Cobordism hypothesis II

Langwen Hui

University of Illinois, Urbana-Champaign

Feb 18, 2025

Outline

1. Dualizability and framings
2. Tangential structure and G-action
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Brief recap

The free generated by a pt is Bord_n^{fr}.

Theorem ([Lur10])

Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category with duals. We have an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \simeq \mathcal{C}^{\sim}$$

given by $Z \mapsto Z(*)$.

One may put a general $\mathcal{C}$ above and replace the RHS by $(\mathcal{C}^{\mathrm{fd}})^{\sim}$.

Dualizability I

sym. mon.

Recall that an (∞, n)-category has duals if

1. Every object admits a dual.
2. Every k-morphism for $1 \leq k \leq n-1$ admits both adjoints.

To say that 1-morphisms $f : X \to Y$ and $g : Y \to X$ are adjoints is to specify 2-morphisms $u : \mathrm{id}_X \to g \circ f$ and $v : f \circ g \to \mathrm{id}_Y$ and such that

$$\left.\begin{aligned} f \simeq f \circ \mathrm{id}_X \xrightarrow{\mathrm{id}_f \circ u} f \circ g \circ f \xrightarrow{v \circ \mathrm{id}_f} \mathrm{id}_Y \circ f \simeq f, \\ g \simeq \mathrm{id}_X \circ g \xrightarrow{u \circ \mathrm{id}_g} g \circ f \circ g \xrightarrow{\mathrm{id}_g \circ v} g \circ \mathrm{id}_Y \simeq g. \end{aligned}\right\}.$$

To say that two k-morphisms are adjoints is similar, with 2-morphisms replaced by $(k+1)$-morphisms.

Dualizability II

Given any $\mathcal{C}$ there is some $\mathcal{C}^{\mathrm{fd}} \to \mathcal{C}$ terminal over $\mathcal{C}$. More explicitly, it is obtained by discarding all non-dualizable objects and non-adjoinable morphisms. An object $x \in \mathcal{C}$ in the essential image called fully dualizable.

Remark. We do not require adjoinability of n-morphisms. That would be equivalent to invertibility since the unit and counit are $(n+1)$-morphisms and hence invertible.

Example

Taking $n = 0$, the only requirement is to have duals for objects, which is equivalent to $\otimes$-invertibility. Thus every grouplike $\mathbb{E}_\infty$-space can be regarded as an (∞, n)-category with duals for arbitrary n.

→ connective spectra.

sym. mon $(\infty, 0)$-cat w/ duals.

Framings I

A k-morphism in $\mathrm{Bord}_n^{\mathrm{fr}}$ for $0 \leq k \leq n$ is equipped with an *n-framing*. This means a choice of trivialization

$$\tau : \underbrace{\underline{TM_k} \oplus \underline{\mathbf{R}}^{n-k}}_{\text{is trivializable.}} \cong \underline{\mathbf{R}}^n.$$

In other words, a lift

hofib$(f) \simeq \Omega BO(n)$

$\simeq O(n)$.

$$\begin{tikzcd}
 & * \arrow[d, "f"] \\
M_k \arrow[ur, dashed, "\exists"] \arrow[r, "TM_k \oplus \underline{\mathbf{R}}^{n-k}"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha"] & \mathrm{BO}(n)
\end{tikzcd}$$

$GL_n(\mathbb{R}) \simeq O(n)$.

and a choice of 2-morphism making the diagram commute.

Framings II

$[X,Y] = \pi_0(\mathrm{Map}_*(X,Y)).$

Using the exact sequence

$$\cdots \to [(M_k)_+, \mathrm{O}(n)] \to [(M_k)_+, *] \to \underbrace{[(M_k)_+, \mathrm{BO}(n)]}_{TM_k \oplus \mathbb{R}^{n-k}.} \to \cdots,$$

if an n-framing exists, the collection of all n-framings forms a $[(M_k)_+, \mathrm{O}(n)]$-torsor.

For manifolds with boundary one requires that the framing is compatible with the chosen framing on boundary.

