Supersymmetric σ-models via superspace

Lagrangian field theory

Fix spacetime $M^d = \mathbb{R}^{1,d-1}$

- field: $\phi(x)$, $\psi_a(x)$, $\cdots$
- Lagrangian: e.g. $\mathcal{L} = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \alpha\phi^2 - \lambda\phi^4$

classical: ϕ s.t. $\frac{\delta S}{\delta\phi} = 0$ equation of motion

quantum: $Z = \int \mathcal{D}\phi\, e^{-iS}$ $\quad S = \int d^d x\, \mathcal{L} \leftarrow$ action

More fields: $\phi_1, \cdots, \phi_N$

$$\mathcal{L} = \frac{1}{2}\sum_n \partial^\mu\phi_n\,\partial_\mu\phi_n - \alpha\sum_n \phi_n^2 - \lambda\left(\sum \phi_n^2\right)^2$$

$$(\phi_1, \cdots, \phi_N) : M^d \to \mathbb{R}^N$$

σ-model (basic version)

Given a Riemann manifold (X, g) and a function $V: X \to \mathbb{R}$ (potential function)

the fields in a σ-model is given by a map

$$\phi: M \to X$$

and the Lagrangian is

$$\mathcal{L} = \frac{1}{2}|d\phi|^2 - \phi^* V$$

$$|d\phi|^2 = g_{ij}(\phi)\,\partial^\mu\phi^i\,\partial_\mu\phi^j$$

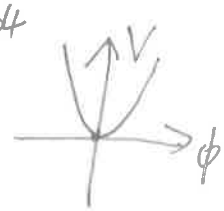
Assume V is bounded below, define moduli space of vacua

$$\mathcal{M}_{vac} := V^{-1}(\text{min of } V)$$

Remark: fields that minimize energy are $\phi(x) \equiv \phi_0 \in \mathcal{M}_{vac}$

e.g. $X=\mathbb{R}$, $L=\frac{1}{2}\partial^\mu\phi\partial_\mu\phi-\alpha\phi^2-\lambda\phi^4$

$\alpha>0$, $\lambda>0$, $M_{vac}=pt$



$\lambda<0$, $M_{vac}=\pm\sqrt{\frac{-\alpha}{\lambda}}$

Superpartical on $M^1=\mathbb{R}^1$

Recall super-Poincaré $P \overset{\text{as vector space}}{\cong} \mathbb{R}^{1,d-1}\oplus so(1,d-1)\oplus S^*$ ← real rep. of $Spin(1,d-1)$

$\{S^*,S^*\}\subset\mathbb{R}^{1,d-1}$ given by

$\Gamma: Sym^2 S^*\to\mathbb{R}^{1,d-1}$

$d=1$, $P=\mathbb{R}\oplus S^*$

$=\langle\partial_t, Q\rangle$

$\{Q,Q\}=2\partial_t$

Fix Riemann manifold X

$\phi: M^1\to X$ bosonic

$\psi: H^0(\phi^*\Pi TX)$ fermionic

(Π: reverse pairity)

$L=\frac{1}{2}|d\phi|^2+\frac{1}{2}\langle\psi,\nabla_t\psi\rangle$ ($\nabla_t\psi$: $\dot\psi^i+\Gamma^i_{jk}\dot\phi^j\psi^k$)

(e.g. $X=\mathbb{R}$, $L=\frac{1}{2}(\partial_t\phi)^2+\frac{1}{2}\langle\psi,\partial_t\psi\rangle$)

η : odd parameter

$\delta\phi=-\eta\psi$

$\delta\psi=\eta\dot\phi$

$\delta L\,dt=\left(\langle-\eta\dot\psi,\dot\phi\rangle+\frac{1}{2}\langle\eta\dot\phi,\dot\psi\rangle+\frac{1}{2}\langle\psi,\eta\ddot\phi\rangle\right)dt$

(assume X flat) $=\left(-\frac{1}{2}\langle\eta\dot\psi,\dot\phi\rangle+\frac{1}{2}\langle\eta\psi,\ddot\phi\rangle\right)dt$

$=-\frac{1}{2}\eta\, d\langle\psi,\dot\phi\rangle$

Let Q be the odd generator of δ (ηQ gen. δ for odd η)

$$[\eta_1 Q, \eta_2 Q]\phi = -2\eta_1\eta_2\dot{\phi}$$

$$[\eta_1 Q, \eta_2 Q]\psi = -2\eta_1\eta_2\dot{\psi}$$

$$\Rightarrow \{Q, Q\} = 2\partial_t$$

Superspace formulation

$$M^{1|1} = \underset{t}{\mathbb{R}} \oplus \underset{\theta}{\Pi\mathbb{R}}$$

$$i: M^1 \to M^{1|1}$$

$$i^* = \text{set } \theta = 0$$

$\Phi: M^{1|1} \to X$ superfield

Some useful vector fields on $M^{1|1}$

∂_t even

$$\left.\begin{aligned} D &= \partial_\theta - \theta\partial_t \\ Q &= \partial_\theta + \theta\partial_t \end{aligned}\right\} \text{odd}$$

