# talk at line: Hyper toric geometry

**Physics context:**

3D $N=4$ SUSY QFT $\rightsquigarrow$ space of vacua — Coulomb branch $\oplus$ higgs branch

(defined by "matter" $V$ & "gauge group" $G$

quaternionic vector space $V = N \oplus N^*$ & group action by $G$

| $(V, G)$ | Higgs | Coulomb |
|---|---|---|
| $\mathbb{C}^{n+1}, U(1)$ | $T^*\mathbb{P}^n$ | $\widetilde{\mathbb{C}^2/\mathbb{Z}_{n+1}}$ |
| $\mathbb{C}^{n+1}, U(1)^n$ | $\mathbb{C}^2/\mathbb{Z}_{n+1}$ | $T^*\mathbb{P}^n$ |

3D mirror symmetry

both higgs & coulomb get hyperkahler structures

$T^*\mathbb{P}^n$, $\widetilde{\mathbb{C}^2/\mathbb{Z}_{n+1}}$ are hypertoric varieties: they carry a large torus action which preserves their hyperkahler structure.

example: Atiyah-Hitchin manifold: classical vacua $\mathbb{R}^3 \times S^1$ recieves quantum corrections

- @ infinity, looks like nontrivial $S^1$ bundle over $S^2$
- Gains nontrivial topology away from infinity

hyper toric structure on $T^*\mathbb{P}^1 \simeq \mathbb{C}^2/\mathbb{Z}_2$

Hyper toric varieties generalize this coulomb branch to other gauge grps.

# Toric Geometry

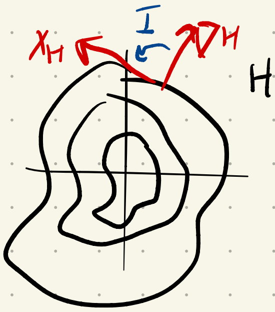
Let $(M, \omega)$ be a kahler manifold

$$g \longleftrightarrow I \quad \text{complex structure } I^2 = -1$$
$$\omega_I \quad \text{nondegenerate 2-form}$$

for any function $H: M \to \mathbb{R}$, define hamiltonian vector field $X_H$

$dH = \omega(X_H, \bullet)$

for kahler manifolds $X_H = I(\nabla H)$




$X_H$ preserves symplectic form:

$\mathcal{L}_{X_H} \omega = 0$ true up to topology


Suppose $U(1)$ acts on $(M, \omega)$ by symplectomorphisms:

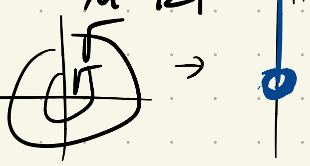
generated by vector field $V$

$U(1)$ is a hamiltonian group action if $V = X_\mu$ for some $\mu: M \to \mathbb{R}$

always true, up to topological constraint

$\mu$ is moment map

$(\mathbb{C}, dx \wedge dy)$ $\mu = |z|^2$

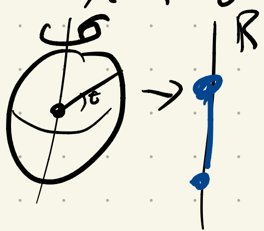



$U(1)$ action: $\theta: z \mapsto e^{i\theta} z$

$(S^2, \omega)$ area form

$U(1)$ rotates sphere abt central axis

$\mu = R\cos\theta$ "height"



similarly, $T = U(1)^k \circlearrowright (M, \omega)$ has infitesimal action $V_t$ vector field for $t \in \underline{t}$ lie algebra of $T$, w/ $V_{t_1} + V_{t_2} = V_{t_1+t_2}$, $[V_{t_1}, V_{t_2}] = V_{[t_1,t_2]} = 0$

this group action is <u>hamiltonian</u> when $V_t = X_{\mu(t)}$ for $\mu(t): M \to \mathbb{R}$
$\mu: M \to \underline{t}^*$

<u>e.g</u> $\mathbb{C}^n = (z_1, \dots, z_n)$ has $U(1)^n$ action rotating each factor

the moment map is $\mu(z_1, \dots, z_n) = (|z_1|^2, \dots, |z_n|^2) \in \underline{t}^* \simeq \mathbb{R}^n$
just rotation moment map for each factor!