Remark. If we specify an orientation (equivalently, an n-orientation) for TM, then the collection of framings compatible with this chosen orientation is a $[(M_k)_+, \mathrm{SO}(n)]$-torsor.

Framings III

Example

1. There are always two n-framings on a point.
2. The tangent bundle TS^1 is trivializable, and its n-framings are torsors for

$\pi_0(L\,O(n)).$

$$\underbrace{[(S^1)_+, O(n)]}_{} = \begin{cases} \mathbf{Z}/2, & n = 1, \\ \mathbf{Z} \rtimes \mathbf{Z}/2, & n = 2, \\ \mathbf{Z}/2 \times \mathbf{Z}/2, & n > 2. \end{cases}$$

$O(2) \simeq SO(2) \rtimes \mathbb{Z}/2$

$\pi_1(SO(n))$

   If we specify an orientation then the $\mathbf{Z}/2$-factors disappear.
3. Existence of n-framing on n-dimensional manifolds is rather restrictive: for example, the only closed 2-manifold that admits a 2-framing is T^2.

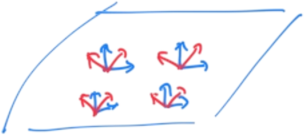
Framings IV

O(n).

We see that there is an $\mathrm{O}(n)$-action on $\mathrm{Bord}_n^{\mathrm{fr}}$. The action of $g \in \mathrm{O}(n)$ on M_k is given by

$$M_k \to * \xrightarrow{g} \mathrm{O}(n) \in [(M_k)_+, \mathrm{O}(n)].$$

More explicitly, g acts by changing all n-framings by g.



g: rotation by some θ.

Framings V

By the equivalence $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \simeq \mathcal{C}^{\sim}$, we get:

$\circlearrowleft O(n)$ $\circlearrowleft O(n)$.

Corollary

There is a canonical $\mathrm{O}(n)$-action on the core $\mathcal{C}^{\sim}$ of any symmetric monoidal (∞, n)-category $\mathcal{C}$ with duals.

Here by $\mathrm{O}(n)$-action we mean a pointed map of $(\infty, 1)$-categories

$* \longmapsto \mathcal{C}^{\sim}$

$$\mathrm{BO}(n) \to \mathcal{S},$$

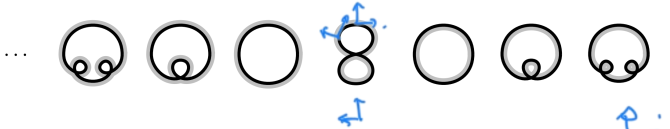
sending the basepoint $* \in \mathrm{BO}(n)$ to $\mathcal{C}^{\sim}$.

Framings VI

Example

1. $O(1) \simeq \mathbf{Z}/2$ acts on $\mathrm{Bord}_1^{\mathrm{fr}} \simeq \mathrm{Bord}_1^{\mathrm{or}}$ by reversing the orientation. For general $\mathcal{C}^{\sim}$, the $\mathbf{Z}/2$-action is the functor $(-)^{\vee}$. $X \mapsto X^{\vee}$.
2. $O(2) \simeq SO(2) \rtimes \boxed{\mathbf{Z}/2}$ acts on $\mathrm{Bord}_2^{\mathrm{fr}}$ and $\mathcal{C}^{\sim}$. The $SO(2)$ part gives a map
$$\{X\} \times SO(2) \simeq \{X\} \times B\mathbf{Z} \xrightarrow{F} \mathcal{C}^{\sim},$$
F(1).
giving an automorphism $S_X : X \to X$, the Serre functor, and assemble to a natural transformation S of $\mathrm{id}_{\mathcal{C}}$.
In $\mathrm{Bord}_2^{\mathrm{fr}}$, the automorphism S_+ rotates the framing on $+$ by 2π.
For general X, S_X turns out to satisfy
$$(S_X \otimes \mathrm{id}_{X^\vee}) \circ \mathrm{coev}_X \simeq \mathrm{ev}^R.$$

The following pictures for $\mathrm{Bord}_2^{\mathrm{fr}}$ are taken from [DSN20].



$\mathbb{Z}$.
$\sqcup$
$\mathbb{Z}$.

