Hyperbolic string art

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Abstract

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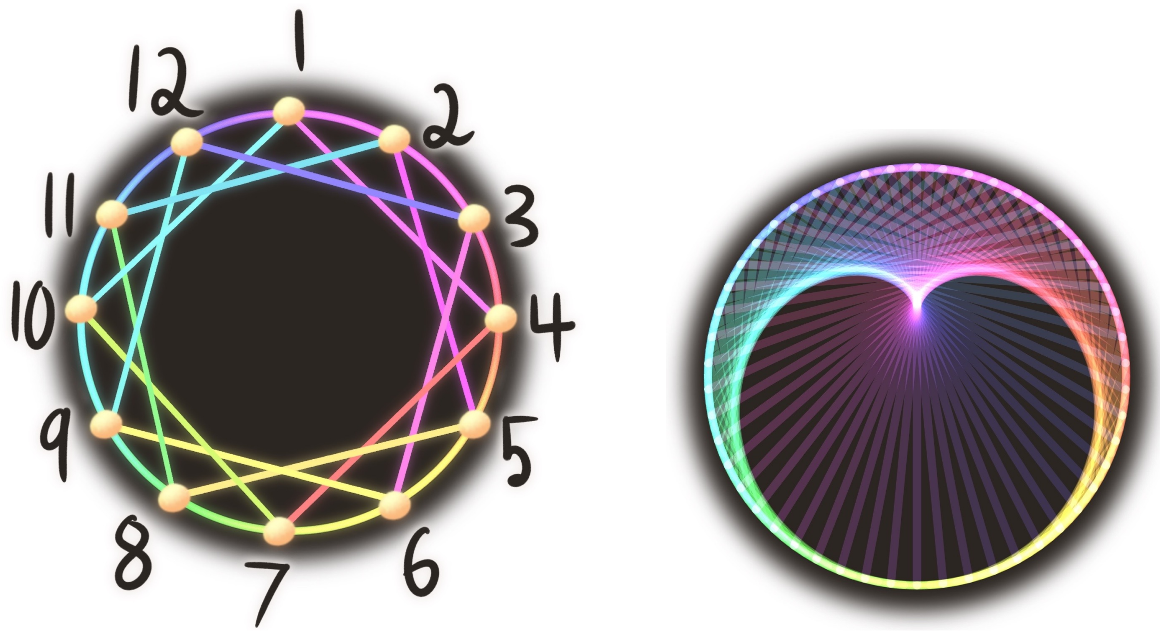
Introduction

String art (or *curve stitching*) is an art form composed of thin stings stretched between points. Placing the strings according to mathematical rules results in aesthetic designs which can illustrate the mathematics of the generating rule. For example, a common classroom activity involves placing equally spaced pins around a circle, labeled 1 to 12. The student stretches strings from the pin labeled i to that labeled $i + 3$ modulo 12 (See figure 1a). The strings accumulate around a smaller central circle, manifesting the rotational symmetry of modular addition. This smaller circle is tangent to all the strings, hence is called the *envelope*. Modular multiplication is similarly illustrated by stretching strings from i to $2i$. The strings once again accumulate around their envelope, the heart-shaped cardioid (Figure 1b).

Unfortunately, the exercise seldom goes much further. Most other common functions (squaring, exponential, etc) produce noisy results without an envelope [[insert noisy results]]. These functions are not natural with respect to the modular structure, so their string art lacks obvious mathematically or aesthetically interesting patterns. So, we change perspective, replacing the integers labeling the pegs with their angle around the circle. The strings are defined by maps from the circle to itself. Modular addition (figure 1a) comes from a rigid rotation, while modular multiplication by two (figure 1b) is the angle doubling map. To make string art, choose some set of input points, and stretch a line from each point to its image under the defining map. This viewpoint suggests several other natural functions, like sine and cosine, which produce aesthetic envelopes (See figure 2).

In this paper, we explore a further leap in perspective, exploring the geometry of the disc which the strings stretch across. We introduce *Hyperbolic string art*, which identifies the disc as the hyperbolic plane. Strings are straight lines across the disc, which represent geodesics (hyperbolic lines[1]) in the Beltrami-Klein model of the hyperbolic plane. The boundary of the disc is not in the hyperbolic plane, rather representing the possible directions a geodesic can point. We will call this the "hyperbolic infinity" or the "circle at infinity" [2]. Hyperbolic geometry suggests several natural functions mapping the circle at infinity to itself. In particular, symmetries of the hyperbolic plane extend to the circle at infinity as Möbius transforms. We will explore the string art coming from these maps, understanding their envelopes and relating them to the underlying symmetries of the hyperbolic plane. Through this, string art visualizes properties and themes of hyperbolic geometry in an artistic way. We also show purely artistic applications, such as fractal string art.

