \langle\nabla_E\varphi,\nabla_E\psi\rangle_E = \langle p_*\nabla_L\hat{\varphi}, p_*\nabla_L\hat{\psi}\rangle_E = p_*h_\flat(p_*\nabla_L\hat{\varphi}, p_*\nabla_L\hat{\psi})g_C(p_*\nabla_L\hat{\varphi}, p_*\nabla_L\hat{\psi}) \tag{26}$$

$$= h_\flat(\nabla_L\hat{\varphi},\nabla_L\hat{\psi})g_\Sigma(\nabla_L\hat{\varphi},\nabla_L\hat{\psi}) = \langle\nabla_L\hat{\varphi},\nabla_L\hat{\psi}\rangle_L = h_\flat(\hat{\varphi},\nabla_L^*\nabla_L\hat{\psi}) \tag{27}$$

$$= p_*h_\flat(\varphi, p_*\nabla_L^*\nabla_L\hat{\psi}) = \langle\varphi, p_*\nabla_L^*\nabla_L\hat{\psi}\rangle \tag{28}$$

We start over C in the first line, pull up to Σ in the second, then push back down to C in the last. Between lines 1 and 2, we used that g_C is the pushforward of g_Σ. This is true because the pullback p^*g_C of g_C to Σ agrees with g_Σ. The lack of branch points means that projection p is a diffeomorphism everywhere. Since g_C has constant negative curvature, so too must p^*g_C. The metric g_Σ is also constant negative curvature, and it defines the same Riemann surface as p^*g_C, so by uniformization the two metrics must be equal.

The identity in equations holds for any choice of φ, so $\nabla_E^*\nabla_E\psi = p_*\nabla_L^*\nabla_L\hat{\psi}$. Together with the potential, we conclude that $p_*H_L = H_E$ as desired. □

In particular, suppose $\hat{\psi}$ is an eigensection of H_L with eigenvalue λ. Then,

$$H_E\psi = p_*H_L\hat{\psi} = p_*\lambda\hat{\psi} = \lambda\psi$$

So the spectrum of H_L is contained in H_E. Likewise, if φ is an eigensection of H_E with eigenvalue λ,

$$H_L\hat{\varphi} = H_Ep_*\hat{\varphi} = H_E\varphi = \lambda\varphi = \lambda\hat{\varphi}$$

Showing the spectrum of H_E is a subset of H_L, thus the two spectra must coincide. Furthermore, the associated eigensections are mapped to each other by pushforward. The the spectral data of H_L and H_E are equivalent.

In effect, we trade a high rank crystal momentum on the low genus curve C, with a rank 1 crystal momentum on Σ. For vector bundles on C arising as pushforward of a line bundle on Σ, the band structure equals the abelian band structure over $\mathrm{Jac}(\Sigma)$. [7] Interestingly, unbranched coverings arose in [33] as clusters. Every N fold unbranched covering of the unit cell arises from a cluster of N contiguous unit cells, representing

[7]Not all vector bundles on C arise as pushforwards from Σ. This is clear by comparing the dimension of moduli of the former $(n^2(g-1)+1)$ and that of the latter $(n(g-1)+1)$.

a curve Σ. This was used to discretize the Jacobian of C. This section indicates the abelian band structure of the cluster gives part of the higher rank band structure on C.

4.2. **Branched covers.** In most applications, holomorphic maps between Riemann surfaces are branched. Our crystal of interest is a branched cover $p : \Sigma \to C$ with branch point set $B \subset C$, and crystal momentum $L \to \Sigma$. We want the pushforward operator of $H_L = \nabla_L^* \nabla_L + \hat{V}$. Since differential operators are local, it suffices to describe the pushforward in a neighborhood of each point. Each point in the complement of the branch points $C' = C \backslash B$ has a neighborhood such that p is an unbranched cover. On this, the reasoning from last section shows H_L pushes forward to $H_E = \nabla_E^* \nabla_E + \hat{V}$, where ∇_E is the pushforward of ∇_L, and the adjoint is with respect to the specific constant negative curvature metric on C whose pullback smoothly extends to all Σ. This glues together to form an operator H_E on $L^2(C', E)$. We wish to extend H_E to a self-adjoint operator $p_* H_L$ on $L^2(C, E)$, whose eigensections are exactly the pushforwards of those of H_L.

This is satisfied by restricting the domain of H_E from $L^2(C', E)$ to bounded sections (with respect to to the flat Hermitian metric). Indeed, if $\hat{\psi}$ is an eigensection of H_L, then elliptic regularity implies it is smooth and in particular bounded on Σ. It's pushforward ψ is an eigensection of H_E on C', and is bounded on C. Conversely, consider a bounded eigensection ψ of H_E on C', with eigenvalue λ. It is the pushforward of a bounded section $\hat{\psi}$ of H_L on $\Sigma' = \Sigma \backslash p^{-1}(B)$. Extending $\hat{\psi}$ by zero to $p^{-1}(B)$ gives a weak solution to the equation $(H_L - \lambda)\hat{psi} = 0$. By elliptic regularity, $\hat{\psi}$ is in the L^2-class of an honest-to-goodness smooth eigenfunction $\hat{\psi}_s$. The pushforward $\psi_s = p_* \hat{\psi}_s$ is a smooth, bounded section of E that agrees with ψ on C', since they are smooth and in the same L^2 class. That is, a bounded eigensection of H_E on C' comes from an eigensection of H_L on Σ. So, this extension of H_E captures the spectrum of H_L. In fact, it is the potential plus the canonical self-adjoint extension of Laplacian on $L^2(C', E)$. This is called the Friedrich's extension, and exists because the Laplacian is non-negative and symmetric [28].

This abstractly characterizes the pushforward operator, but is not very elucidating. To better describe the behavior at branch points, we treat the pushforward of the flat connection as a parabolic connection. For the sake of clarity, we turn to the simple case where p is a double cover. When the base curve is $\mathbb{P}^1$, this is the hyperelliptic case described in section 3.2. First, the pushforward connection. Transporting around a branch point locally interchanges the sheets of Σ, which is an involution σ. Following section 3.2.1, the sections even and odd under σ push forward to a frame of E on an open set around the branch point. The monodromy is the action of σ, which in this frame is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This comes from a logarithmic flat connection $p_* \nabla_L$ on E with simple pole at the branch point. Its reside is

$$\operatorname{Res}(p_* \nabla_L) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

This defines a parabolic point E at the base point, with weights $(0, 1/2)$ and a distinguished line of odd sections of L. To summarize, the pushforward connection is a flat, parabolic connection ∇_E on E, with parabolic points with weights $(0, 1/2)$ at each branch point and a distinguished line in E.

Next we need the adjoint ∇_E^*, which involves both the Riemannian metric on Σ and a Hermitian metric on E. The Riemannian metric is described in section 3.2.1. It is the unique one with constant negative curvature and cone points with angle π at every branch point. Next, the Hermitian metric on E is the pushforward of the flat Hermitian metric $h^\flat$ on L, induced by the standard Hermitian metric on $\mathbb{H} \times \mathbb{C}$. This is a singular metric: Introducing a conformal coordinate w on C around a branch point, the double cover Σ gets the coordinate z, related by $w = z^2$. The holomorphic sections of L given by 1 and z are respectively even and odd under interchange of sheets, and pushforward to a holomorphic frame (e_e, e_o) of E. We see $p_* h\flat(e_e, e_e) = h^\flat(1, 1) = 1$, while $p_* h^\flat(e_0, e_0) = h^\flat(z, z) = z^2 = w$. This means $p_* h^\flat$ is singular along the distinguished line, so is adapted to the parabolic structure.

To summarize, the operator $\nabla_L^* \nabla_L$ pushes forward to $\nabla_E^* \nabla_E$. The parabolic connection ∇_E is adapted to the parabolic structure on E, as is the Riemannian metric and the Hermitian metric $p_* h^\flat$ used to define the adjoint. The Hamiltonian associated to a rank 2 Higgs bundle (E, ϕ) is $H = \nabla_E^* \nabla_E + V$, where E has parabolic points at the zeros and poles of $\det(\phi)$, and distinguished line defined by the matrix of ϕ at these points. The extension to rank>2 Higgs bundles is yet to come, but we expect the story to be similar. The only difficulty we foresee is describing the parabolic structure in terms of the Higgs field.

Remark 5. For a hyperelliptic curve, this relates the line bundle on with a rank 2 parabolic vector bundle on $\mathbb{P}^1$ and an associated logarithmic connection. Up to a shift of degree, these are called Fuschian systems.

4.3. **Application to band theory.** Packaging the crystal lattice and abelian crystal momentum as the spectral data of a Higgs bundle lets us parameterize crystals using the Hitchin moduli space $\mathcal{M}^{\text{Higgs}}$. As described in section (**CITE** section), this has a fibration with base parameterizing spectral curves, and fibers their Jacobians. Interpreting this with hyperbolic crystals, the base parameterizes crystal lattices and the fiber gives the space of abelian crystal momenta on that lattice. Once we assign a potential V on C, each Higgs bundle gives the Hamiltonian discussed in section 4.2. As an example, every genus 2 curve is hyperelliptic, so arises from a rank 2 parabolic Higgs bundle on $\mathbb{P}^1$. The base of the Hitchin fibration contains all possible genus 2 crystals, and the fibers give all possible rank 1 crystal momenta. Specifying a potential V on $\mathbb{P}^1$, the Hamiltonian describes the moduli space of genus 2 crystals with spatial inversion symmetry, and potential on the fundamental domain given by V.

