Extending TQFT

https://arxiv.org/abs/1004.2307

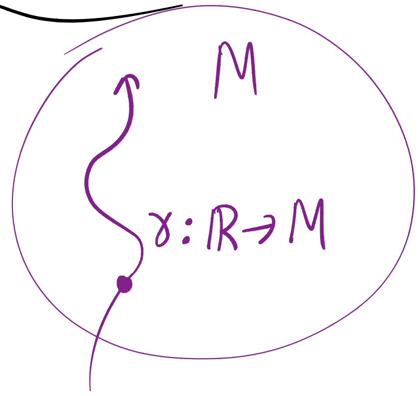
From section 2.1-2.2 of Kapustin, Topological field theory, higher categories, & their applications

Level -1: every point gets a # $\quad Z^0(\cdot) = n$

Level 0: Quantum mechanics

Classically, a particle is a point moving through spacetime

Quantum mechanically, we integrate over all possible trajectories



$$\underbrace{Z^1(|)}_{\text{what QFT assigns to } \mathbb{R}} = \int_{Maps(\mathbb{R},M)} e^{i\underbrace{S(\gamma)}_{}} \text{ "action functional" } \in \mathbb{C}$$

$Z^1(\bullet) = \mathcal{H}$ is a hilbert space: The space of Quantum states

This is the "vector space of boundary conditions"

imagine the source manifold is $\underset{v_0 \qquad\qquad v_1}{\overset{[0,t] \in \mathbb{R}}{[\text{————}]}}$: To define the partition fn, we specify

$v_0, v_1 \in \mathcal{H}$: starting & ending quantum states

$Z\big(\underset{v_0}{[}\text{——}\underset{v_1}{]}\big) \in \mathbb{C}$ measures the transition amplitude from v_0 to v_1 after evolving from time 0 to t

if evolution is governed by hamiltonian $H: \mathcal{H} \circlearrowleft$, then $Z^1\big(\underset{v_0}{[}\text{——}\underset{v_1}{]}\big) = \langle e^{i \cdot t \cdot H} v_0, v_1 \rangle$

Putting these together, we define
$$Z^1([\text{——}]) : Z^1(\bullet) \to Z^1(\bullet)$$
$$\mathcal{H} \longmapsto \mathcal{H}$$
$$Z^1([0,t]) = e^{itH}$$

we say the 1D QFT is topological if it is independent of length on the source: i.e. independent of t. in this case, this means $H=0$.

$\Rightarrow Z^1(\bullet\text{——}\bullet) = \text{identity}$

path integral over possible states

$$Z(\underset{S^1}{\bigcirc}) = \sum_{e_i \text{ o.n.b of } \mathcal{H}} Z(\bullet\text{—}\bullet) = \sum_{e_i} \langle e_i, Id\, e_i \rangle = Tr(Id) = \dim \mathcal{H}$$

In summary: $Z^1(\bullet) = \mathcal{H}$ $\quad Z^1(\underbrace{\bullet \cdots \bullet}_{n}) = \mathcal{H}^{\otimes n}$

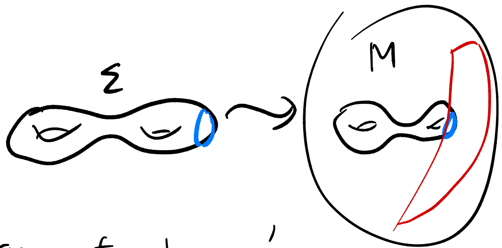
$Z^1(\,\mathbf{I}\,) : Z^1(\bullet) \to Z^1(\bullet)$

$Z^1(O) = \mathrm{Tr}(Z^1(\,\mathbf{I}\,)) = \dim \mathcal{H} = Z^0(\bullet)$ "dimensional reduction"

Level 1: String theory

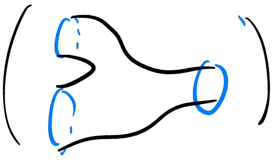
$Z(\underset{\text{surface}}{\Sigma}) \in \mathbb{R}$

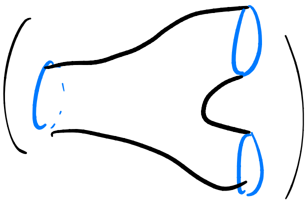
defines a 0d TQFT



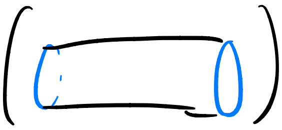
$\underset{S^1}{Z^2(O)}$ gives hilbert space $\mathcal{H}$ "space of boundary conditions" (branes)

