2/8

(Follows Kac's expository paper & Carmeli, Caston, Fioresi's text)

First Definitions

- A super vector space is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$
  - $\exists$ parity-change functor $\Pi V = V_1 \oplus V_0$
- A superalgebra " algebra $A = A_0 \oplus A_1$
- If the superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ has product $[\, , \,]$ s.t.

  $[x,y] = -(-1)^{|x||y|}[y,x]$

  $[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]$

  it is called a Lie superalgebra
- A homomorphism of super things must preserve the grading
  - the category of SVS's admits an inner hom

    $\underline{\mathrm{Hom}}(V,W) = \mathrm{Hom}(V,W) \oplus \mathrm{Hom}(V, \Pi W)$ & inner dual $V^* = \underline{\mathrm{Hom}}(V,k)$
  - tensor category via $(V \otimes W)_0 = V_0 \otimes W_0 \oplus V_1 \otimes W_1$, $(V \otimes W)_1 = V_0 \otimes W_1 \oplus V_1 \otimes W_0$

Ex: Any Lie algebra $\mathfrak{g}_0$ is a (purely even) Lie superalgebra

Ex: $\mathfrak{gl}(m|n) \simeq \mathfrak{gl}(k^{m|n}) := \underline{\mathrm{Hom}}(k^{m|n}, k^{m|n})$

- $k^{m|n} = k^m \oplus k^n$
- $\mathfrak{gl}(m|n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; \begin{matrix} A \in \mathfrak{gl}(k^m) & B \in \mathrm{Mat}_{m \times n} \\ C \in \mathrm{Mat}_{n \times m} & D \in \mathfrak{gl}(k^n) \end{matrix} \right\} = \underbrace{\left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}}_{\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)} \oplus \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}$
- $[X,Y] = XY - (-1)^{\bar{X}\bar{Y}} YX$

Def: A Lie superalgebra is of classical type if it is simple, $\mathfrak{g}_1 \neq 0$, and the representation $\mathfrak{g}_0 \curvearrowright \mathfrak{g}_1$ is completely reducible.

Thm: A Lie superalgebra of classical type is $\simeq$ to one of the following:

(1) $A(m,n)$

(2) $B(m,n)$, $C(n)$, $D(m,n)$

(3) $F(4)$, $G(3)$

(4) $D(2,1;\alpha)$ (5) $P(n)$, $Q(n)$

$\mathfrak{sl}(m+1|n+1)$ if $m \neq n$

$A(m,n)$: $\mathfrak{sl}(m|n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} = X \in \mathfrak{gl}(m|n) \;\middle|\; \begin{array}{l} \mathrm{str}(X) = 0, \text{ i.e.} \\ \mathrm{tr}(A) - \mathrm{tr}(D) = 0 \end{array} \right\}$

- simple unless $m = n$, in which case we must quotient by $\langle I_{2m} \rangle$

B, C, D: $\mathfrak{osp}(m|2n)$ = matrices in $\mathfrak{gl}(m|n)$ that leave an even, nondegenerate, supersymmetric bilinear form invariant

- supersymmetric means $\langle u,v \rangle = (-1)^{|u|\cdot|v|} \langle v,u \rangle$ & $\langle v_0, v_1 \rangle = 0$
- $\mathfrak{g}_0 = \mathfrak{so}(m) \oplus \mathfrak{sp}(2n)$
- $B(m,n) = \mathfrak{osp}(2m+1, 2n)$ $\quad m \geq 0, \; n > 0$
- $C(n+1) = \mathfrak{osp}(2, 2n)$ $\quad n > 0$
- $D(m,n) = \mathfrak{osp}(2m, 2n)$ $\quad m \geq 2, \; n > 0$

$D(2,1;\alpha)$: Family of $(9|8)$-dim'l deformations of $D(2,1)$ depending on a parameter $\alpha \in k - \{0,-1\}$

$F(4)$: $(24|16)$-dim'l with even part $\mathfrak{sl}(2) \oplus \mathfrak{so}(7)$

$G(3)$: $(17|14)$-dim'l with even part $\mathfrak{sl}(2) \oplus G_2$

$P(n)$, $Q(n)$ harder to define

Definition A.1.4. We define the *strange series* $P(n)$ as

$$P(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right\} \subset \mathrm{gl}(n+1|n+1),$$

where $A \in \mathrm{sl}(n+1)$, B is symmetric and C skew-symmetric.

The *strange series* $Q(n)$ is defined as follows. Set

$$q(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\};$$

$\mathrm{sq}(n)$ are the matrices in $q(n)$ with $\mathrm{tr}(B) = 0$ and $Q(n-1) = \mathrm{psq}(n) = \mathrm{sq}(n)/kI_{2n}$, i.e.,

$$Q(n-1) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid B \in \mathrm{sl}_n \right\} / kI_{2n}.$$

$Q(n)$ is a subquotient of

Ex: $\mathfrak{q}(n)$ = subsuperalgebra of $\mathfrak{gl}(n|n)$ leaving invariant an odd automorphism p s.t. $p^2 = -1$.