The graph of the spectrum of these Hamiltonians gives a band structure over $\mathcal{M}^{\text{Higgs}}$. Restricted to any fiber, this agrees with the band structure over the abelian Brillouin zone of that curve. Hence, we have constructed a sort of moduli space of band structure. To make this more suggestive, we could write this as a universal object. On $\mathcal{M}^{\text{Higgs}} \times C$, there is a universal Hamiltonian which restricts on $\{(E, \phi)\} \times C$ to the Hamiltonian defined above. The band structure is then a single submanifold of $\mathcal{M}^{\text{Higgs}} \times C \times \mathbb{C}$ constant constant along the C direction, derived from the universal Hamiltonian. This point of view turns Band theory into a moduli problem.

The space of possible band structures is not finite dimensional, but in some limits it is finite. For example, the tight binding limit gives a finite dimensional Hilbert space, so there is a finite dimensional moduli space S of all tight binding models. These are described in 5.2. The Hamiltonian depends on a point in $\mathcal{M}^{\text{Higgs}}$, giving a band structure over $\mathcal{M}^{\text{Higgs}} \times S$. This finite dimensional structure encodes all of this class of tight binding band structures.

We describe this band structure in detail for Euclidean crystals in section 6. Motivated by this, we speculate on the branching of band structure along high symmetry branes on $\mathcal{M}^{\text{Higgs}}$ in section 7.3.

5. Higgs bundles as complex momenta

In the preceding section, we used the spectral data of a Higgs field to parameterize different crystals, which we designate the *Crystal moduli interpretation*. However, Higgs fields naturally arise in a rather different context: They define an imaginary crystal momentum. We call this the *Complex momentum interpretation*. To motivate this, the spectral curve of a Higgs bundle is tantalizingly similar to a band structure (In fact, this observation first prompted the nascent study of hyperbolic band theory.) However, techniques from (complex) algebraic geometry do not naturally apply to the real eigenvalues of the Hamiltonian, so we need to complexify. The Hamiltonian is self-adjoint because the crystal momenta are unitary representations $\pi_1(\Sigma) \to U(N)$. Allowing non-unitary crystal momenta $\pi_1(\Sigma) \to GL(n, \mathbb{C})$ permits non self-adjoint Hamiltonians, thus complex energies. Just as the $U(N)$ representations parameterize holomorphic vector bundles, $GL(n, \mathbb{C})$ representations parameterize Higgs bundles.

5.1. **Complex crystal momenta.** Let us once again look to the Euclidean case for inspiration. A crystal momentum is a unitary character of the lattice $\chi_k : \Gamma \to U(1)$, where translating by a lattice vector γ multiplies the phase by $\chi_k(\gamma)$. The crystal momentum is determined by a vector $k \in \mathbb{R}^2$, with associated Hamiltonian $H_k = (i\nabla + ik) \cdot (i\nabla + ik) + V$. A complex crystal momentum is a character $\chi_k : \Gamma \to \mathbb{C}^*$, so a lattice translation multiplies both the phase and the amplitude. The space of such characters is not $U(1)^2$, but $\mathbb{C}^{*2}$. The complex crystal momentum is determined by $k \in \mathbb{C}^2$. The associated H_k has the same form, but is no longer self-adjoint. Indeed, we compute the Hermitian conjugate

$$H_k = -\nabla^2 - 2k \cdot \nabla - k \cdot k + V$$
$$H_k^\dagger = -\nabla^2 - 2k \cdot \nabla - k \cdot k + V$$

H_k for complex k is not self-adjoint so may have complex eigenvalues, which means an eigenstate gains or loses total amplitude over time. This is frowned upon by physics, but for now we bight our tongue and consider the mathematical consequences.

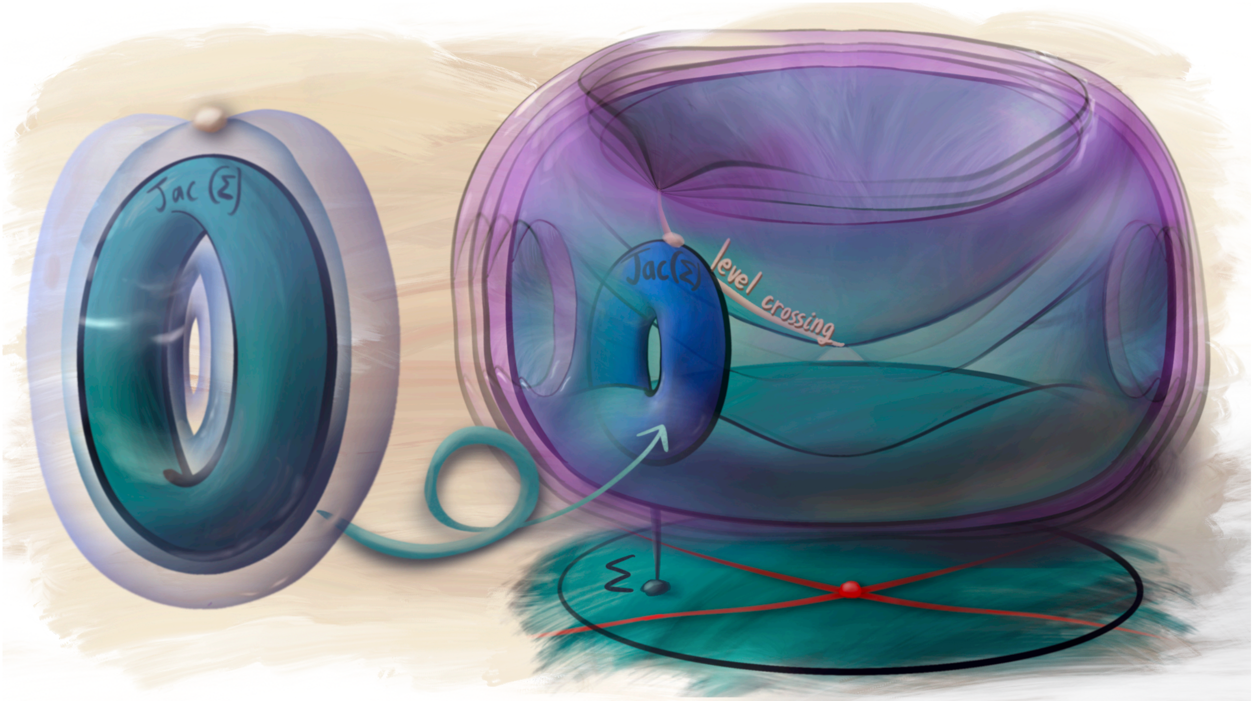


FIGURE 9. The universal band structure over the moduli space of Higgs bundles. Each point on the base gives a spectral curve, representing a hyperbolic crystal. The Hamiltonian defines a band structure over the fiber, its Jacobian. This glues together to a band structure over all $\mathcal{M}^{\text{Higgs}}$. (**NOTE**: remove level crossing from picture)

$H_k^\dagger = H_{\bar{k}}$.

(**NOTE**: One could complexify the hamiltonian by demanding that the adjoint be complex antilinear or complex linear. The first is the hermitian adjoint, and it results in a hamiltonian with a real spectrum. The second is what we describe below, and is what we want physically. The hamiltonian H_ρ then varies holomorphically with the irriducible representation ρ. This implies the spectrum also varies holomorphically, away from degenerate points. So, the Bloch variety is analytic and indeed a variety. (this is just reletive to the complex structure induced on $\mathcal{M}^{\text{Higgs}}$ by the charecter variety))

Abelian crystal momenta on a hyperbolic crystals are complexified in much the same way. We replace unitary representations $\pi_1(\Sigma) \to U(1)$ with general ones $\pi_1(\Sigma) \to \mathbb{C}^*$. Decomposing $\mathbb{C}$ into phase $U(1)$ and scaling $\mathbb{R}^{>0}$ splits the charecter variety

$$\mathrm{Hom}(\pi_1(\Sigma), \mathbb{C}^*)/\mathbb{C}^* \cong \mathrm{Hom}(\pi_1(\Sigma), U(1))/U(1) \times \mathrm{Hom}(\pi_1(\Sigma), \mathbb{R}^{>0})/\mathbb{R}^{>0}$$

The first factor is the Jacobian $U(1)^{2g}$, and the second is $(\mathbb{R}^{>0})^{2g} \cong \mathbb{C}^g$. We can express this as

a point in this character variety uniquely defines a flat connection $D_{(L,\phi)} = \mathrm{d} + k$ on a line bundle L, where k is a harmonic one-form. The splitting of the character variety is induced by the decomposition $k = k_r + ik_i$, where $k_r = \frac{1}{2}(k + \bar{k})$ is the real part and $k_i = \frac{1}{2i}(k - \bar{k})$ is the imaginary part. The real part k_r controls the $U(1)$ part of the monodromy, while k_i controls the $\mathbb{R}^{>0}$ part. The Higgs field is the unique holomorphic one-form ϕ such that $k_r = \phi + \bar{\phi}$. Likewise, every rank 1 Higgs bundle (L, ϕ) gives a flat connection $D = \nabla_L + i(\phi + \bar{\phi})$. This is the flat connection from the underlying line bundle L with an added skew-Hermitian part based on the Higgs field. The correspondence between rank 1 Higgs bundles and flat connections is the abelian version of the nonabelian Hodge correspondence. For more on rank 1 Higgs bundles, see [16].