(in general, a moment map for $G \circlearrowright M$ is $\mu: M \to \underline{g}^*$ s.t $X_{\mu(g^*)}$ generates $G$ action
$\mu$ intertwines $G \circlearrowright M$ with the coadjoint action of $G$ on $\underline{g}^*$)

## <u>Symplectic Reduction</u>

in symplectic geometry, groups act twice

consider $T = U(1)^k \circlearrowright (M^{2n}, \omega)$ symplectic group action. want to form the space of orbits. This is not a manifold usually.

however, $T$ abelian $\Rightarrow$ $\mu(g \cdot x) = Ad_g \mu(x) = \mu(x)$ for $g \in T$. orbits of $T$ lie in level sets $\mu^{-1}(\xi)$ for $\xi \in \underline{t}^*$. might as well restrict to level sets.

(for general $G$, for $\mu^{-1}(\xi)$ to be invariant, need $Ad\ G$ to act trivially on $\xi$. so, need $\xi \in Z(\underline{g}^*)$, the center)

Define <u>symplectic</u> <u>reduction</u> $M /\!/_{\xi} T = \mu^{-1}(\xi)/T$

$\xi$ is regular value of $\mu$ $\Rightarrow$ $T$ acts freely on $\mu^{-1}(\xi)$
$\Rightarrow \mu^{-1}(\xi)/T$ smooth manifold!

$$\dim \mu^{-1}(\xi)/T = \dim(\mu^{-1}(\xi)) - \dim T = \underbrace{\dim M}_{2n} - \underbrace{\dim \underline{t}^*}_{k} - \underbrace{\dim T}_{k} = 2n - \underline{\underline{2k}}$$

dimension reduced twice! first by moment map, then by group.

$M /\!/ T$
2 slashes, b.c $T$ acts twice.

$\mu^{-1}(\xi)/T$ inherits a symplectic structure from $M$.

**Example:** $\mathbb{C}^2$ carries $U(1)$ action $(z_1, z_2) \mapsto e^{i\theta}(z_1, z_2)$ w/ moment map $\mu(z_1, z_2) = |z_1|^2 + |z_2|^2$

$$\mathbb{C}^2 /\!/_1 U(1) = \mu^{-1}(1)/U(1) = S^3/U(1) = \mathbb{P}^1 \simeq S^2$$

hopf fibration!

$$\begin{array}{ccc} U(1) & \hookrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

(show hopf fibration toy, show website)

# Toric Manifolds

Def: a toric manifold is a symplectic mfld $(M,\omega)$, $\dim M = 2n$ w/ a half-dimensional torus action $T = U(1)^n \circlearrowright (M,\omega)$

this is the maximal dimension of torus since in symplectic geo, grps act twice,

Basic example: $\mathbb{C}^n$ w/ natural $U(1)^n$ action scaling each factor

big family of other examples:

say $T^k \simeq U(1)^k$ acts on $\mathbb{C}^{n+k}$ as subtorus of $U(1)^{n+k}$.

fits into s.e.s $\underset{\dim k}{T^k} \hookrightarrow \underset{\dim n+k}{U(1)^{n+k}} \twoheadrightarrow \underset{\dim n}{U(1)^{n+k}/T^k} = T^n$

then, $\underset{\dim = 2n}{\mathbb{C}^{n+k}/\!/T^k}$ has residual $\underset{\dim = n}{T^n}$ action. so $\mathbb{C}^{n+k}/\!/T^k$ is toric!

example ($\mathbb{P}^n$): for the diagonal embedding $U(1) \hookrightarrow \overbrace{U(1) \times \cdots \times U(1)}^{n+1}$, we get the group action $U(1) \circlearrowright \mathbb{C}^{n+1}$ where $\theta: \vec{z} \mapsto e^{i\theta}\vec{z}$

moment map is $\mu = \sum |z_i|^2$, so $\mu^{-1}(1)/U(1) = \mathbb{P}^n$

Thm (Delzant correspondence) all compact toric manifolds are $\mathbb{C}^{n+k}/\!/T^k = \mu^{-1}(\xi)/T^k$ for some choice of $T^k \subset U(1)^{n+k}$, $\xi \in \mathfrak{t}^{k*}$

This information is encoded in a Delzant Polytope in $\mathbb{R}^n$

Polytope w/ some rationality properties

or...

$$\begin{tikzcd}
& \mathbb{C}^{n+k}/\!/T^k \arrow[dl, leftrightarrow, "(1)"'] \arrow[dr, leftrightarrow, "(3)"] & \\
\text{Toric Manifolds} \arrow[rr, leftrightarrow, "(2)"] & & \text{Delzant Polytopes}
\end{tikzcd}$$

This root gives a very explicit construction for any toric manifold!