[1]In this paper, we will refer to hyperbolic lines as geodesics, and Euclidean lines as lines
[2]hyperbolic infinity is sometimes called the set of *ideal points*



(a) 12 *pins,* $i \to i + 3$ *modulo* 12

(b) 50 *pins,* $i \to 2i$ *modulo* 50

Figure 1: *String art illustrating modular arithmetic.*

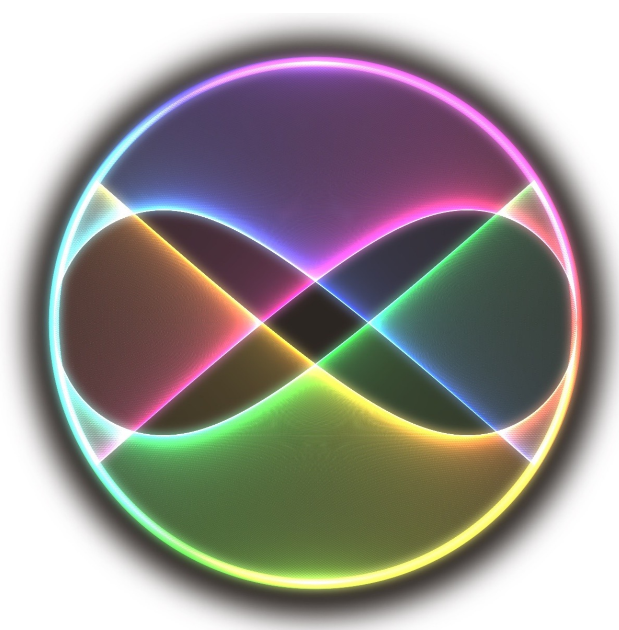


Figure 2: *String art for the function* $\theta \to 2(\theta + \frac{1}{2}\sin(\theta)$

String art in the hyperbolic plane

Let us first recap necessary aspects of hyperbolic geometry (For reference, see chapter 2 of [1]). The hyperbolic plane is a two dimensional plane with constant negative curvature. Just like the Euclidean plane, there are lines, angles, distances, and an axiomatic approach to geometry analogous to Euclid's. However, the negative curvature forces geodesics to curve away from one another, possibly never intersecting. This is a failure of Euclid's fifth postulate, as there are infinitely many geodesics thru a given point parallel to a given geodesic. We draw the hyperbolic plane on Euclidean paper using *models*. These are like flat maps of the spherical earth, and like maps every model has its own uses and drawbacks. Let us start with the *Beltrami-Klein model*, where the hyperbolic plane is drawn as the interior of a unit disc, and geodesics are drawn as straight lines. A fixed hyperbolic distance looks smaller the closer it is to the boundary of the disc, vanishing at the boundary. In fact, the bounding circle is infinitely far from any point inside the disc, hence its moniker "the circle at infinity".

Note that every line passing thru the interior of the unit disc intersects the boundary circle twice, and moreover is uniquely defined by its intersection points with the boundary. Translating to hyperbolic geometry, each geodesic is uniquely defined by its two asymptotic directions, each given by points at hyperbolic infinity. This manifests a creed of hyperbolic geometry: *an object in the hyperbolic plane is uniquely determined by its behavior at hyperbolic infinity.* This stands in contrast to Euclidean space, where the forward and backwards direction of a line are necessarily opposite, and many lines share the same asymptotic direction. The core insight of hyperbolic string art is to treat the boundary circle as hyperbolic infinity, and treat strings as hyperbolic geodesics.

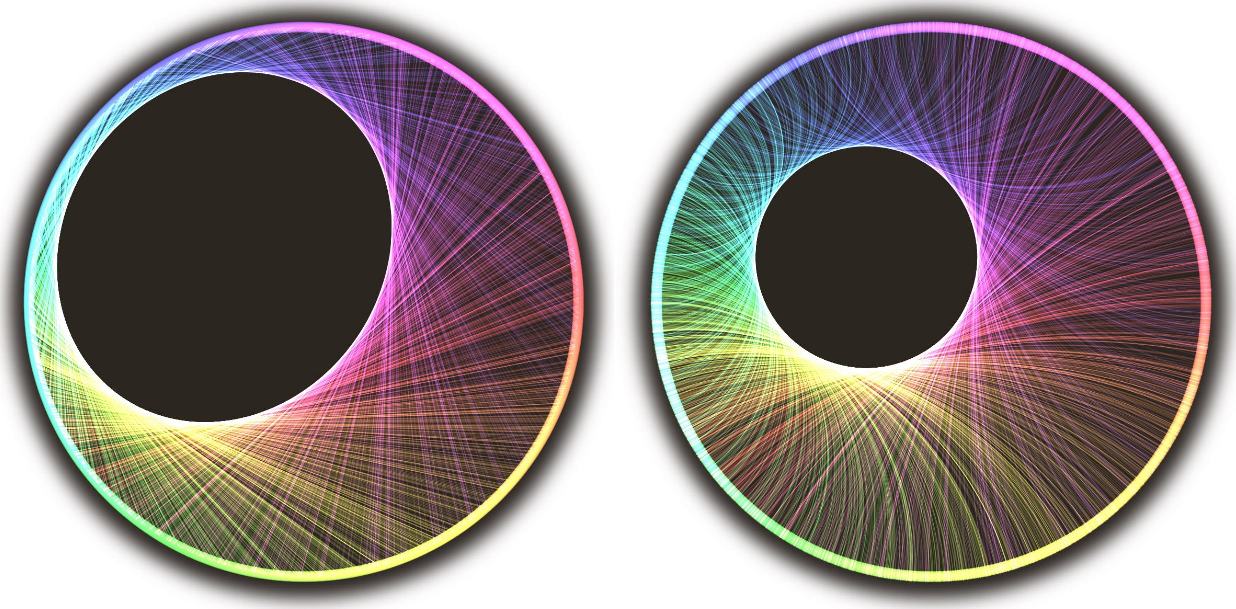
The hyperbolic plane carries similar symmetries (or isometries) to the Euclidean plane, including rotation around a point and translation along a direction. Unlike Euclidean space, hyperbolic translations and rotations mix together, as translation along a loop results in a net rotation. This mixing is captured in the symmetry group of the hyperbolic plane $SL(2,\mathbb{R})$, which consists of 2×2 real matrices with determinant one. Hyperbolic symmetries send geodesics to other geodesics, hence naturally act on the space of asymptotic directions. They restrict to *Möbius transforms* on the circle at infinity. In fact, hyperbolic symmetries are uniquely defined by their action at infinity, manifesting the above hyperbolic creed.