If $\partial\Sigma^2 = \overline{N}^1_{\text{in}} \sqcup M^1_{\text{out}}$, then $Z(\Sigma) : Z(N^1) \to N(M)$. Z only depends on the topology of Σ

$Z^2($  $)$: $Z^2(OO) \to Z^2(O)$ $\quad \mathcal{H}^{\otimes 2} \to \mathcal{H}$ product

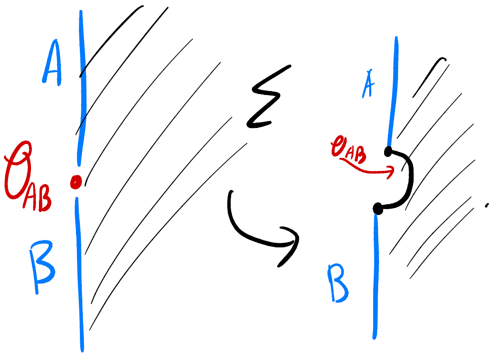
$Z^2($  $)$: $Z^2(O) \to Z^2(OO)$ $\quad \mathcal{H} \to \mathcal{H}^{\otimes 2}$ coproduct

$\mathcal{H}$ is a hopf algebra

$Z^2($  $)$: $Z^2(O) \to Z^2(O)$ dimensional reduction

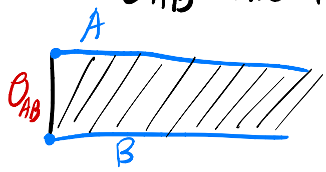
$Z^1(\,\vdash\!\!\dashv\,) : Z^1(\bullet) \to Z^1(\bullet)$

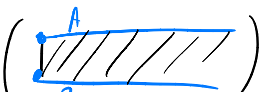
What happens when 2 boundary conditions meet?



$\mathcal{O}_{AB}$ gives data necessary to give a well-defined path integral

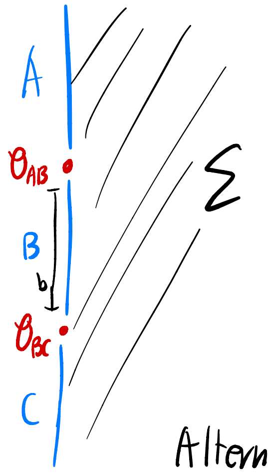
the data $\mathcal{O}_{AB}$ forms a vector space: Expanding it out...



$Z^2($  $) = Z^2(\mathbf{I}^A_B) = V_{AB}$ hilbert space

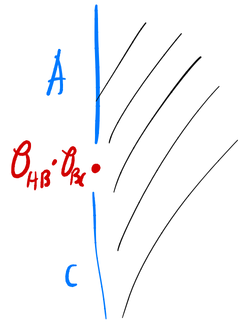
$Z^1_{AB}(\,\text{—}\,) = Z^1_{AB}(\bullet) = V_{AB}$

there is a "fusion product" on these vector spaces



limit as length $b \to 0$ $\longrightarrow$

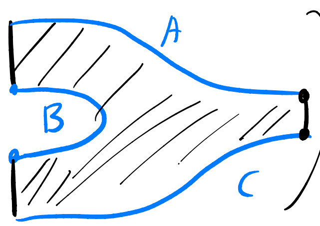
exists as TQFT is invariant under b



"combining defects"

$$V_{AB} \otimes V_{BC} \to V_{AC}$$

Alternative Hyroglyph

$Z^2\Big($  $\Big) : Z^2\left(\begin{matrix} I^A_B \\ I^B_C \end{matrix}\right) \longrightarrow Z^2\left(I^A_C\right)$

$$V_{AB} \otimes V_{BC} \longrightarrow V_{AC}$$

so we see what 2D TQFT assigns to a point:

$Z^2(\bullet)$ = "category of boundary-changing local operators"

objects = boundary conditions $\in Z^2(O)$

morphisms $A \to B$ = elements of V_{AB}

composition = "fusion product" $V_{AB} \times V_{BC} = V_{AC}$

---

level 2: fully extended 3D TQFT

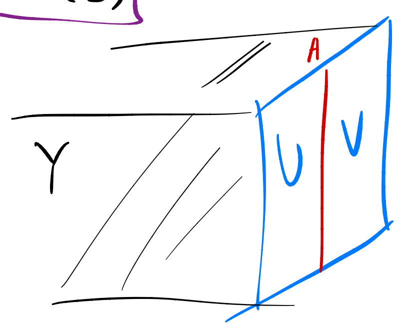
expect	$Z^3(Y^3)$	$Z^3(\Sigma^2)$	$Z^3(S^1)$	$Z^3(pt)$
	number	vector space	category	2-category

