What's the deal with BPS?

BPS Equations?

BPS state?

BPS monopole?

BPS solitons?

BPS invariants?

BPS Black holes?

BPS algebra?

BPS Quiver?

BPS lumps?

BPS thermodynamics?

BPS Limit?

BPS Charges?

BPS Operators?

BPS Cohomology?

BPS Moduli spaces?

BPS Branes?

BPS Strings?

BPS Particles?

BPS Structure?

BPS Crystal?

BPS vortex?

BPS q-series?

BPS indicies?

BPS Skyrmions?

BPS spectrum?

BPS Corner?

BPS Graphs?

BPS Bound?

BPS wilson loop?

Part 1: Representation theory

Source: Neitzke BPS Lectures, Day 1

Supersymmetry: Super-Lie algebra ($\mathbb{Z}_2$-graded) $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$

We want super-reps $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$

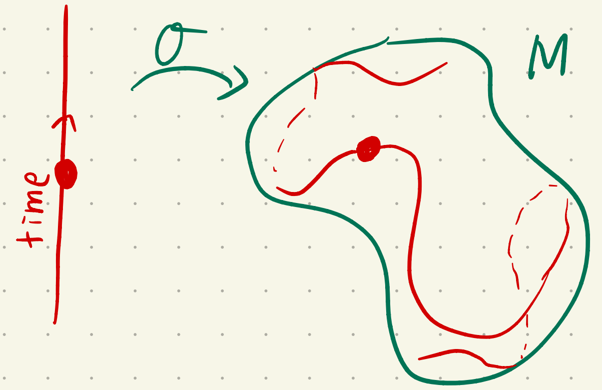
Roughly: a BPS state lives in an irrep anhilated by part of $\mathcal{A}_1$

Vibes - supersymmetric quantum mechanics

i.e $\underset{\text{space}}{0} + \underset{\text{time}}{1}$-D $\mathcal{N} = 1$ nonlinear sigma model ordinary QM: $\mathcal{H} = L^2(M)$

$H = \Delta = d^* d$

hamiltonian laplacian



supersymmetric QM:

$\mathcal{A} = \overset{\mathcal{A}_0}{\mathbb{C}^2} \oplus \overset{\mathcal{A}_1}{\mathbb{C}^2}$ Generators: $\mathcal{A}_0 = \langle H, Q\bar{Q} \rangle$, $\mathcal{A}_1 = \langle Q, \bar{Q} \rangle$

Relations: $[Q, \bar{Q}] = H$, all else vanish

$a \wedge d = d \wedge a$ even

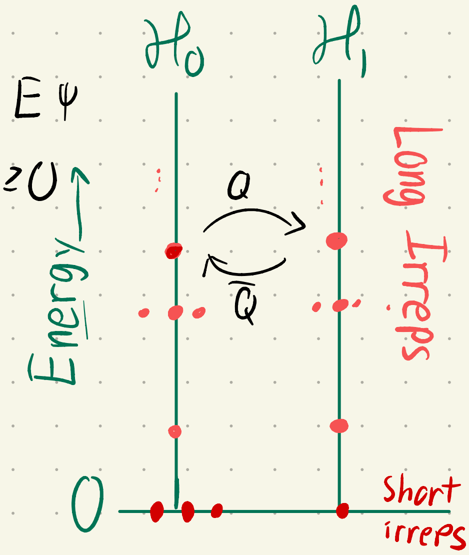
$a \wedge d = -d \wedge a$ odd

representation: $\mathcal{H} = \underbrace{\Omega^\bullet(M)}_{\text{forms}} = \bigoplus_k \underbrace{\overset{\mathcal{H}_0}{\Omega^{2k}(M)}}_{\text{even deg.}} \oplus \underbrace{\overset{\mathcal{H}_1}{\Omega^{2k+1}(M)}}_{\text{odd deg.}}$

$\underset{\Omega^k \mapsto \Omega^{k+1}}{Q = d}$ $\underset{\Omega^k \mapsto \Omega^{k-1}}{\bar{Q} = d^*}$ $H = d^* d + d d^*$ hodge laplacian!

irreps: classified by action of center $\langle H \rangle$ $H\psi = E\psi$

note $\langle \psi, H\psi \rangle = \langle \psi, Q\bar{Q}\psi \rangle + \langle \psi, \bar{Q}Q\psi \rangle = \|\bar{Q}\psi\|^2 + \|Q\psi\|^2 \geq 0 \Rightarrow E \geq 0$



$E > 0$: $HQ\psi \overset{[Q,H]=0}{=} QH\psi = EQ\psi$

$H\bar{Q}\psi = \bar{Q}H\psi = E\bar{Q}\psi$

Vector space generated by ψ: span$(\overset{\psi}{\psi}, \overset{d\psi}{Q\psi}, \overset{d^*\psi}{\bar{Q}\psi}, \overset{dd^*\psi}{Q\bar{Q}\psi})$

acted on by clifford algebra of 2 elements

unique irrep w/ dimension 2

$E = 0$ (saturates bound): $(d^*d + dd^*)\psi = 0 \Leftrightarrow d\psi = d^*\psi = 0$ hodge theory

ψ & $Q\psi$ in same vect space!

anhilated by $Q, \bar{Q}$ $\Rightarrow$ irrep has dimension 1

in fact, 1-D irreps give cohomology of M! $\ker \Delta: \Omega^k \to \Omega^k \simeq H^k(M)$

- rigid under deformations

these are "~~BPS~~" states "super symmetric"

Vibes: irrep saturate bound $\Rightarrow$ smaller dimension

$\Rightarrow$ more rigid $\Rightarrow$ Topological info

$\mathcal{N}=2$ $d=4$ SUSY

symmetries of spacetime: $\mathbb{R}^{3,1}\rtimes Spin(3,1)$, algebra $\mathfrak{iso}(3,1)$

Coleman-Mandula theorem: consistant QFT has symmetry $\mathfrak{iso}(3,1)\times\mathfrak{g}$

Haag-Łopuszański-Sohnius thm: supersymmetric QFT algebra $\mathcal{A}=\mathcal{A}_0\oplus\mathcal{A}_1$

$\mathcal{A}_0=\mathfrak{iso}(3,1)\oplus\mathbb{C}\oplus\mathfrak{g}$ $\quad\mathcal{A}_1=\langle Q^1,\bar{Q}^1,Q^2,\bar{Q}^2\rangle$

Where $\mathcal{A}_0$ acts on $\mathcal{A}_1$ via $Spin(3,1)\simeq su(2)\oplus su(2)$ $\quad Q^1,Q^2\in S^+$

Q spin $1/2$, live in representations $\boxed{S^+}$ or $\boxed{S^-}$ $\quad\bar{Q}^1,\bar{Q}^2\in S^-$

$[Q^1,Q^2]=Z$ $\qquad[\bar{Q}^1,\bar{Q}^2]=\bar{Z}$ $\qquad\boxed{[Q^1,\bar{Q}^1]=[Q^2,\bar{Q}^2]=P\to M}$

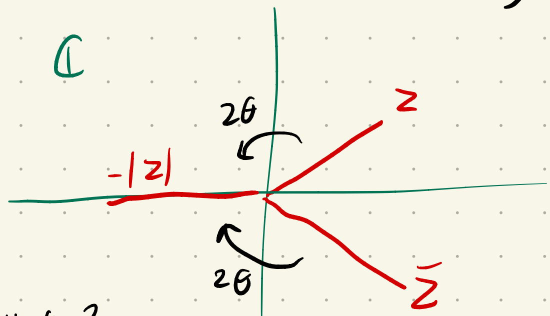
$Z\in\mathbb{C}\subset\mathcal{A}_0$
"central charge"

P momentum, transforming as a vector under $Spin(3,1)$

Reps classified by action of central element of rep:
Casimir invariant P^2. Say P^2 acts by M^2 (mass)
Choose rest frame, w/ $P=(M,0,0,0)$
Q: Bound on M?

trick: rotate Q^1 into $\bar{Q}^2$ $\qquad Q^\theta=\frac{1}{\sqrt{2}}(e^{i\theta}Q^1+e^{-i\theta}\bar{Q}^2)$

$[Q^\theta,\bar{Q}^\theta]=M+\frac{1}{2}(e^{2i\theta}Z+e^{-2i\theta}\bar{Z})$
$=M-|Z|$
@ optimal angle



BUT... $Q^\theta=\bar{Q}^{\theta\,\dagger}\Rightarrow\langle\psi|[Q^\theta,\bar{Q}^\theta]\psi\rangle=\|Q^\theta\psi\|^2+\|\bar{Q}^\theta\psi\|^2\geq 0$
a. la hodge theory

so $M-|Z|\geq 0\Rightarrow\boxed{M\geq|Z|}$ BPS Bound

w/ $M=|Z|$ whenever $Q^\theta\psi=\bar{Q}^\theta\psi=0$ BPS state

Non-BPS state: Ψ acted on by clifford algebra $\{Q^1, \bar{Q}^1, Q^2, \bar{Q}^2\}$ 8 elements

including spinor indicies, irreps dimension $2^{8/2} = \underline{16}$ Long irrep

BPS state: 2 of the 4 generators of $\mathcal{A}$, anhilate Ψ

$\Rightarrow$ Ψ acted on by 2-generator Clifford algebra (4 w/ spinors)

$\Rightarrow$ irrep dimension $2^{4/2} = \underline{4}$

Part 2: monopoles

introduction to S-Duality in $N=2$ field theories

BPS: Bogomol'nyi - Prasad - Sommerfield

Worked on monopoles, not supersymmetry!

classical Yang-mills theory: manifold $\mathbb{R}^{3,1}$ w/ G-bundle
field: Connection D_A, curvature $F \in \Omega^2(\mathrm{ad}(G))$

$\mathcal{L} = \|F\|^2 = \mathrm{tr}\, F \wedge *_4 F \xRightarrow[\text{lagrange}]{\text{Euler}} D_A *_4 F = 0$ YM eqs

$*$ on $\mathbb{R}^{3,1}$

e.g electromagnetism $G = U(1)$: F 2-form, $D_A = d_4$

look for static solution $\frac{\partial}{\partial t} F = 0$:

split space & time: $\mathbb{R}^{3,1} = \mathbb{R}^3 \times \mathbb{R}^1$

$$F = \mathcal{B} + E \wedge dt$$

$*$ on $\mathbb{R}^3$

$\mathcal{B} = *B$ magnetic field two-form

$\mathcal{E} = *E$ electric field two-form

d on $\mathbb{R}^3$

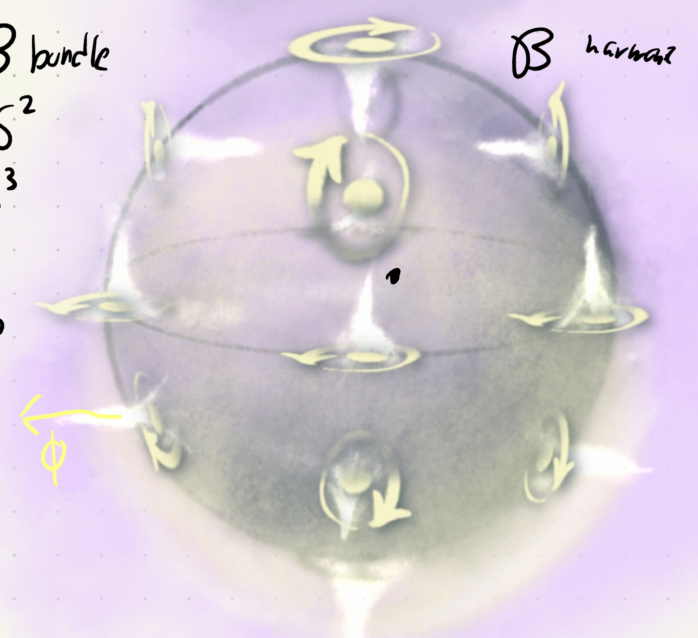
YM eqs: $d\mathcal{B} = d*\mathcal{B} = 0$
$d\mathcal{E} = d*\mathcal{E} = 0$