Let us form string art from a hyperbolic symmetry. First we choose a symmetry, such as rotation around a point. This acts on the circle at infinity via a Möbius transform. Then we draw the string art for this map, shown in figure 3a. The envelope forms an ovoid shape, surrounding the pivot point of the rotation.

However, The Beltrami-Klein model obfuscates hyperbolic shapes, because it distorts hyperbolic angles. To understand the envelope's shape, we switch to the *Poincare disc model*, which is conformal, meaning it preserves hyperbolic angles. The Poincare disc model is again based on the unit disc, but geodesics are drawn as circular arcs which intersect the boundary circle at right angles. Using these circle arcs to render the strings from figure 3a results in figure 3b. In the Poincare disc model, the envelope appears to be a perfect circle. The center of the envelope circle is indeed the original pivot point of the hyperbolic rotation. None of these observations are coincidences. In the next section, we will explain why the envelope is a hyperbolic circle, and discuss the envelopes for all other types of Möbius transforms.

Möbius transform taxonomy through string art

Symmetries of the hyperbolic plane fall into 3 classes: Rotations, translations, and parabolic transforms. Their associated Möbius transforms are called elliptic, hyperbolic, and parabolic respectively. These are illustrated in figure 6: The Orange arrows circumnavigating the border circles show how the Möbius transform acts on each point. An elliptic Möbius transform has no fixed points, and comes from a hyperbolic rotation about the point marked in purple. Moving to column 2, a parabolic Möbius transform has one fixed point, shown by the orange dot. The corresponding symmetry of the hyperbolic plane is also called a parabolic transform,



(a) *Beltrami-Klien model*

(b) *Poincare model*

Figure 3: *String art of a Möbius transform, shown in two different models of hyperbolic space.*

as it is not as easily described as rotation or translation. Shown in column 3, a hyperbolic Möbius transform pulls points away from a repelling fixed point, and towards an attracting fixed point. This is the boundary action of hyperbolic translation along a geodesic which starts at the repelling point and ends at the attracting point called the *axis* of the translation. (marked in purple).

Algebraically, this trichotomy arises from the conjugacy classes of the associated elements of $SL(2, \mathbb{R})$. Each conjugacy class is uniquely defined by its trace. When the trace is > 2 (resp. $= 2, < 2$), the associated Möbius transform is elliptical (resp. parabolic, hyperbolic). Geometrically, this represents symmetries of the hyperbolic plane up to change in perspective. All hyperbolic rotations by θ degrees are conjugate to one another by a translation: First translate one pivot point to another, then rotate by θ, then translate back. θ defines a conjugacy class, and is uniquely determined by the trace. Similarly, hyperbolic translations are conjugate by rotations, and the only invariant under conjugacy is the length of translation. Finally, all parabolic transforms are conjugate. Each class of Möbius transforms can be understood from a single family of basic transforms, such as rotation around a fixed point or translation along a fixed line. We will use this technique to study the string art of Mobius transforms. These are also shown in figure 6, with their envelopes shown in green. The descriptions of each envelope are defined and justified below.

Elliptic Möbius transforms

As mentioned above, elliptic Möbius transforms all arise from hyperbolic rotations. We start with the simplest rotation, pivoting around the center of the Poincare disc. This acts on the boundary circle by rigid rotation. All the strings are thus circular arcs of the same radius. The envelope is the circle with radius equal to the smallest distance between one string and the center. To see this, consider the distance function to the center of the Poincare disc. At the minimum distance along one string, the string must be tangent to the level set of the distance function, which is a circle (this is a geometric realization of the method of Lagrange multipliers). By rotational symmetry, every string has the same minimal distance, so all strings are tangent to the same level set. Hence, the envelope is a circle. Note that this argument also justifies the circular envelope in string art for modular addition, shown in figure 1a. Since the Poincare projection is conformal, the circular envelope

