Coulomb Branch Seminar Notes

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1 What is RW Theory?

Let X be a hyperkähler manifold. It has a riemannian metric such that the holonomy of the Levi-Civita connection lies in an $\mathrm{Sp}(n)$ subgroup of $\mathrm{SO}(4n)$, where $\dim_{\mathbb{R}} X = 4n$. We thus have that $(T_X)^{\mathbb{C}} = T_X \otimes_{\mathbb{R}} \mathbb{C} = V \otimes S$, where V is a rank $2n$ complex principal $\mathrm{Sp}(n)$-bundle, and S is a trivial rank two bundle. The Levi-Civita connection on $(T_X)^{\mathbb{C}}$ reduces to a $\mathrm{Sp}(n)$ connection on V times the trivial connection on S.

Start with a 6 dimensional spacetime $\mathbb{R}^6$ with a euclidean flat metric. We pick bosonic fields $\Phi : \mathbb{R}^6 \to X$ and fermions $\psi : \mathbb{R}^6 \to \mathcal{S}_+ \otimes \Phi^*(V)$ where $\mathcal{S}_+$ is a spin bundle of $\mathbb{R}^6$ and V is the $\mathrm{Sp}(n)$ bundle over X above. Note that $\mathcal{S}_+$ is a rank 4 complex vector bundle. The supersymmetric lagrangian of this theory is

$$L = \int_{\mathbb{R}^6} \mathrm{d}^6 x \left(\frac{1}{2}(\mathrm{d}\,\Phi, \mathrm{d}\,\Phi) + \frac{i}{2}(\psi, D\psi) + \frac{1}{24}\Omega(\psi^4). \right),$$

where D is the Dirac operator $\mathcal{S}_+ \otimes \Phi^* V \to \mathcal{S}_- \otimes \Phi^* V$ and $(,)$ denotes the metrics on $T^*{}_{\mathbb{R}^6} \otimes \Phi^* T_X$ and $\mathcal{S}_- \otimes \Phi^* V$.

We want to reduce this theory to a dimension in which we can topologically twist it. Dimension 3 is the greatest one for which this works; in this case, reduction means we take the fields to independent of the coordinates x^i where $i > 3$. We are then left with a theory on $\mathbb{R}^3$ with an $\mathrm{SO}(3) \times \mathrm{SO}(3)$ symmetry. Call these $\mathrm{SO}(3)_E$ and $\mathrm{SO}(3)_N$, where E stands for "euclidean" and N for "internal." (need to mod out by $\mathbb{Z}_2$ if we have fermions)

The supercharges transform as two copies of $(\mathbf{2}, \mathbf{2})$ under $\mathrm{SU}(2)_E \times \mathrm{SU}(2)_N$. Let $\mathrm{SU}(2)'$ be a diagonal subgroup, and define a new action of rotations by thinking of $\mathrm{SU}(2)'$ as the rotation generators. Thus, the supercharges now transform as $\mathbf{1} \oplus \mathbf{3}$ under $\mathrm{SU}(2)'$. In particular, we are left with two $\mathrm{SU}(2)'$-invariant supercharges. Call these Q_A with $A \in \{1, 2\}$; they obey $\{Q_A, Q_B\} = 0$ [Does this follow because otherwise there would be an additional R-symmetry?].

Now, since we have $\mathrm{SU}(2)'$-invariant supercharges with ${Q_A}^2 = 0$, the theory can be generalized from flat spacetime to an arbitrary riemannian 3-manifold so that Q_A are still conserved and with a metric dependence of the form $\{Q_A, \ldots\}$. In this case, if we restrict to Q_A-invariant observables, we manifestly get a topological field theory.

Since the fermions in the untwisted theory are in $(\mathbf{2}, \mathbf{2})$, in the twisted theory they transform as $\mathbf{1} \oplus \mathbf{3}$; hence, they are given by a 0-form η and a 1-form χ_μ which both take values in $\Phi^* V$.

Thus, start with M an arbitrary riemannian 3-manifold with local coordinates x^μ, $\mu \in \{1, 2, 3\}$, and X a hyperkähler (it's actually sufficient to take any holomorphic symplectic) target manifold of complex dimension $2n$ as before.