We want monopoles! solutions w/ non zero electric / magnetic charge, w/ $e = \int_{S^2@\infty} \mathcal{E}$, $m = \int_{S^2@\infty} \mathcal{B}$

But this is cohomologically impossible: needs a puncture / field singularity

alternativly, $\mathcal{B}$, $\mathcal{E}$ curvatures of connections on seperate $U(1)$ bundles on $\mathbb{R}^3$
($\mathcal{B}$ inherits F's $U(1)$ bundle, $\mathcal{E}$ uses $*F$'s bundle)
e, m are 1st Chern class of bundles @ S^2_∞

for now: suppose $\mathcal{E}=0$, only consider $\mathcal{B}$ bundle $L \to S^2$



U(1) Bundle of magnetic charge 2

issue: L topologically nontrivial, doesn't extend to $\mathbb{R}^3$

solution (t'Hooft-Polyakov):

extend across 0 by enlarging Gauge group

realize U(1) bundle as subbundle of trivial SO(3) bundle $\tilde{L} \to \mathbb{R}^3$

use **Higgs Mechagnism**

$$\mathcal{L} = \|\tilde{F}\|^2 + \|D_{\tilde{A}}\phi\|^2 - V(\phi)$$
$$= \mathrm{tr}\left(\tilde{F} \wedge * \tilde{F} + D_{\tilde{A}}\phi \wedge * D_{\tilde{A}}\phi\right) - V(\phi)$$

Yang-Mills-Higgs Lagrangian

for SO(3) connection $\tilde{A}$ w/ curvature $\tilde{F}$

the higgs field ϕ is a section of $\tilde{L}$'s adjoint bundle ($\phi(x) \in \underline{so}(3)$)

($\vec{\phi}$ transforms as a vector in the adjoint rep of SO(3))

energy density is $T_{00} = \|\tilde{F}\|^2 + \|D_{\tilde{A}}\phi\|^2 + V(\phi) \geq 0$ if $V(\phi) \geq 0$

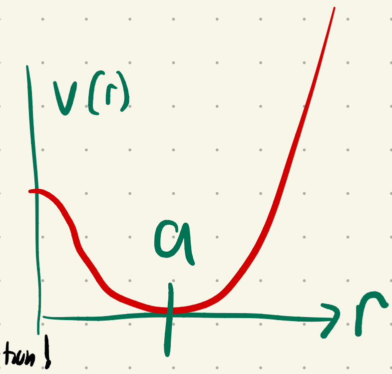
a finite energy field configuration has $T_{00} \to 0$ as $r \to \infty$

$T_{00}=0 \Rightarrow \tilde{F}=0, \quad D_{\tilde{A}}\phi=0, \quad V(\phi)=0$

Vacuum manifold $\mathcal{M}$: moduli of solutions to $T_{00}=0$. $\tilde{F}=D_{\tilde{A}}\phi=0 \Rightarrow \phi$ const, so $\mathcal{M}=V^{-1}(0)$

Choose $V(\phi) = \lambda(\mathrm{tr}(\phi^2)-a^2)^2$: $V(\phi)=0 \Rightarrow \mathrm{tr}\,\phi^2=a^2$,

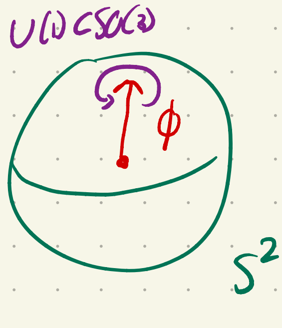
$\mathcal{M} = V^{-1}(0) = S^2$



as $r \to \infty$, field energy $\to 0$, so field configuration $\to \mathcal{M}$.

Spontanious symmetry breaking: $\phi \in S^2$ chooses a direction!

Residual symmetry is stabilizer of $\phi = U(1) \subset SO(3)$



choice of ϕ on S^2_∞ fixes stabalizing U(1) bundle L over S^2_∞: $m = c_1(L) = \deg(\phi: S^2_\infty \to S^2)$

In terms of curvature: magnetic field 2-form $\mathcal{B} = \overset{\text{invariant inner product}}{\langle \phi, \tilde{F} \rangle}$ $\tilde{F}$: $so(3)$-valued 2-form

Check YM-equations @ ∞, using $D_A\phi = 0$

$d\mathcal{B} = d\langle \phi, \tilde{F}\rangle = \langle \cancel{D_A\phi}, \tilde{F}\rangle + \langle \phi, \cancel{D_A\tilde{F}}\rangle = 0$ ✓

$d{*}\mathcal{B} = d\langle \phi, *\tilde{F}\rangle = \langle \cancel{D_A\phi}, *\tilde{F}\rangle + \langle \phi, \cancel{D_A{*}\tilde{F}}\rangle = 0$ ✓

Symmetry Breaking from $G \to H$: ϕ valued in symmetric space G/H

or, Reduction of structure group G-bundle $\to$ H-bundle, classified by global sections of quotient G/H-bundle. either way, ϕ classified by $\pi_2(G/H)$

Dyons: need ϕ for each $\mathcal{B}, \mathcal{E}$. take complex $\phi \in \Gamma(\underline{ad}(\tilde{L}) \otimes \mathbb{C})$ section of complexified adjoint bundle

then, $F = \langle \phi, \tilde{F}\rangle$ decomposes as $F = \mathcal{B} + i\mathcal{E}$

Total charge: $m + ie = \int_{S^2_\infty} \mathcal{B} + i\mathcal{E} \overset{\text{Stokes}}{=} \int_{\mathbb{R}^3} d\mathcal{B} + i d\mathcal{E} = \int_{\mathbb{R}^3} d\langle \phi, \tilde{F}\rangle = \int_{\mathbb{R}^3} \langle D_{\tilde{A}}\phi, \tilde{F}\rangle$

Q: what is "mass" (total energy)? $M = \int T_{00} = \int \|\tilde{F}\|^2 + \|D_{\tilde{A}}\phi\|^2 + V(\phi)$

Rotate electric & magnetic charges into one another:

$\|\tilde{F} + e^{i\theta} D_{\tilde{A}}\phi\|^2 = \|\tilde{F}\|^2 + \|D_{\tilde{A}}\phi\|^2 + e^{i\theta}\langle D_{\tilde{A}}\phi, \tilde{F}\rangle + e^{-i\theta}\langle D_{\tilde{A}}\bar{\phi}, \tilde{F}\rangle = \|\tilde{F}\|^2 + \|D_{\tilde{A}}\phi\|^2 + \mathrm{Re}\, e^{i\theta}\langle D_{\tilde{A}}\phi, F\rangle$

$$M = \int \|\tilde{F} + e^{i\theta} D_{\tilde{A}}\phi\|^2 + V(\phi) + \mathrm{Re}\, e^{i\theta}\langle D_{\tilde{A}}\phi, \tilde{F}\rangle \geq \int \mathrm{Re}\{e^{i\theta}\langle D_{\tilde{A}}\phi, \tilde{F}\rangle\}$$

$\geq \mathrm{Re}\{e^{i\theta}(m + ie)\}$ $\forall\ \theta$. choosing optimal θ, we get

$$\boxed{M \geq \sqrt{e^2 + m^2}}$$ BPS Bound!

Saturate BPS bound $\Rightarrow$ global minima of YM action $\int \|\tilde{F}_4\|^2$ on $\mathbb{R}^{3,1} \simeq S^3 \times \mathbb{R}$

$\Rightarrow \tilde{F}_4$ is (anti-) self-dual $\Rightarrow$ static solutions satisfy Bogolomnyi equations:

$\tilde{F} = *D_{\tilde{A}}\phi$ - BPS state

Bottom line:

- can make $U(1)$ monopoles w/o singularities, By breaking $SO(3)$
- topological twisting @ ∞ = electric & magnetic charge
- Twisting enforces minimal total energy - BPS bound
- BPS states satisfy nice diffeqs ☺

Part 3: Putting it together

Olive-Witten 1978

Why are these both called the same thing?

they both used the trick rotating charges into one another – there must be more

Like $N=1$ example: $Q \leftrightarrow d$, BPS $\leftrightarrow \ker \Delta$

Represent SUSY in superfields: Superspace formalism

super spacetime $= \mathbb{R}^{3,1} \times \underbrace{Cl(\theta^1, \bar\theta^1, \theta^2, \bar\theta^2)}_{\text{Clifford algebra}}$

$N=2$ is 1 chiral superfield + 1 vector superfield (superfns on superspace)

fields:

spin 1	A_μ	gauge field (curvature F)	F
spin ½	λ	ψ fermions	& auxiliary fields (integrated out)
spin 0	ϕ	scalar Higgs field	D

$$\mathcal{L} = \int \underbrace{\tfrac{1}{g^2}\|F\|^2}_{YM} + \underbrace{\|D_A\Phi\|^2}_{\text{kenetic higgs, coupled to gauge}} - \underbrace{\mathrm{Tr}\,[\phi,\phi^\dagger]^2}_{V(\Phi)}$$

Yang-Mills-Higgs

$$- \underbrace{i\bar\lambda D_A\lambda - i\bar\psi D_A\psi}_{\text{free fermions}} - \underbrace{i[\bar\lambda,\bar\psi]}_{\text{Yukawa coupling}}\phi + h.c$$

SUSY terms

$= \mathrm{Im}\,\mathrm{Tr}\int \tau\Psi^2$ for $N=2$ chiral superfield Ψ

$\Phi = \phi + i\phi^\dagger$ is complex higgs field, ϕ scalar $\Rightarrow \mathcal{E}$; $\phi^\dagger$ pseudoscalar $\Rightarrow \mathcal{B}$

Global energy minimizers: $\lambda = \psi = F = 0$, $D_A\Phi = 0$, $V(\Phi) = 0$

for $G = SO(3)$, $[\phi,\phi^\dagger] = 0$ iff $\phi = \lambda\phi^\dagger$. Vacuum Manifold $\mathcal{M} \simeq \mathbb{R}P^{2}$

equivently, $\mathcal{M}$ chooses a cartan subalgebra of $\mathfrak{so}(3)$

$\Rightarrow$ associated $U(1)$ bundle is choice of maximal torus in $SO(3)$

$\mathbb{R}P^2 = S^2/\pm 1$, so $\pi_2(\mathbb{R}P^3) \simeq \mathbb{Z}$

Classical solutions of $SO(3)$ $N=2$ $\Leftrightarrow$ t'Hooft-Polyakov monopole

SUSY generators as differential operators:

$$Q^I = \frac{\partial}{\partial \theta^I} - i\,\theta^I \cancel{\partial} \qquad \bar{Q}^I = \frac{-\partial}{\partial \bar\theta^I} + i\,\cancel{\partial}\,\bar\theta^I$$

vector field: rotates real space into superspace

noether's theorem $\Rightarrow$ conserved current

$$S^1_\mu = F\gamma_\mu\lambda + (D_A\phi\,\gamma_\mu + D_A\phi^\dagger\gamma_5)\psi + \gamma_\mu\gamma_5[\phi,\phi^\dagger]\lambda$$

$S^2_\mu = S^1_\mu$ acted on by R-symmetry $\binom{\lambda}{\psi} \mapsto \binom{\psi}{-\lambda}$

w/ $Q^1 = \int_{\mathbb{R}^3} S^1_0$, $Q^2 = \int_{\mathbb{R}^3} S^2_0$

$$Z = [Q^1, Q^2] = \int_{\mathbb{R}^3}[S^1_0, S^2_0] = \ldots = \int_{\mathbb{R}^3} d(\langle\phi, F\rangle + \gamma_5\langle\phi^\dagger, F\rangle) = \delta e + \delta_5 m$$

$\Rightarrow$ E-charge $e = g\cdot n_e$, B-charge $m = \frac{1}{g}\cdot n_m$

$$\boxed{M \geq |Z| = \left|-i\langle\phi\rangle\left(n_e + i\frac{4\pi}{g^2}n_m\right)\right|}$$

Shows classical BPS Bound Holds in Quantum theory

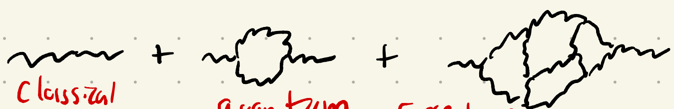
in general, for $\mathcal{L} = \operatorname{Im}\operatorname{Tr}\int \underbrace{\mathcal{F}}_{\text{holo. polynomial}}(\Psi)$, $Z = a\,n_e + a_D\,n_m$

$a_D = \partial\mathcal{F}/\partial a$

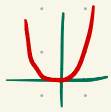
Classically, all particles are BPS monopoles

e.g. W-boson has charge e, classical mass $\langle\phi\rangle e$, so $M = \langle\phi\rangle|z|$

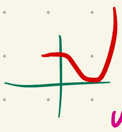
Q: what are quantum masses of particles?