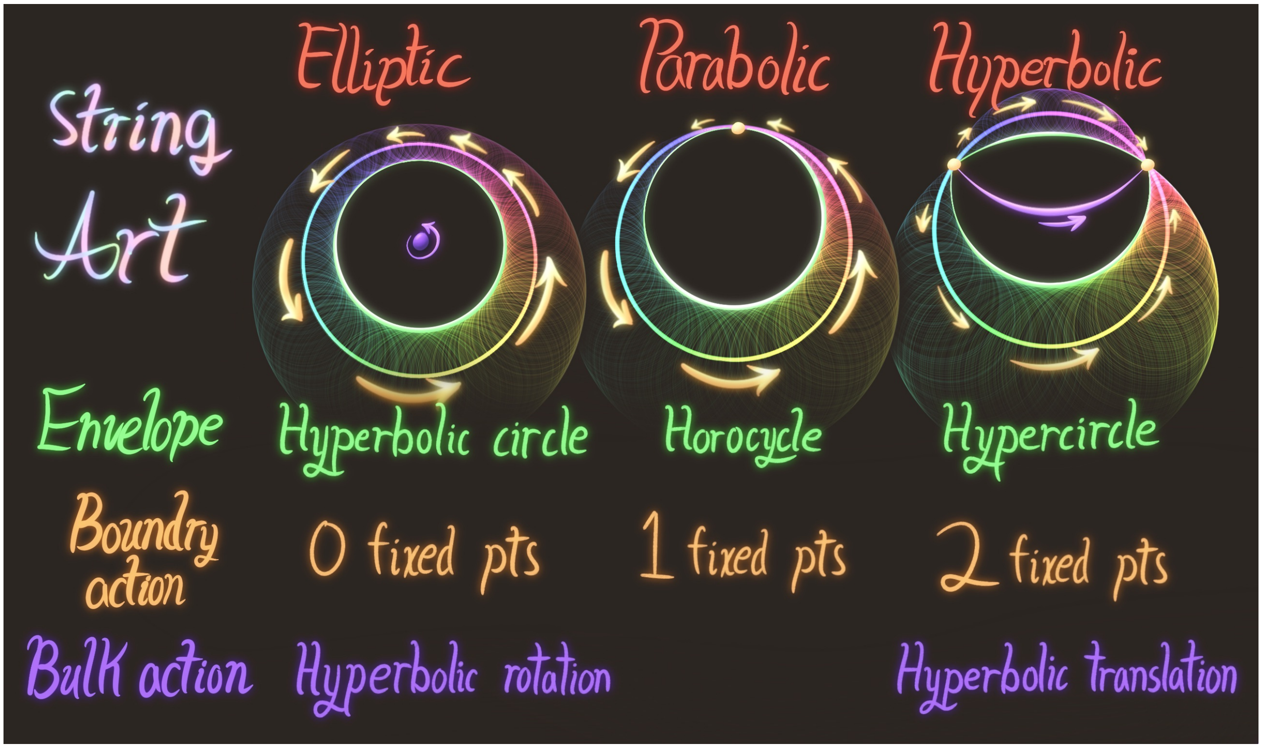


Figure 4: *Description of string art for each type of Möbius transform.*

is the image of a hyperbolic circle.

Now we extend the simple case of rigid rotation R_0 to all elliptic Möbius transforms. Let us translate the string art of rigid rotation by a hyperbolic translation T_p sending the center of the Poincare disc to a point p. Since T_p is a hyperbolic symmetry, it sends geodesics to geodesics. After translating, the geodesics form string art from a new function f, which maps the start of each translated geodesic to the endpoint. Symbolically, if d is a point at hyperbolic infinity, the function f maps $T_p(d)$ to $T_p(R_0(d))$. Labeling $T_p(d)$ as d', we see

$$f : d' \to T_p R_0 T_p^{-1}(d')$$

So, the image is the string art for the Möbius transform $T_p R_0 T_p^{-1}$, which is a hyperbolic rotation around the point p. Every elliptic Möbius transform is conjugate to R_0 in this way.

The image under the translation T_p of the envelope for rigid rotation R_0 gives the envelope for rotation around p, $T_p R_0 T_p^{-1}$. However, the envelope for R_0 is a hyperbolic circle. The translation T_p is an isometry, so it sends hyperbolic circles to hyperbolic circles. Thus, *The envelope of any elliptic Möbius transform is a hyperbolic circle.* This is drawn in the Poincare disc as an Euclidean circle, as seen in column 1 of figure 6.

Parabolic Möbius transforms

We establish the string art for parabolic transforms following the blueprint of the above argument. First, we choose a specific model of hyperbolic space and specific parabolic transform with a very simple description. In particular, it will be realized by an Euclidean isometry of the model. Then, we find the envelope, and describe it intrinsically with hyperbolic geometry. Finally, we describe the string art of all other parabolic Mobius transforms by conjugating the base case. We notice that our intrinsic description of the envelope is preserved by isometries, and conclude that all envelopes of parabolic transforms share the same intrinsic description.

