Talk notes: Abelian coloumb branches :)

Recall: gauge group G, representation $M \circlearrowleft G$,

- get 3D, $\mathcal{N}=4$ SUSY gauge theory
  - moduli space of vacua (classical solutions to equations of motion)

Colomb branch + Higgs branch

coloumb branch: Gauge group G breaks to maximal torus $U(1)^{\text{rank } G} = T < G$

$$\mathcal{M}_C^{\text{classical}} = (\mathbb{R}^3 \times S^1)^{\text{rank } G} / \mathrm{Weyl}(G)$$

Weyl(G) — residual gauge group on T

Higgs branch: $\mathcal{M}_H = T^*M /\!/\!/\!/ G$ hyperKahler quotient

Quantum theory: QFT topological $\Rightarrow$ equals low energy effective theory

$\Rightarrow$ nonlinear SUSY σ-model w/ target $\mathcal{M}_C^{\text{quantum}}$ (hyperKahler!!)

topology & Geometry of $\mathcal{M}_C$ controlled by rep. $M \circlearrowleft G$

Abelian gauge theories

(G, M)	$\mathcal{M}_H$	$\mathcal{M}_C^{\text{classical}}$
$(U(1), \mathbb{C}^{n+1})$	$T^*\mathbb{P}^n$	$\widetilde{\mathbb{C}^2/\mathbb{Z}_{n+1}}$
$(U(1)^n, \mathbb{C}^{n+1})$	$\widetilde{\mathbb{C}^2/\mathbb{Z}_{n+1}}$	$T^*\mathbb{P}^n$

3D mirror symmetry

$\widetilde{\mathbb{C}^2/\mathbb{Z}_n}$ resolution of singularity of $\mathbb{C}^2/\mathbb{Z}_n$

$(U(1), \mathbb{C}^n)$ coloumb branch:

$\mathcal{M}_C^{\text{classical}} = \mathbb{R}^3 \times S^1 \simeq \mathbb{C} \times \mathbb{C}^*$ as holomorphic symplectic manifold

coordinates: $\underset{\varphi}{\mathbb{C}} \times \underset{v^\pm}{\mathbb{C}^*}$ $\quad v^\pm$ are exponentiated coordinates: $v^\pm = e^{\pm z}$

holo. symplectic form $\Omega = d\varphi \wedge d \log v^+$

ring of holo. fns $\mathbb{C}[\mathcal{M}_C^{\text{classical}}] = \mathbb{C}[\varphi, v^+, v^-]/v^+v^- = 1$

$\mathcal{M}_C^{\text{quantum}}(U(1):\mathbb{C}^n)$: hyper kahler manifold, with modified metric & topology
"N- centered taub-NUT space"

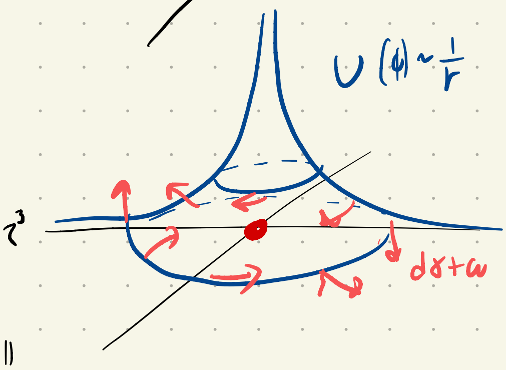
Geometry:

$\mathbb{R}^3 \times S^1$, but shrink circle fibers to points at n points $\{\vec{m}_1, \dots, \vec{m}_n\} \subset \mathbb{R}^3$

coordinates $(\vec{\phi}, \gamma)$

metric $ds^2 = U(\phi)\, d\vec{\phi} \cdot d\vec{\phi} + \dfrac{(d\gamma + \omega)^2}{U(\phi)}$