Figure: Some framed circles.

Figure: The Serre functor.

Figure: Evaluation and coevaluation maps.

Understanding $\mathrm{Bord}_2^{\mathrm{fr}}$ II

Take a symmetric monoidal $(\infty, 2)$-category $\mathcal{C}$. What is the condition that a fully dualizable $X \in \mathcal{C}$ must satsify?
The minimal amount of data is a $X^\vee$ and associated 1-morphisms

$$\mathrm{ev} : X \otimes X^\vee \to 1, \quad \mathrm{coev} : 1 \to X \otimes X^\vee$$

and adjoints of ev and coev, and further adjoints of these... This is an infinite amount of data.

Understanding $\mathrm{Bord}_2^{\mathrm{fr}}$ III

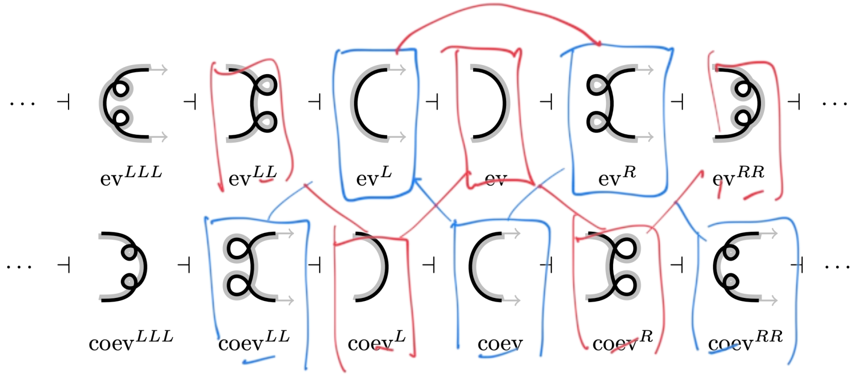


Figure: Adjoints of ev and coev, taken from [DSN20].

Understanding $\mathrm{Bord}_2^{\mathrm{fr}}$ IV

However, it turns out that one only needs ev^L and ev^R to exist. Consider the universal example of $\mathrm{Bord}_2^{\mathrm{fr}}$. We may obtain the Serre functor S_+ via

$$S_+ \cong (\mathrm{id}_+ \sqcup \mathrm{ev}) \circ (\tau_{+,+} \sqcup \mathrm{id}_-) \circ (\mathrm{id}_+ \sqcup \mathrm{ev}^R)$$

and similarly S_+^{-1}. All other adjoints of ev and coev can be obtained by applying S_+ and S_+^{-1}.

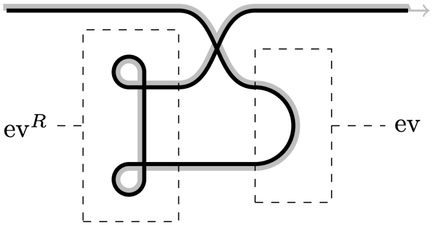


Figure: The Serre functor from ev^L and ev^R, taken from [DSN20].

Tangential structure and G-action I

Instead of compatible n-framings, fix a map

$$\mathrm{B}G \xrightarrow{\zeta} \mathrm{BO}(n).$$

This classifies an n-dimensional vector bundle ζ over $\mathrm{B}G$. A G-structure on M_k is the data of:

1. A map $f : M_k \to \mathrm{B}G$ lifting $M_k \to \mathrm{BO}(n)$.
2. A choice of isomorphism $f^*\zeta \simeq TM_k \oplus \underline{\mathbf{R}}^{n-k}$.

In other words,

$$\begin{tikzcd} & \mathrm{B}G \arrow[d, "\zeta"] \\ M_k \arrow[ur, dashed, "f"] \arrow[r, "TM_k \oplus \underline{\mathbf{R}}^{n-k}"'] & \mathrm{BO}(n) \end{tikzcd}$$

$G = e \Rightarrow \mathrm{B}e \simeq *$.

Tangential structure and G-action II

Requiring that for $k \leq n$ compatible G-structure on all k-morphisms and that these G-structures are preserved by k-morphisms for $k > n$, we get a bordism category Bord_n^G.

