Square roots in supersymmetry

Review: The "supersymmetry algebra" is a supersymmetric extention of The poincare algebra

note: focus on $d=3$ in preparation for 3D mirror symmetry

Poincare algebra: $\underline{iso}(2,1) := \underline{so}(2,1) \ltimes \mathbb{R}^{2,1}$

$[\mathbb{R}^{2,1}, \mathbb{R}^{2,1}] = 0$
$[\underline{so}, \underline{so}] \subset \underline{so}$
$[\underline{so}, \mathbb{R}^{2,1}] \subset \mathbb{R}^{2,1}$

Super poincare algebra: $\underline{siso}(3,1) \simeq \underline{g}_0 \oplus \underline{g}_1$ Lie superalgebra, $\underline{g}_0$ even, $\underline{g}_1$ odd

for any lie superalgebra: $[\underline{g}_0, \underline{g}_1] \subset \underline{g}_1 \Rightarrow$ Ad defines rep $\underline{g}_0 \circlearrowright \underline{g}_1$

Spin-Statistics theorem (Requirement of physics)

$\underline{g}_1$ is a spinor representation of $\underline{so}(2,1)$

Square root #1: spinors $= \sqrt{\text{vectors}}$

anecdote about applicability of $U(1) \subset \mathbb{C}'$ for 2D rotations
$S^3 C$

idea: start with $\mathbb{R}$, & introduce 2 square roots of -1, 1 sq. root of $+1$ $\mathbb{R}^{2,1}$

let $\langle , \rangle$ be the quadradic form w/ signature $+1,-1,-1$ on $V = \mathbb{R}^{2,1}$

clifford algebra $$Cl_{2,1} := \frac{\mathbb{R} \oplus V^{\otimes 1} \oplus V^{\otimes 2} \oplus \cdots}{\langle v \otimes v - \langle v,v \rangle \rangle} = \frac{\mathbb{R}[\overbrace{e_0, e_1, e_2}^{\text{Basis of } \mathbb{R}^{2,1}}]}{\langle e_\mu e_\nu = -e_\nu e_\mu \ (\mu \in 0 \cdots 2),\ e_0^2 = 1,\ e_i^2 = -1 \ (i \in 1 \cdots 2) \rangle}$$

as a vector space, $Cl_{2,1} \simeq \Lambda^* \mathbb{R}^{2,1}$

as an algebra, new multiplication structure: $v \cdot v = \langle v,v \rangle \in \Lambda^0 \mathbb{R}^{3,1} \simeq \mathbb{R}$

in fact, $Cl_{2,1} \simeq \mathbb{C}[2]$ 2×2 complex matricies

$\hookrightarrow Cl_{2,1}$ natural 2-dim'l compex representation Δ - the spinor rep.

Δ has $Cl_{2,1}$-invariant real structure $\Rightarrow$ Defines 2D Real rep S.

The spin group $Spin(2,1)$ is generated by norm 1, invertible elements of $Cl_{2,1}$

$Spin(2,1) \simeq SL(2,\mathbb{R})$ inherits spinor rep S: in fact, S is standard rep of $SL(2,\mathbb{R})$

$Spin(2,1)$ is a simply connected, Double cover of $SO(2,1)$

$\Rightarrow Lie(Spin(2,1)) \simeq Lie(SO(2,1))$, so $\underline{so}(2,1)$ inherits rep. Δ $\boxed{\text{spinor rep of } \underline{so}(2,1)}$

Let V denote standard rep $SO(2,1) \curvearrowright V$

have Symmetric pairing $\Gamma: S^* \otimes S^* \to V$: $V \simeq \mathrm{Sym}^2(S^*)$

this is map $\phi: S^* \otimes V^* \to S$, which eats a vector & spinor & spits out spinor

defined by $\phi(s,v) = v \cdot s$, w/ $v \in V \hookrightarrow Cl(V)$ acting on S by clifford representation

motto $\boxed{\text{The spinor rep is the square root of standard rep}}$

Verbal Remark: all these properties have analogs in other dimensions / signatures