we can understand a 3D TQFT using dimensional reduction

$Z^3(\Sigma \times S^1) \rightsquigarrow Z^2(\Sigma)$ effective 2D TQFT

$Z^3(S^1) \rightsquigarrow Z^2(pt)$ category of branes

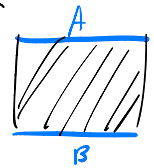
use this to define $Z^3(S^1)$

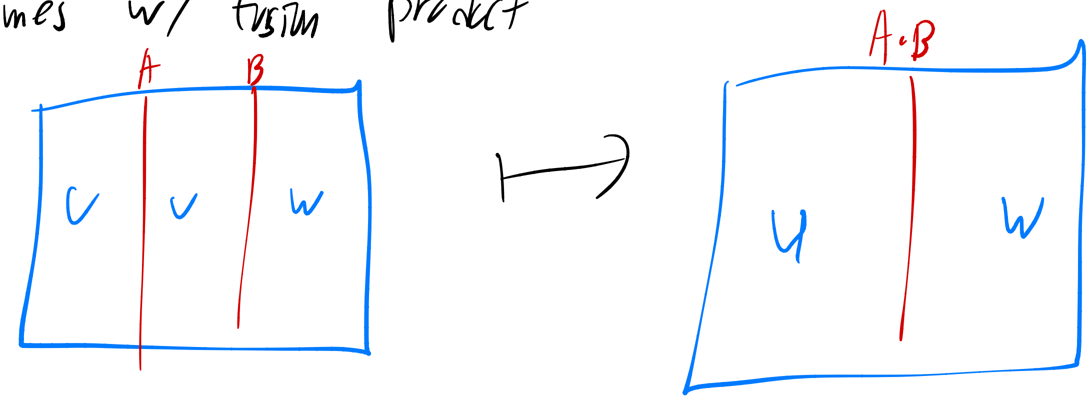


$\partial Y = \Sigma$

U, V are boundary conditions on Y

a line operator A converts between boundary conditions

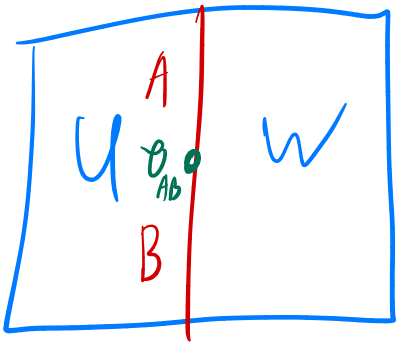
as before, $A \in W_{UV} = Z^3\Big($  $\Big)$ vector space comes w/ fusion product



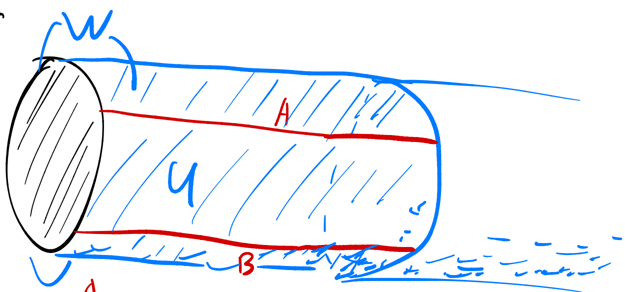
$$W_{UV} \otimes W_{VW} \longmapsto W_{UW}$$

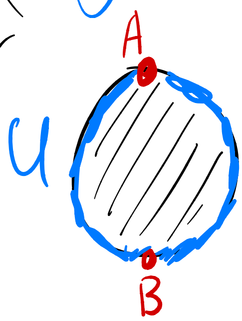
$\Rightarrow Z(\,|\,)$ gives a category of line operators