$G \to O(n)$.

Example

1. $G = \{e\}$: recover $\mathrm{Bord}_n^{\mathrm{fr}}$.
2. $G = \mathrm{O}(n)$ with $\mathrm{BO}(n) \xrightarrow{\mathrm{id}} \mathrm{BO}(n)$: get Bord_n.
   no action
3. $G = \mathrm{SO}(n)$ with $\mathrm{BSO}(n) \to \mathrm{BO}(n)$: get $\boxed{\mathrm{Bord}_n^{\mathrm{or}}}$. Note that there is a residual $\mathbf{Z}/2$-action generated by $(-)^\vee$ here (orientation reversal).

Tangential structure and G-action III

Theorem ([Lur10])

Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category with duals. We have an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^G, \mathcal{C}) \simeq (\mathcal{C}^{\sim})^{hG}.$$

given by $Z \mapsto Z(*)$.

$$G \downarrow$$

- G acts on $\mathcal{C}^{\sim}$ through composing $\mathrm{O}(n) \circlearrowright \mathcal{C}^{\sim}$ with $G \to \mathrm{O}(n)$
- $(-)^{hG}$ is **homotopy fixed point functor**: the limit of the functor $\mathrm{B}G \to \mathcal{S}$.

$$(\mathcal{C}^{\sim})^{hG} \simeq \lim_{BG} F$$

$$F = \begin{tikzcd} BG \arrow[r, "\mathcal{C}^{\sim} \text{ w/ } G\text{-action.}"] & \mathcal{S} \\ * \arrow[u] \arrow[ur, "\mathcal{C}^{\sim}"'] & \end{tikzcd}$$

Tangential structure and G-action IV

Example

! $F: BG \to \mathcal{S}$

If $BG \to * \to BO(n)$, we have

$\lim_{BG} F$

objects in C w/ a G-action

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^G, \mathcal{C}) \simeq (\mathcal{C}^{\sim})^{hG} \simeq \mathrm{Map}(BG, \mathcal{C}^{\sim}).$$

Thus a functor Z from LHS corresponds to an object $Z(*) \in \mathcal{C}$ with G-action.

Let's specialize to $n = 2$ and $\mathcal{C} = \mathrm{Cat}_{(\infty,1)}$. Then Z is given by some fully dualizable $\mathcal{D} \in \mathrm{Cat}_{(\infty,1)}$ with a G-action up to homotopy. Assume that G is connected. Then

$\Omega G \simeq \Omega^2 BG$ based at $\mathrm{id}_{\mathcal{D}}$.

$X \leftarrow \Omega X$

connected $B\Omega X$?! X.

$$\underbrace{\mathrm{Map}_{\mathbb{E}_1}(G, \mathrm{Aut}(\mathcal{D}))}_{G\text{-action on } \mathcal{D}} \simeq \mathrm{Map}_{\mathbb{E}_2}(\Omega G, \Omega \mathrm{Aut}(\mathcal{D}))$$
$$\simeq \mathrm{Map}_{\mathbb{E}_2}(\Omega G, \mathrm{Aut}(\mathrm{id}_{\mathcal{D}})).$$

$C_* \Omega G$ $\quad$ $\Leftarrow \mathrm{End}(\mathrm{id}_{\mathcal{D}})$ $\quad$ $HH^{\cdot}(\mathcal{D})$

Recall that we previously encountered $G = S^1$ in a different context. If $\mathcal{D}$ is stable (e.g. dg) this may be further linearized. See [Tel14, Theorem 2.5].

