Supersymmetry algebra (to a physicist)

Recall: a Lie Super-algebra is a $\mathbb{Z}_2$ graded vector space $\underline{g} = \underline{g}_0 \oplus \underline{g}_1$ equipped with a $\mathbb{Z}_2$-graded bracket $[,]$ & $\mathbb{Z}_2$-graded Jacobi identity

1) $[e,e'] = -[e',e] \in \underline{g}_0$ for $e, e' \in \underline{g}_0$
2) $[o,o'] = [o',o] \in \underline{g}_0$ for $o, o' \in \underline{g}_1$
3) $[o,e] = [e,o] \in \underline{g}_1$

3) implies $ad_e = [e, \bullet] : \underline{g}_1 \to \underline{g}_1 \Rightarrow ad_- : \underline{g}_0 \to \underline{gl}(\underline{g}_1)$ is a natural representation of $\underline{g}_0$ on $\underline{g}_1$

Physics intuition

Quantum field theory studies "fields" on Riem manifold M^4; "space time" (four-dim'l)

start w/ local model: functions $\mathbb{R}^{3,1} \to X$ "space of fields"

$\mathbb{R}^{3,1} := \mathbb{R}^4$ w/ indefinite signature bilinear form $\langle , \rangle$: orthonormal basis $\langle e_0, e_0 \rangle = -1$, $\langle e_i, e_i \rangle = 1$, $i = 1,2,3$

symmetries of QFT = symmetries of $\mathbb{R}^{3,1}$ $\oplus$ symmetries of X

$\underline{iso}(3,1)$ (lie algebra of isometries) (1a) $\quad \mathcal{I}$ "internal supersymmetry"

What is supersymmetric extention?

Physicist's supersymmetry algebra h

$$\underline{h}_0 = \underline{iso}(3,1) \oplus \mathcal{I}_0 \quad (3)$$
$$\underline{h}_1 = S \oplus \mathcal{I}_1$$

where $\underline{iso}(3,1) \circlearrowright S$ is spin representation (2) $\quad \mathcal{I} = \mathcal{I}_0 \oplus \mathcal{I}_1$

$\mathcal{I}_0 \circlearrowright \mathcal{I}_1$ defines lie superalgebra

note $h = \underbrace{(\underline{iso}(3,1) \oplus S)}_{s\,\underline{iso}(3,1) \text{ super Poincare algebra}} \oplus \mathcal{I}$ (1b)

Def $\underline{siso}(3,1)$: The isometry group of $\mathbb{R}^{3,1}$ is $ISO(3,1) \simeq SO(3,1) \ltimes \mathbb{R}^{3,1}$

Where $SO(3,1)$ acts on $\mathbb{R}^{3,1}$ in standard way

$$\mathrm{Lie}(ISO(3,1)) = \underline{iso}(3,1) \simeq \underbrace{\underline{so}(3,1)}_{\text{is } \underline{sl}(2,\mathbb{C})} \oplus \mathbb{R}^{3,1} :$$

$[\underline{so}, \underline{so}] \in \underline{so}$
$[\mathbb{R}^{3,1}, \mathbb{R}^{3,1}] = 0$
$[\underline{so}, \mathbb{R}^{3,1}] \in \mathbb{R}^{3,1}$

Spinor representations: $\underline{sl}(2,\mathbb{C})$ has weights $\cdots -3/2, -1, -1/2, 0, 1/2, 1, 3/2 \cdots$

by theorem of heighest weight, reps are classified by $n/2 \in \mathbb{Z}/2$

the spinor representation has heighest weight $1/2$.

Construction of spinor rep: Simplest rep of $\underline{so}(3,1)$ not coming from $SO(3,1)$

uses clifford algebras: consider vector space $C\ell(3,1) \simeq \Lambda^*(\mathbb{R}^{3,1})$

w/ product $e_i \cdot e_j = e_i \otimes e_j - \langle e_i, e_j \rangle$: essentially, $\{e_0, e_0\} = 0$, $[e_i, e_j] = 0 \quad j \in 1,2,3$

$C\ell(3,1) \otimes \mathbb{C} \simeq \mathbb{C}[4]$ 4x4 matrices, with define the 4 complex-dim'nl spinor representation $C\ell(3,1) \circlearrowleft S$

$Spin(3,1) :=$ units of $C\ell(3,1)$ w/ norm 1: acts on $C\ell(3,1)$ by conjugation

$Spin(3,1)$ is double cover of $SO(3,1)$, so lie algebra is $\underline{so}(3,1)$

$Spin \circlearrowleft C\ell(3,1) \circlearrowleft S$, so S is the spinor representation of $Spin(3,1)$

infinitessimally gives rep of $\underline{so}(3,1)$ described above.

Physics input:

1. symmetry splits into spacetime $\oplus$ internal, with no mixing
   - 1a. Coleman-Mandula Theorem (no supersymmetry)
   - 1b. Haag-Łopuszański-Sohnius Theorem (w/ supersymmetry)

   Consequence of QFT axioms. Roughly: invariance under $\underline{iso}(3,1)$ is extremely constraining, any other symmetry would enforce a trivial QFT

2. in superpoincare algebra, odd part is spinor representation

   Follows spin-statistics theorem: spin $n \in \mathbb{Z}$ rep $\Rightarrow$ commuting field (even)
   spin $n + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$ rep $\Rightarrow$ anticommuting field (odd)

3. $\mathcal{I}_0$ is compact (Killing form is negative definite)

   ensures there are no "norm zero" states $e \in \mathfrak{g}$ s.t. $\langle e, e \rangle = 0$

Superconformal algebras

https://arxiv.org/pdf/hep-th/9712074.pdf

upgrade poincare algebra $SO(3,1) \ltimes \mathbb{R}^{3,1}$

$\Downarrow$

conformal algebra $SO(4,2) \ltimes \mathbb{R}^{3,1}$

in general: conformal transforms on $\mathbb{R}^{d-1,1}$ are $so(d,2)$

conformal supersymmetry algebra = lie superalgebra w/ $so(d,2) \subset \underline{g}_0$, $\underline{g}_1$ carries spinor rep of $so(d,2)$

for what d does this exist? check Kac classification

Kac's classification

$B_m = \underline{sl}(2m+1, \mathbb{C})$

$D_m = \underline{sl}(2m, \mathbb{C})$

$C_2 \simeq Sp(2) = SO(5)$

$G = G_0 + G_1$	G_0	G_0 Rep on G_1
	d=6 $B(4,n)$	triality: vector rep $\leftrightarrow$ spinor rep
$B(m,n)$ $B(m,2)$, $d=3$	$B_m + C_n$	$vector \times vector$
$D(m,n)$ $D(m,2)$, $d=3$	$D_m + C_n$	$vector \times vector$
$D(2,1,\alpha)$	$A_1 + A_1 + A_1$	$vector \times vector \times vector$
$d=5$ $F(4)$	$B_3 + A_1$ $B_3 = \underline{sl}(7)$	$spinor \times vector$
$G(3)$	$G_2 + A_1$	$spinor \times vector$
$Q(n)$	A_n	$adjoint$

$G = G_0 + (G_1 \oplus G_{-1})$	G_0	G_0 Rep on G_{-1}
$A(m,n)$	$A_m + A_n + C$	$vector \times vector \times C$
$A(m,m)$	$A_m + A_n$	$vector \times vector$
$C(n)$	$C_{n-1} + C$	$vector \times C$

Implication: <u>no</u> superconformal algebras for $d > 6$

Supersymmetries and their representations
W. Nahm

sciencedirect.com/science/article/abs/pii/0550321378902183