For higher rank crystal momenta, the story is similar. The Hamiltonian again has the form $H_E = \nabla_E^* \nabla_E + V = (\mathrm{d} + iA)^*(\mathrm{d} + iA) + V$ Where now A is a $\mathfrak{u}(n)$-valued 1-form. To complexify, we replace $\mathfrak{u}(n)$ with $\mathfrak{u}(n) \otimes \mathbb{C} \cong \mathfrak{gl}(n, \mathbb{C})$. For an arbitrary matrix, $M \in \mathfrak{gl}(n, \mathbb{C})$ the Hermitian part is $(M + M\dagger)/2$ and the skew-Hermitian part is $(M - M\dagger)/2i$. Thus, $\mathfrak{gl}(n, \mathbb{C})$ splits into $\mathfrak{u}(n) \oplus \mathfrak{u}(n)$, where the first factor contains Hermitian matrices and the second skew Hermitian. Denoting the adjoint with respect to the Hermitian metric h by $\dagger_h$ Split A into Hermitian part $A_I = A_I^{\dagger_h}$ and skew Hermitian part $A_R = -A_R^{\dagger_h}$. The adjoint of the Hamiltonian is

$$H_A^\dagger = (-\mathrm{d} - iA^{\dagger_h})(-\mathrm{d} - iA^{\dagger_h})^* + V = H_{A^{\dagger_h}}$$

So H_A is self adjoint whenever $A = A^{\dagger_h}$, or $A_R = 0$. For a Higgs field ϕ on E, the associated flat connection is $D = D_E + \phi + \phi^{\dagger_h}$ where D_E is the flat connection for the holomorphic structure of E. For any skew Hermitian one form A_R, there is a unique holomorphic one form ϕ satisfying $i(\phi + \phi^{\dagger_h}) = A_R$. So, the decomposition $\mathfrak{gl}(n, \mathbb{C}) \cong \mathfrak{u}(n) \oplus \mathfrak{u}(n)$ defines the vector bundle and the Higgs field.

In total, we take a Higgs pair (E, ϕ) and send it to its canonical flat section, then take the Laplacian of that. Since ϕ is the complex part of momentum, the only pure real part of the spectrum is over the $\phi = 0$. On parabolic Higgs bundles, this is only defined on the complement of the parabolic locus. However, to match with the branch points in section 4, we take the Friedrichs extension of $H_E = D^*_{(E,\phi)}.D_{(E,\phi)} + V$ across the parabolic points. Note that we need parabolic Higgs bundles and not twisted Higgs bundles, because the nonabelian Hodge theorem does not apply in the latter case.

5.2. **Bloch variety and tight binding models.** In the complex momentum interpretation of Higgs fields, each Higgs bundle (E, ϕ) has a flat connection $D_{(E,\phi)}$, and a Hamiltonian $H_{(E,\phi)} = D^*_{(E,\phi)} D_{(E,\phi)} + V$. Therefore we can build a band structure, by graphing the spectrum of $H_{(E,\phi)}$ as (E, ϕ) varies across the moduli space of Higgs bundles $\mathcal{M}^{\text{Higgs}}$. More specifically, we can assemble the eigenstates of H_k into a master Bloch function $\psi(x, k, E)$ satisfying

$$H_k \psi(x, k, E) = E\psi(x, k, E)$$

The band structure is the set of pairs $k \in \mathcal{M}^{\text{Higgs}}$, $E \in \mathbb{C}$ where this equation has a solution. We call this set the Bloch locus $\beta \subset \mathcal{M}^{\text{Higgs}} \times \mathbb{C}$. This is also the zero locus of the equation $\det(H - E)$ (for some suitably regularized determinant) meaning β is a codimension 1 analytic object in $\mathcal{M}^{\text{Higgs}} \times \mathbb{C}$.

Remark 6. Physics lends some expected properties of the band structure, such as the 'avoided crossing' phenomena, where a perturbation lifts degeneracies at level crossings. This is formalized by the Von Neumann-Wigner theorem, which states that generic band structures have codimension 2 level crossings [39]. This is usually cited for the Hermitian operators with real energies, but the same reasoning applies to generic operators with complex energies, for which branching is complex codimension 2. In algebraic geometry, a morphism of algebraic varieties branches on a codimension 1 subvariety, by the purity of the branch locus (**NOTE**: cite). Therefore, if the Bloch locus is a variety and has generic band structure, it cannot be smooth. In fact, the crossing locus is necessarily singular. This is apparent in the canonical example of a perturbatively stable level crossing, the Dirac point of graphene, where the crossing is a conical singularity.

This is especially nice when the band structure is a finite branched cover, which happens when the Hamiltonian is finite dimensional. In condensed matter, such

This happens in finite dimensional models of a crystal, and in particular tight binding models. [8] In their simplest iteration, we assign a vector space of states to each cell of the crystal, and assume each cell only interacts with its neighbors. That is, the states are tightly bound to their cells. The Hamiltonian for each site splits into an *on-site matrix* M that couples each cell to itself, *hopping matrices* J that couple each cell to its neighbor. As the lattice is periodic, we assume every site has the same Hamiltonian.

Tight binding models are especially important to the working condensed matter physicist, because they provide a simple means to realize quantum materials experimentally or numerically. For example, hyperbolic crystals were experimentally realized by constructing a tight binding model with sites connected like a $\{3, 7\}$ hyperbolic tiling [32]. The theory of hyperbolic tight binding models was briefly treated in [**maciejko_hyperbolic_2020**], and further built upon in [33]. Connections between sites are single cell translations. Each site has $2g$ connections corresponding to the cycles $\{A_i, B_i\}$ of Σ, each with their own hopping matrix J_{A_i}, J_{B_i}. Hopping the opposite direction is governed by $J_\gamma^\dagger$ $(* \quad *)$, so for a trivial crystal momentum the full Hamiltonian is

$$H = M + \sum_{\gamma \in A_i, B_i} J_\gamma + J_\gamma^\dagger$$

[8]One can also discretize space, and use a discrete Laplacian for the Hamiltonian, see [41]. This approach is useful in different regimes from the tight binding model.

For an abelian Bloch state with complex crystal momentum k, and associated representation $\chi_k : \Gamma \to \mathbb{C}^*$. The new Hamiltonian is

$$H_k = M + \sum_{\gamma \in A_i, B_i} \chi_k(\gamma) J_\gamma + \chi_k(\gamma)^{-1} J_\gamma^\dagger$$

We can see H_k is a linear function of χ_k, χ_k^{-1} for $2g$ parameters $\chi_k \in \mathbb{C}^*$. The momentum space is then $(\mathbb{C}^*)^{2g}$, and the Bloch locus is the zero set of the characteristic polynomial $\det(H_k - E)$. In particular, this is a finite degree polynomial in χ and χ^{-1}, so the Bloch locus is algebraic!

5.2.1. *Tight binding limit, multi-atomic crystals, and quivers.* Tight binding models arise from kinetic plus potential Hamiltonians in the limit of infinitely deep potential wells. Imagine the potential came from a crystal, with potential wells at each atom of the crystal. For deeper wells, the eigenstates localize more about each atom and overlap with each other less. In the limit, the only significant interactions occur between atoms of neighboring cells. The relevant part of Hilbert space restricts to a finite dimensional subspace of ”atomic orbitals”, so we have a tight binding model.

Going one step further, we split the vector space of the cell into vector spaces attached to each ”atom”. Whenever the Hamiltonian has a nonzero element between two atomic vector spaces, these atoms have a ”bond”. We can associate this configuration of atoms with Hamiltonian to a quiver with representation. Each atom in the unit cell gives a node with the corresponding atomic vector space. The Hamiltonian maps these vector spaces amongst themselves, so we add arrows accordingly to get a quiver representation. The atomic component of the Hamiltonian gives an arrow from an atom to itself, and the bonds give arrows to and from its endpoints. Because the Hamiltonian is self-adjoint, the matrix attached to these bond arrows must be conjugates of each other. We must also include bonds passing from one unit cell to a neighbor, see figure 10. Moreover, this embeds the quiver into the Riemann surface of the lattice.

A nonzero abelian complex crystal momentum k acts by modifying the Hamiltonian, and thus the quiver representation. For each arrow passing from the unit cell to a cell related by γ, the associated matrix is multiplied by $\chi_k(\gamma)$. This gives an action of $(\mathbb{C}^*)^{2g}$ on the moduli space of quiver representations. The Bloch variety lives over such an orbit, and is always algebraic. For a nonabelain crystal momentum $\rho : \Gamma \to GL(n, \mathbb{C})$, the Hamiltonian lives on the direct sum of n copies of the quiver, which are intertwined (∗ word choice? ∗)according to ρ along bonds passing between crystal cells. Quivers and their representations are commonly studied in algebraic geometry and high energy physics, so the tight binding model from a quiver could bridge these fields with hyperbolic crystallography. We speculate on one possible relationship in section 7.1.

5.3. **Crystal moduli interpretation vs. complex momentum interpretation.** We have encountered two natural interpretations of Higgs bundles within hyperbolic band theory. The first, detailed in section 4, treats the spectral curve of a Higgs bundle as the crystal unit cell and the associated spectral line bundle as an abelian crystal momentum. Then, the moduli of Higgs bundles is a moduli space of crystal data. Alternatively, in this section we described Higgs bundles as giving the complex part of crystal momentum. Ultimately, these interpretations are two natural ways to send a Higgs bundle (E, ϕ) and potential V to a Hamiltonian. In the complex momentum interpretation, we get the Laplacian of the flat connection associated to the Higgs field, plus the potential:

$$H_{\text{complex}} = (\nabla_E + \phi + \phi^\dagger)^*(\nabla_E + \phi + \phi^\dagger) + V$$

In the crystal moduli interpretation, we get the Laplacian of the flat connection on a line bundle, plus the lift of the potential

$$H_{\text{crystal}} = \nabla_L^* \nabla_L + \hat{V}$$

This pushes forward to to the self-adjoint extension of the operator

$$H_{\text{crystal}} = \nabla_{(E,P)}^* \nabla_{(E,P)} + V$$

Where we emphasize that $\nabla_{(E,P)}$ is the flat connection of the parabolic bundle with underlying bundle E, and parabolic data denoted by P. We have two different Hamiltonians defined on C, each giving a different band structure. This is real in the crystal moduli interpretation, and complex in the complex momenta interpretation. The relationship between these two Hamiltonians is still mysterious.