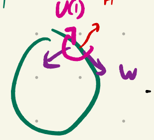
this is hard :($m(p)$ = 

classical quantum corrections

but... for $a^2 = 0$, $V(\phi) =$  $\Rightarrow$ no symmetry breaking!

particles: tangent space to $\langle\phi\rangle = 0$ is $\underset{\substack{\text{value of } a^2\\ \text{massless scalar higgs}}}{\mathbb{R}} \oplus \underset{\text{massless vector gauge}}{\mathfrak{so}(3)}$

Break symmetry: $\langle\phi\rangle > 0$:  $\underset{\text{massive scalar higgs}}{\mathbb{R}} \oplus \underset{\substack{\text{massive vector gauge}\\ W^+, W^-,\\ \text{charge } e}}{T_\phi S^2} \oplus \underset{\substack{\text{massless vector gauge}\\ \text{unbroken symmetry,}\\ \text{photon}}}{\mathfrak{u}(1)}$



dim irrep of massless $W^\pm$ is 4

if mass $W^\pm > \langle\phi\rangle e$, then dimension irrep is 16! $\Rightarrow$ mass $= \langle\phi\rangle e$

i.e - BPS states rigid under deformation.

Once BPS state (e.g massless) $\Rightarrow$ always BPS state.

$\Rightarrow$ We know mass Exactly! No quantum corrections!

Takeaways:

Supersymmetric state $\Leftrightarrow$ satisfies diffeq

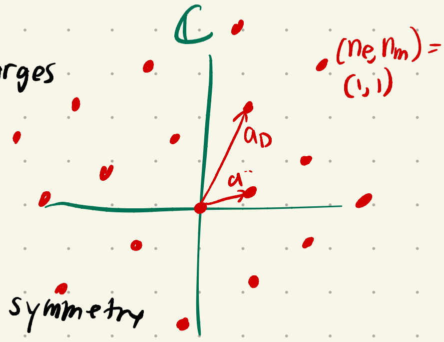
minimal mass (BPS) $\Leftrightarrow$ nice rep, nice diffeq

$N=2$ SUSY $\approx$ Monopoles, central charge $Z \approx$ electric / magnetic charge

Deformation invariance of BPS $\Rightarrow$ Powerful, exact infrences

Looking forward:

$Z = a n_e + a_D n_m$ w/ $(n_e, n_m) = \#$(electric, magnetic) charges



possible central charges form lattice in $\mathbb{C}$

Electric / Magnetic Duality: $a, a_D \mapsto -a_D\ a$ exact quantum symmetry

So, charges only defined up to $SL(2,\mathbb{Z})$: All is in lattice

a & a_D coupling constants, value depends on $u = \langle \mathrm{tr}\, \phi^2 \rangle$

So, family of lattices for $u \in \mathbb{C}$ Integrable System!

When a, a_D give degenerate lattice, $\exists\, n_e, n_m$

s.t Z is zero. $M \geq |Z| = 0 \Rightarrow \exists$ massless dyons @ singularities of lattice family

Larger gauge group: $V = [\phi, \phi^\dagger]^2$ breaks symmetry to stabalizer

a cartan. Associated grp is max torus $U(1)^r$ (r rank), charges $\vec{n}_e, \vec{n}_m$

$Z = \vec{a} \cdot \vec{n}_e + \vec{a}_D \cdot \vec{n}_m$ 2r-Dimnl lattice = Higher Dimension complex torus

$\vec{a}, \vec{a}_D$ periods of higher genus Riemann surface (chiral) super fields: $\Phi(x,\theta) = \underset{\text{spin } 0}{\overset{\text{higgs field}}{A(x,\theta)}} + \underset{\text{spin } 1/2}{\overset{\text{fermionic}}{\theta\psi(x,\theta)}} + \underset{\text{spin } 1}{\overset{\text{Auxiliary field}}{\theta\theta F(x,\theta)}}$

(vector) super fields: $V(x,\theta \cdot\cdot) = \underset{\text{Gauge field}}{\overset{\text{spin } 1}{\theta\bar{\theta} A_\mu}} + \theta^2\bar{\theta}\lambda + \theta^2\bar{\theta}^2 D$

$$\Big(\underset{\substack{\text{higgs}\\ \text{field}}}{\overset{\text{spin } 0}{\underline{\phi}}},\ \overset{\text{spin } 1/2}{\psi, \lambda},\ \underset{\substack{\text{Gauge field,}\\ \text{curvature } F}}{\overset{\text{spin } 1}{\underline{A_\mu}}},\ \underset{\substack{\text{Auxiliary fields,}\\ \text{integrated out}}}{\underline{F, D}} \Big)$$

Seeing the world thru BPS states

Fact: (100% true, no opinions)
BPS states are the primary way that Supersymmetry (& hence all physics) influences modern mathematics

today: BPS $\Rightarrow$ seiberg-witten solution
BPS $\simeq$ complex geometry (?)

Next week: Everything was BPS states all along!

Seiberg-Witten solution

Exact, low energy effective theory (LEET) for
$\mathcal{N}=2$ Supersymmetric $SU(2)$ Yang-Mills

Review: $\mathcal{N}=2$ SUSY $\Rightarrow \mathcal{L} = \operatorname{Im}\int \mathcal{F}(\Psi)$ — Ψ holds all $\mathcal{N}=2$ fields, $\mathcal{F}$ holomorphic "Pre-Potential"

high energy (free) theory: $\mathcal{F}(z) = z^2$

low energy: $\mathcal{F} = \underbrace{\mathcal{F}_{1\text{-loop}}}_{z^2 \ln\left(\frac{z^2}{\Lambda^2}\right)} + \underbrace{\sum \mathcal{F}_{\text{instaton}}}_{\mathcal{F}_k \cdot \left(\frac{\Lambda}{z}\right)^k}$

Perturbitavly exact — Quantum corrections (hard :!)

Seiberg-Witten 1994: indirectly find $\mathcal{F}$ using BPS states

(1) free theory BPS states:

clasically:

Ψ contains W_μ (vector multuplet): A_μ gauge field (curvature F), ψ fermions

Φ (Chiral multuplet): λ fermions, ϕ scalar Higgs field

(& Auxiliary fields)

(note: notation changed to mostly follow Seiberg-Witten 1994)

$$\mathcal{L} = \operatorname{Im}\int \mathcal{F}(\Psi) = \operatorname{Im}\left[\int \mathcal{F}'(\Phi)\bar{\Phi} + \int \underbrace{\mathcal{F}''(\Phi) W_\mu W^\mu}_{N=1 \text{ YM w/ coupling constant } \tau = \mathcal{F}'' \ (\operatorname{im}\tau > 0)}\right]$$

$\mathcal{F}(z)=z^2$:

$$= \int \underbrace{\tau F \wedge * F}_{\text{YM}} + \underbrace{\|D_A \phi\|^2}_{\text{kenetic higgs, coupled to gauge}} - \underbrace{\operatorname{Tr}[\phi, \phi^\dagger]^2}_{V(\Phi)} \quad \text{Yang-Mills-Higgs}$$

$$- \underbrace{i\bar{\lambda} D_A \lambda - i\bar{\psi} D_A \psi}_{\text{free fermions}} - \underbrace{i[\bar{\lambda}, \bar{\psi}]}_{\text{Yukawa coupling}}\phi + \text{h.c} \quad \text{SUSY terms}$$

Classical solution = Static Yang-mills-higgs = **MONOPOLE!**

low energy ⇒ spontanious symmetry breaking $SU(2) \to U(1)$ @ ∞

$F \mapsto \overset{\text{electric}}{E} + i \overset{\text{magnetic}}{B}$, monopole charges $n_e = \int_{S^2_\infty} E$ $\quad n_m = \int_{S^2_\infty} B$ (topological!) integers!

energy $\boxed{\mathcal{E} \geq |n_e + \tau n_m|}$ "BPS-Bound"

Quantumly:

statement about representations of SUSY algebra

$\boxed{M \geq |Z|}$ $\qquad [Q^1, Q^2] = Z$ ← central charge

$M = |Z| \Rightarrow$ Supersymmetric in half the generators

$\Rightarrow$ lives in smaller irrep.

$\Rightarrow$ Deformation invariant (usually)

Classical = Quantum: (Olive-Witten 1978)

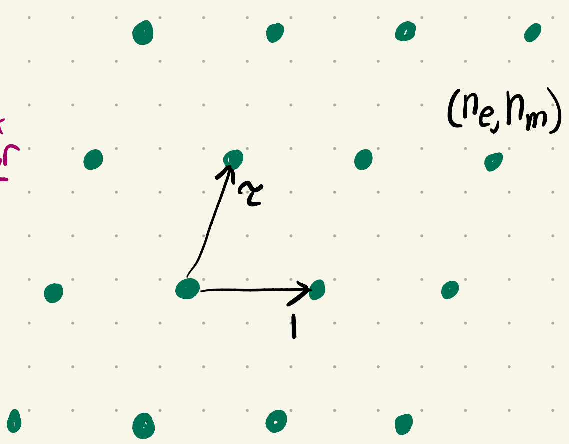
Superspace formalism manifests Z, M as differential operators

for $\underline{\mathcal{L} = \int \Psi^2}$ $\qquad Z: \int_{S^2_\infty} E + \tau \int_{S^2_\infty} B = n_e + \tau n_m$

conserved charges: $\qquad M: \int \|E + iB\|^2 + \|D_A \phi\|^2 + V(\phi) = \mathcal{E}$

BPS monopole $\mathcal{E} = |n_e + \tau n_m| \iff$ BPS state $M = |Z|$

$\boxed{Z \propto n_e + \tau n_m}$



"charge lattice" $\Gamma \simeq \mathbb{Z}^2$ $\quad$ for rank r, $\Gamma = \mathbb{Z}^r$

$Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$

Dirac quantization for dyons:

$\langle (e_1, m_1), (e_2, m_2) \rangle := e_1 m_2 - e_2 m_1 \in \mathbb{Z}$

Defines symplectic structure on Γ

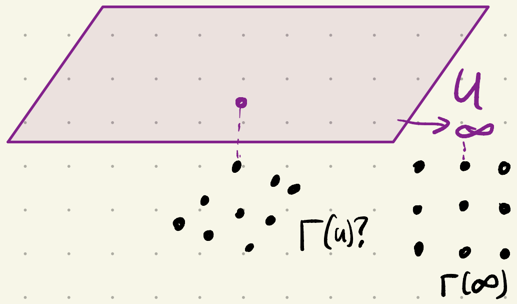
② extend to non-free theories

Classical vacua paramertized by invariant polynomial of Higgs field, $u = \mathrm{tr}\,\phi^2$

Quantum vacua parameritized by $\langle \mathrm{tr}\,\phi^2 \rangle$

$\Rightarrow$ moduli space is $\mathbb{C}$ (the u-plane!)

Coloumb branch:
Parameritizes U(1) theories



Asymptotic freedom: free theory as $u \to \infty$

Deform Γ as from ∞ to finite u...