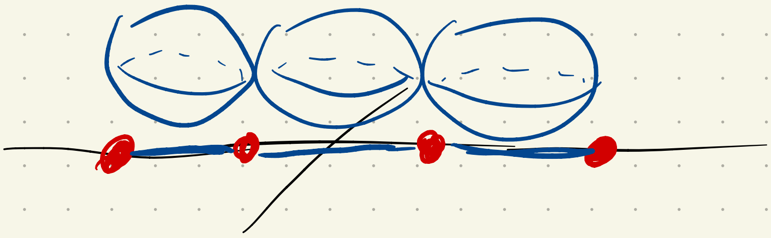
$U(\phi) = 1 + \sum \dfrac{1}{|\phi - m_i|}$ ω 1-form, w/ $d\omega = *dU$



$d\gamma + \omega$ connection 1-form for nontrivial $U(1)$ bundle on $\mathbb{R}^3 - \{m_i\}$. on S^2_∞, has chern class n

- fibers shrink to point over $\vec{\phi} = \vec{m}_i$
- fibers wrap into nontrivial S^1 fibration asymptotically

- Topology: Deformation retracts onto chain of spheres



$v^\pm$ extended from $\mathbb{C}^*$ to $\mathbb{C}$,
$\dfrac{1}{\log(v^+v^-)}$ measures size of fiber @ φ

Algebra: Ring of holomorphic functions

$$\mathbb{C}[\mathcal{M}_C] = \mathbb{C}[\varphi, v^+, v^-] \big/ v^+ v^- = \prod (\varphi + m_i)$$

w/ holo. symplectic form $\Omega = d\varphi \wedge d\log v^+$

Remark: mathematically, $\mathcal{M}_C := \operatorname{spec}(\mathbb{C}[\mathcal{M}_C])$. Braverman, Finkleberg, Nakajima noticed that $\mathbb{C}[\mathcal{M}_C]$ is a sort of equivariant cohomology for the affine grassmanian for $G = U(1)$

$(U(1), \mathbb{C}^n)$ Higgs branch: $\mathcal{M}_H = T^*\mathbb{C}^n /\!/\!/\!/ U(1)$ hyperkahler quotient

<u>Hyperkahler geometry</u> has 3 anticommuting parallel complex structures I, J, K s.t. $I^2 = J^2 = K^2 = IJK = -1$ "quaternionic geometry"

Example: $T^*\mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}^* \simeq \mathbb{H}$. I induced by multiplication by i, etc...

each complex structure gives symplectic form $\omega_I = \langle \cdot, I \cdot \rangle$, ω_J, ω_K analagous

combine these as $\omega_{\mathbb{R}} := \omega_I$, $\omega_{\mathbb{C}} := \omega_J + i\omega_K$

Example: choose coordinates $(z_i, \tilde{z}_i)$ for $\mathbb{C} \oplus \mathbb{C}^*$, then, $\omega_{\mathbb{R}} = \underbrace{\sum dx \wedge dy}_{\text{standard form on } \mathbb{C}^n} + \underbrace{\sum d\tilde{x} \wedge d\tilde{y}}_{\text{standard on } \mathbb{C}^{n*}}$

$\omega_{\mathbb{C}} = dz \wedge d\tilde{z} = (dx + i dy) \wedge (d\tilde{x} + i d\tilde{y})$

canonical holomorphic symplectic form on $T^*\mathbb{C}$

$T \simeq U(1)^n \curvearrowright (M, g)$ is a <u>hyperhamiltonian</u> group action when it is hamiltonian for ω_I, ω_J, & ω_K. Thus has moment maps μ_I, μ_K, μ_J

$\mu_{\mathbb{R}} = \mu_I : M \to \mathfrak{t}^*$, $\mu_{\mathbb{C}} = \mu_J + i\mu_K : M \to \mathfrak{t}^* \otimes \mathbb{C}$

example: the usual action $U(1) \curvearrowright \mathbb{C}$ extends to $T^*\mathbb{C}$ as $U(1) \curvearrowright \mathbb{C} \oplus \mathbb{C}^*$,