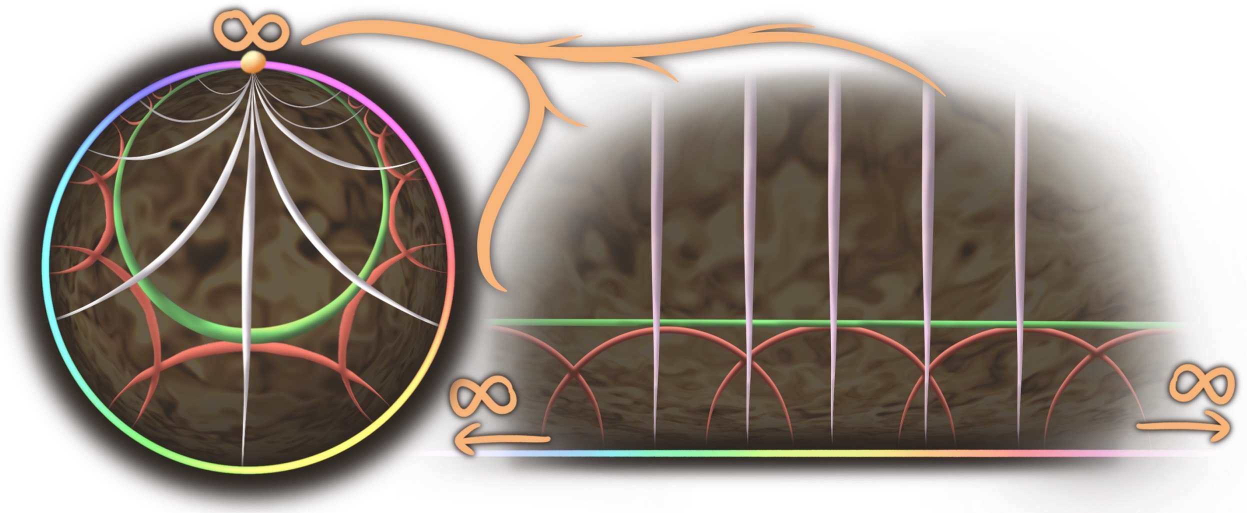


Figure 5: *String art for a parabolic Möbius transform in the Poincare disc and half-plane models. The rainbow boundary is hyperbolic infinity, the red curves are geodesic strings, the green curve is the envelope of the strings, and the white shows normal geodesics to the envelope.*

We will describe our base parabolic transform in the Poincare half-plane model. This consists of the complex numbers with real part > 0, endowed with a metric which scales down distances near the real line, and is invariant under real translation. Hyperbolic infinity consists of the real line, along with the point at infinity in the complex plane. Geodesics consist of semicircles intersecting the real line orthogonally. Möbius transforms act on hyperbolic infinity by fractional linear transforms:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \qquad A : x \longrightarrow \frac{ax + b}{cx + d}$$

All parabolic Möbius transforms are conjugate to those of the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : x \longrightarrow x + 1$$

which are induced by a translation of the hyperbolic plane, $z \longrightarrow z + 1$.

The strings for this action are semicircles stretching from x to $x + 1$ on the real line. The envelope is a horizontal line at height $1/2$. Indeed, every string has maximal height $1/2$, so the level set of the height function at $1/2$ is tangent to every string. Returning to the hyperbolic plane, this horizontal line is a *Horocycle*: A curve whose perpendicular geodesics all converge to the same point at hyperbolic infinity. Indeed, the normal geodesics of a horizontal line are vertical lines (infinitely large semicircles), which converge at infinity in the half plane model. Converting back to the Poincare disc model, this horocycle is a circle tangent to the top of the boundary circle.

All parabolic Möbius transforms are conjugate to that described above. Following the argument for elliptic transforms, the string art of every parabolic transform are related by hyperbolic symmetries. These symmetries preserve horocycles, so *the envelope of a parabolic Möbius transform is a horocycle*. These are drawn in the Poincare model as circles tangent to the boundary circle, as seen in figure 6.

Hyperbolic Möbius transforms

For hyperbolic Möbius transforms, we follow the same strategy as above. We describe the base hyperbolic translation in the *band model*. This consists of the strip of complex numbers z with Re $z \in (-1, 1)$, endowed

