Coloumb Branches, Day 1

today we discussed why we care about coloumb branches & organized the future of the seminar.

Question: How do we think about quantum field theory?

Swapnil, John, Chan: a functor Cobd $\longrightarrow$ Vect

Teleman: The category-Theoretic notion of QFT Doesn't yet extend to non-topological QFTS. That would be like assigning a "Length" to each arrow in the category - "metric category theory".
Goes on to describe SUSY QFTs, and some papers we could go though together...

Coleen: a machine which converts pictures to numbers. E.g.
- TQFTs turn Knot diagrams to topological invariants
- Perturbative QFTs turn feynmann diagrams to amplitudes

Elliot: we Define Physics as the field of math studying integrals of the form

$$\int_M e^{S(x)/\hbar}:$$

$S(x)$ is the action
$\hbar$ is quantum parameter
M is some (probably undefined) space w/ measure, e.g the space of connections on a principle bundle G.

- Looking @ $\hbar \to 0$ gives critical points of $S(x)$
  - ↳ For $S(A) = F_A \wedge * F_A$, where A is a connection & F_A curvature the extrema satisfy $d_A * F_A = 0$ (Yang-Mills equations)
- the solutions to this are classical field theory.

- Quantum field theory incorporates all possible solutions weighted according to $e^{S(x)}$.

> QFT is a collection of heuristics & intuitions which describe how to solve $e^{S(x)}$, sometimes with very deep relations between different ways of writing the problem.

- sometimes, these predict relationships between mathematically precise objects - thats where physics influences math.

Primer on Coloumb branches

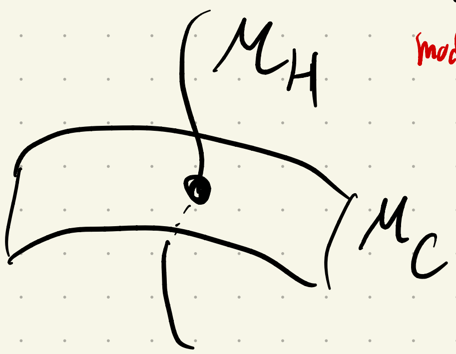
We'll be learning about coloumb branches, a structure arising from a supersymmetric quantum field theory.

Physically, this represents a sector of the moduli space of vacua, forming a (possibly singular) manifold.

formed by Renormalization: Start w/ a SUSY (supersymmetry) QFT, and zoom way out. then, the solutions are forced to be low energy. a moduli space of vacua

SUSY $\xrightarrow{\text{renormalize}}$ $\mathcal{M}$ moduli space of vacua

$\mathcal{M} = \mathcal{M}_H$ "+" $\mathcal{M}_C$ two components — Higgs branch $\mathcal{M}_H$ & Coloumb branch $\mathcal{M}_C$ connecting at a point.

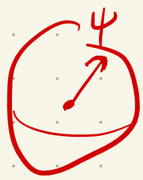


moduli space example: $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$, $S(\varphi) = \|d\varphi\|^2 + (\|\varphi\|^2 - 1)^2$

Global minima: $d\varphi = 0$, $\|\varphi\| = 1$

$\Rightarrow$ φ is constant unit vector.

moduli space of solutions is S^2

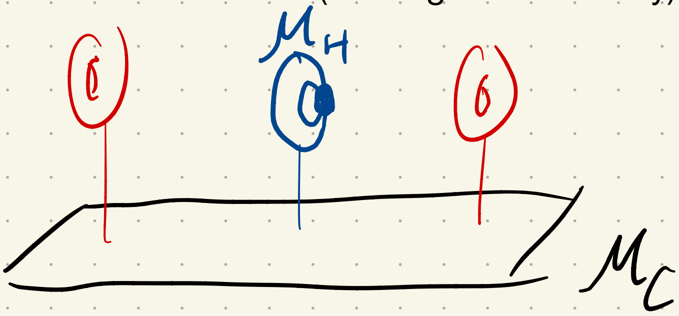


this arises as a moduli space of vacua for certain Yang-Mills equations, which arise from supersymmetry. This is classical - not $\mathcal{M}_H$ or $\mathcal{M}_C$