- Complex properties are periodic mod 2 } Bott
- Real properties are periodic mod 8 } Periodicity
- Differences: Real spinors are only easy to find in $n = 1,2,3 \ \%8$
- pairing Γ only exists for signature $(d-1,1)$
- SUSY tends to resists uniform description, makes life hard.

full description of super Poincare algebra:

$\underline{s\mathfrak{iso}}(2,1) \simeq \underline{\mathfrak{g}_0} \oplus \underline{\mathfrak{g}_1}$

$\underline{\mathfrak{g}_0} = \underline{\mathfrak{iso}}(2,1) \simeq \underline{\mathfrak{so}}(2,1) \oplus \mathbb{R}^{2,1}$

$\underline{\mathfrak{g}_1} = \underbrace{S^* \otimes \cdots \otimes S^*}_{\mathcal{N} \text{ copies}}$

e.g, $\mathcal{N}=1$ SUSY has one copy of S^*

Schematics for lie bracket:

choose $J \in \underline{\mathfrak{so}}(2,1)$ $P \in \mathbb{R}^{2,1}$ $Q \in S^*$

$\underbrace{\begin{aligned}&[J,J] \in \underline{\mathfrak{so}}(2,1)\\ &[J,P] \in \mathbb{R}^{2,1}\\ &[P,P] = 0\end{aligned}}_{\underline{\mathfrak{iso}}(2,1)}$

$[P,Q] = 0$

$[J,Q] \in S^*$ spinor rep

$\underbrace{\{Q_1,Q_2\} = \Gamma(Q_1,Q_2) \in \mathbb{R}^{3,1}}$

$Q^2 \in \mathbb{R}^{2,1}$

$[R,Q] \in S^* \otimes \cdots \otimes S^*$

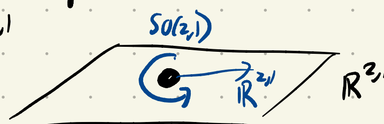
$R \in \mathrm{End}(S^* \otimes \cdots \otimes S^*)$

R-symmetry

Square root #2: Supersymmetry $= \sqrt{\text{translation}}$

Want to represent $\underline{s\mathfrak{iso}}$ as an action on some space

motivation: $\{\text{Poincare group}\} / \{\text{stabalizer}\} \simeq SO(2,1) \ltimes \mathbb{R}^{2,1} / SO(2,1) \simeq \mathbb{R}^{2,1}$



introducing ... super space! $\mathcal{N}=1$

coset space $\{\text{Super-Poincare Grp}\} / SO(2,1)$

affine space based on super-vector space $\overbrace{\mathbb{R}^{3,1}}^{\mathfrak{g}_0} \oplus \overbrace{\Pi S^*}^{\mathfrak{g}_1}$

Parity change

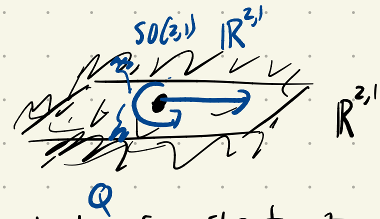
see Freed "5 lectures on super symmetry"

Ring of functions: $\underbrace{\overbrace{\mathbb{C}[x_0,x_1,x_2}^{C^\infty_{\mathbb{C}}(\mathbb{R}^{2,1})}}_{\text{ordinary functions}}, \underbrace{\theta \,]}_{\text{super functions}}$ nilpotent!! $\theta^2 = \bar{\theta}^2 = 0$

θ "grassman number"

$\mathbb{C}[\theta]$ w/ $\theta^2=0$ is grassman (exterior) algebra

<u>Def:</u> super-minkowski space = $\mathrm{Spec}\left(\mathbb{C}[x_0,x_1,x_2,\theta]\right)$



"cloud of nilpotent elements"

Action on superspace

Guiding example: we have action $\mathrm{Isom}(\mathbb{R}^{2,1}) \simeq SO(2,1) \ltimes \mathbb{R}^{2,1} \curvearrowright \mathbb{R}^{2,1}$

infetesimal action defined by vector fields $\underline{iso}(2,1) \longrightarrow \Gamma(T\mathbb{R}^{3,1})$.