Use BPS states:

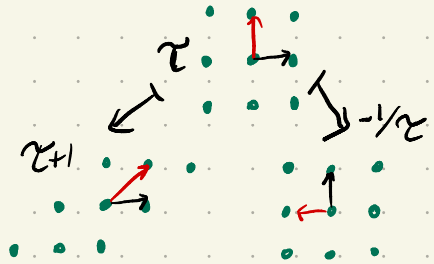
<u>NOTE</u>: for $\mathcal{L} \neq \int \psi^2$, Z is not topologically quantized! no longer charge!

but... BPS states don't deform $\Rightarrow$ formula $Z = n_e + \tau n_m$ still holds!

Does every (n_e, n_m) in the lattice deform identically?

Use Electric-Magnetic Duality:

<u>Exact</u> quantum symmetry $\tau \mapsto \tau + 1$, $\tau \mapsto -1/\tau$



($\mathcal{L} = \int F \wedge * F$, Dual theory uses $F_D = *F$ as fundamental field, impose $D_{A_D} F_D$ externally)

Gives $SL(2,\mathbb{Z})$ (generally, $Sp(2,\mathbb{Z})$) action on Γ

$\Rightarrow$ charge lattice must stay as a lattice!

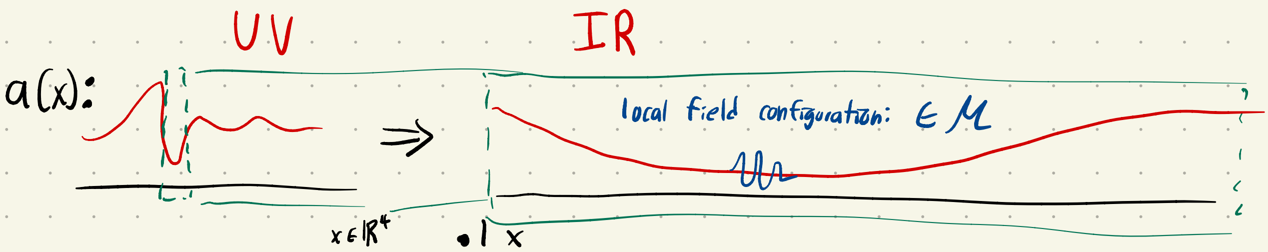
BPS state properties & duality $\Rightarrow \exists$ symplectic lattice over the Coloumb branch

③ Constrain lattice as function of u

relies on geometry of moduli of vacua $\mathcal{M}$. Recall $\Phi = \begin{pmatrix} a & \\ & -a \end{pmatrix}$, $u = a^2$

Cute trick: each pt in $\mathcal{M}$ has a different value of a.

Pretend a is a field <u>valued</u> in $\mathcal{M}$! σ-model! $a: \mathbb{R}^4 \to \mathcal{M}$



a(x) locally constant

$\Rightarrow$ locally is a vacuum state

Kenetic term in Φ induces metric on $\mathcal{M}$: $\tau_{ij} = \mathrm{im}\,\dfrac{\partial^2 \mathcal{F}(\Phi)}{\partial \Phi_i \partial \Phi_j}$ (rank > 1)

Using $a \in \Phi$ as (local) coord. on $\mathcal{M}$, get $ds^2 = \mathrm{im} \underbrace{\frac{\partial^2 \mathcal{F}(a)}{\partial a^2}}_{\tau(a)} da\, d\bar{a}$

ds^2 Kahler, w/ Kahler potential $\mathrm{im}\left(\bar{a} \frac{\partial \mathcal{F}}{\partial a}\right)$

(But it's really special)

Dual coordinates: Choose a symplectic basis α, β of Γ (defines n_e & n_m)

$a(u) = Z(\alpha)$, $a_D(u) := Z(\beta)$ (a & a_I coords of $Z \in \Gamma^*$)

Duality $\Rightarrow$ $a \in \Phi$, $a_D \in \Phi_D$ where Φ_D is lagrange multiplier enforcing $dF_D = 0$

$\mathcal{N}=2 \Rightarrow \Phi_D = \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi}$, so $\boxed{a_D = \frac{\partial \mathcal{F}}{\partial a}}$ in rank >1, $a_D{}^i = \frac{\partial \mathcal{F}(a)}{\partial a^i}$

$a(u)$, $a_D(u)$ both holo. coords of $\mathcal{M}$: special coordinates

$\tau = \frac{\partial a_D}{\partial a}$, $\omega = d\,\mathrm{Re}\{a\} \wedge d\,\mathrm{Re}\{a_D\}$ "flat Darboux coordinates"

$\tau = \frac{\partial^2 \mathcal{F}}{\partial a^2}$

Note: Now, $\mathcal{F}$ is holomorphic fn on $\mathcal{M}$!

But, $\tau(u = a^2)$ is lattice parameter of $\Gamma(u)$ on $\mathcal{M}$

$\hookrightarrow$
- the Charge lattice Γ varies holomorphically on $\mathcal{M}$
- solving $\Gamma(u)$ solves $\mathcal{F}(a)$, & hence $\mathcal{L}^{eff} = \int \mathcal{F}(\Phi)$

Pit stop

Physics Input
- Electric-Magnetic Duality
- BPS state invariance
- Holomorphicity of $\mathcal{N}=2$

SL(2,ℝ) action $\longrightarrow$ Reduction to SL(2,ℤ)

Coloumb branch Data
- holo. lattices $\Gamma(u)$ $\cup$ holo. V.B $\Gamma \otimes \mathbb{R} \downarrow \mathcal{M}$
- symplectic pairing $\Gamma \otimes \Gamma \to \mathbb{Z}$

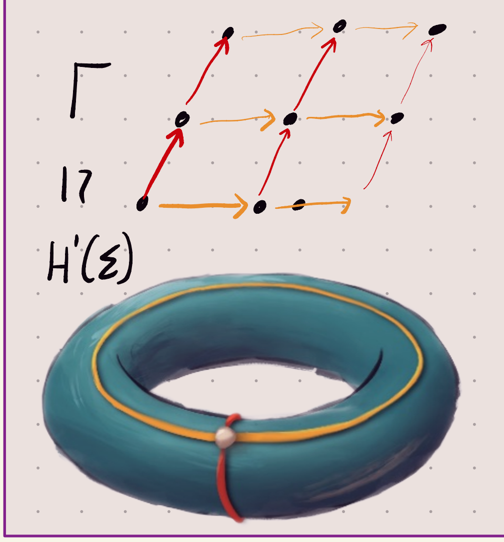
$\Gamma \subset \mathbb{C} \Rightarrow \mathbb{C}/\Gamma$

④ use Riemann surfaces:

Find $\Sigma(u)$ s.t $\Gamma(u) = H^1(\Sigma, \mathbb{Z})$

"Seiberg-Witten curve"

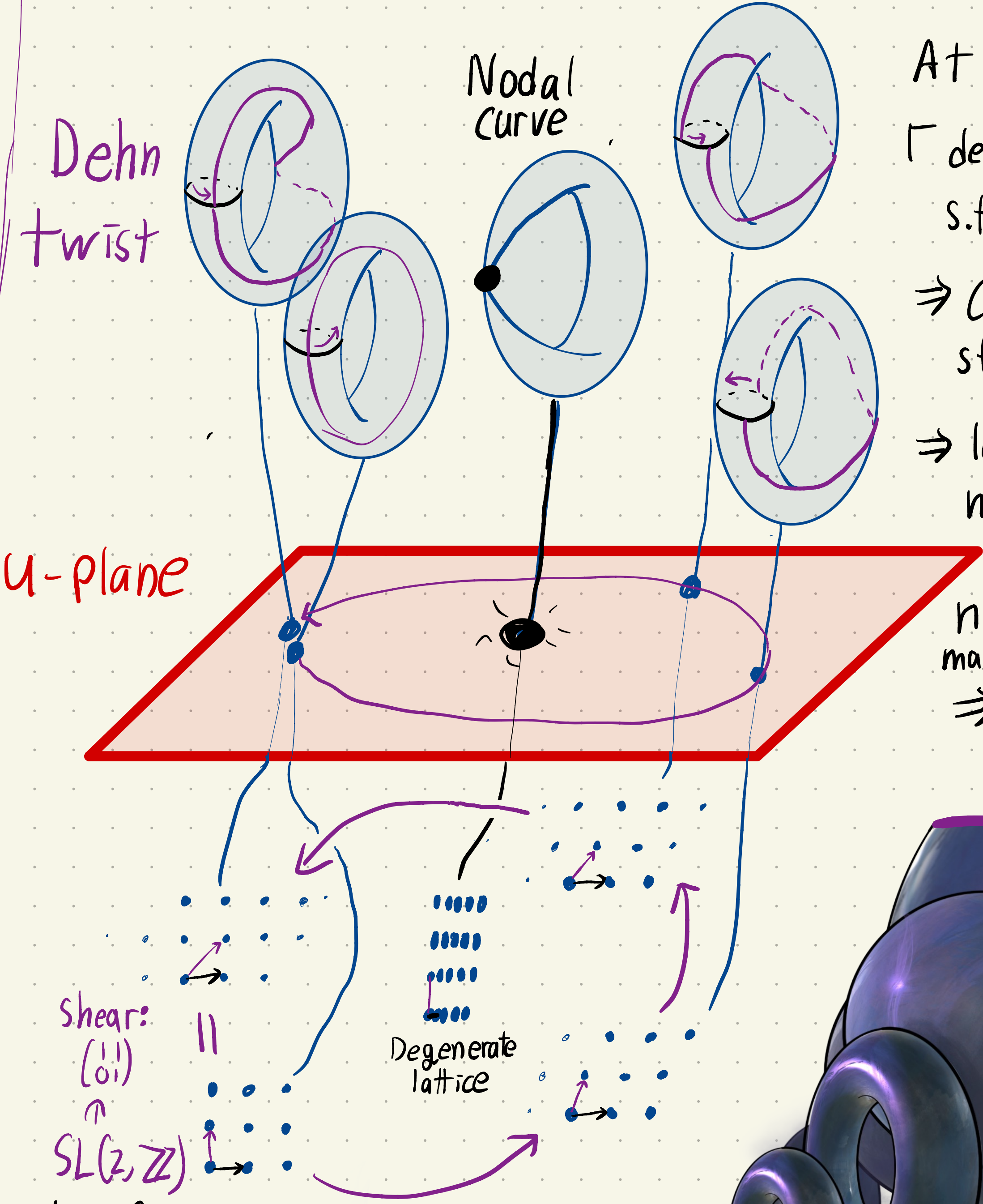
$\langle , \rangle$ = intersection pairing



for $\Gamma \simeq \mathbb{Z}^2$, $\Sigma(u)$ is 1D torus

$\Sigma(u)$ glues together into Elliptic surface

determined by monodromy about singular fibers



At singularity:

Γ degenerates $\Rightarrow \exists\, \gamma$ s.t $Z(\gamma) = 0$

$\Rightarrow$ Corresponding BPS state becomes massless

$\Rightarrow$ low energy approx. breaks !! no simple $U(1)$ theory

need to couple to new massless monopoles

$\Rightarrow$ SW equations

Kodira Classification

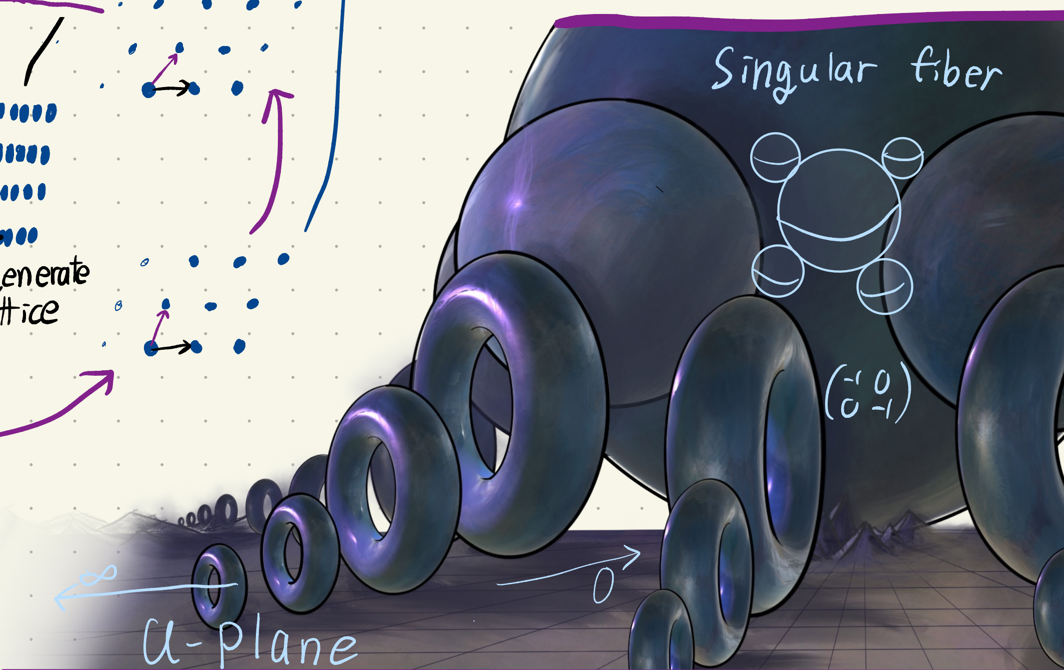
monodromy

$\shortparallel$

Singular fiber topology

$\shortparallel$

ADE classification



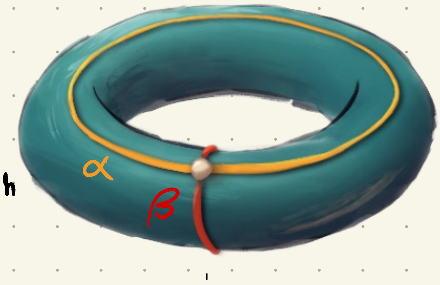
SW Curve: $\boxed{y^2 = (x-1)(x+1)(x-f(u))}$ elliptic curve function of pt. on $\mathcal{M}$

Double-Branched cover of $\mathbb{P}^1$

Idea: find $\lambda(u) \in H^1(\Sigma(u), \mathbb{C})$ s.t for $\gamma \in \Gamma \simeq H_1(\Sigma)$, $Z(\gamma) = \int_\gamma \lambda_{sw}$ "Seiberg-Witten Differential"

For Homology symplectic basis α, β $a = \int_\alpha \lambda_{sw}$

$a_D = \int_\beta \lambda_{sw}$



but, τ is the ratio of periods of the unique holo 1-form λ_h

$$\frac{\int_\alpha \lambda_h}{\int_\beta \lambda_h} = \tau = \frac{da_D}{da} = \frac{\partial a_D/\partial u}{\partial a/\partial u} = \frac{\int_\alpha \partial_u \lambda_{sw}}{\int_\beta \partial_u \lambda_{sw}}$$

(note: can't take a_D, a periods of λ_h, b.c $\lambda_h \in H^{1,0}(\Sigma)$, so cannot repoduce some Zs)

Choose λ_{sw} satisfying $\boxed{\partial_u \lambda_{sw} = \lambda_h}$! higher rank, identifying $H^{1,0}(\Sigma) \simeq T^*\mathcal{M}$ by Kodira-Spencer, $\nabla_v \lambda_{sw} = \lambda_1(v)$

Canonical choice : $y^2 = (x-1)(x+1)(x-f(u)) = P_u(x) \Longrightarrow$ quadradic differential $\phi_u = P_u(x)\, dz^2$

Defines spectral curve $\Sigma_u \subset T^*\mathbb{P}^1$

$T^*\mathbb{P}^1$ has tautological 1-form λ_{tot} w/ double pole above ∞

then $\boxed{\lambda_{sw} = \lambda_{tot}\big|_{\Sigma_u}}$ (restriction, not pull back)

$\lambda_{sw}(u)$ family of mero. forms w/ zero residue, having the same pole structure, so $\partial_u \lambda_{sw}$ is holomorphic

5 find monodromies

monodromies preserve physics, but Γ (& thus $\begin{pmatrix} a \\ a_D \end{pmatrix}$) defined up to $SL(2,\mathbb{Z})$

Or, $\mathrm{aut}(T^2) \simeq SL(2,\mathbb{Z})$

So, see how a & a_D change as you go in loops

β-fn gives asymptotic value of $\mathcal{F}$: $\mathcal{F}(a) \simeq a^2 \ln\left(\frac{a^2}{\Lambda^2}\right)$ large a

$a_D = \frac{\partial \mathcal{F}}{\partial a} \sim a\ln(a) + a$. In monodromy $\ln(a) \mapsto \ln(a+1)$, so

$$a_D \mapsto -a_D + 2a \qquad \begin{pmatrix} a_D \\ a \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}$$
$$a \mapsto -a$$

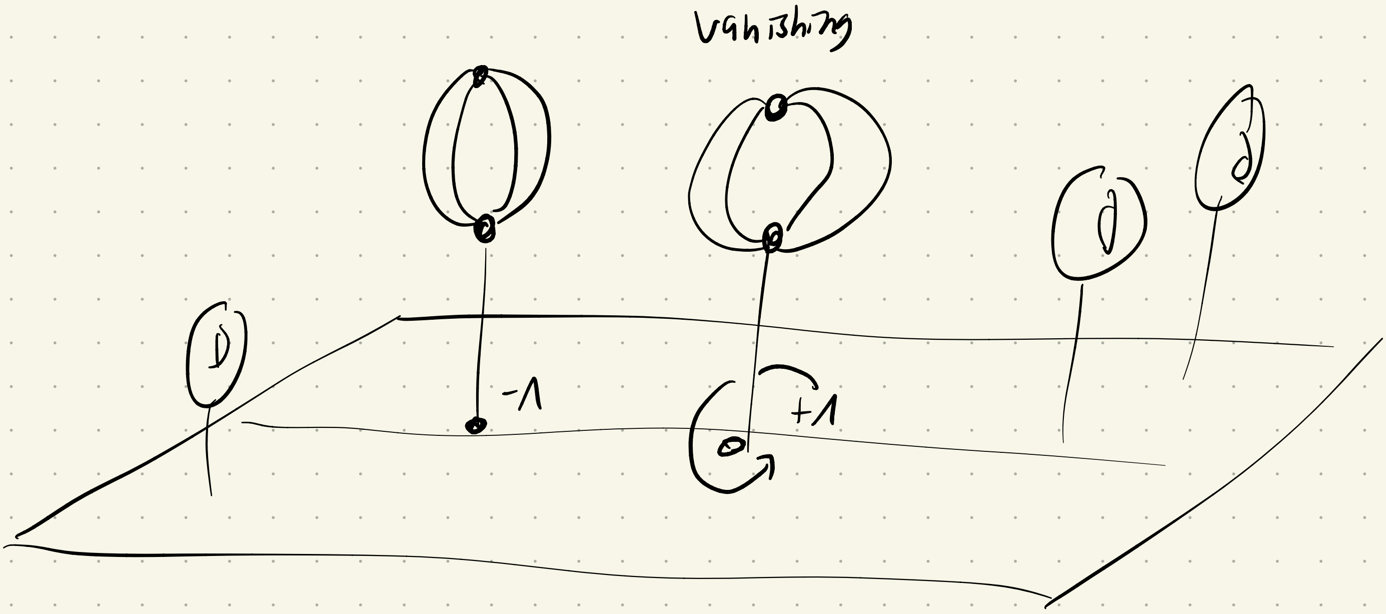
SW suggest 2 other singular pts – $u = \pm\Lambda$.

Weakly coupled $\Rightarrow$ not asymptotically free "hard"

Switch to dual coordinates: now strongly coupled, & free!

get structure

$$y^2 = (x-1)(x+1)(x-u)$$



a unified Picture of BPS states

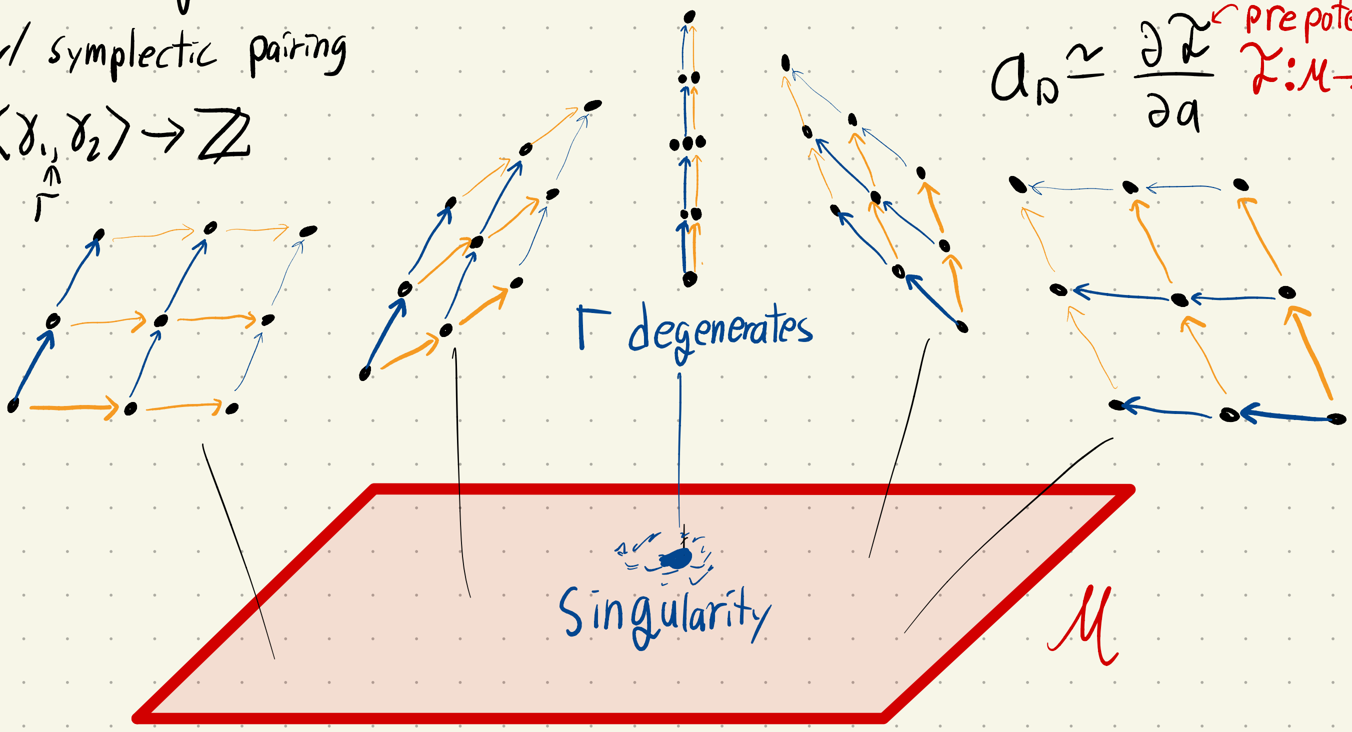
Review: Seiberg-Witten found low energy effective theory for $\mathcal{N}=2$ SYM using BPS states & coloumb branch geometry

"Charge lattice" $\Gamma(u) = a(u)n_e + a_D(u)n_m$ holomorphic in u

w/ symplectic pairing

$\langle \gamma_1, \gamma_2 \rangle \to \mathbb{Z}$
$\hat{\Gamma}$

$a_D \simeq \frac{\partial \mathcal{F}}{\partial a}$ ← prepotential $\mathcal{F}: \mathcal{M} \to \mathbb{C}$



Coulomb branch / u-plane / moduli of vacua

In general: coloumb branch $\mathcal{M}$ (dim. n), codim. 1 singular locus $\mathcal{B} \subset \mathcal{M}$

rank $2n$ vector bundle $V \to \mathcal{M}' = \mathcal{M} \setminus \mathcal{B}$ with:

- symplectic structure
- "charge lattice" $\Gamma(u) \subset V|_u$
- flat connection preserving Γ

(reduces structure group from $Sp(n,\mathbb{R})$ to $Sp(n,\mathbb{Z})$)

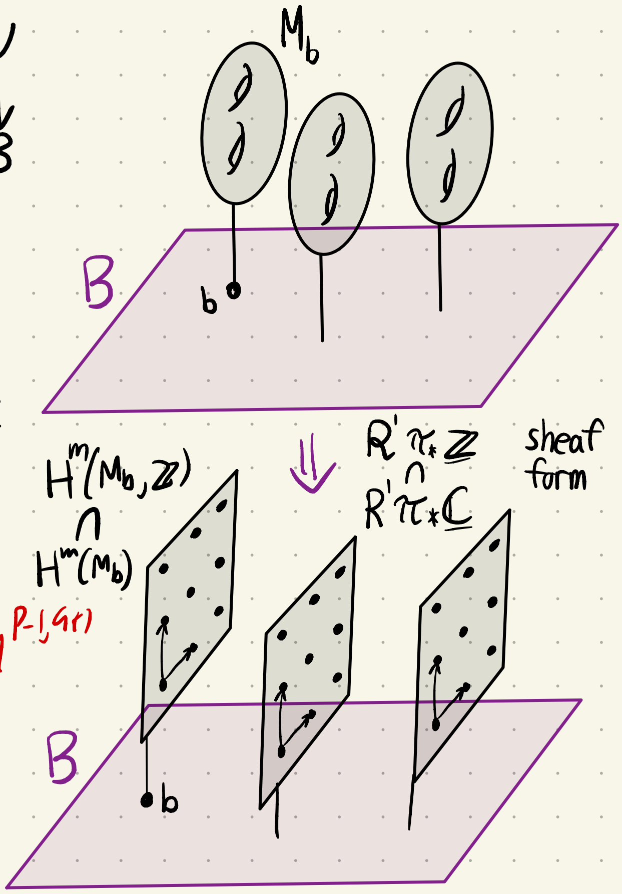
- central charge $Z(u): \Gamma(u) \to \mathbb{C}$ giving local "duality coordinates" $a_i(u)$, $a_{D_i}(u)$ w/ $Z = \sum n_e^i a_i + n_m^i a_{D_i}$ satisfying $a_{D_i} = \frac{\partial \mathcal{F}}{\partial a_i}$

Defines "special Kahler metric" $\frac{\partial a_i}{\partial a_{D_j}} dz_i \otimes d\bar{z}_j$ on $\mathcal{M}$

Storminger, "special geometry"
https://link.springer.com/article/10.1007/BF02096559

Think of This as variations of hodge structure

Idea: family of complex mflds $M_b \hookrightarrow U \to B$



cohomology bundle H^m w/ fibers $H^m(M_b,\mathbb{C})$

- lattice $H^m(M_b,\mathbb{Z})$
- flat "gauss-Manin" connection ∇

- Hodge structure on H^m: filtration $\mathcal{F}^\ell H^m = \bigoplus_{\substack{p+q=m \\ p\geq \ell}} H^{p,q}$ "weight m"
  each $\mathcal{F}^\ell H^m$ Holomorphic
- Griffiths transversality:

$$\nabla_V : \mathcal{F}^\ell H^m \to \mathcal{F}^{\ell-1} H^m \qquad H^{p,q} \simeq H^{p-1,q+1}$$

a variation of hodge structure is any vector bundle w/ these structures

Coloumb branch geometry gives "polarized" VHS of weight 1 on $\mathcal{M}$

complex v.b H, holomorphic subbundle $H^{1,0} \subset H_{\mathbb{C}}$, lattice $H_{\mathbb{Z}}$

note: Griffith's Transversality says $\nabla_V : H^{1,0} \to H$, automatically satisfied

Polarization: $\langle \cdot,\cdot \rangle : H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}$ antisymmetric

$H^{0,1} = H^1/H^{1,0}$ $\quad \langle H^{1,0}, H^{1,0} \rangle = 0$ $\quad$ if $\beta \in H^{0,1}$, $\bar{\beta} \in H^{1,0}$

$Q(\alpha,\beta) = \langle \alpha, \bar{\beta} \rangle$ nondegenerate on $H^{0,1}$

weight 1 VHS	coloumb branch
$H_{\mathbb{Z}}$	charge lattice Γ $\quad \Gamma \simeq \mathbb{Z}^{2g}$
$H_{\mathbb{C}}$	$\Gamma \otimes \mathbb{C}$ $\quad \mathbb{C}^{2g}$
$H_{\mathbb{C}}/H^{1,0}$	V
Polarization	dirac quantization $\langle \cdot,\cdot \rangle$

weight 1 polarized hodge structure
$\|$
abelian variety (Projective complex torus)

$$J(H) = H_{\mathbb{C}} / (H^{1,0} + H_{\mathbb{Z}})$$

Polarization $\Leftrightarrow$ ample line bundle on $J(H)$

weight 1 Polarized VHS $\Leftrightarrow$ family of abelian varieties

Geometric realization of VHS: Seiberg-Witten Curve $\Sigma(u)$

$V = H^1(\Sigma, \mathbb{R}) \qquad \Gamma = H_1(\Sigma, \mathbb{Z}) \qquad V_u / \Gamma_u^* \simeq \mathrm{Jac}(\Sigma_u)$

symplectic pairing on Γ $\simeq$ intersection pairing on $H_1(\Sigma, \mathbb{Z})$

original Seiberg-Witten: $G = SU(2) \Rightarrow \mathcal{M} = \mathbb{C}$. so, $\dim H^1(\Sigma, \mathbb{R}) = 2$,

$\Rightarrow \Sigma(u)$ is genus 1, so is an elliptic curve $\quad y^2 = x(x-1)(x-u)$

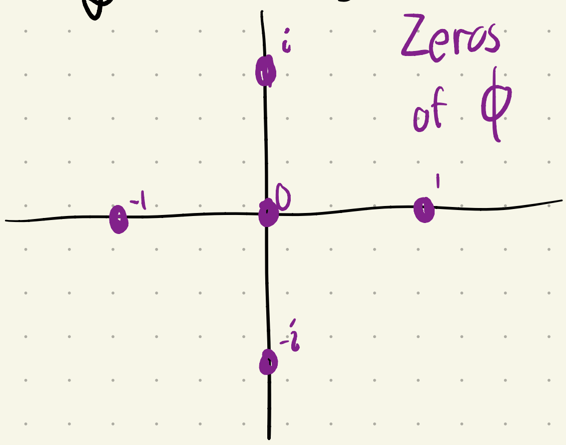
for $G = SU(n+1)$, $\mathcal{M} = \mathbb{C}^n$, so $\dim H^1(\Sigma, \mathbb{R}) = 2n$, & Σ is genus n.

in fact, Σ is <u>hyperelliptic</u> $\quad y^2 = \prod_{i=0}^{n} (x - a_i)$

Invariantly $\phi = \prod^n (x - a_i)\, dx^2$ defines a quadratic <u>differential</u> on $\mathbb{P}^1$

$\Sigma \subset T^*\mathbb{P}^1$ is the <u>spectral curve</u> of ϕ

e.g. $G = SU(3)$



Zeros of ϕ

$\rightsquigarrow$

∞

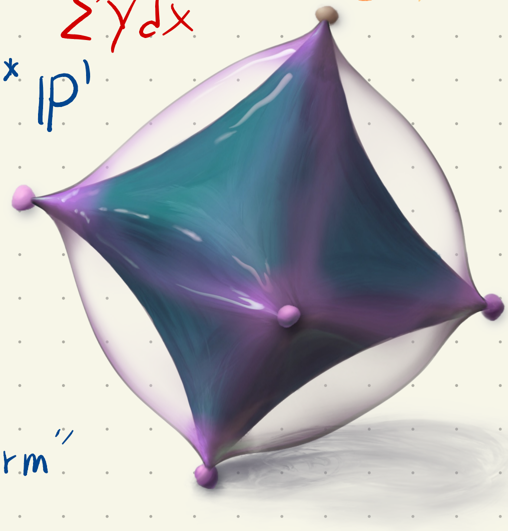
$\Sigma\, y\, dx$

$\Sigma \subset T^*\mathbb{P}^1$

solution to

$\lambda_{tot}^2 - \phi = 0$

"tautological 1-form"



note ϕ defines $SL(2, \mathbb{C})$ higgs field $\Phi = \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}$, w/ $\Sigma = \{(x, y) \mid \det(\Phi(x) - y) = 0\}$
graph of e. values

Special Kahler structure: central charge $Z : \Gamma \to \mathbb{C}$

$\Gamma \simeq H_1(\Sigma, \mathbb{Z}) \Rightarrow Z \in H^1(\Sigma, \mathbb{C})$, realized by closed form λ_{sw}

α_i, β_i basis of $H_1(\Sigma)$, $\quad a_i = \int_{\alpha_i} \lambda_{sw}$

$a_{D_i} = \int_{\beta_i} \lambda_{sw}$

a_i, a_{D_i} satisfy $a_{D_i} = \dfrac{\partial \mathcal{F}}{\partial a_i} \Rightarrow$

for any basis λ_i of holomorphic 1-forms,

Period matrix

$$\frac{\int_{\beta_i} \lambda_j}{\int_{\alpha_k} \lambda_j} = \tau_{ik} = \underbrace{\frac{d a_{D_i}}{d a_k}}_{\text{metric coeficients}} = \frac{\partial a_{D_i} / \partial u_j}{\partial a_k / \partial u_j} = \frac{\int_{\beta_i} \partial_j \lambda_{sw}}{\int_{\alpha_k} \partial_j \lambda_{sw}}$$

① $\boxed{Z(\gamma) := \int_\gamma \lambda_{sw}}$

② $\boxed{\partial_{u_j} \lambda_{sw} = \lambda_j}$

u_j coords on $\mathcal{M}$

Coordinate Invariant formulation

② gives isomorphism $\begin{array}{ccc} T\mathcal{M} & \to & H^{1,0}(\Sigma) \\ v & \longmapsto & \nabla_v \lambda_{sw} \end{array}$

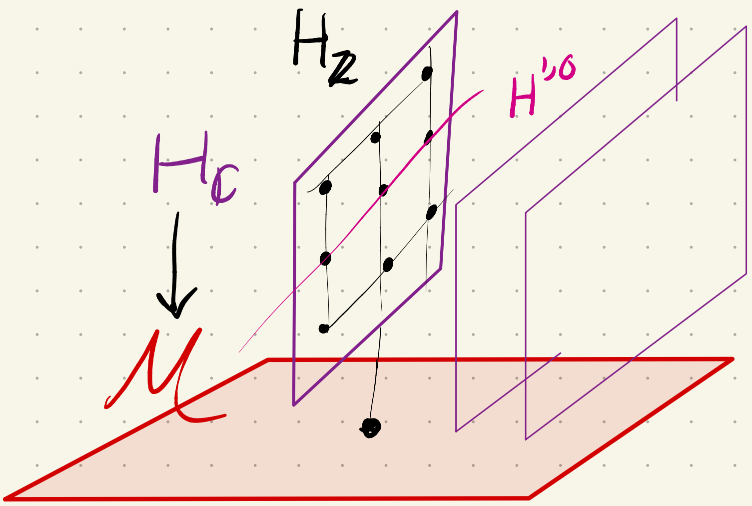
for a section λ_{sw} of $H_{\mathbb{C}}$, $\rho_\lambda : v \mapsto \nabla_v \lambda_{sw}$ gives map $T\mathcal{M} \to H_{\mathbb{C}}$

λ_{sw} is an abstract Seiberg-Witten differential if image of $\rho_{\lambda_{sw}}$ is $H^{1,0}$