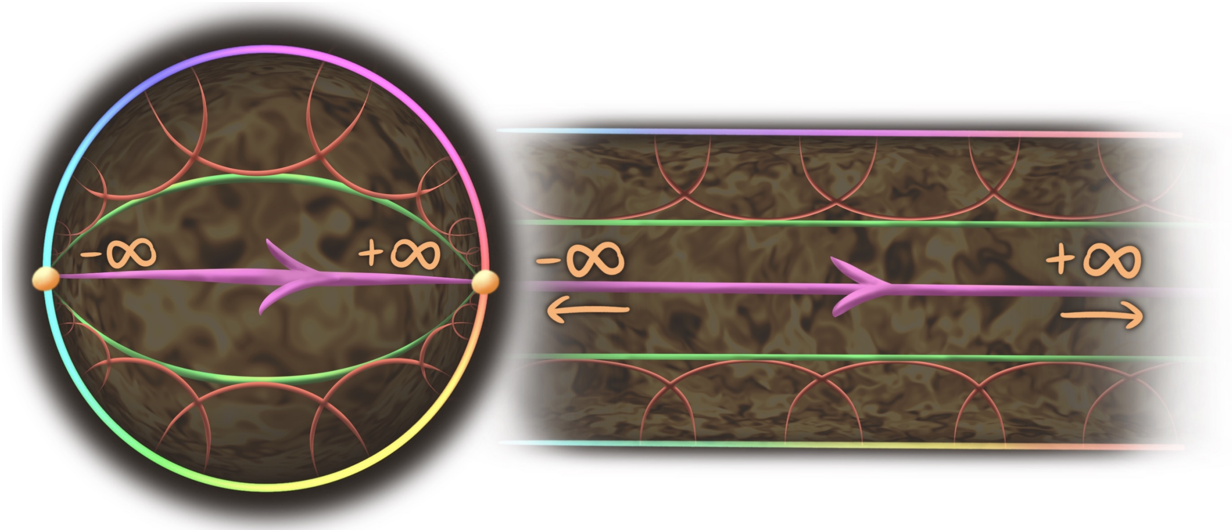


Figure 6: *String art for a hyperbolic Möbius transform in the Poincare disc model and the band model. The rainbow boundary is hyperbolic infinity, the red curves are geodesic strings, the green curve is the envelope of the strings, and the purple geodesic is the axis of the hyperbolic translation.*

with a constant negative curvature metric that scales distances to zero at the top and bottom line. Moreover, the metric is translation invariant under $z \to z + b$ for real b. Hyperbolic infinity consists of the lines $\mathrm{Re}\, z = \pm 1$, along with two points at infinity of the real line, $\pm\infty$. A geodesic stretching between two points on the same boundary line is not a semicircle, but does still form a convex Euclidean shape which does not cross the real axis. In particular, each geodesic has a unique point closest to the middle of the band.

Every hyperbolic translation is conjugate to an Euclidean translation $z \to z + b$ in the band model. The fixed points of this action are $\pm\infty$, and the geodesic connecting the two fixed points is the real axis.[3] The string art consists of geodesics connecting z with $z + b$ for $\mathrm{Re}(z) = \pm 1$. Every string is the same shape, so has the same minimal distance to the real axis. Hence, the envelope consists of two parallel lines, with the same distance to the real axis. This envelope is the locus of points of fixed hyperbolic distance to the geodesic connecting $\pm\infty$, which is called a *Hypercircle*.

Converting back to the Poincare disc model, this hyperbolic Möbius transform has fixed points at the left and right extremes of the unit disc, and the geodesic connecting these two points is again the real axis. The hypercircle is drawn as two circle arcs connecting the fixed points of the hyperbolic Möbius transform, mirror to one another about the real axis (see figure xxx). In general, hypercircles about a given geodesic are drawn in the Poincare disc are drawn as a pair of circular arcs sharing the endpoints as the given geodesic. See the string art for the hyperbolic Möbius transform in figure 6.

Once again, we realize the string art of all hyperbolic Möbius transforms by acting on the string art described above by a hyperbolic symmetry. Since hyperbolic symmetries preserve the hyperbolic distance to geodesics, the image of the envelope is still a hypercircle. Hence, *the envelope of any orientation preserving hyperbolic Möbius transform is a hypercircle*. (See figure 7c for a non-orientation preserving transform, which does not have an envelope). The only conjugacy invariant of hyperbolic translations is the translation amount, which controls the angle of the envelope at infinity. For example, small translations will yield strings that stay far from the translation axis, so the envelope will have a wide angle.

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[3]Compare this to the parabolic case, where the parabolic transform was realized as an Euclidean translation with single fixed point at Euclidean infinity. The hyperbolic construction realizes a hyperbolic transform as an Euclidean translation, with two fixed points at Euclidean infinity.

Generalities about Möbius transforms

Figure 6 summarizes the results of this section. All Möbius transforms share a uniform description for their envelopes in the Poincare disc model:

- Elliptic transforms have circular envelopes that remain within the boundary circle.
- Parabolic transforms have circular envelopes tangent to the boundary circle.
- Hyperbolic transforms have circular envelopes intersecting the boundary circle.

The points of intersection with the boundary circle are the fixed points of the transform. The origin of the two components of the hyperbolic transform envelope is clear after extending the strings to full circles (as in Figure 6). Then, every envelope has two components, each circular, one being the inversion of the other about the boundary circle. For elliptic and parabolic transforms this second envelope lies outside the disc, but for hyperbolic transforms both envelopes intersect the disc.

This description of string art reconstructs the hyperbolic symmetry from its action at infinity. Starting with an elliptic Möbius transform, you plot a few strings, and find an envelope circle tangent to all the strings. The center of this circle gives the pivot of hyperbolic rotation, and the radius determines the angle. In fact, only three strings are needed to find the envelope, as three circles have a unique common tangent fourth circle. This is not surprising, as a Mobius transform is uniquely defined from its action on three points, and the transform uniquely determines the hyperbolic symmetry. However, it does give an explicit hyperbolic compass-and-straightedge construction for the pivot point in terms of the boundary action. Likewise, the axis and length of a hyperbolic translation fall out from the string art. This shows again that the action at hyperbolic infinity determines the whole action.