$\theta : (z, \tilde{z}) \mapsto (e^{i\theta} z, e^{-i\theta} \tilde{z})$

for $\omega_{\mathbb{R}}$, this has moment map $\boxed{\mu_{\mathbb{R}} = |z|^2 - |\tilde{z}|^2}$

for $\omega_{\mathbb{C}}$, $\boxed{\mu_{\mathbb{C}} = z\tilde{z}}$

<u>Hyperkahler reduction</u>: choose regular value $\xi = (\xi_{\mathbb{R}}, \xi_{\mathbb{C}}) \in \mathfrak{t}^* \oplus \mathfrak{t}^* \otimes \mathbb{C}$

then $M /\!/\!/\!/_{\xi} T = \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) /\!/_{\xi_{\mathbb{R}}} T = \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) \cap \mu^{-1}(\xi_{\mathbb{R}}) / T$

In hyperkahler geo groups act 4 times!

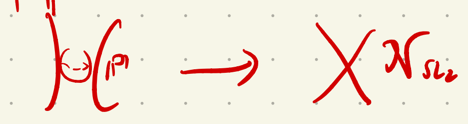
example: take $U(1)$ acting diagonally on $\mathbb{H}^n$. Then, $\mathbb{H}^n /\!/\!/\!/_{(1,0)} U(1) = T^*\mathbb{P}^n$

$\mathbb{H}^2 /\!/\!/\!/_{(1,0)} U(1) = T^*\mathbb{P}^1$

however, $\mathbb{H}^2 /\!/\!/\!/_{(0,0)} U(1) = \mathcal{N}_{sl_2} = \left\{ \begin{bmatrix} w & u \\ v & -w \end{bmatrix} \middle| \; w^2 + uv = 0 \right\} \simeq \mathbb{C}^2/\mathbb{Z}_2$

is the nilpotent cone.

$\underbrace{\mathbb{H}^2 /\!/\!/\!/_{(1,0)} U(1)}_{T^*\mathbb{P}^1} \rightsquigarrow \mathbb{H}^2 /\!/\!/\!/_{(0,0)} U(1)$ is a symplectic resolution of singularities!



Hyperkahler metric on $T^*\mathbb{P}^1$ is the atiyah-hitchin metric.

$(U(1)^n, \mathbb{C}^{n+1})$ Higgs branch: $T^*\mathbb{C}^{n+1} /\!/\!/\!/ U(1)^n$

General Hypertoric Varieties are hyper Kahler quotients of a vector space $T^*\mathbb{C}^{n+k}$ by torus $T^n = U(1)^n$

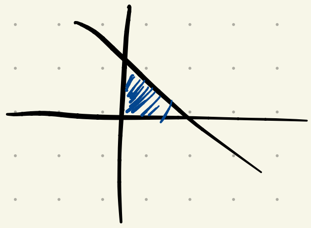
hyperplane arrangement

$$T^n \hookrightarrow U(1)^{n+k} \curvearrowright T^*\mathbb{C}^{n+k}, \quad \xi \in \mathfrak{t}^{*n} \Rightarrow T^*\mathbb{C}^{n+k} \underset{(\xi,0)}{/\!/\!/\!/} T^n = \mathcal{M}(\mathcal{A})$$

Combinatorial Data (hyperplane arrangement) identical to symplectic toric variety $\mathbb{C}^{n+k} \underset{\xi}{/\!/} T^k = \mu^{-1}(\xi)/T^k$

embedding $T^n \hookrightarrow U(1)^{n+k}$ & moment map $\xi \in \mathfrak{t}^{*n} \Rightarrow n+k$ hyperplanes in $\mathfrak{t}^{*k}$

e.g $T^*\mathbb{P}^2 = T^*\mathbb{C}^3 \underset{(1,0)}{/\!/\!/\!/} U(1) \Rightarrow$



Polytope $\leftrightarrow$ toric variety

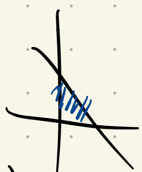
Hyperplane arrangement $\leftrightarrow$ hypertoric variety

Hypertoric varieties are never compact

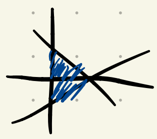
Remark: this is fundamentally different from toric varieties, because the level sets $\mu^{-1}(\xi, 0)$ are not separating

core $\mathcal{L}(\mathcal{A})$: union of toric varieties for all compact polytopes in arrangement.