(1): shown above

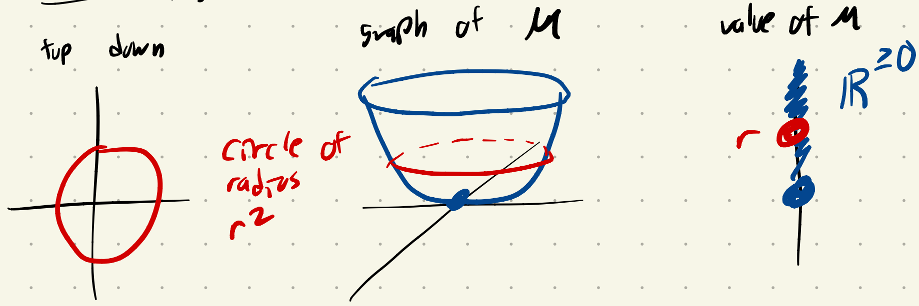
(2) (Atiyah, Guillemin-Sternberg)

the polytope $\Delta$ is the image of $\mu$!

idea: decompose $M$ into orbits of $T$ & value of $\mu$:

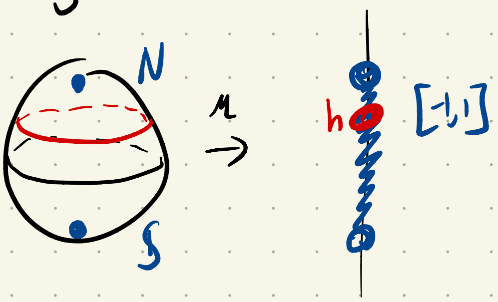
$\dim T = \frac{\dim M}{2} \Rightarrow \mu^{-1}(\xi) \simeq T$ for regular $\xi$

eg $\mathbb{C}$,

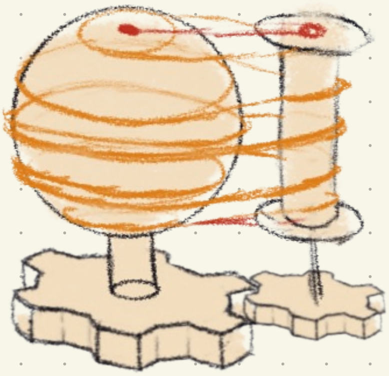



$\mathbb{C} = S^1 \times \mathbb{R}^{>0} + \{pt\} \times \{0\}$

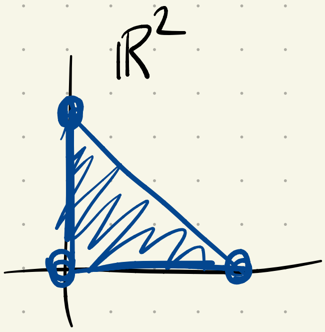
$S^2$




$S^2 = S^1 \times (-1,1)$
$\sqcup \{N\} \times \{1\}$
$\sqcup \{S\} \times \{-1\}$



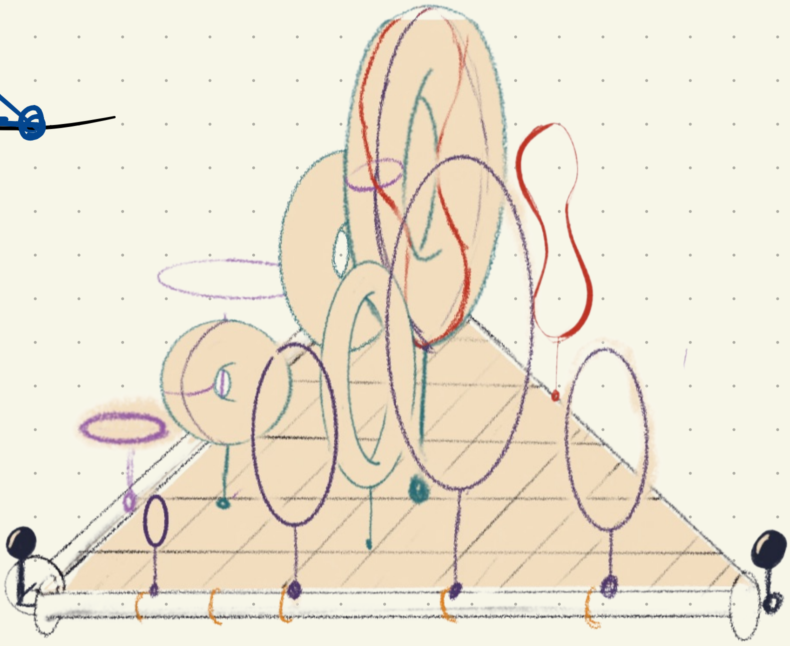
$\mathbb{P}^2 \xrightarrow{\mu}$




faces $= T^2$ fibers

edges $= S^1$ fibers

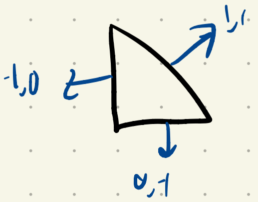
vertices $=$ pt fibers



Atiyah-Guillemin-Sternberg convexity theorem:

- the image of $\mu$ is a convex polytope $\Delta$ in $\mathbb{R}^n$
- the vertices of $\Delta$ are the images of the fixed points of torus action

The resulting polytope has faces w/ normal vectors $n_i \in \mathbb{Z}^n = \mathfrak{t}^\vee \subset \mathfrak{t}^*$
satisfying a couple nice properties
"Delzant polytope"




rationality: above,
simplicity: $n$ edges @ each vertex
smoothness: each $v_i$ @ a vertex form a basis (if not, orbifold pt)

③ $\mathbb{C}^{n+k} /\!\!/_{\xi} T^k \longleftrightarrow$ polytope

we have a SES $T^k \hookrightarrow U(1)^{n+k} \longrightarrow T^n$, w/ $\xi \in \mathfrak{t}^{k*} = (\xi_1, \dots, \xi_k)$

each $U(1)$ factor of $T^k$ embeds into $U(1)^{n+k}$ as

$$e^{i\theta} \mapsto (e^{iv^1\theta} z_1, \dots, e^{iv^{n+k}\theta} z_{n+k}), \quad \text{w/ } \vec{v} \in \mathbb{Z}^{n+k}$$

so, the embedding is described by integer vectors $\vec{v}_1, \dots, \vec{v}_k$

passing to dual lie algebras ... $\mathfrak{t}^{*k} \twoheadleftarrow \mathbb{R}^{n+k} \xleftarrow{\iota} \mathfrak{t}^{*n}$ note that order swaps

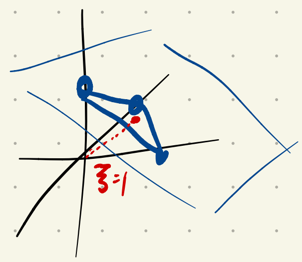
$\iota$ defines a dimension $n$ affine subspace $A$ of $\mathbb{R}^{n+k}$, & $\xi$ gives the displacement from the origin $A = \{x \in \mathbb{R}^{n+k} \mid \forall_i \ x \cdot \vec{v}_i = \xi_i\}$

Example: $\mathbb{P}^2$

$$\mathbb{P}^2 = \mathbb{C}^3 /\!\!/ U(1) = \mu^{-1}(1)/U(1) \Leftarrow U(1) \xhookrightarrow{\text{diagonal}} U(1)^3 \twoheadrightarrow U(1)^2 \quad \text{w/ } \xi = 1$$

$n=3$, $k=1$ $\qquad \vec{v} = (1,1,1)$

$$\mathfrak{t}^* \twoheadleftarrow \mathbb{R}^3 \hookleftarrow \mathfrak{t}^{*2}$$




hyperplane $x_1 + x_2 + x_3 = 1$

Or, we can work intrinsically on $A \simeq \mathfrak{t}^{*n}$:

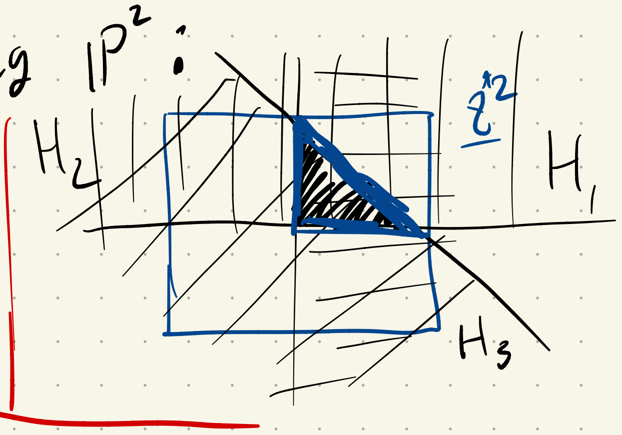
each coordinate half plane intersects $A$ in a halfplane

$$H_i = \{x \in \mathfrak{t}^{*n} \mid \iota(\vec{x})\, \vec{v}_i - \xi_i = 0\}$$

then, Delzant polytope $= \bigcap_{i=1}^{n+k} H_i$

thus, $\mathbb{C}^{n+k} /\!\!/ T^k \longrightarrow$ Delzant polytope

e.g. $\mathbb{P}^2$:




every polytope can be constructed in this way, so we have Polytope $\to \mathbb{C}^{n+k} /\!\!/ T^k$.
This is the delzant construction.
I prefer the symplectic cut procedure, it gives a much more hands-on way of seeing the resulting toric manifold.