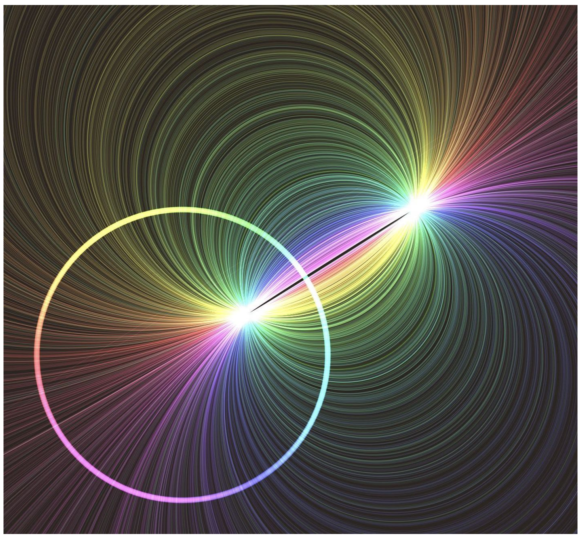
Some Möbius sit apart from the above classification. For example, inversion around a point (Figure 7a) comes from rotation, where the envelope circle reduces to radius zero. There are also orientation-reversing hyperbolic transforms (Figure 7c). One such transform is reflection around a geodesic, such as Figure 7b, which shows reflection across the geodesic connecting the two circles of size zero. Or, there are singular Möbius transforms, like those that send every point to zero (Figure 7d).

The arguments establishing the shape of the envelopes are extrinsic, relying crucially on the Euclidean geometry and symmetries of various models of the hyperbolic plane. One can formalize these arguments into proofs without much trouble. We can also prove these results intrinsically, without reference to any model. The intrinsic proof is is more elegant, but requires more background. It is given in the appendix of an online, extended version of this paper. [2]

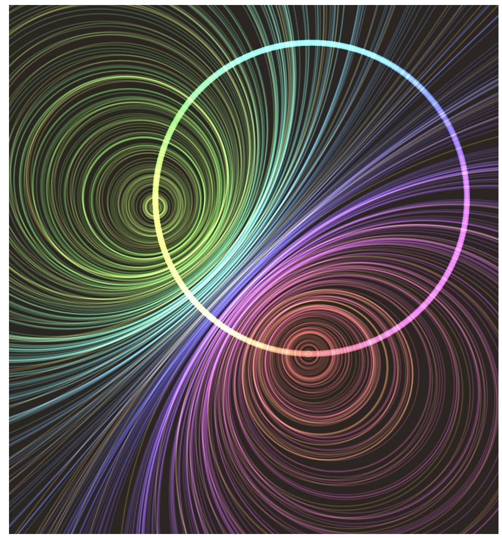
Aesthetic applications

On top of the mathematical content, string art for Möbius transforms can look striking. The visual effect is stronger in the Poincare disc model than the Beltrami-Klein model, as orthogonal circles move details to the center which would otherwise be relegated to the edges of the disc. This was noted in [3], which (in our language) investigated string art for multiplication and addition functions in the Poincare disc model. The visuals are further amplified when the strings are extend from circular arcs to full circles, shown in Figure 7.

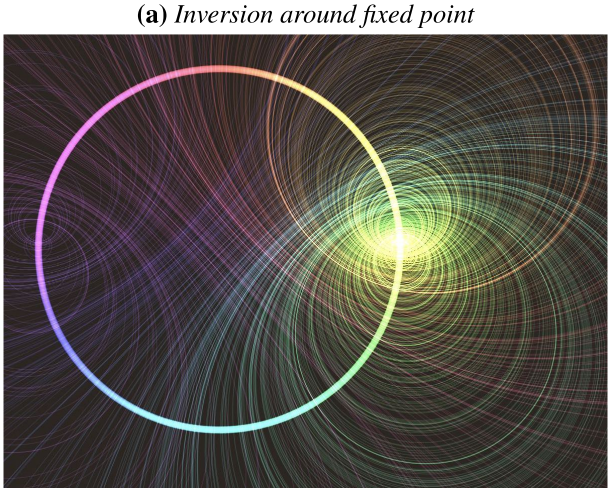
We can also choose functions purely for their artistic merit. Figure 8 shows string art inspired by the Mandelbrot set, a fractal built from the dynamics of the holomorphic map $z \to z^2 + c$. The Mandelbrot set plots the results of repeatedly squaring a complex number and applying a parabolic Möbius transform. Restricting from the Riemann sphere to the unit circle, $z \to z^2$ multiplies angles by 2, and shifting by c is a Möbius transform. A hyperbolic analog is the map $\theta \to 2$ Möbius(θ). Figure 8 shows the string art for this function, iterated various numbers of times.