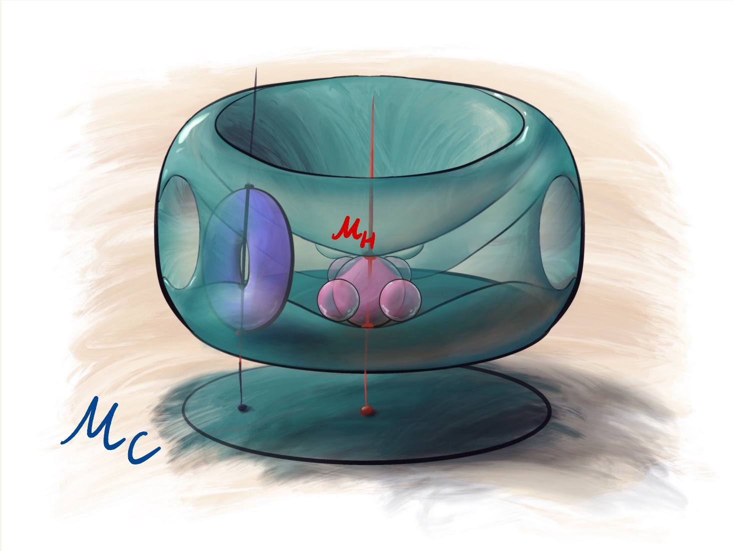
Supersymmetry endows the coloumb branch with all sorts of structure, like a hyperkahler metric. Historically, we understand this only thru specific examples, but those examples are rich and varied. They include:

torus fibrations (Seiberg-Witten theory)

$\mathcal{M}_C$ is the base of a holomorphic symplectic torus fibration. The Higgs branch $\mathcal{M}_C$ is a singular fiber



Here's another example, coming from Hitchin systems:



This structure shows the integrable aspects of SUSY, which connects it with conformal field theory, algebraic geometry, & many other areas

coloumb branches also form quiver varieties of various forms:
- ALE hyperkahler spaces a.la kronheimer
- gauge theoretic moduli spaces of instatons (The Atiyah-Hitchen moduli space of su2 instatons, moduli spaces of ASD yang-mills connections, etc)
- springer resolutions and other critters from geometric representation theory

It also ties to topological quantum field theory. Each supersymmetric algebra has 2 topological twists, & the resulting theories are mirror dual with swapped Higgs & coloumb branches. These can coincide w/ well known topological invariants, like Reshitikian-Turyev invariants on a 3-mfld

$$RT(\mathcal{M}_C, M^3) \xleftrightarrow[\text{symmetry}]{\text{mirror}} I(\mathcal{M}_H, M^3)$$

some other topological invariant built from M^3 & the higgs branch

These examples suggest that coloumb branches are more than mere hyperkaheler manifolds. They usually connect to gauge theoretic moduli spaces. They often connect to geometric representation theory. They have subtle dualities (Mirror symmetry) which shed new light on gauge theory and representation theory.

In short, Coloumb branches manifest the magic of supersymmetry

All this begs for a general, unifying desecription coloumb branches and their geometry. In this seminar, we will try to understand the couloumb branches of 3d n=4 supersymmetric quantum field theories

PLans:
- about 50% of people are here because they hope to study things related to coloumb branches. The other 50% are here for cultural enrichment
- We will go over the basics & try to actually understand things
- Will meet 2-3, go on break for tea, then resume for 3:30-4:00

Future talks:

Next week: introduction to supersymmetry
follow HKLR "Hyperkahler metrics & supersymmetry"

possibilities for future talks:

- Seiberg-Witten theory & the first coloumb branch
- Review Quivers & ADHM construction
  - ↳ Kronhiemer ALE hyperkahler metrics
  - ↳ 3D mirror symmetry for these spaces (Seiberg-Intrilligator)

more to decide later.