λ_{sw} gives special Kahler structure: for basis α_i, β_i of $H_{\mathbb{Z}}$,

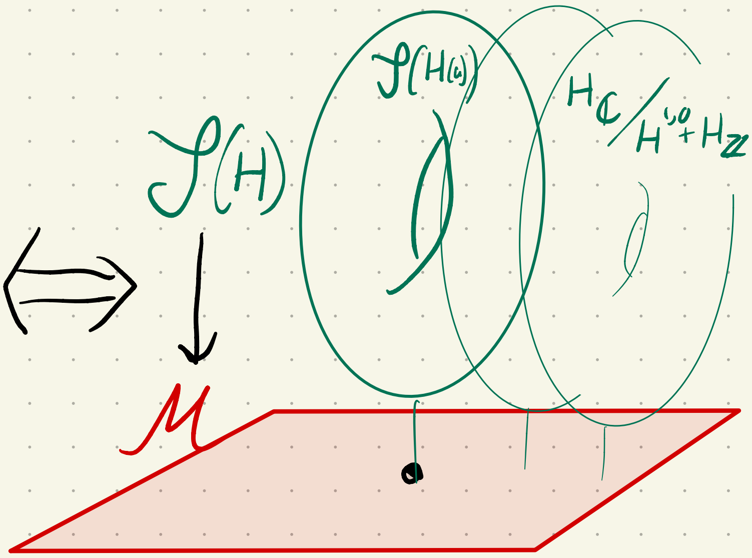
- $a_i := \lambda_{sw}(\alpha_i)$, $a_{D_i} := \lambda_{sw}(\beta_i)$ are coordinates on the base,
- $da_i / da_{D_i} = \nabla_{\partial u_i} \lambda_{sw}(\alpha_i) / \nabla_{\partial u_i} \lambda_{sw}(\beta_i)$ gives the period matrix, as $\nabla_{u_i} \lambda \in H^{1,0}$

Something better: weight 1 VHS $H \Leftrightarrow \mathcal{J}(H) = H_{\mathbb{C}} / H^{1,0} + H_{\mathbb{Z}}$ family of abelian varities over $\mathcal{M}$



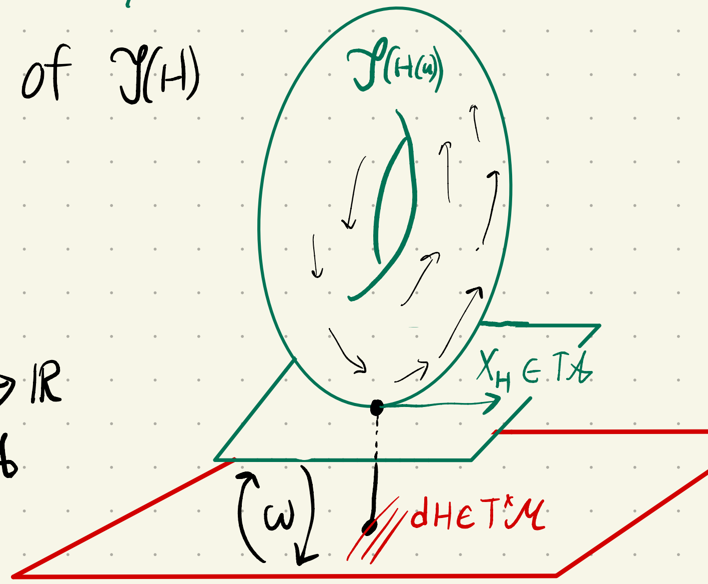
Variation of Hodge structure

$\Longleftrightarrow$



Family of abelian varietys

Def] $\mathcal{J}(H)$ is Algebraically complete integrable system
if $\exists$ symplectic form ω s.t fibers of $\mathcal{J}(H)$
are Lagrangian ($\omega|_{T\mathcal{A}(u)}=0$)



ω identifys $T^*\mathcal{M} \simeq T\mathcal{J}(H(u))$
by $\omega(X_H,-)=dH$ for $H:\mathcal{M}\to\mathbb{R}$
Hamiltonian vector fields from $\mathcal{M}$ span $T\mathcal{A}$
(likewise, $T\mathcal{M}\simeq T^*\mathcal{J}(H(u))$)

Thm: $\mathcal{J}(H)$ is an integrable system iff H has S.W. Differential,
w/ correspondence realized by $\boxed{\omega=d\lambda_{sw}}$

"Aspects of Calabi–Yau Integrable and Hitchin Systems", florain beck
https://arxiv.org/abs/1809.05736

Idea is, $T\mathcal{J}(H_u)\simeq H_{\mathbb{C}}/H^{1,0}=H^{0,1}$, so $T^*\mathcal{J}(H_u)\simeq H^{0,1^*}\simeq H^{1,0}$
the isomorphism $\rho_{\lambda_{sw}}$ equates $T\mathcal{M}$ w/ the cotangent space of the fiber
this describes a unique symplectic form w/ Lagrangian fibers

We can realize H_u geometrically as $H^1(\mathcal{J}(H_u))$ (Polarization identifies $\mathcal{J}(H_u)$ w/ it's dual)
abstract S.W differential comes from true 1-form λ_{sw} on $\mathcal{J}(H)$
this is tautological 1-form (identifying $H_u\simeq T^*_u\mathcal{M}$, & quotenting to $\mathcal{J}(H)$)
so $\omega=d\lambda_{sw}$ comes from standard symplectic form on $T^*\mathcal{M}$, which has Lagrangian fibers

The Seiberg-Witten differential is Exactly the structure needed to promote rank 1 polarized VHS to an integrable system

Canonical choice of λ_{sw}:

$T^*\mathbb{P}^1$ has tautological 1-form λ_{tot}, mero. w/ double pole above ∞

then $\boxed{\lambda_{sw} = \lambda_{tot}|_{\Sigma_u}}$ "abelian Differential of the 2nd kind"

λ_{tot} (& thus λ_{sw}) has residue 0 @ pole $\Rightarrow$ gives well-defined element in $H^1(\Sigma, \mathbb{C})$

Rank 1: λ_{sw} meromorphic w/ a single double-pole, so $\partial_u \lambda_{sw}$ is holo.

Higher rank: deformation of spectral curve

lives in holo. normal bundle $N\Sigma \subset T(T^*\mathbb{P}^1)$

for section v,

$$\nabla_v \lambda_{tot} = \underbrace{i^* \mathcal{L}_v \lambda_{tot}}_{\text{WTS this is holomorphic}} \qquad \mathcal{L}_v \lambda_{tot} = \underbrace{i_v d\lambda_{tot}}_{i_v \omega} + d i_v \lambda_{tot}$$

identifies v w/ holo 1-form living on Σ

BPS states are geodesics

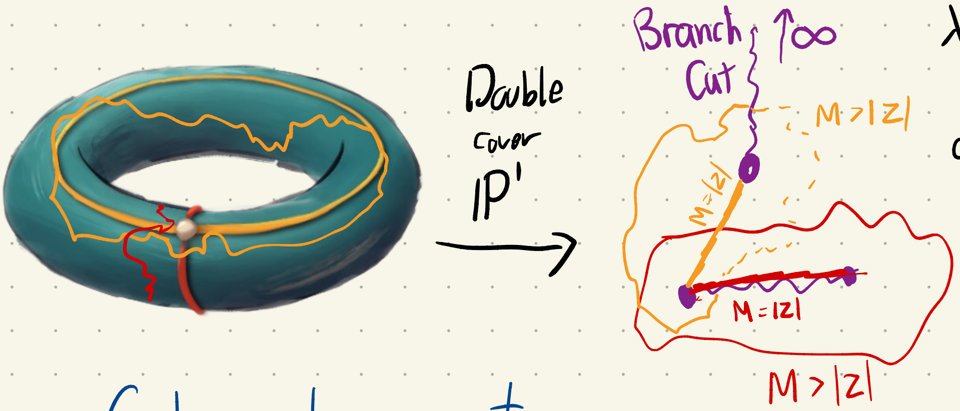
charge classified by element $[\gamma]$ in homology $H^1(\Sigma, \mathbb{Z})$

Central charge $Z = \int_\gamma \lambda_{SW}$, mass $M \overset{?}{=} |Z|$

$M^2 \overset{?}{=} \left|\int_\gamma \lambda_{sw}\right|^2 \leq \int_\gamma |\lambda_{sw}|^2$ w/ equality when λ_{sw} has constant phase

↑ metric on Σ

can calculate BPS masses as minimal lengths in a homology class under metric $|\lambda_{sw}|^2$ (since λ_{sw} meromorphic, $|\lambda_{sw}|^2$ is flat!)



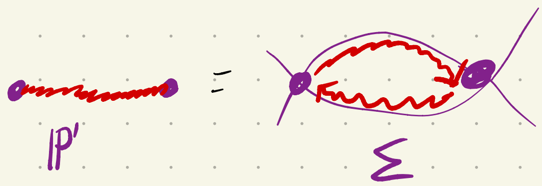
λ_{sw} is tautological 1-form from spectral curve defined by quadratic differential $\phi = \sqrt{z(z-1)(z-u)}\, dz^2$

$\Downarrow$

metric $|\lambda_{sw}|^2$ is lift of $|\phi|^2$

Schematic picture

States	Paths $\gamma: S^1 \to \Sigma$
Charge	Homology class
mass	length in metric $|\lambda_{sw}|^2$
BPS States	geodesics (global energy minimizers)



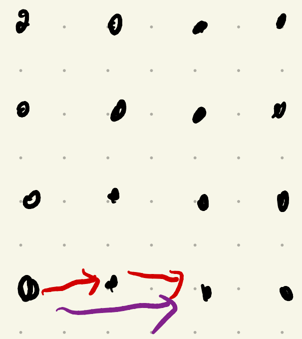
BPS state counts:

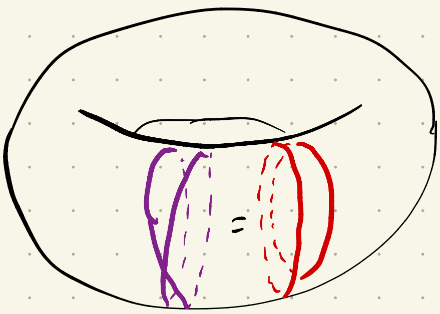
every $H^1(\Sigma)$ class has a geodesic