e.g:



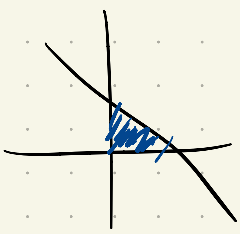
core $= \mathbb{P}^2 \subset T^*\mathbb{P}^2$

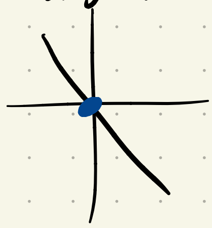


core $= \mathbb{P}^2 \sqcup \mathbb{P}^2$, glued along copy of $\mathbb{P}^1$

- $\mathcal{L}(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$ is
  - Lagrangian submanifold
  - Deformation retract

$T^*\mathbb{C}^{n+k} \underset{(\xi,0)}{/\!/\!/\!/} T^n$ $\xrightarrow{\text{symplectic resolution}}$ $T^*\mathbb{C}^{n+k} \underset{(0,0)}{/\!/\!/\!/} T^n$ singular



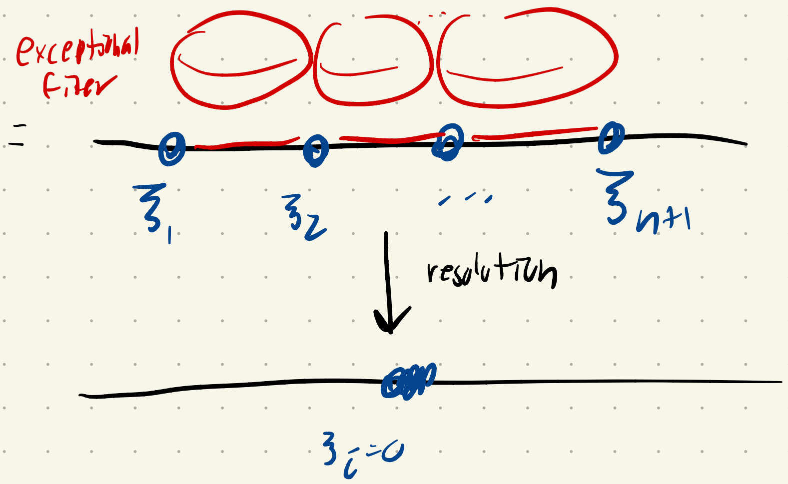


Core $\mathcal{L}(\mathcal{A})$ is exceptional fiber of resolution of singularities for $\mu_{\mathbb{C}}^{-1}(0) \cap \underbrace{\mu_{\mathbb{R}}^{-1}(0)}_{\text{singular}} / G$

$$\mathcal{M}_H\left(U(1)^n, \mathbb{C}^{n+1}\right) = T^*\mathbb{C}^{n+1} \underset{(\xi,0)}{////} U(1)^n \qquad \xi = (\xi_1, \dots, \xi_n)$$

"masses"

$$\underset{\{\vec\theta \mid \theta_0 + \dots + \theta_n = 0\}}{U(1)^n} \hookrightarrow \underset{(\theta_0, \dots, \theta_n)}{U(1)^{n+1}} \longrightarrow \underset{\theta_0 + \dots + \theta_n}{U(1)}$$



$= \widetilde{\mathbb{C}^2/\mathbb{Z}_{n+1}}$

$= T^*\mathbb{C}^{n+1} \underset{(0,0)}{////} U(1)^n$

$= \left(S^{n+1} T^*\mathbb{C}\right)^{\mathbb{Z}_{n+1}}$ $\mathbb{Z}_n$ - invariant sets of n pts on $\mathbb{C}^2$

$= \mathbb{C}^2/\mathbb{Z}_{n+1}$ orbifold singularity

mirror symmetry

To summarize:

Higgs branch of $(U(1)^n, \mathbb{C}^{n+1})$

$= \widetilde{\mathbb{C}^2/\mathbb{Z}_{n+1}}$ with hyperkahler metric

Exceptional fiber is chain of n spheres, Dynkin graph of type A_n

(similar constructions relate all finite subgroups $\Gamma \subset SU(2)$ to dynkin diagrams of ADE type (Kronheimer))

$\xi = (\xi_1, \dots, \xi_{n+1})$

Coulomb branch of $(U(1), \mathbb{C}^{n+1})$

$= \mathbb{R}^3 \times S^1$ with $n+1$ singular fibers

$= \widetilde{\mathbb{C}^2/\mathbb{Z}_{n+1}}$ with hyperkahler metric

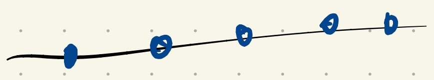
Thm (Kronheimer): all hyperkahler metrics on $\widetilde{\mathbb{C}^2/\mathbb{Z}_{n+1}}$ w/ same cohomology class are isometric

$\Longrightarrow$ same Hyperkahler metric!!

$\Longleftrightarrow$ singular fibers at $\vec m_1 = (\xi_1, 0, 0), \dots, \vec m_{n+1} = (\xi_{n+1}, 0, 0)$

$\mathcal{M}_H\left(U(1), \mathbb{C}^{n+1}\right)$ = Hypertoric variety

$U(1)^n \hookrightarrow U(1)^{n+1} \longrightarrow U(1)$



$\widetilde{\mathbb{C}^2/\mathbb{Z}_{n+1}}$ $\xleftrightarrow{\text{gale duality}}$

$\mathcal{M}_H\left(U(1), \mathbb{C}^{n+1}\right)$ hypertoric variety

$U(1)^\vee \hookrightarrow U(1)^{n+1,\vee} \longrightarrow U(1)^{n\vee}$ dual sequence

$T^*\mathbb{P}^n$

Coloumb branches for general (G,M)

fundamental philosophy of lie theory: $(G, \mathrm{Ad}) \longleftrightarrow (T, W)$ maximal torus, weyl group

$$\mathcal{M}_C^{classical}(G,M) = (\mathbb{R}^3 \otimes \mathfrak{t}^*) \times T / W = (\mathbb{R}^3 \times S^1)^{\mathrm{rank}(G)} / W$$

where W acts freely on $\mathfrak{t}^*$, (on weyl chambers $\mathfrak{t}^* - \Delta$), get abelian theory

$$\mathcal{M}_C^{abel} = (\mathbb{R}^3 - \Delta) \times S^1 / W$$

this is hyperkahler quotient, by mirror symmetry w/ $\mathcal{M}_H$

[BDG17] (physics) $\mathbb{C}[\mathcal{M}_C] \subset \mathbb{C}[\mathcal{M}_C^{abel}]$ coulomb branch determined by holo. fns on $\mathcal{M}_C^{abel}$ which extend over walls of weyl chamber

[BFN 16] (math) $\mathbb{C}[\mathcal{M}_C]$ = equivariant cohomolgy of affine grassmanian

[Bielawski-Fasculo 23] (math) explicitly describes $\mathcal{M}_C(G,M)$!

classically, $\mathcal{M}_C^{classical}(G,M) = \mathcal{M}_C^{classical}(T,M)/W$ = hypertoric variety / weyl group

$\mathcal{M}_C(T,M)/W = (S^{|W|}\mathcal{M}_C(T,M))^W$ W-invariant configurations of $|W|$ points

$\xrightarrow{\text{resolve singularities}}$ $\mathrm{Hilb}^W(\mathcal{M}_C(TM))$ ring-theoretic object

$W\text{-Hilb}(\mathcal{M}_C^{abel})$

<u>Thm</u>: All BFN coulomb branches are W-Hilbert schemes of hypertoric varieties, or toric hyperkahler quotient of $W\text{-Hilb}(\mathcal{M}_C^{abel})$ ☒

Example: $\mathbb{C}^2/\mathbb{Z}_n$ & $\widetilde{\mathbb{C}^2/\mathbb{Z}_n}$

represent $\mathbb{Z}_n$ in $SU(2)$ as $\xi \mapsto \begin{bmatrix} \xi & \\ & 1/\xi \end{bmatrix}$ for $\xi = e^{2\pi i/n}$ generator of $\mathbb{Z}_n$.