now we can put point operators on the line, a defect of line operators

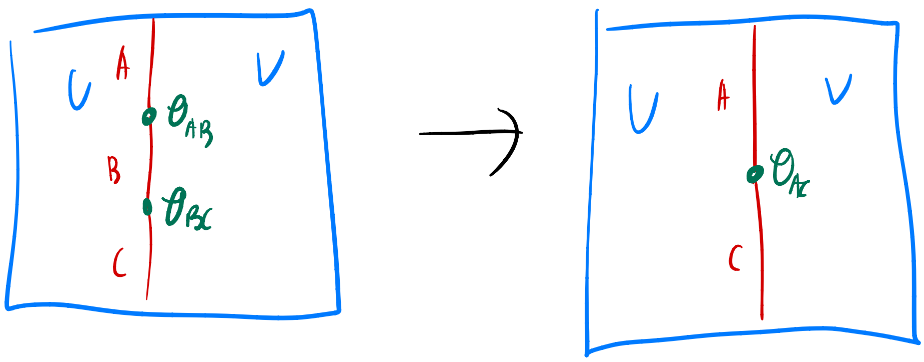


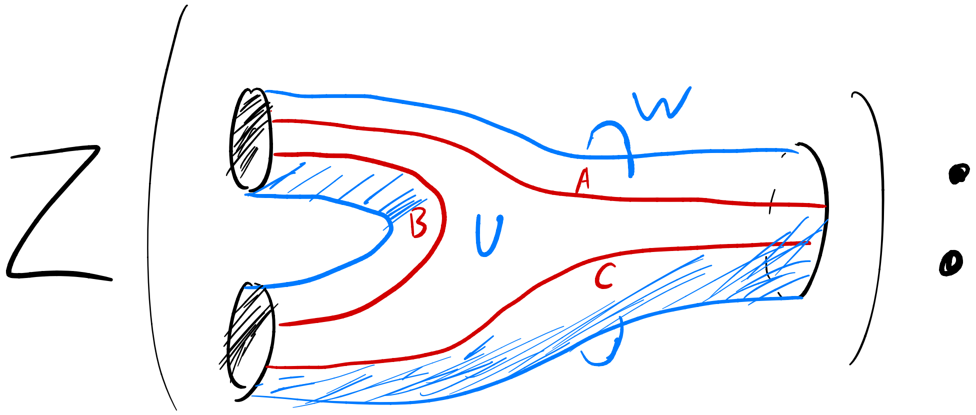
let V_{AB} be the vector space of operators interpolating A & B

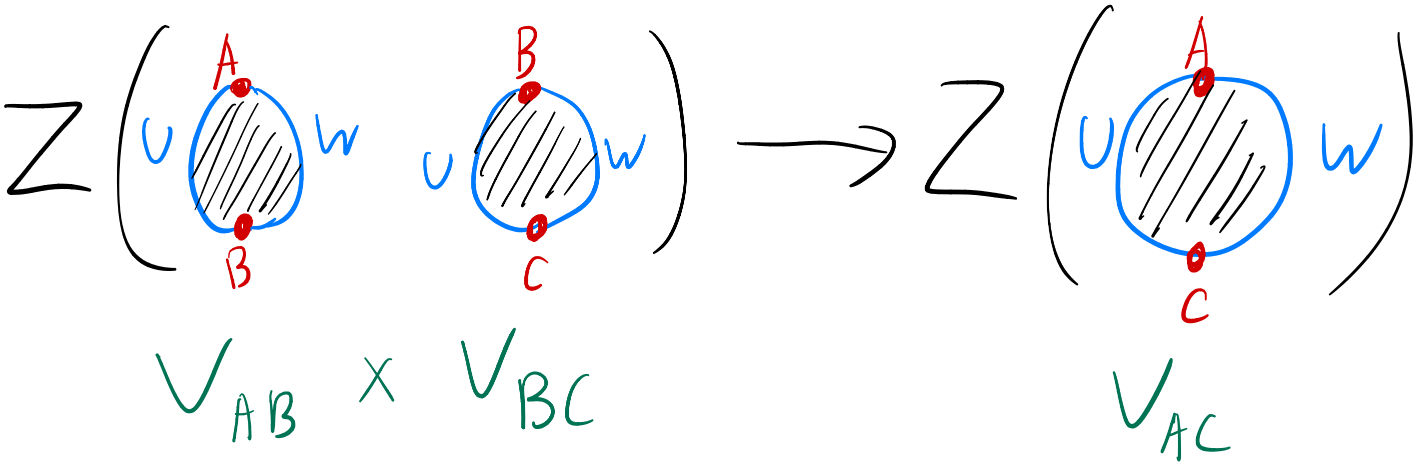
$V_{AB} = Z^3\Big($  $\Big)$

$= Z^3\Big($  $\Big)$

These have a fusion product $V_{AB} \times V_{BC} \to V_{AC}$

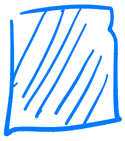




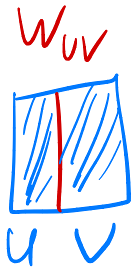


$Z^3(pt) =$ 2 category with:

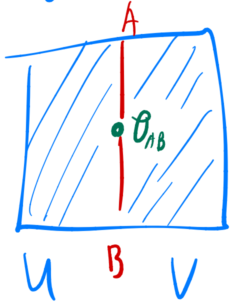
Objects: boundary conditions



Morphisms: vector space of line defects



2-Morphism: for $A, B \in W_{UV}$,
vector space of point defects V_{AB}



Cobordism hypothesis (Baez-Doran, proven by Lurie):
A fully extended TQFT is determined by its value at a point.