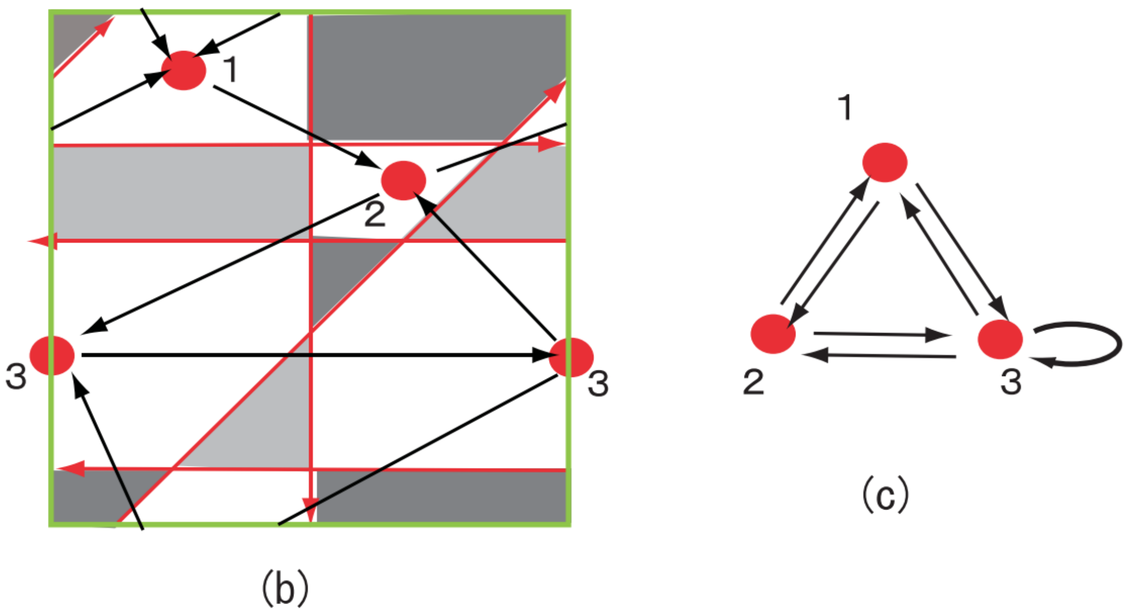


FIGURE 10. (**NOTE**: From [47]. Redo)

Remark 7. We could have called the crystal moduli interpretation the "de Rahm" one, and the complex momentum interpretation the "Betti" one. The crystal moduli is based on Higgs bundles and the Hitchin fibration, while the complex momentum is based on the character variety.

6. Euclidean crystals through Higgs bundles

Let us see this formalism in action and apply it to familiar 2D Euclidean crystals. We will find that Higgs bundles consolidates the standard band theory canon into an algebraic geometry package.

A Euclidean crystal is defined by a lattice $\Gamma = \langle 1, \tau \rangle \subset \mathbb{C}$, whose unit cell $\Sigma = \mathbb{C}/\Gamma$ is a genus one Riemann surface. As discussed in section 2.1.1, a periodic potential and the standard flat metric on $\mathbb{C}$ define the Hamiltonian $\Delta + V$. The abelian Bloch states are classified by a flat line bundle $L \to \Sigma$, and the Hamiltonian is $H_L = \nabla_L^* \nabla_L + V$ acting on sections of L. The band structure is the spectrum of H_L as L varies over the moduli space of flat line bundles, $\mathrm{Jac}(\Sigma)$. Since the genus is 1, $\mathrm{Jac}(\Sigma)$ is isomorphic to Σ. The Jacobian is a group with distinguished identity, making it an elliptic curve.

We wish to understand the band structure using Higgs bundles. To start, we describe the Higgs bundle associated to Σ by the crystal moduli interpretation. Genus 1 curves are the prototypical example of hyperelliptic curves, so this follows section 3.2. The Branched covering $\Sigma \to \mathbb{P}^1$ is defined by the equation $\lambda^2 = P(z)$, for a degree $2g + 2 = 4$ polynomial $P(z)$. The branch points are at the roots of $P(z)$, which we take to be $0, 1, ,$ and m. The location of m uniquely determines Σ. The root m is related to the lattice $\langle 1, \tau \rangle$ through the modular lambda function, $\lambda(\tau) = m$.

A Higgs field with genus 1 spectral curve lives on a rank 2 bundle E over $\mathbb{P}^1$, valued in the line bundle $K(D)$ for a a degree $g + 3 = 4$ divisor D. Alternatively, these are strongly parabolic Higgs fields with 4 parabolic points. These exhaust the $2g+2$ branch points (A property unique to genus 1 spectral curves), so the spectral curve is totally determined by D. Up to automorphism of $\mathbb{P}^1$, the points in D are $0, 1, \infty$ and m, so the Higgs field has determinant proportional to

$$\det(\phi) \propto \frac{\mathrm{d}z^2}{z(z-1)(z-m)}$$

which is the meromorphic quadratic differential defining the spectral cover.

Next, consider the crystal momentum defined by a line bundle L over Σ. As a holomorphic bundle, this pushes forward to a degree -2 vector bundle E on $\mathbb{P}^1$. By stability E is either $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ or $\mathcal{O}\oplus\mathcal{O}(-2)$. The trivial line bundle must push forward to a bundle with a nontrivial holomorphic section, which must be $\mathcal{O}\oplus\mathcal{O}(-2)$. Every other line bundle pushes forward to $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. This vector bundle gets a parabolic structure from the elliptic involution, with parabolic points at the branch locus $D=\{p_0,p_1,p_\infty,p_m\}$. The parabolic structure of E and associated meromorphic connection follow the structure described in section 4.2. Uniquely to this situation, the parabolic structure of the Higgs field and that induced by the Higgs field on E are the same, since the parabolic divisors coincide. Consequently, the Higgs field with given parabolic structure has no control of the line bundle on Σ, and thus no control of the spectral data. In fact, the space of Higgs bundles is one-dimensional.

We can convert the parabolic connection on $E=\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ to a pair of coupled meromorphic differential equations on $\mathbb{C}$, also called a Fuschian system. These are equivalent to a flat connection on a trivial rank 2 bundle over $\mathbb{P}^1\backslash D$, with prescribed monodromies. This bundle is $E\otimes\mathcal{O}(1)\cong\mathcal{O}\oplus\mathcal{O}$. To keep the parabolic degree zero we change the weights from $(0,\frac{1}{2})$ to $(-\frac{1}{4},\frac{1}{4})$. On the level of flat connections, the monodromies are replaced with

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

This admits an explicit description, described in [20]. Since E is trivial, the eigenlines of each parabolic point define 4 lines in $\mathbb{C}^2$. The moduli space of parabolic structures is that of 4 marked points on $\mathbb{P}^1$, or simply $\mathbb{P}^1$. Denote this parameter by u. The connection ∇^u and the Higgs field ϕ^u are explicitly given by

$$\begin{gathered}
\nabla^u = \mathrm{d} + A_0^u\frac{\mathrm{d}z}{z} + A_1^u\frac{\mathrm{d}z}{z-1} + A_m^u\frac{\mathrm{d}z}{z-m} \\
A_0^u = \frac{1}{4}\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad A_1^u = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_m^u = \frac{1}{4}\begin{pmatrix} -1 & 2u \\ 0 & 1 \end{pmatrix} \\
\Phi^u = \Phi_0^u\frac{\mathrm{d}z}{z} + \Phi_1^u\frac{\mathrm{d}z}{z-1} + \Phi_m^u\frac{\mathrm{d}z}{z-m} \\
\Phi_0^u = \begin{pmatrix} 0 & 0 \\ 1-u & 0 \end{pmatrix}, \qquad \Phi_1^u = \begin{pmatrix} u & -u \\ u & -u \end{pmatrix}, \qquad \Phi_m^u = \begin{pmatrix} -u & u^2 \\ -1 & u \end{pmatrix}
\end{gathered} \tag{29}$$

The parameter m controls the torus, while the parameter u controls the line bundle on the torus.

6.1. **Description of moduli space.** The moduli space of these Higgs fields with specified parabolic divisor is a Hitchin system [11, §8]. In the parameterization of equations 29, this consists of Higgs bundles $B\Phi^u$ for some $B\in\mathbb{C}$ The Hitchin base consists of meromorphic quadratic differentials

$$\mathcal{B} = \left\{ \frac{B}{z(z-1)(z-m)}\mathrm{d}z^2 \middle| B\in\mathbb{C} \right\} \cong \mathbb{C}$$

and fibers are the Jacobian of the specified elliptic curve. This base does *not* parameterize different lattices, which are already determined by the parabolic structure. Both the base and fiber are one dimensional, making the moduli space a complex surface. This is the simplest nontrivial Hitchin like system, earning it the moniker "toy model" $\mathcal{M}^{\text{toy}}$ [18].