$\Rightarrow$ representation by differential operators on fns: $\underline{iso}(2,1) \curvearrowright C^\infty(\mathbb{R}^3)$

supersymmetric extention: $\underline{siso}(3,1)$ acts on super functions $\Phi(x,\theta) \in \mathbb{C}[\vec{x},\theta]$

vector fields induced from left-invariant fields on $SISO(3,1)$

	$\underline{iso}(2,1)$	$\underline{siso}(2,1)$
$\mathbb{R}^{2,1}$	∂_i translation	∂_i translation (real space only)
$\underline{so}(2,1)$	$x_i\partial_j - x_j\partial_i$ rotation	$x_i\partial_j - x_j\partial_i + \theta_b\frac{\partial}{\partial\theta_a} - \theta_a\frac{\partial}{\partial\theta_b}$ rotation (real & super)
S^*	n/a	$D_a = \frac{\partial}{\partial\theta_a} - \Gamma^\mu_{ab}\theta^b\partial_\mu$ rotates between real & super

invariantly, $\Gamma^\mu_{ab}\theta^b\partial_\mu = \sum_\mu \underbrace{\partial_\mu \bullet \theta}_{\text{clifford action}} = \not{\partial}\theta$ ← treating $\theta \in S$ "Dirac operator"

note $\{D_a,D_b\} = D_aD_b + D_bD_a = \cancel{\frac{\partial}{\partial\theta_a}\frac{\partial}{\partial\theta_b}}^{\,0} + \cancel{\Gamma^\mu_{ac}\theta^c\,\Gamma^\nu_{ad}\theta^d}^{\,0}\partial_\mu\partial_\nu - 2\Gamma^\mu_{ab}\partial_\mu + D_bD_a$

$= 2\Gamma^\mu_{ab}\partial_\mu$

or, $D_aD_a = \Gamma^\mu_{aa}\partial_\mu$: $\boxed{\text{Supersymmetry} = \sqrt{\text{translation}}}$

A soup of superfields:

$N=1$: write $\Phi(x,\theta) = f(x) + g(x)\theta$

more invariantly, decompose into irreps

"multiplet"

$$\Phi = \begin{cases} A = \Phi\big| & \text{scalar} \\ \psi_a = D_a\Phi\big| & \text{spinor} \\ F = D^2\Phi = D_aD^a\Phi & \text{scalar} \end{cases}$$

↰ $\theta=0$ part

$\mathcal{N}=2$: $\Psi(\vec{x},\theta^1,\theta^2) = a(x) + b(x)\theta^1 + c(x)\theta^2 + d(x)\theta^1\theta^2$

again: simplify using D^1, D^2: Define $D \equiv (D^1+iD^2)/2$ think ∂

$\bar{D} \equiv (D^1-iD^2)/2$ $\bar{\partial}$

decompose Ψ into irreps

$\bar{D}_a\Phi = 0 \Rightarrow \Phi \in \text{im } \bar{D}_a\bar{D}^a$ "chiral" - holomorphic

$D_a\bar{\Phi} = 0 \Rightarrow \bar{\Phi} \in \text{im } D_a D^a$ "antichiral"

G linear $\Rightarrow G \in \text{im } D_a \bar{D}^a$

Φ is a complexified $\mathcal{N}=1$ real superfield.

splitting into real & imaginary real superfields $\Phi^{i\in 1,2}$, we get a SUSY action

Q_a: $\Phi \mapsto D\Phi$, $\Phi^i \mapsto D_a\Phi^j I^i{}_j$ for $I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ an almost complex structure

Supersymmetry algebra $\Rightarrow$ I is integrable

Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric -Model
Luis Alvarez-Gaume and Daniel Z. Freedman

$\mathcal{N}=4$: 4 $\mathcal{N}=1$ superfields, each belonging to a complex pair.

SUSY algebra $\Rightarrow$ 3 integrable, compatible complex structures

Quaternionic structure!

$\mathcal{N}>4$: pattern continues, $\mathcal{N}$ $\sqrt{-1}$ matrices forming a clifford algebra.

Square root #3: extended SUSY $\Rightarrow \sqrt{-1}$

$\mathcal{N}=2 \Rightarrow$ can choose holomorphic fields $\Rightarrow$ supersymmetry rotates real & imaginary components by 90°

each extra supersymmetry adds a $\sqrt{-1}$