Kähler Quantization & Physics

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MATH 868C

Kähler Quantization

(X,ω) Kähler (L,h) pos line bundle $\Theta(h)=\omega$

$$\mathcal{H}_\omega = \{\varphi \mid \omega + i\partial\bar\partial\varphi > 0\}$$

$$\mathcal{H}_\omega^k = \{\text{inner products on } H^0(X, L^k)\}$$

$$\lim_{k\to\infty} \mathcal{H}_\omega^k \to \mathcal{H}_\omega \qquad 1/k = \hbar\,?$$

Classical mechanics

States of physical system

ω 2-form

closed

nondegenerate

Symplectic geometry: (X, ω)

Kahler

Hamiltonian H

↳ flow preserving H

rotates 90°

$dH = i_{X_H}\omega$

States: $p \in X$

observables: $f \in C^\infty(X)$

Dynamics: $\dot{p} = X_H p$

$\dot{f} = X_H f = \{,\}$

Poisson algebra

Quantum mechanics

States: Hilbert space $\mathcal{H}$ norm 1 & $v \sim e^{i\theta} v$

e.g. $L^2(\mathbb{R})$ (wave fn) $\Rightarrow \mathbb{CP}^n$

Observables: $\mathrm{Herm}(\mathcal{H})$ $A(\psi) = \langle \psi, A\psi \rangle$

dynamics: $i\dot{\psi} = H\psi$ $i\dot{A} = [H, A]$

schrodinger

ω_{FS}:

$$X_{\underbrace{\langle \psi, H\psi \rangle}_{\mathbb{CP}^n \to \mathbb{R}}} = -iH$$

schrodinger

$\omega_{FS} \Leftrightarrow$ Quantum Dynamics

Quantization:

$$C^\infty(X) \longrightarrow \mathrm{herm}(\mathcal{H})$$

$$\{,\} \longrightarrow [,]$$

impossible !!

Geometric Quantization

(X, ω) Kahler, $\underline{(L, h)}$ $\Theta(h) = \omega$

wavefn

Hilbert space $\mathcal{H} = H^0(X, L) \cap L^2$ holomorphic sections

$$\langle s, s' \rangle = \int_X \frac{\omega^n}{n!} h(s, s') \quad !!$$

$f \in C^\infty$

$$[Q_f, Q_g] = Q_{\{f,g\}} \, !$$

$\Updownarrow$

chern connection

$$Q_f = \hbar \left(\nabla_{X_f} - f \right) : H^0(X, L) \to H^0(X, L)$$

Semiclasical limit 

$$(X, \omega, L, h) \longmapsto (X, \underline{k\omega}, \underline{L^k}, h^k)$$

bigger 'more particles'

Kodira: $X \overset{\phi}{\hookrightarrow} \mathbb{C}P^{d_k - 1}$

$$x \longmapsto [s_1(x), \dots, s_{d_k}(x)]$$

$$\Phi_k^* \omega_{F.S} \xrightarrow{k \to \infty} \omega$$

$\mathbb{C}P^{d_{k-1}}$ ω_{FS}

quantum dynamics approximates classical!