The total space $\mathcal{M}^{\text{toy}}$ admits an explicit description. It is almost $\mathbb{C}\times\mathrm{Jac}(\Sigma)/\langle\pm1\rangle$, where $\langle\pm1\rangle$ multiplies the Higgs field by ±1 and acts by $L\to L^{-1}$ on $\mathrm{Jac}(\Sigma)$. This is singular at the fixed points, where the Higgs field is zero and the line bundle is a 2-torsion point. $\mathcal{M}^{\text{higgs}}$ is obtained by resolving these singularities and blowing up at these 4 points. The resulting moduli space has regular fibers everywhere on the base $\mathcal{B}$ except zero, where the fiber is a central $\mathbb{P}^1$ with four satellite $\mathbb{P}^1$s. The satellite $\mathbb{P}^1$s are located at $0,1,\infty$ and m.

By forgetting the Higgs field, we map from $\mathcal{M}^{\text{toy}}$ to the moduli space of parabolic structures on the underlying vector bundle E. This is determined by the four eigenlines at the parabolic points, giving a point in $\mathbb{P}^1$. In the above parameterization, this point is u. In particular, this moduli space is realized in $\mathcal{M}^{\text{toy}}$ as the central $\mathbb{P}^1$ in the zero fiber of the Hitchin map.

We can split $\mathcal{M}^{\text{toy}}$ into strata corresponding to the underlying holomorphic bundle. Those with $E=\mathcal{O}\oplus\mathcal{O}(-2)$ come from the trivial line bundle on each Jacobian, and defines the "small stratum" [11]. The

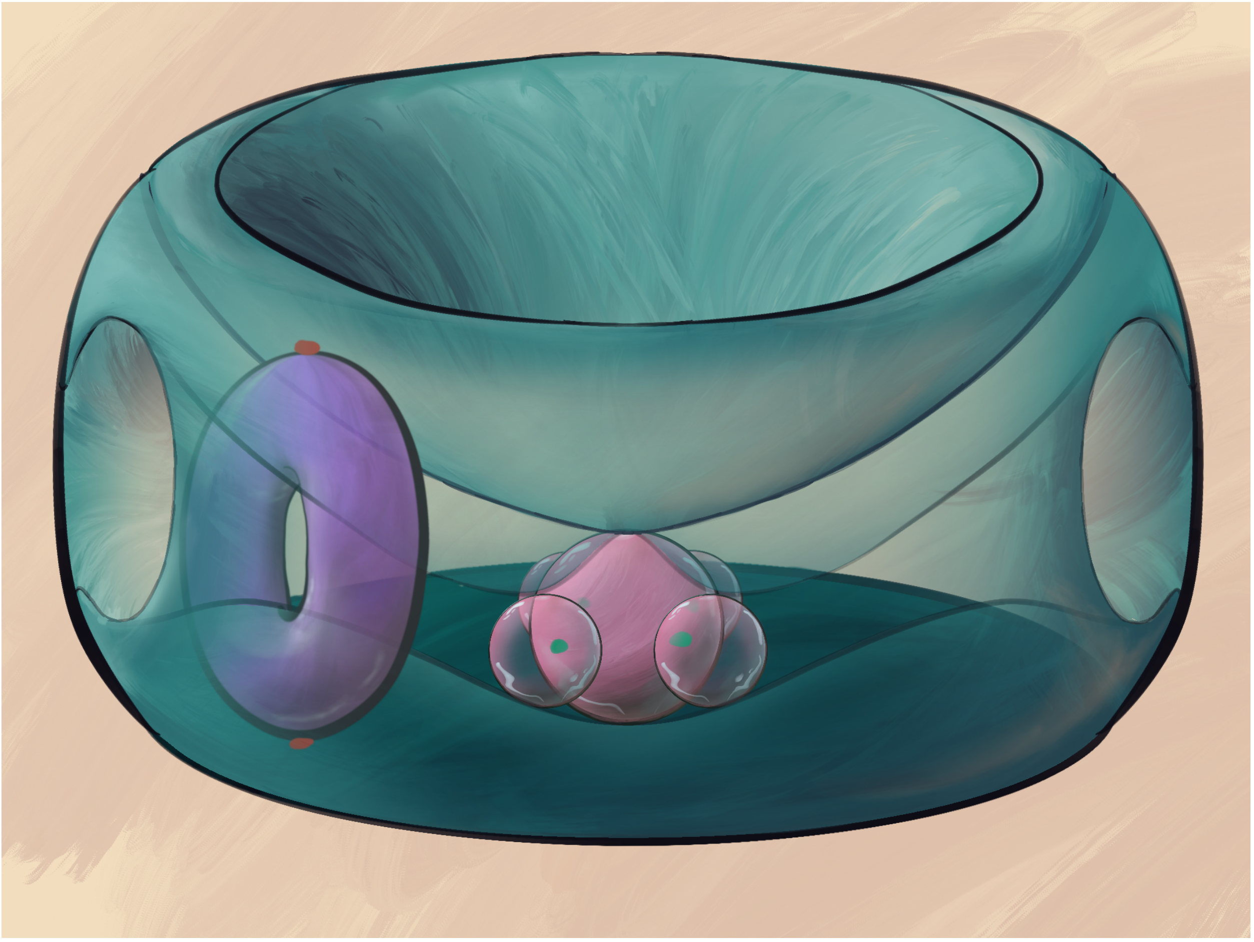


FIGURE 11. The moduli space of Higgs bundles on $\mathbb{P}^1$ with 4 parabolic points, or Hausel's toy model $\mathcal{M}^{\mathrm{toy}}$.

rest have $E = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, defining the "large stratum". A Higgs bundle in the small stratum lives in

$$\phi \in \begin{pmatrix} \mathcal{O}(2) & \mathcal{O}(4) \\ \mathcal{O} & \mathcal{O}(2) \end{pmatrix}$$

And every one is conjugate to

$$\phi \in \begin{pmatrix} 0 & P(z) \\ 1 & 0 \end{pmatrix}$$

This is the twisted analog of the Hitchin section. It also coincides with the theta divisor on each Jacobian.

Alternatively, we could study the moduli space of Twisted Higgs bundles. This amounts to not fixing the parabolic points. The twisting line bundle $K(D) = \mathcal{O}(2)$ is dual to the canonical bundle $K = \mathcal{O}(-2)$, bestowing these the nickname "co-Higgs field". The moduli space of such bundles was studied in [42]. The Hitchin base is

$$\mathcal{B} = H^0(\mathbb{P}^1, \mathcal{O}(2)) \oplus H^0(\mathbb{P}^1, \mathcal{O}(2)^{\otimes 2}) \cong \mathbb{C}^7$$

The fibers are the Jacobians of the spectral curve defined by $\det(\phi)$. This describes every Euclidean crystal lattice, but is redundant. We want a moduli space between the two extremes, giving all Euclidean lattices and line bundles without redundant dimensions. To achieve this, we only specify the locations of 3 parabolic points ($0, 1$ and ∞) and let the fourth vary. This amounts to replacing the moduli space with $\mathcal{M}^{\mathrm{toy}} \times \mathcal{M}^{\mathrm{ell}}$, where $\mathcal{M}^{\mathrm{ell}}$ is the moduli space of elliptic curves.

Let us review the form of $\mathcal{M}^{\mathrm{ell}}$. An elliptic curve is defined by a lattice $\langle 1, \tau \rangle$ in $\mathbb{C}$ for a parameter τ in the upper half plane $\mathbb{H}$. The moduli space of lattices is found by quotienting the action of the modular group, giving $\mathcal{M}^{\text{ell}} = \mathbb{H}/(2,\mathbb{Z})$. As a topological space this is $\mathbb{P}^1$ with one puncture, but it also has an orbifold structure. The orbifold group of a given lattice is the isomorphism group of that lattice. First, every lattice has $\mathbb{Z}_2$ inversion symmetry about its center, corresponding to the elliptic involution. τ along the imaginary axis defines a rectangular lattice with extra $\mathbb{Z}_2$ symmetry, so these are $\mathbb{Z}_2$ orbifold points of $\mathcal{M}^{\text{ell}}$. For τ on the unit circle, the unit cell is a rhombus so there is a Z_2 symmetry from swapping 1 and τ. At $\tau = i$, it is a square lattice with a further $\mathbb{Z}_4$ symmetry, meaning this is an orbifold point of the dihedral group D_4. Likewise at $\tau = e62\pi i/3$ defines an equilateral triangular lattice, which yields a D_3 orbifold point. Finally as $\tau \to \infty$, $\mathcal{M}^{\text{ell}}$ has a cusp. In this limit, the elliptic curve becomes nodal.