But might not be simple

Shortest path consists of other geodesics

$H^1(\Sigma, \mathbb{Z})$





shortest $(2,0)$ geodesic is 2 $(1,0)$ geodesics

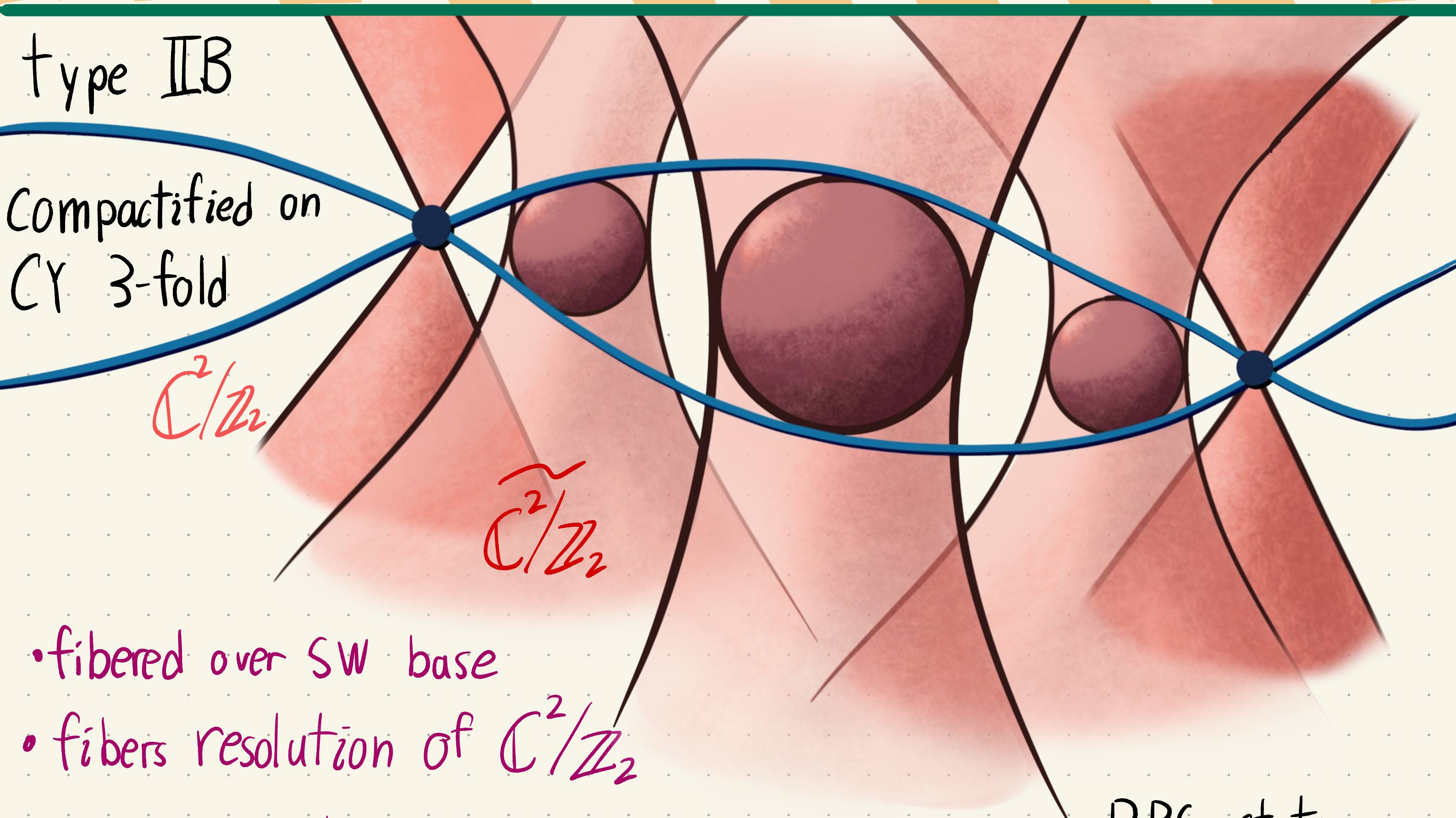
$\Rightarrow$ no BPS $(2,0)$ state, only $(1,0)$

Seiberg-Witten thru String theory

"Geometric Engineering"

type IIB

Compactified on CY 3-fold



- fibered over SW base
- fibers resolution of $\mathbb{C}^2/\mathbb{Z}_2$
- SW curve traced by North/South poles of exceptional divisors $H^{2,1}$

BPS state

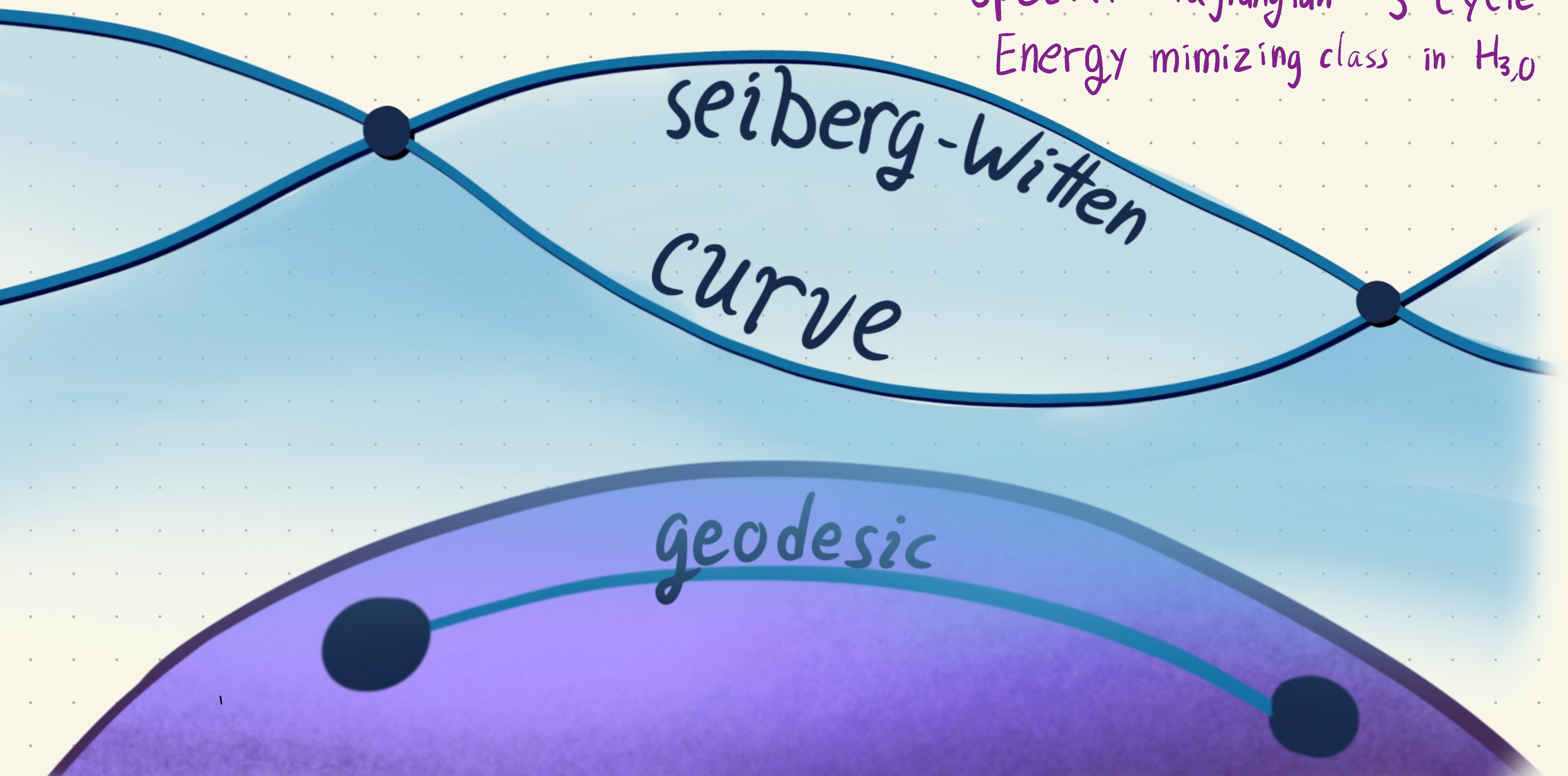
=

Geodesic

=

special lagrangian 3-cycle

Energy mimizing class in $H_{3,0}$



CY 3-fold X defined by $zw = y^2 - P(x)$

family of Calabi-Yau 3-folds X_u paramertized by a the moduli $\mathcal{M}$ of complex structures on X

Variation of Hodge structure:

$V_{\mathbb{C}} = H^3(X_u, \mathbb{C})$ $\qquad$ $V_{\mathbb{C}} / V^{1,0} + V_{\mathbb{Z}}$ is the intermediate jacobian

$V_{\mathbb{Z}} = H^3(X_u, \mathbb{Z})$ $\qquad$ $\mathcal{J}(V) = \mathcal{J}^2(X_u)$

$V^{1,0} = H^{3,0}(X_u) \oplus H^{2,1}(X_u)$

$\langle a, b \rangle = \int a \wedge \bar{b}$

want to calculate period matrix for $\lambda_i \in H^{2,1}(X_u)$, reletive to $H_3(X)$.

use unique holomorphic 3-form $\Omega \in H^{3,0}(X_u) \simeq \mathbb{C}$

deformations of Ω live in $H^{3,0} \oplus H^{2,1}$, & actually give an isomorphism

$T\mathcal{M} \simeq H^1(X, TX) \to H^{2,1}(X, \mathbb{C})$

So, Ω is like "Seiberg-Witten form"

string theory states = lagrangian 3-cycles $\mathcal{L}$, charge $\Gamma \in H_3(X)$

mass bound $M = |\int_\Gamma \Omega| \leq \int_{\mathcal{L}} |\Omega_1|^2 + |\Omega_2|^2$ for $\Omega = \Omega_1 + i\Omega_2$

minimized when $\Omega_2|_{\mathcal{L}} = 0$ "special lagrangian submanifold"

These are the exceptional Divisors fibering above a geodesic

so, $\mathcal{L} \simeq S^3$ or $S^1 \times S^2$

BPS states = energy minimizing representative of $H_3(X)$

Massless BPS states

Vanishing cycles

Monodromy given by Picard-Lefshetz

BPS monopole $\Rightarrow$ saturate YM energy

$H^*(\mathcal{M}_{monopole})$

BPS states are extremal, saturating some bound

Quantum: $M \geq |Z|$ "Central charge" $[Q_1, \bar{Q}_2] = Z$

$M = |Z| \Rightarrow$ Supersymmetric in half the generators

$\Rightarrow$ lives in smaller irrep.

$\Rightarrow$ Deformation invariant (usually)

Monopoles: Static Yang-mills-higgs solution

low energy $\Rightarrow$ Spontanious symmetry breaking $SU(2) \to U(1)$ @ ∞

$F \mapsto (\underset{\text{electric}}{E}, \underset{\text{magnetic}}{B})$, monopole charges $n_e = \int_{S^2_\infty} E$ $\quad n_m = \int_{S^2_\infty} B$ (topological!) integers!

$\mathcal{L} \sim \operatorname{Im} \int \tau (E + iB) \wedge * (E + iB)$

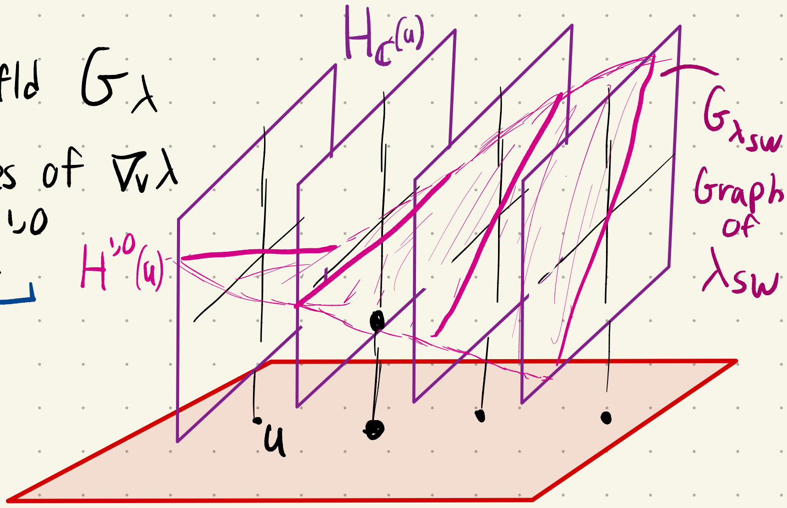
energy $M \geq |n_e + \tau n_m|$

What's going on??

Graph of λ is an immersed submfld G_λ

in flat coords, TG_λ holds possible values of $\nabla_v \lambda$

λ is Seiberg-Witten when $TG_\lambda = H^{1,0}$



Canonically identify $T^*\mathcal{J}(H_u)$ with $(H_{\mathbb{C}}/H^{1,0})^* \simeq H^{0,1}(u)^*$

$H^{0,1*} \simeq H^{1,0}$, using the polarization

$\rho_{\lambda_{SW}}: T\mathcal{M} \to H^{1,0}$ is section of $T^*\mathcal{M} \times H^{1,0} \simeq T^*\mathcal{M} \times T^*\mathcal{J}(H_u) \simeq T^*\mathcal{J}(H)$

adding massive hypermultiplets = adding punctures to Σ

$$H^1(\Sigma - \{p_i\}, \Sigma) = H^1(\Sigma) \oplus \underbrace{\langle p_1, \dots, p_n \rangle}_{\text{"flavor charges"}}$$

naturally captured by "mixed Hodge structure" https://arxiv.org/abs/2107.11180