In general, we could have

$$1 \to K \to G \to H \to 1$$

image of G in O(n)

where $H \subset \mathrm{O}(n)$ is the image of $G \to \mathrm{O}(n)$. For any $X \in \mathcal{S}$ with $\mathrm{O}(n)$-action,

$$X^{hG} \simeq (X^{hK})^{hH} \simeq \mathrm{Map}(BK, X^{hH}).$$

$X = C^{\sim}$

Example at $n = 0$

For $n = 0$, recall that a symmetric monoidal $(\infty, 0)$-category with duals is a grouplike $\mathbb{E}_\infty$-space (i.e. connective spectrum). The free such object with one generator is

$$\Omega^\infty \Sigma^\infty S^0 \simeq \operatorname{colim}_n \Omega^n S^n.$$

The cobordism hypothesis translates to the familiar statement that

$$\mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}, E) \simeq \Omega^\infty E$$

where $\mathbb{S}$ is the sphere spectrum and E is any spectrum.
Further, every grouplike $\mathbb{E}_\infty$-space X admits compatible actions of $\mathrm{O}(n)$ for all n, hence a map $\mathrm{BO} \to \mathrm{Aut}(X)$. Taking $X = \Omega^\infty \Sigma^\infty S^0$ yields the J-homomorphism.

Example at $n = 1$ I

In the case $n = 1$ we have $\mathrm{O}(1) \simeq \mathbf{Z}/2$ and $\mathrm{SO}(1) \simeq \{e\}$, and $\mathrm{Bord}_1^{\mathrm{or}} \simeq \mathrm{Bord}_1^{\mathrm{fr}}$ has objects and 1-morphisms as the classical oriented bordism 1-category.

There are higher morphisms: for example, the 1-morphism $S^1 : \emptyset \to \emptyset$ has a $\mathrm{Diff}^+(S^1) \simeq \mathbb{T}$-worth of automorphisms.

The $\mathbb{T}$-action shows up when the target category $\mathcal{C}$ has higher morphisms.

Example

When $\mathcal{C}$ is the Morita $(\infty, 1)$-category $\mathrm{Alg}_{\mathcal{D}}^{\circ}$ with non-invertible 2-morphisms discarded. If $Z(+) \simeq A$ then

$$Z(S^1) \simeq \mathrm{HH}(A)$$

for some dualizable A. Hochschild homology is equipped with its canonical $\mathbb{T}$-action.

Example at $n = 1$ II

Similarly, Bord_1 has the same objects and 1-morphisms as the unoriented bordism 1-category. In this case, the 1-morphism $S^1 : \emptyset \to \emptyset$ has a $\mathrm{Diff}(S^1) \simeq \mathbb{T} \rtimes \mathbf{Z}/2$-worth of automorphisms.
Being a fixed point of $\mathrm{O}(1)$-action on $\mathcal{C}^{\sim}$ amounts to a choice of an equivalence

$$Z(*) \simeq Z(*)^{\vee},$$

i.e., a non-degenerate pairing $Z(*) \otimes Z(*) \to \mathbf{1}$. If the target is $\mathrm{Vect}_{\mathbf{C}}$, this means that $V := Z(*)$ is equipped with the additional data of a non-degenerate symmetric bilinear form

$$\langle -, - \rangle : V \otimes V \to \mathbf{C}.$$

Example at $n = 2$

Recall that $\mathrm{O}(2) \simeq \mathrm{SO}(2) \rtimes \mathbf{Z}/2$ acts on $\mathrm{Bord}_2^{\mathrm{fr}}$ and $\mathcal{C}^{\sim}$. The $\mathrm{SO}(2)$ part is an automorphism $S : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$, the Serre functor, and assemble to a natural transformation S of $\mathrm{id}_{\mathcal{C}}$.

Example ([Cos07])

The bounded derived category

$\mathcal{C} := \mathrm{Cat}_{\mathrm{perf}}(\mathbb{C})$.

$$\mathcal{D} := D^b(\mathrm{Coh}(X))$$

X smooth proper $/\mathbb{C}$.

is a fully dualizable object in $\mathrm{Cat}_{\mathrm{perf}}(\mathbf{C})$. The Serre functor acts as $(-) \otimes K_X[\dim_X]$ and satisfies

$$\mathrm{Map}_{\mathcal{D}}(C, S(D)) \simeq \mathrm{Map}_{\mathcal{D}}(D, C)^{\vee}.$$

A trivialization of S amounts to a trivialization of K_X, i.e. a Calabi–Yau structure on X. See also [Tel16].

References

[Cos07] K. Costello. “Topological conformal field theories and Calabi–Yau categories”. In: *Advances in Mathematics* (2007).

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