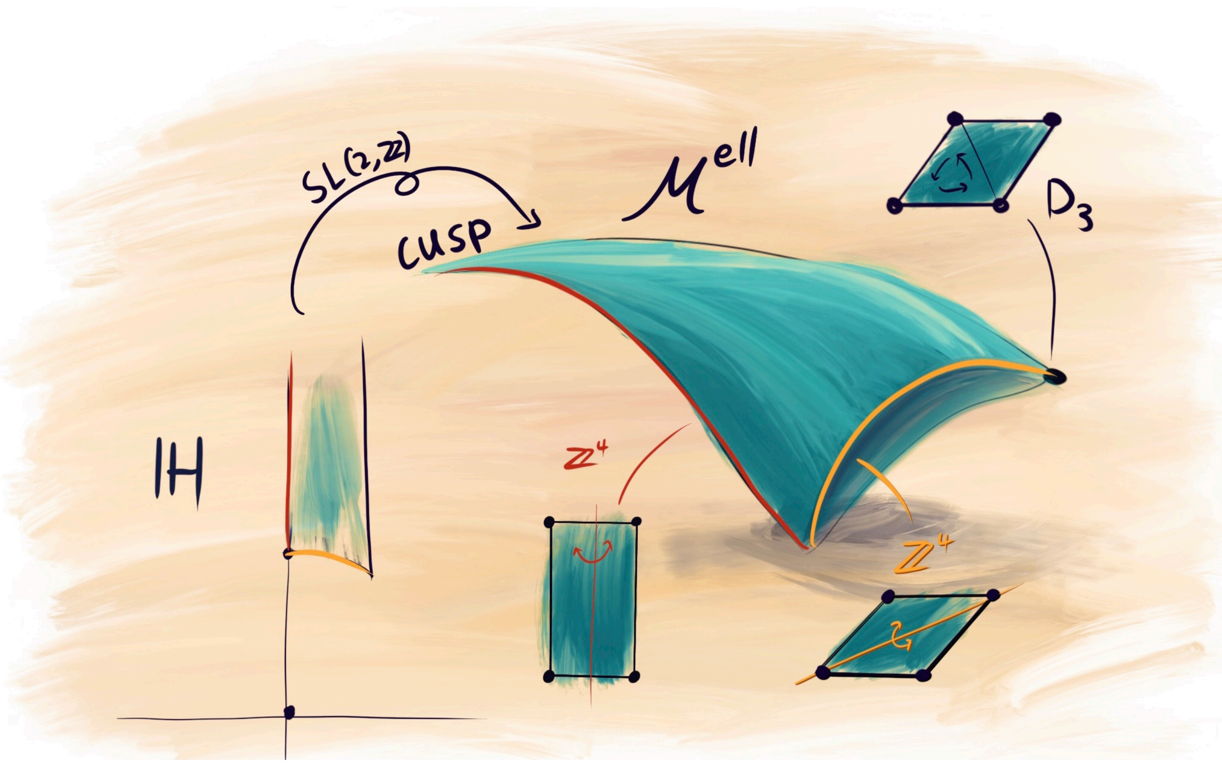


FIGURE 12. Illustration of the moduli space of elliptic curves $\mathcal{M}^{\text{ell}}$. The orbifold points of $\mathcal{M}^{\text{ell}}$ come from lattices with large isomorphism groups (**NOTE**: I did this really quick, I can make it a lot better)

6.2. **Description of band structure.** The band structure of Euclidean crystals is well understood, giving us the opportunity to explicitly describe the band structure over the whole Higgs moduli space. For simplicity, we will do this with the potential equal to zero, known as the empty lattice approximation. First, let us fix an elliptic curve with parameter τ. If the crystal Σ carries metric g, the crystal momentum space $\text{Jac}(\Sigma) \cong \Sigma$ carries a dual metric g'. This is the standard metric on $\mathbb{C}$ quotient the reciprocal lattice $\Gamma^* = \langle 1, \tau/||\tau||^2\rangle = \langle 1, -1/\tau\rangle$. Note that $\tau \to -1/\tau$ is an action of $(2,\mathbb{Z})$, so the dual lattice is equal up to scale to the original. As Riemann surfaces, Σ equals $\text{Jac}(\Sigma)$, as elliptic curves are self-dual abelian varieties. For a free particle, the dispersion relation is quadratic, so $E(k) = ||k||^2$. Taking into account periodicity, these parabolas are centered on each lattice point in Γ^* resulting in a many sheeted band structure over the unit cell. Level crossings occur at any point equidistant from two lattice points, with energy at crossing given by the square of that distance. The 2-torsion points, given by the half lattice $\Gamma^*/2$, are always at least 2-fold degenerate. For example, the center of the lattice is always equidistant from kitty-corner lattice points.

The band structure over each individual Jacobian glues together quite simply on $\mathcal{M}^{\text{toy}}$. Each regular fiber is isomorphic, so the band structure outside the zero fiber is identical. Note that there is a natural flat connection between the fibers from the Gauss-Manin connection (or in this case, the Picard-Fuchs connection), which preserves this band structure. The monodromy around the central fiber sends L to L^{-1}, meaning it acts by the elliptic involution. Luckily, the band structure is also invariant under this transform. There is a well defined limit approaching the central $\mathbb{P}^1$ fiber, which was obtained from $\mathrm{Jac}(\Sigma)/(L \to L^{-1})$. At the satellite $\mathbb{P}^1$s located at each 2-torsion point, the band structure is constant and equal to the spectrum above the associated point on the Jacobian.

Next, let us vary the elliptic curve parameter around $\mathcal{M}^{\text{ell}}$. There is extra branching when the lattice is symmetric. For example, in a rectangular lattice the center is equidistant from all lattice points in the unit cell, meaning it is at least 4-fold degenerate. This occurs along the ZZ_2 orbifold points of $\mathcal{M}^{\text{ell}}$. In general, the band structure on $\mathcal{M}^{\text{toy}} \times \mathcal{M}^{\text{ell}}$ branches along orbifold points of $\mathcal{M}^{\text{ell}}$.

Approaching the cusp of $\mathcal{M}^{\text{ell}}$ translates to a lattice stretched infinitely long in the imaginary direction. The dual lattice limits to $\langle 1, 0\rangle$, which is singular. Approaching this limit, the band spacing becomes closer and closer to zero, as the dual lattice points get closer and closer together. At the limit there is a continuous band of positive in the unbounded direction of the crystal, while the bounded direction still has discrete bands. Instead of describing this more carefully, we are content to leave the cusp out of $\mathcal{M}^{\text{ell}}$ and have a noncompact parameter space.

6.3. **Band structure with complex momenta.** Now we will look at the band structure in the complex momentum interpretation. This assigns a different Hamiltonian to the parabolic Higgs field on $\mathbb{P}^1$. Namely, we add a factor of $i(\phi+\phi^\dagger)$ to the flat parabolic connection on $\mathbb{P}^1\backslash D$. With the parameterization In genus 1 the parabolic structure of the Higgs field equals that induced by the orbifold, so this is the pushforward of a complex flat connection on Σ. In the parameterization of equation 29, the connection is $\nabla^u + iB(\Phi^u + {\Phi^u}^\dagger)$ for $B \in \mathbb{C}$ a point on the Hitchin base. This is the pushforward of the connection for a rank one Higgs bundle (L, ϕ) on Σ, so their Laplacians have the same spectrum. On the torus, ϕ holomorphic implies it is a constant one-form $B\mathrm{d}z$. The complex Brillouin zone is $\mathbb{C}/\Gamma^* \times \mathbb{C}$. The free particle complex dispersion relation is $E(k_x, k_y) = k_x^2 + k_y^2$ where $K_x, k_y \in \mathbb{C}$ are the complex momenta. Organizing into a single vector $\vec{k} = \vec{k}_r + i\vec{k}_i$ with $\vec{k}_r, i\vec{k}_i$ real, we have

$$E(\vec{k}) = \left(||\vec{k}_r||^2 - ||\vec{k}_i||^2\right) + 2i\left(\vec{k}_r \cdot \vec{k}_i\right)$$

The band structure is the graph of these dispersion relation in $\mathbb{C} \times \mathbb{C}$, centered around the points in the integer span of $k_x = 1 + 0i$ and $k_y = -1/\tau + 0i$.

7. Speculations

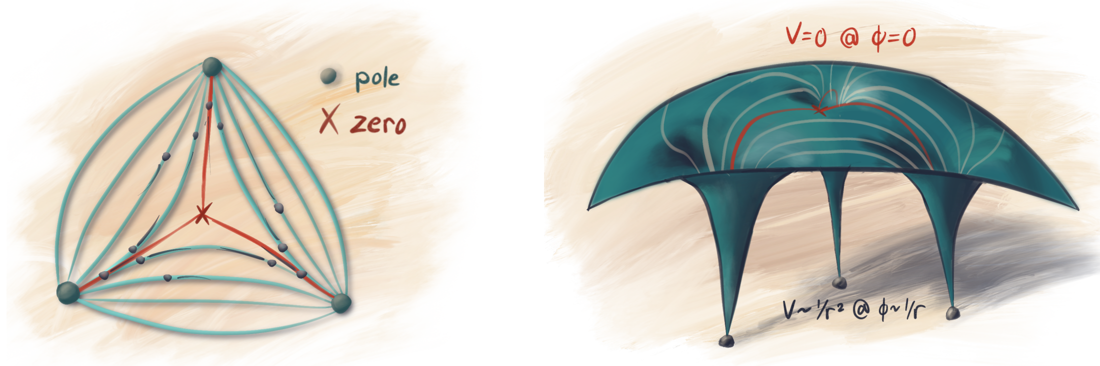
So far, we have argued that Higgs bundles are a natural way to view hyperbolic band theory. Now we guess how they are useful. Higgs bundles sit at the center of a highly interconnected web of mathematics and physics, spanning integrable systems, supersymmetric field theory, the geometric Langlands correspondence, and others. Higgs bundles serve as an ambassador, facilitating relationships between these fields and thus an exchange of ideas. We suspect they will play a similar role for band theory, weaving it into the already expansive web of ideas. In this section, we speculate about these connections and the possible implications for band theory.

7.1. **Connections to supersymmetric field theory.** Higgs bundles are no strangers to physics, especially high-energy physics. This connection is most familiar from from the work of Sieberg and Witten, where a Hitchin-type moduli space arises in the low energy limit of $\mathcal{N} = 2, d = 4$ supersymmetric Yang-Mills (SYM) theory. This suggests an analogy between such theories and hyperbolic band theory. We recall the dictionary between supersymmetric field theories and Hitchin systems in the first two columns of table 1. See [38] for a readable overview, and [9] for a detailed discussion emphasising integrable systems. In the last two columns, we recall the dictionary between Hitchin systems and hyperbolic crystals in the crystal moduli interpretation built up in this paper. The transitive property of dictionaries relates supersymmetric field theories with hyperbolic crystals. In particular, the Sieberg-Witten curve of the effective theory maps to a hyperbolic crystal.