$\mathbb{Z}_n$ acts on $\mathbb{C}^2$ w/ fixed point $(0,0)$, so $\mathbb{C}^2/\mathbb{Z}_n$ has an orbifold singularity at $(0,0)$. How do we resolve it?

construct it as a hyperkähler variety! following Kronheimer...

(1) find all irreducible reps of $\mathbb{Z}_n$

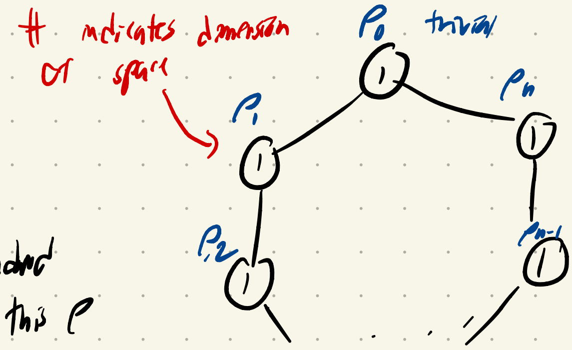
pontryagin dual

$\mathbb{Z}_n$ abelian, so all reps 1 dimensional & $\hat{\mathbb{Z}}_n \simeq \mathbb{Z}_n$

define rep $\rho_k : \alpha \longmapsto \xi^k$ for α generator of $\mathbb{Z}_n$, $\xi = e^{2\pi i/n}$ nth root of unity

$k = 0, \dots, n$

label each w/ a vertex



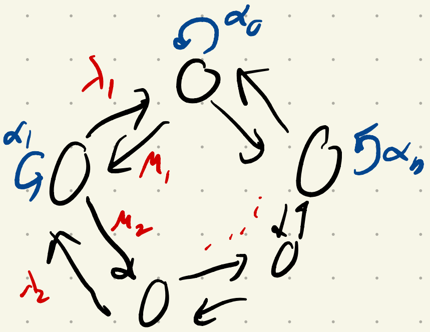
(2) form the mackay graph

$\mathbb{Z}_n \subset SU(2)$ carries the induced standard 2D rep, shown above. call this ρ

draw a line between ρ_i & ρ_j if $\rho_i \otimes \rho$ contains ρ_j as an irreducible component

$\rho = \begin{bmatrix} \xi & \\ & \xi^{-1} \end{bmatrix} = \rho_1 \oplus \rho_{-1}$, so $\rho_i \otimes \rho = \underbrace{\rho_{i+1} \oplus \rho_{i-1}}_{\text{mod } n+1}$ ⇒ get cycle graph

(3) construct quiver representation variety



replace the lines w/ double arrows, & label each arrow from $\rho_i \to \rho_j$ w/ matrix $\mathrm{Hom}(\rho_i, \rho_j)$

total vector space is $\overbrace{\bigoplus_{i<j} \mathrm{Hom}(\rho_i, \rho_j)}^{V} \oplus \overbrace{\left(\bigoplus_{i<j} \mathrm{Hom}(\rho_j, \rho_i) \right)}^{V^*} = V \oplus V^* = T^* V$

since $\dim \rho_i = \dim \rho_j = 1$, $\mathrm{Hom}(\rho_i, \rho_j) \simeq \mathbb{C}$. There are $n+1$ edges, so $V \simeq \mathbb{C}^{n+1}$

Gauge Group: rotation of each ρ_i by $\alpha_i \in U(\rho_i) \simeq U(1)$

this induces action on $\mathrm{Hom}(\rho_i, \rho_j)$:

if $\lambda_i \in \mathrm{Hom}(\rho_i, \rho_{i+1})$, then $\alpha_i \lambda_i \alpha_{i+1}^{-1} \in \mathrm{Hom}(\alpha_i \rho_i, \alpha_{i+1} \rho_{i+1})$