To summarize:

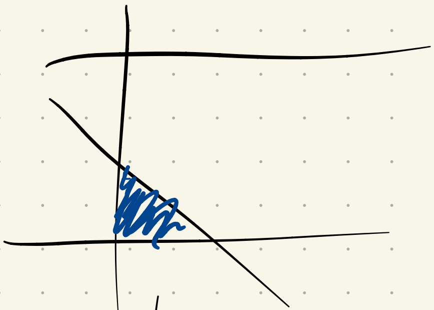
Toric manifolds $\longleftrightarrow$ Convex Polytopes

Geometric properties $\longleftrightarrow$ combinatorics

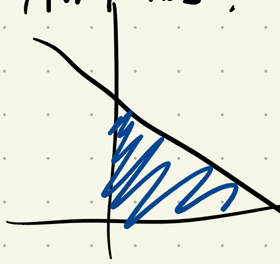
all geometry captured by the combinatorics of the half-plane arrangement in particular the polytope, so this is the perfect testing ground for geometric conjectures.

## Deformations of toric manifolds:

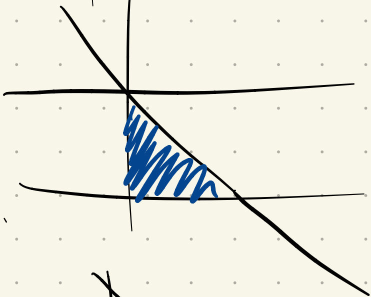
governed by changing value of moment map = affine positions of hyperplanes!



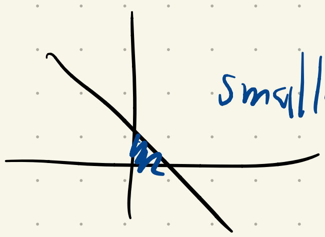
$\mathbb{P}^2$



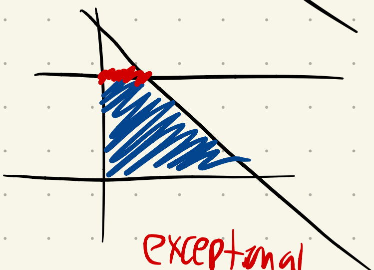
$\mathbb{P}^2$



still $\mathbb{P}^2$

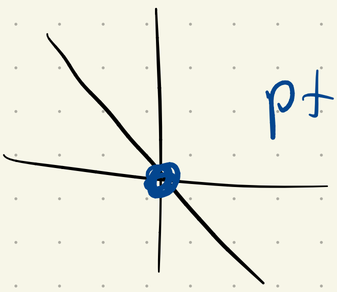


smaller $\mathbb{P}^2$

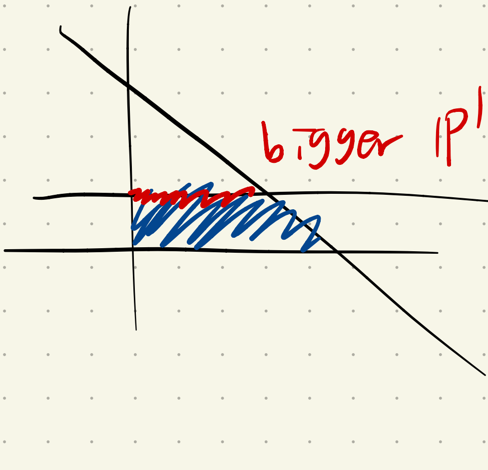


exceptional divisor $= \mathbb{P}^1$

$\mathbb{P}^2$ w/ blowup @ 1 pt

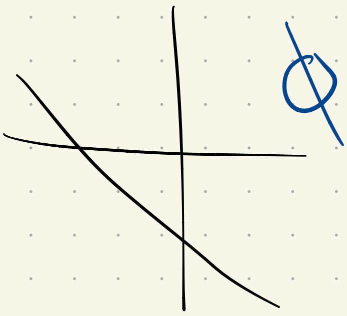


pt



bigger $\mathbb{P}^1$

$\mathbb{P}^2$ w/ blow up @ 1 pt



$\emptyset$