A similar analogy is visible in the large Higgs field limit. This was studied using supersymmetric field theories by Gaiotto, Moore and Neitzke, giving a conjectural picture of the asymptotics of the Hitchin metric [13]. In the complex momentum interpretation of hyperbolic crystals, the large-scale limit is a large (imaginary) momentum limit, a semiclassical regime. This is governed by the associated classical system,

TABLE 1. Dictionary between $\mathcal{N} = 2$ SYM theory, Hitchin systems, and hyperbolic band theory.

$\mathcal{N} = 2$ SYM theory	Hitchin systems	Hyperbolic band theory
Moduli of vacua	Hitchin Base	Family of hyperbolic crystals
Sieberg-Witten curve	Spectral curve	Hyperbolic crystal
lattice of EM charges	Regular Hitchin fiber	Abelian Brillouin zone



(A) label 1

(B) label 2

FIGURE 13. WKB

which replaces the quantum Hamiltonian $\nabla_L^* \nabla_L + V$ on C with a function on T^*C. The classical Hamiltonian on T^*C is

$$H = \langle (p + A + i\Re(\phi), (p + A + i\Re(\phi)\rangle + V$$

where p is the tautological one-form, the inner product $\langle , \rangle$ is induced by the Riemannian metric, A is the vector potential of L, and $\Re(\phi) = \phi + \bar{\phi}$ is the real part of the Higgs field. The geometric approach to WKB analysis replaces the quantum state evolving under the quantum Hamiltonian with a Lagrangian submanifold evolving under the classical Hamiltonian. For a nonzero Higgs field, the vector potential A and thus $H = H_r + iH_i$ are complex. The Hamiltonian flow is $X_H = X_{H_r} + JX_{H_i}$ for complex structure J. On a Kähler manifold, JX_{H_i} is just the gradient $-\nabla H_i$, so H_i always decreases along the Hamiltonian flow. In other words, the imaginary part of the Hamiltonian is dissipative.

(**NOTE**: After some thought, my explanation for the next 2 paragraphs does not work.. In particular, the dynamics don't reduce to the imaginary part equals zero: In fact, the flow preserves the symplectic form, I think. I haven't given up on this, but I don't think I can find convincing reasoning. Question: What do we do about the rest of this section?

In the large Higgs field limit, the imaginary part of the vector potential $i(\phi + \bar{\phi}) = i\Re(\phi)$ dominates the Hamiltonian. Decomposing the Hamiltonian into real and imaginary parts,

$$H_r = \langle (p + A), (p + A)\rangle - \langle \Re\phi, \Re\phi\rangle + V$$

$$H_i = 2\langle p, (\phi + \bar{\phi})\rangle = 2(\langle p, \phi\rangle + \langle p, \bar{\phi}\rangle) = 2\Re(\langle p, \phi\rangle)$$

First, the real locus of the Hamiltonian consists of the momenta p such that $\langle p, \Re\phi\rangle$ is purely imaginary. The momentum dependence of the real Hamiltonian is purely in the quadratic kinetic energy term, meaning Hamilton's equations say $p = \dot{x}$. The restriction means the trajectories of particles $\gamma(x)$ satisfy $\Re(\phi(\dot{\gamma})) = 0$, which are called *WKB curves*[**gaiotto_wall-crossing_2011**]. On these curves, the real Hamiltonian has potential dominated by the term $-\langle \Re\phi, \Re\phi\rangle$. This is maximal when $\Re\phi = 0$, which happens at the zeros of ϕ. Inversely, it diverges at poles of ϕ. So, it acts like a force pushing from zeros, to poles. Together with the dissipative dynamics, the semiclassical behavior at large Higgs fields is constrained to WKB curves and has a deep potential at the poles of the Higgs field.

7.1.1. *BPS crystals.* The deep potential well suggests a sort of tight binding model. The states are eigenfunctions of the Hamiltonian, or semiclassically, Lagrangians fixed under the Hamiltonian flow. Since we are in complex dimension one, these these Lagrangians are level sets of the Hamiltonian. For a real energy E, the locus $H_r = E, H_i = 0$ defines a double cover Σ of C, with branching at the poles and zeros of ϕ. In the large ϕ limit, we can ignore V, so these equations become *Holomorphic*. This defines a holomorphic double cover with branching at zeros and poles, so it must equal the spectral curve of ϕ. For a given energy, the level sets live on the set of trajectories between a pole and a zero. In other words, for the WKB triangulation, the geometric WKB eigenstates in the tight binding limit are attached to a edge. They interact near the zeros and pooles, giving rise to bonds. Following section **??**, this is encoded by a quiver with a node for each edge of the WKB triangulation and arrows describing hwo edges connect. This is exactly the BPS quiver associated to a supersymmetric field theory.)

Once we have the BPS quiver, the BPS spectrum is found by solving a quiver quantum mechanics problem. Specifically, these correspond to ground states of $\mathcal{N} = 4$ supersymmetric quantum mechanics on the moduli of quiver representations. When this is embedded on a 2-torus, we can lift to the universal cover $\mathbb{R}^2$, and the associated covering quiver. We get an action by $U(1)^2$, corresponding to translating on this universal cover (**NOTE**: explain better)[37, 40]. Note that this is *Exactly* the modification of the Hamiltonian when changing the crystal momentum of a Euclidean crystal. Now, the ground states in supersymmetric quantum mechanics localize around these fixed points, and can be computed with equivariant De Rahm cohomology. This indicates BPS states are based around tight binding models which are gauge invariant under a changing crystal momentum – that is, those with a constant band structure.

Remark 8. The fixed points of the torus action are uniquely described as a crystal melting state. Confusingly, those crystals are not related to those described here. Similarly, the quiver quantum mechanics problem used to find the BPS spectrum is essentially unrelated to our quantum mechanics problem, which we use to find the Hamiltonian's spectrum.

7.2. **Topological materials.** It would be remiss to not mention the application of these ideas to topological materials, which have attracted enormous interest in recent decades [17]. These are materials whose properties are protected by some topological invariant. For instance, topological insulators are gaped materials characterized by the topological type of a vector bundle over momentum space, whose fiber is the span of eigenfunctions with energies below the gap. Some topological invariants of the bundle are necessarily trivial in small dimensions. For instance, The second Chern class is a 4-form, so must vanish on a 3-dimensional momentum space. Physical euclidean crystals have this restriction, so cannot realize all types of topological materials. In our situation, momentum space is the moduli space of vector bundles on a genus g curve, which can have arbitrarily large dimension. This is an alternative to constructing a model 4-dimensional euclidean crystal [10], or using synthetic dimensions [48].

We can ask about the topological invariants of this vector bundle over any parameter space of Hamiltonians. Following the theme of this paper, we consider the moduli space of Higgs bundles. This space deformation retracts to the fiber above 0 in the Hitchin base (the nilpotent cone), which thus encodes the topology of the whole space. Any Topological invariants can be measured from the band structure over the nilpotent cone. The nilpotent cone has maximally degenerate spectral curves. In the crystal moduli interpretation, this suggests all topological properties can be seen in the degenerating limit of crystals.

Topological materials ubiquitously exhibit a bulk-edge correspondence, where topological aspects of the bulk band structure determines topologically protected edge states. One can build these edge states using imaginary momenta, whose eigenstates exponentially decay away from the boundary of the crystal. [10]. In the complex momentum interpretation, this role is played by Higgs bundles. The protected edge states should correspond to non contractible cycles in the Bloch variety over Hitchin moduli space.

7.3. **High symmetry momenta.** Crystals in physics often have rich point group symmetries. Much of the standard band theory canon focuses on bands and their crossings at high symmetry momenta, or momenta fixed under large subgroups of the point group. For hyperbolic crystals, the point groups are the automorphisms of the associated Riemann surface [34]. These act on vector bundles by pullback, thus defining an automorphism on the moduli space of vector bundles (momentum space). For degree zero line bundles, the Torelli theorem says any automorphism of $\mathrm{Jac}(\Sigma)$ preserving its principle polarization comes from an automorphism of Σ and possibly the dualizing map $L \mapsto L^*$ (also called the Kummer involution). On the

Jacobian's universal cover $\mathbb{C}^g$ this acts by inversion, $k \mapsto -k$. The symmetries of momentum space are those of position space, with inversion.[9]

On the holomorphic differential (i.e vector potential) A associated to L, the Kummer involution acts by $A \mapsto -A$ so corresponds to time reversal. Since the magnetic field is zero, the Hamiltonian is time-reversal invariant and the band structure is preserved by the Kummer involution. It's fixed points, the 2-torsion points of the Jacobian, are important in band theory. On the universal cover $\mathbb{C}^g$, these are momenta k where $2k$ is a lattice point, which are face centers of the Brillouin zone. They are often called Time Reversal Invariant momenta (TRIMs). For spin-1/2 systems, Kramer's theorem says the Hamiltonian is at least 2-fold degenerate at such points.

We can extend theses symmetries to the moduli space of Higgs bundles $\mathcal{M}^{\text{Higgs}}$. In the crystal momentum interpretation, a fiber of the Hitchin fibration is the Jacobian of a crystal, and the fiberwise time-reversal (Kummer involution) extends to all $\mathcal{M}^{\text{Higgs}}$. For rank 2 trace-free Higgs bundles, this is the holomorphic involution $(E, \phi) \to (E, -\phi)$ [44]. It's fixed points are the $SL(2, \mathbb{R})$ Higgs bundles, first described in [22]. One connected component is the Hitchin section, which corresponds to a trivial line bundle on each spectral curve, thus the zero momentum point of each Jacobian. The $SL(2, \mathbb{R})$ Higgs bundles are highly symmetric points of the extended momentum space $\mathcal{M}^{\text{Higgs}}$.