$$[D, D] = -2\partial_t$$

$$[Q, Q] = 2\partial_t$$

$$[D, Q] = 0$$

$$[\partial_t, D] = [\partial_t, Q] = 0$$

$$\mathcal{L} = -\tfrac{1}{2}\langle D\Phi, \partial_t\Phi\rangle$$

$$\int d\theta\,(a + b\theta) = b \Rightarrow \int d\theta = \partial_\theta$$

Idea: ① construct $\mathcal{L}$ on $M^{1|1}$ such that

$\int dt\, d\theta\, \mathcal{L}$ is invariant.

then define $L = \int d\theta\, \mathcal{L}$ "Berezin integral"

$\Rightarrow \int dt\, L$ is invariant

② "Taylor expand in θ.

$\Phi = \phi + \theta\psi$ "

Better way, $\phi = i^*\Phi$, $\psi = i^* D\Phi$

$$\int d\theta\, \mathcal{L} = -\tfrac{1}{2}\partial_\theta\langle -\theta\dot{\phi} + \psi, \dot{\phi} + \theta\dot{\psi}\rangle$$

$$= \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\langle\psi, \dot{\psi}\rangle$$

Superspace formulation $\longleftrightarrow$	Component formulation
$M^{1\|1}$	M^1
Φ	ϕ, ψ
$Q\Phi$	$Q\phi, Q\psi$
$\mathcal{L}$	L

3d $\mathcal{N}=1$ SUSY σ-model

Recall $\mathrm{Spin}(1,2) \simeq SL(2,\mathbb{R})$ $S=$ funda. rep of $SL(2,\mathbb{R})$

$\dim_{\mathbb{R}} S = 2$

Notation: $\mathfrak{p}^{d|s}$, $s=$ # of supercharges

$\mathcal{N}=*$: copies of S^*

$s = \mathcal{N}\cdot \dim_{\mathbb{R}} S^*$

$$\mathfrak{p}^{3|2} = V \oplus so(V) \oplus S^*$$

$$\Gamma : \mathrm{Sym}^2 S^* \xrightarrow{\simeq} V$$

$$M^{3|2} = V \oplus \Pi S^*$$

y^{11}, y^{12}, y^{22} θ^1, θ^2

$$\theta^a \otimes \theta^b \longrightarrow y^{ab} \quad a,b \in \{1,2\}$$

$$\left(y^{11} = \frac{x^0 + x^1}{2},\ y^{22} = \frac{x^0 - x^1}{2},\ y^{12} = \frac{x^2}{2} \right)$$

$$\partial_{ab} = \frac{\partial}{\partial y^{ab}} \qquad i : V \to M^{3|2}$$

$$D_a = \frac{\partial}{\partial \theta^a} - \theta^b \partial_{ab} \qquad \Phi : M^{3|2} \to X$$

$$Q_a = \frac{\partial}{\partial \theta^a} + \theta^b \partial_{ab}$$

$$[D_a, D_b] = -2\partial_{ab}$$

$$[D_a, Q_b] = 0$$

$$\left.\begin{aligned} \phi &= i^* \Phi \\ \psi_a &= i^* D_a \Phi \\ F &= -\tfrac{1}{2} \varepsilon^{ab}\, i^* D_a D_b \Phi \end{aligned}\right\} \text{ scalar multiplet}$$

$\varepsilon^{12} = 1,\ \varepsilon^{21} = -1$

$$\psi \in C^\infty(N, \phi^* \Pi TX \otimes S)$$

$-\eta^a Q_a \rightsquigarrow$

$$\delta\phi = -\eta^a \psi_a$$

$$\delta\psi_a = \eta^b(\partial_{ab}\phi - \varepsilon_{ab} F)$$

$$\delta F = \eta^a \left[(\not{D}\psi)_a + \tfrac{1}{3} \varepsilon^{bc} R(\psi_a, \psi_b)\psi_c \right]$$

$$(\not{D}\psi)_a = -\varepsilon^{bc} \partial_{ab} \psi_c \quad \text{Dirac operator}$$

(compare $\not{D}\psi = \gamma^\mu \partial_\mu \psi$ in x^0, x^1, x^2)

$$\gamma^0 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},\ \gamma^1 = \begin{pmatrix} & -1 \\ -1 & \end{pmatrix},\ \gamma^2 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\mathcal{L} = \frac{1}{4}\varepsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle$$

$$\rightsquigarrow L = \frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle\psi \not{D}_\phi \psi\rangle + \frac{1}{12}\varepsilon^{ab}\varepsilon^{cd}\langle\psi_a, R(\psi_b,\psi_c)\psi_d\rangle + \frac{1}{2}|F^2|$$

Remark: ① F only algebraic, <u>auxiliary field</u>

equation of motion $\Rightarrow F=0$

② We will focus on $L_{bos} = L|_{\psi=0} = \frac{1}{2}|d\phi|^2$

More SUSY, Kähler/hyperkähler manifold.