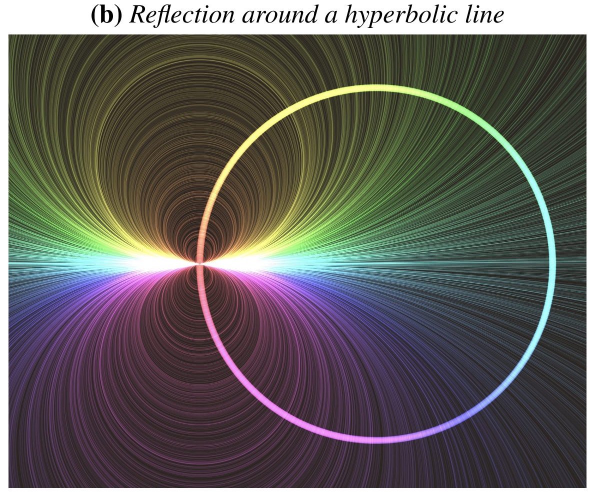
(a) *Inversion around fixed point*



(b) *Reflection around a hyperbolic line*



(c) *An orientation-reversing hyperbolic transform*



(d) *The zero map*

Figure 7: *String art of exceptional Möbius transforms*

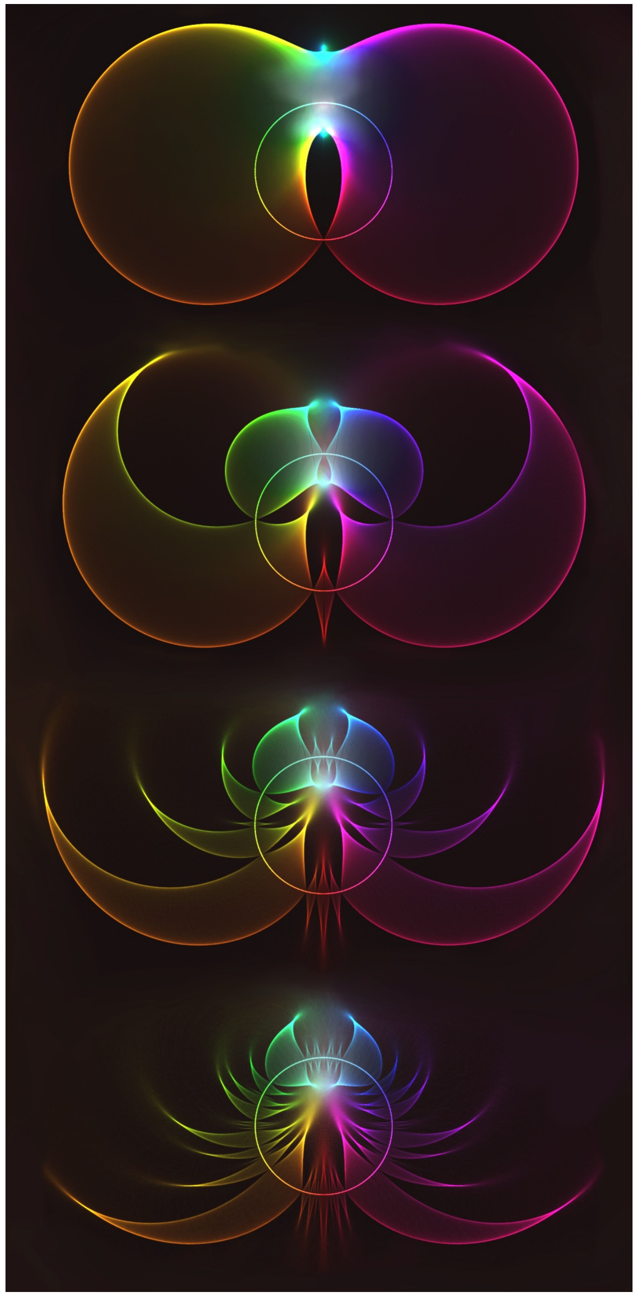


Figure 8: *A fractal derived from the string art of an iterated Möbius transform and angle doubling map,* $\theta \to 2$ *Möbius*(θ). *From the top down, we depict 1, 3, 5, and 7 iterations.*

Summary and Future work

Hyperbolic string art reconstextualizes string art in the framework of hyperbolic geometry, suggesting new, natural functions and ways to render the strings. In this paper, we explored only the simplest functions in hyperbolic geometry. We found that the envelopes of string art are natural hyperbolic objects, which reflect the symmetries of the hyperbolic plane. They visualize hyperbolic geometry in a novel way, without sacrificing artistic effect. However, hyperbolic geometry is not satisfied with mere symmetries. Maps from the circle at infinity to itself can capture much more information.

Deforming a hyperbolic lattice induces a fractal map on the circle at infinity, and the space of such maps captures the space of deformed lattices. In fact, it contains all deformation spaces of all lattices, and thus every possible hyperbolic structure on a closed surface. Potentially, drawing the string art for these maps could help visualize this space. Indeed, the boundaries of this space are paramertized by geodesic laminations, objects which already look similar to hyperbolic string art. Finally, some areas of physics need to understand the space of hyperbolic surfaces in great detail. The space of maps from hyperbolic infinity to itself is thus a central object in, ironically enough, string theory. Perhaps one day, string art can help us visualize string theory.

Acknowledgements

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