In general, any automorphism of the base curve gives an automorphism of the Hitchin moduli space, whose fixed points are especially symmetric momenta. These high symmetry loci have an extensive line of mathematical research, see for instance [44, 15, 7, 43, 14]. They are generally branes on $\mathcal{M}^{\text{Higgs}}$, submanifolds classified by their relation to the three kähler structures composing the hyperkähler metric on $\mathcal{M}^{\text{Higgs}}$ [21]. Since this symmetry is reflected in the band structures, it connects with the hyperkähler geometry of $\mathcal{M}^{\text{Higgs}}$. In view of section 7.1, hyperbolic band theory relates in a separate matter to the work of Gaiotto, Moore and Neitzke, and consequently the asymptotics of the hyperkähler metric.

7.4. **Geometric Langlands correspondence.** (**NOTE**: I don't know if this section is worth including, might be too speculative (espically the last part))
The space of abelian crystal momenta for a genus g surface is a g-dimensional complex torus $\text{Jac}(\Sigma) = \mathbb{C}^g/\Lambda$. This is also the momentum space for the dual $2g$-dimensional real torus $\mathbb{R}^{2g}/\Lambda^*$, equivalently a Euclidean crystal in $\mathbb{R}^{2g}$ with lattice Λ^*. The possibility of encoding high dimensional Euclidean crystals in hyperbolic crystals is enticing. In the crystal moduli interpretation of Higgs bundles, it takes a new flavor. There is a global map between the moduli of G-Higgs bundles and that of ${}^LG$-Higgs bundle where ${}^LG$ is the Langlands dual group, which fiberwise sends a torus to it's dual [19, 8]. This is an instance of SYZ mirror symmetry[10], which plays a large role in the geometric Langlands correspondence [12].

Physics has already found its way to geometric Langlands through S-duality of $\mathcal{N} = 4$ supersymmetric gauge theories [27]. The structures on each side of the geometric Langlands correspondence are interpreted as branes on the Hitchin moduli space, mapped to one another by mirror symmetry. In section 7.3, we discussed the crystallographic interpretation of some branes on Hitchin moduli space. These branes are related by mirror symmetry [6], perhaps bringing the geometric Langlands correspondence into the sphere of hyperbolic band theory.

7.5. **Fractional quantum Hall states.** The logical next step with hyperbolic band theory is to apply a magnetic field [46, 26, 1]. Our techniques should carry over without much trouble to integer quantum hall states, where the magnetic flux through a unit cell is an integer multiple N_B of the hyperbolic area. It amounts to replacing degree zero bundles with degree N_B bundles on the Riemann surface. (Alternatively, bundles of nonzero degree arise from projective unitary representations of the fundamental group [3], relating to the magnetic translation operators constructed in [1]). Fractional quantum Hall (FQH) states, where $N_B = q/n$ is rational, are necessarily multi-particle interacting states. One approach uses approximate ground states called Laughlin wavefunctions. These capture the qualitative aspects of FQH states, such as quasiparticles with fractional statistics (**CITE** Does this need a citation?). On compact Riemann surfaces, Laughlin states are holomorphic sections of a line bundle on the n^{th} symmetric product of the surface. [29, 31]. This line bundle is naturally twisted by the n^{th} exterior tensor product of an abelian crystal momentum[30], so Laughlin wavefunctions naturally fit within hyperbolic band theory.

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[9] These are the automorphisms of $\text{Jac}(\Sigma)$ as a principally polarized abelian variety. The crystallographic meaning of the principle polarization or the associated theta divisor are still unclear.

[10] This is a prime example of nominative determinism, as the letters "SYZ" are themselves mirror symmetric.

What about nonabelian crystal momenta? Perhaps these correspond to nonabelian FQH states on Σ, where the quasiparticles are vector-valued. These are best described through an effective Chern-Simons theory on Σ. Nonabelian Laughlin wavefunctions are the conformal blocks of a Chern-Simons theory [36], which are characterized sections of the prequantum line bundle over the Chern-Simons phase space [45]. For complex Chern-Simons theory, this phase space is the character variety for homomorphisms $\pi_1(\Sigma) \to SL(n, \mathbb{C})$, equivalently the moduli space of trace-free Higgs bundles. The holomorphic sections of the prequantum line bundle are nonabelian theta functions, which restrict on each fiber to ordinary theta functions [4].

There are seemingly two ways to obtain FQH states on a hyperbolic crystals. We could follow physics and consider interacting multi-particle states and study their effective field theory. Or, we could construct crystals from spectral covers, construct theta functions on each crystal's Jacobian, then glue those together to a nonabelian theta function on Hitchin moduli space. Guided by aesthetic sensibilities, we guess the resulting states are in some sense *the same*.

7.5.1. *Vortices.* (**NOTE**: I don't know if anything I suggest in this section is convincing. I leave this to your discression.)The effective field theory description also applies to abelian FQH states. One popular choice is the *abelian Higgs model*, which couples the $U(1)$ gauge connection A with a scalar field a represented as a quadratic differential. The Quasiparticles are classical configurations of this theory, which satisfy the abelian vortex equation

$$\bar{\partial}_A a = 0 \tag{30}$$

$$F_A = (1 - |a|^2)\omega, \tag{31}$$

where F_A is the curvature of A, $\bar{\partial}_A$ is the associated holomorphic structure, and ω is the kähler form on Σ. Hitchin noted that this is equivalent to Hitchin's equations for rank 2 Higgs bundles on the Hitchin section [22, §11]. A vortex defines the Higgs bundle

$$\phi = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$$

with 2-fold spectral curve branching at zeros of a, carrying the trivial line bundle. Following the crystal moduli interpretation this defines a family of hyperbolic crystals, hinting at a relationship between abelian FQH states on C and abelian band theory of double covers Σ. The analogy between FQH states and vorticies holds in the nonabelian situation. A nonablian vortex equation is induced from Hitchin's equations by restricting to the fixed points of the natural $U(1)$-action rotating the Higgs field. The moduli space of vorticies is the fixed locus in the Hitchin moduli space, which corresponds to a locus of complex nonabelian crystal momentum.

These effective field theories have a particle-vortex duality, where the vortex solutions are replaced by the fundamental fields and vice versa (**NOTE**: cite). This results from S-duality of Chern-Simons theory, which in gauge theoretic contexts manifests the geometric Langlands correspondence [27]. This has been applied to the integer/fractional quantum Hall effect [25] [24]. Perhaps two hyperbolic particle-vortex dual states corresponds are related by mirror symmetry of the moduli of Higgs bundles.

7.6. **Putting everything together.** The subjects of the last 5 sections are already linked in a well-connected web weaved around Higgs bundles (figure 14). We described how hyperbolic matter concepts tie in with each of these, mediated by Higgs bundles. It connects to the web at several points, but the relationship between these connections is mysterious. It would be wonderful if every path through these fields ended up at the same place, so they all fit in the same story. Perhaps the high-symmetry branes in $\mathcal{M}^{\text{Higgs}}$ match the asymptotic hyperkähler structure seen in the semiclassical large Higgs field limit. Perhaps these branes are mapped to other high-symmetry branes by sending the Jacobian to its Euclidean crystal, reflecting the geometric Langlands correspondence. Perhaps this map relates two families of hyperbolic crystals, whose associated fractional quantum hall states are related by particle-vortex duality. All this is to say, perhaps this diagram (figure 14) commutes.

The only evidence so far is aesthetic, so we cannot call this a conjecture in good conscience. Instead we conclude with:

Daydream 1. *This diagram commutes*

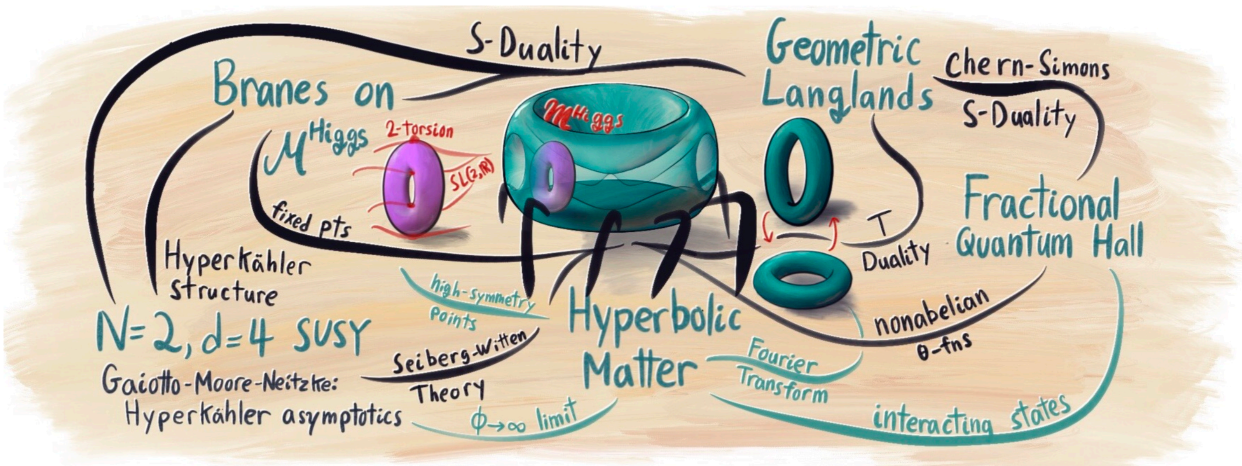

FIGURE 14. (**NOTE**: Rerender this for more friendly layout. Add shadow under $\mathcal{M}^{\text{Higgs}}$ and Jac.)

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