Def'n: A Kähler manifold is a Riemann manifold (M,g) with complex structure I such that $\nabla I = 0$ and $g(IY_1, IY_2) = g(Y_1, Y_2)$

Remark: $\omega(Y_1,Y_2) = g(IY_1, Y_2)$ is a closed, non-degenerate 2-form i.e. M is symplectic

Fact: There exists a real function K on M such that

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^{\bar{j}}}$$

$z^i, \bar{z}^{\bar{j}}$ are local coordinates on M and $g = g_{i\bar{j}}\, dz^i \otimes d\bar{z}^{\bar{j}}$

Def'n: A Hyperkähler manifold is a Riemann manifold (M,g) with 3 complex structures I, J, K that Kähler w.r.t. g. and

$$I^2 = J^2 = K^2 = IJK = -1.$$

4d $\mathcal{N}=1$ SUSY

"Proposition": The target manifold X of a 4d $\mathcal{N}=1$ SUSY σ-model must be Kähler.

$Spin(1,3) \simeq SL(2,\mathbb{C})$

S = funda. rep. of $SL(2,\mathbb{C})$, $\dim_{\mathbb{R}} S = 4$

$\mathfrak{p}^{4|4} = V \oplus \mathfrak{so}(1,3) \oplus S^*$, $\Gamma: Sym^2 S^* \to V$

$S_{\mathbb{C}} \simeq S' \oplus S''$, $S'' = \overline{S'}$ $\quad \theta^a \otimes \bar\theta^{\dot b} \mapsto y^{a\dot b}$

$V_{\mathbb{C}} \simeq S'^* \otimes S''^*$

$$M^{4|4} = \underset{y^{a\dot b}}{V} \oplus \underset{\theta^a,\ \bar\theta^{\dot b}}{\Pi S^*} \qquad a, b \in \{1,2\}$$

$$\partial_{a\dot b} = \frac{\partial}{\partial y^{a\dot b}} \qquad \left(\begin{aligned} y^{1\dot 1} &= \frac{x^0+x^1}{2} \\ y^{2\dot 2} &= \frac{x^0-x^1}{2} \\ y^{1\dot 2} &= \frac{x^2+ix^3}{2} \\ y^{2\dot 1} &= \frac{x^2-ix^3}{2} \end{aligned}\right)$$

$$D_a = \frac{\partial}{\partial\theta^a} - \bar\theta^{\dot b}\partial_{a\dot b}$$

$$\bar D_{\dot a} = \frac{\partial}{\partial\bar\theta^{\dot a}} - \theta^b\partial_{b\dot a}$$

$$[D_a, \bar D_{\dot b}] = -2\partial_{a\dot b}$$

Observation:

① We need X to have complex structure

② Superfield $\Phi: M^{4|4} \to \mathbb{C}$ will lead to too many fields

$\rightsquigarrow$ require Φ to be <u>chiral</u>

i.e. $\bar D_{\dot a}\Phi = 0$

to see why X must be Kähler:

(1) $4d\ \mathcal{N}=1 \xrightarrow[\text{reduction}]{\text{dimensional}} 3d\ \mathcal{N}=2$

(2) 3d $\mathcal{N}=1$ SUSY σ-model can be classified.

(3) require addition SUSY.

$\Rightarrow$ complex structure is Kähler.

Given X Kähler, construct 4d $\mathcal{N}=1$ SUSY

K: Kähler potential

$\Phi: M^{4|4} \to X$ superfield.

$$\mathcal{L} = K(\Phi, \bar\Phi)$$

$$L = \int d^4\theta\, K(\Phi, \bar\Phi)$$

$$\left.\begin{aligned} \phi &= i^* \Phi \\ \psi_a &= \frac{1}{\sqrt{2}} i^* D_a \Phi \\ F &= -\frac{1}{2} \varepsilon^{ab} i^* D_a D_b \Phi \end{aligned}\right\} \text{chiral multiplet}$$

$$\Phi = \phi + \theta^a \cdots \quad + \underline{i \bar{\theta} \sigma^\mu \theta \partial_\mu \phi}$$

$$\overline{\partial^\mu \phi^j} \partial_\mu \phi^i g_{i\bar{j}} \text{ in } L$$

SUSY with 8 supercharges $\quad$ ($\mathcal{N}=(1,0)$ to be precise)

(3d $\mathcal{N}=4$, or 4d $\mathcal{N}=2$ or 6d $\mathcal{N}=1$)

$\mathrm{Spin}(1,5) \simeq SL(2,\mathbb{H})$

8 real dim rep.

The target X must be hyperkähler

Remark: usually in component formulation since chiral condition is too complicated.