- If $p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $\mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \;\middle|\; A, B \in \mathfrak{gl}(k^n) \right\}$
- $\mathfrak{q}(n)_0 = \mathfrak{gl}(n)$

Def: A representation (or $\mathfrak{g}$-module) of $\mathfrak{g}$ is a (purely even) hom $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ for a svs V

- A submodule is $\mathbb{Z}_2$-graded
- Irreducible means no nontrivial submodules

Schur's Lemma: Let $V = V_0 \oplus V_1$ be a svs, and $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ an irrep. Let $\mathrm{End}(\rho) = \{ T \in \mathfrak{gl}(V) \mid [T, \rho(x)] = 0 \;\; \forall x \in \mathfrak{g} \}$. Then either:

(1) $\mathrm{End}(\rho) = \langle 1 \rangle$

(2) $\dim V_0 = \dim V_1$, and $\mathrm{End}(\rho) = \langle 1, A \rangle$ for an invertible $A \in \mathfrak{gl}(V)$,

Ex: Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{g}_0 = \langle x \rangle$, $\mathfrak{g}_1 = \langle \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \rangle$

- $[\alpha_i, \beta_i] = e$ $\forall i$, and all other brackets vanish.
- Consider the Grassmann superalgebra $\Lambda(n)$ generated by n linearly independent odd variables $\xi_1, \dots, \xi_n$
  - Say it is commutative because the commutative super bracket satisfies $[a,b]=0$ $\forall a,b \in \Lambda(n)$
- For $c \in k^*$, define the representation ρ_c of $\mathfrak{g}$ in $\Lambda(n)$:

$$\rho_c(\alpha_i) = \frac{\partial}{\partial \xi_i} \qquad \rho_c(\beta_i) = c\,\xi_i \qquad \rho_c(e) = c$$

- This representation is irreducible! (why?) 2^n-dim'l, even though $\mathfrak{g}$ is solvable!
  - $\rho_c(e) = c$ means an irrep is spanned by monomials
  - $\rho_c(\alpha_i)$, $\rho_c(\beta_i)$ move us down, up (respectively)

Ex: Define $\mathfrak{g}' = \mathfrak{g} \oplus \langle \gamma \rangle$ for $[\mathfrak{g}, \gamma] = 0$ & $[\gamma, \gamma] = e$

- Define $\Lambda'(n) = \Lambda(n) \otimes k[\varepsilon]$ for ε an odd variable s.t. $\varepsilon^2 = c \in k^*$
- Define the rep ρ_c' of $\mathfrak{g}'$ in $\Lambda'(n)$ by

$$\rho_c'(x)(u \otimes v) = (\rho_c(x)\,u) \otimes v \qquad \text{for } x \in \mathfrak{g},\ u \in \Lambda(n),\ v \in k[\varepsilon]$$
$$\rho_c'(\gamma)(u \otimes v) = c(1 \otimes \varepsilon)(u \otimes v)$$

- $\dim(\text{this irrep}) = 2^{n+1}$, even though $\mathfrak{g}'$ is solvable!

Exercise: These examples satisfy cases (1) + (2), resp, of Schur's lemma.

Elliot's talk:

Quantum field theory: "fields" on $(M, \langle , \rangle)$ Riemannian mfld

- $M \to X$ (X is "space of fields")
- Local model $M = \mathbb{R}^{3,1} = (\mathbb{R}^4, \langle , \rangle)$ where $\langle , \rangle$ has signature $-,+,+,+$
- Symmetries of QFT: symmetries of $\mathbb{R}^{3,1}$ $\oplus$ symmetries of X

infinitesimal

$$\text{symmetries of } \mathbb{R}^{3,1} = \mathrm{iso}(3,1) = \text{Poincaré algebra} \cong \underset{\text{rotations}}{\mathrm{so}(3,1)} \ltimes \underset{\text{translations}}{\mathbb{R}^{3,1}}$$

symmetries of X = $\mathfrak{g}$ some other Lie group

Def: A supersymmetry algebra is a type of LSA $\mathfrak{h} = \underset{\text{"super Poincaré"}}{\mathrm{siso}(3,1)} \oplus \underset{\text{"internal symmetries" i.e. "symmetries of X"}}{I}$

- $\mathrm{siso}(3,1) =: \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$
  - $\mathfrak{g}_0 = \mathrm{iso}(3,1)$
  - $\mathfrak{g}_1$ = Spin representation $S \curvearrowleft \mathfrak{g}_0$

Spin reps:

- $\mathfrak{iso}(3,1) \simeq \mathfrak{so}(3,1) \ltimes \mathbb{R}^{3,1}$
  - $[\mathfrak{so}, \mathfrak{so}] \subset \mathfrak{so}$
  - $[\mathbb{R}^{3,1}, \mathbb{R}^{3,1}] = 0$
  - $[\mathfrak{so}, \mathbb{R}^{3,1}] \subset \mathbb{R}^{3,1}$
- $\mathfrak{so}(3,1) \simeq \mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
- reps of " classified by HW $\quad -3/2 \quad -1 \quad -1/2 \quad 0 \quad 1/2 \quad 1 \quad 3/2$
  - spin rep is the HW $\lambda = \frac{1}{2}$ rep!
- it does NOT come from a rep of $SO(3,1)$ (bc its not simply connected)