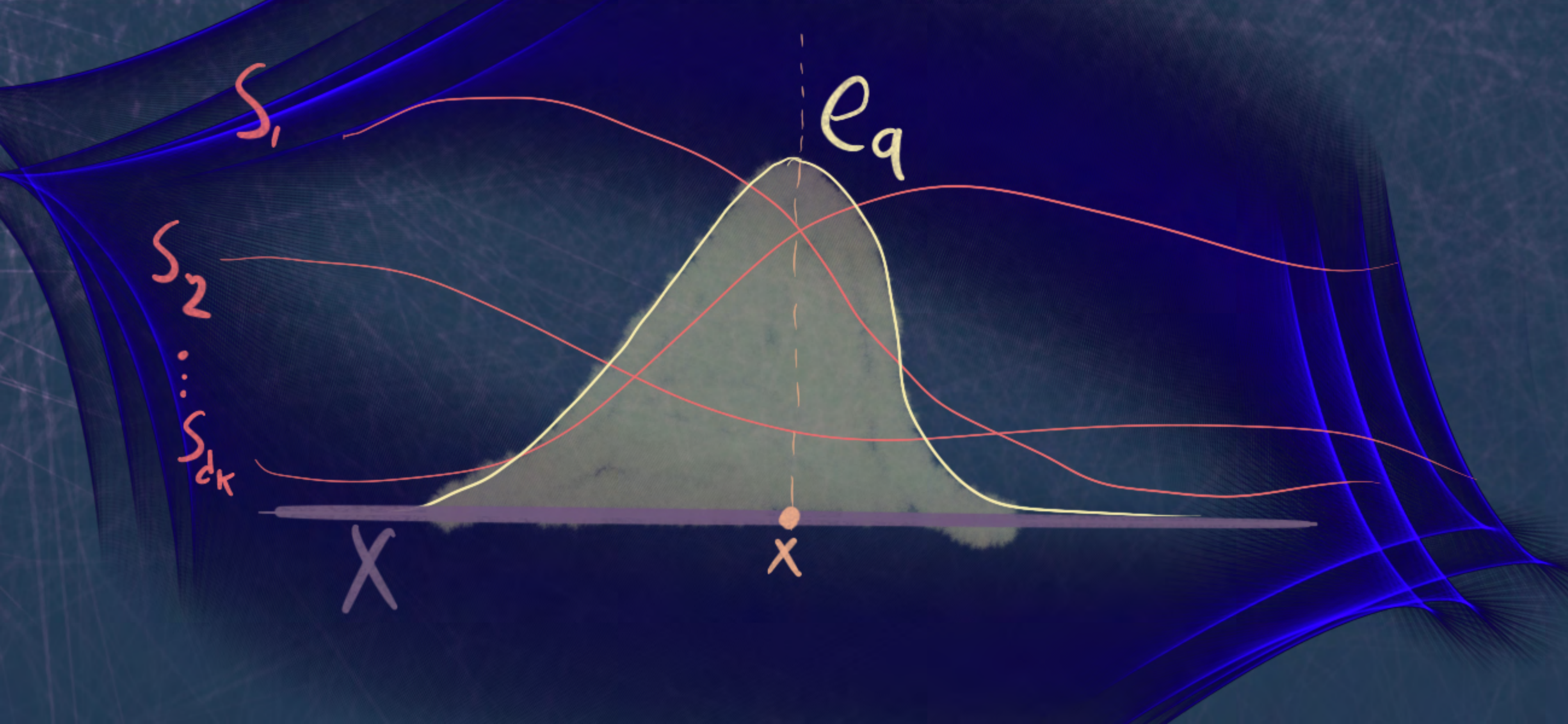
Coherent states like me, hopefully

$\phi(x) \in \mathcal{H} = H^0(X, L).$ what is it? $\pi : L \to X$

$s \mapsto s(x)$ almost linear: choose $q \in \pi^{-1}(x)$, $s \mapsto \frac{s(x)}{q}$

Rietz representation: $\exists e_q \in \mathcal{H}$ $\langle s, e_q \rangle = \frac{s(\pi(q))}{q}$

e_q Coherent state best approx. for $\delta(x)$



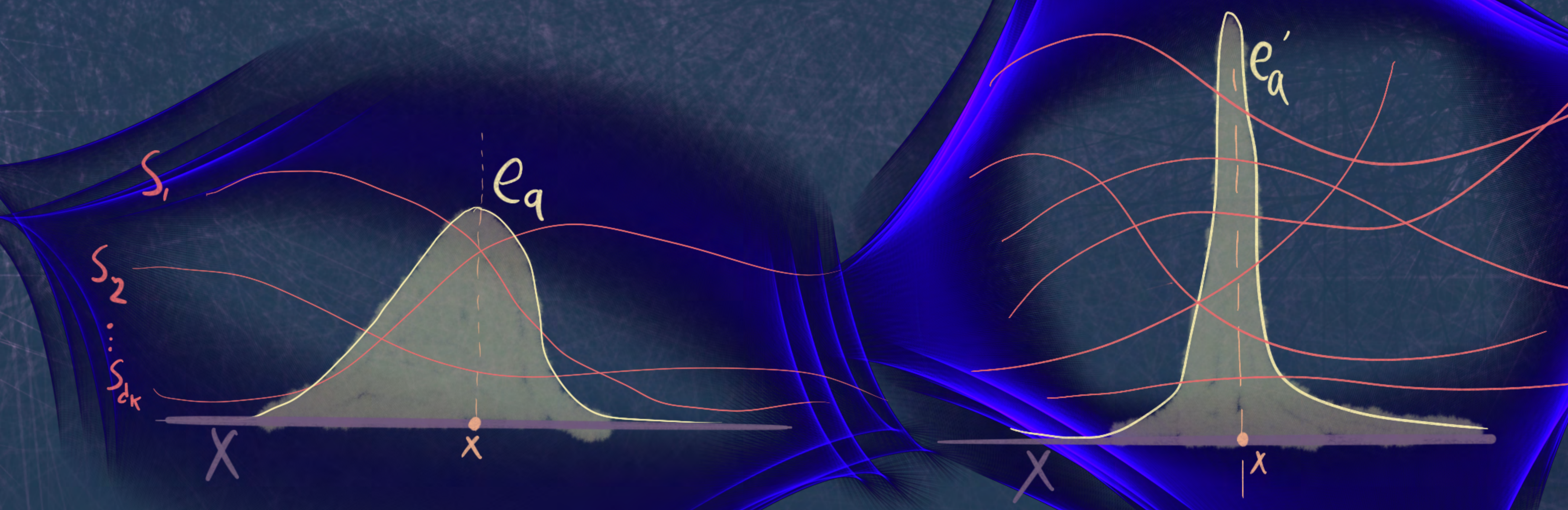
$q^2 \| e_q \|^2$

$\Sigma h(s_i(x), s_i(x))$

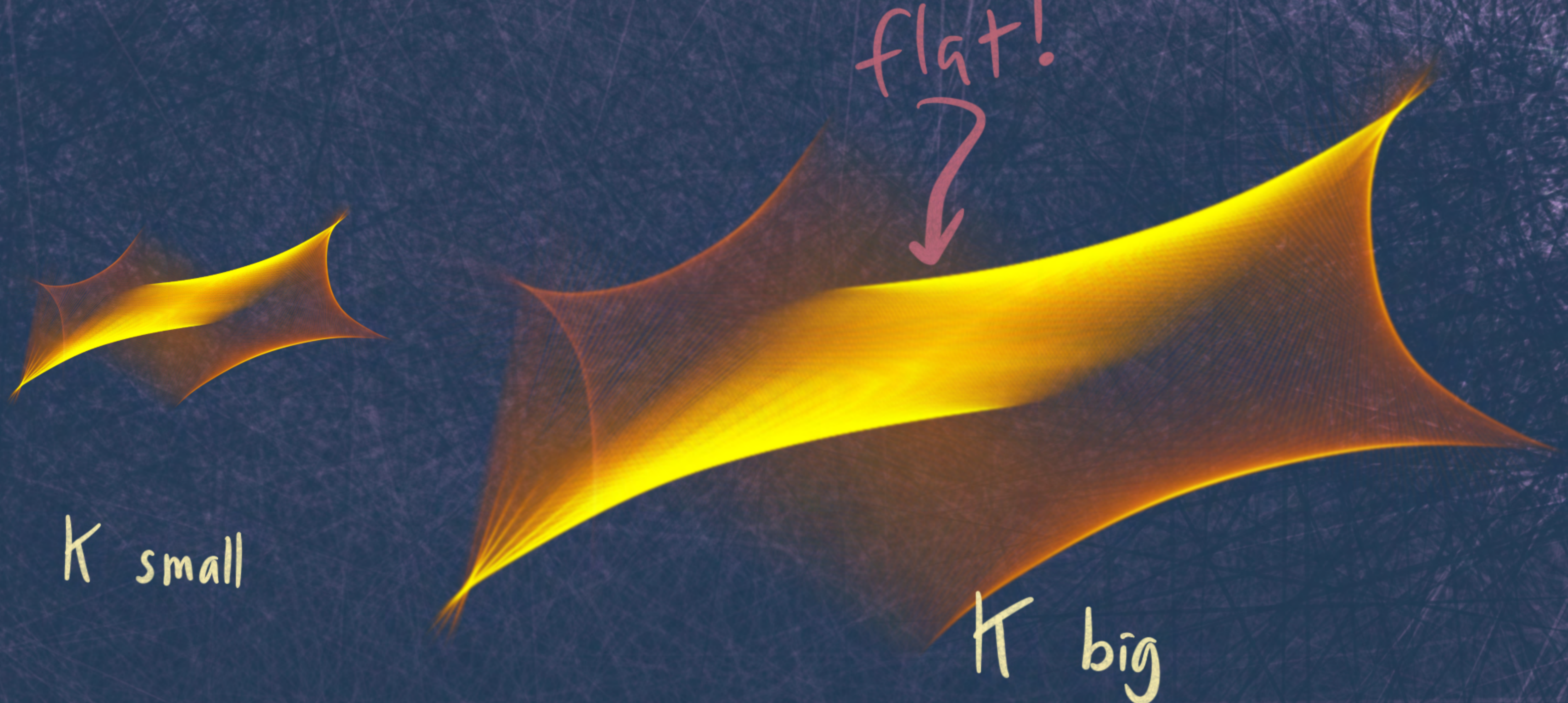
$= FS(U)!$

$k \to \infty$

- larger $k \Rightarrow$ more narrow peaks
- $k \to \infty \Rightarrow$ dirac deltas (? no)
- better 'resolution'



$\omega \mapsto \kappa\,\omega$



References

- Rawnsley, Caden, Gutt: Quantization of Kähler manifolds I: geometric interpretation of Berezin's quantization
    - Connects geometric quantization and coherent states
- Douglas, Klevtsov, Bergman Kernel from Path Integral
    - Computes Bergman Kernel with path integral (Different physical realization)
- Szekelyhidi An introduction to extremal kahler metrics
    - Kahler quantization from math prespective