The BRST transformation $\varepsilon^A Q_A$ acts on the fields according to

$$\delta_\varepsilon \phi^I = 0, \quad \delta_\varepsilon \phi^{\bar{I}} = \eta^{\bar{I}}, \quad \delta_\varepsilon \eta^{\bar{I}} = 0, \quad \delta_\varepsilon \chi^I = \mathrm{d}\,\phi^I.$$

You can check that this forces ${\delta_\varepsilon}^2 = 0$. The (BRST invariant) action is given by

$$S = \int_M (\mathscr{L}_1 + \mathscr{L}_2),$$

where

$$\begin{aligned}
\mathscr{L}_1 &= \delta_\varepsilon (g_{I\bar{J}} \chi^I \wedge \star d\phi^{\bar{J}}) = g_{I\bar{J}} d\phi^I \wedge \star d\phi^{\bar{J}} - g_{I\bar{J}} \chi^I \wedge \star \nabla \eta^{\bar{J}} \\
\mathscr{L}_2 &= \frac{1}{2}\Omega_{IJ} \left(\chi^I \wedge \nabla \chi^J + \frac{1}{3} R^J_{KL\bar{M}} \chi^I \wedge \chi^K \wedge \chi^L \wedge \eta^{\bar{M}} \right).
\end{aligned}$$

Ω is the holomorphic symplectic form on X, g the metric on X, R the curvature of X, and ∇ is the pullback of the Levi-Civita connection from X.

Fix one of the $\mathbb{P}^1$-many complex structures on X, and let ϕ^I be local holomorphic coordinates in this structure. Then, we can pick a basis $Q, \overline{Q}$ for the two supercharges so that they act by

$$\begin{aligned} \delta\phi^I &= \eta^I, \quad \delta\bar{\phi}^{\bar{I}} = 0, \\ \delta\eta^I &= 0, \quad \delta\chi^I_\mu \ = \mathrm{d}\bar{\phi}^I, \end{aligned}$$

for Q and

$$\begin{aligned} \bar{\delta}\phi^I &= 0, \quad \bar{\delta}\bar{\phi}^{\bar{I}} = g^{\bar{I}J}\varepsilon_{JK}\eta^K, \\ \bar{\delta}\eta^I &= 0, \quad \bar{\delta}\chi^I_\mu = -\partial_\mu\phi^I \end{aligned}$$

for $\overline{Q}$.

Note that this theory is topological: Only the BRST exact piece $\mathscr{L}_1$ depends on the metric on M; hence, the partition function and correlation functions of BRST-invariant observables are independent of the metric. It follows that if we restrict our attention to BRST-invariant operators, the theory is topological. Equivalently, we can say that the algebra of topological operators is equal to the cohomology of the BRST operator. After picking a particular complex structure on X, $\varepsilon^{IJ}g_{J\bar{K}}$ establishes an isomorphism $\Omega^{p,0} \cong \Omega^{0,p}$. Let $\omega \in \Omega^{p,0}$ which is closed as both a $(p,0)$-form and a $(0,p)$-form.

Remark. The grading here is the "ghost number," where we can axiomatize the ghost number to be 0 for the usual fields, 1 for the ghosts, -2 for the antighosts, and -1 for the antifields. This arises because the theory has a $\mathbb{Z}/2\mathbb{Z}$ symmetry which multplies all fermions by -1. If the lagrangian contained only the $\mathscr{L}_1$ term, this could be extended to a $U(1)$ symmetry, so that χ has charge -1 and η charge 1. This symmetry is broken by $\mathscr{L}_2$ down to $\mathbb{Z}/2\mathbb{Z}$. We can formally restore it by assigning Ω charge 2.

Proposition 1. The algebra of topological observables in RW theory is isomorphic to

$$\oplus_p H^p(\mathbb{O}_X).$$

Proof. Let ω be a $(0,p)$-form, not necessarily closed. Define

$$\begin{aligned} \mathbb{O}_\omega &= \frac{1}{p!}\omega_{\bar{I}_1,\ldots,\bar{I}_p}\eta^{\bar{I}_1}\wedge\cdots\wedge\eta^{\bar{I}_p} \\ &= \frac{1}{p!}\omega(\eta^{\bar{I}_1}\wedge\cdots\wedge\eta^{\bar{I}_p}). \end{aligned}$$

Then

$$\delta_\varepsilon\mathbb{O}_\omega = \mathbb{O}_{\bar{\partial}\omega}.$$

Therefore, BRST cohomology is isomorphic to Dolbeault cohomology. The algebra structure is given by the ordinary exterior product of differential forms. □

This proposition and the fact that it holds in the quantum theory is one of the main reasons why RW theory is rich. It turns out that there are no quantum corrections to the statement of Proposition 1.

Remark. The advantage of having a BRST symmetry inside the supersymmetry algebra is that in the quantum theory massive cancellations become possible. Here's the gist.

1. When we try to calculate some 1-loop Feynman diagrams, we find that we need only consider Feynman diagrams whose order of contribution is $\hbar^{2n}$, since the integral related to the $4n$ bosonic zero modes goes as $(2\pi\hbar)^{-2n}$. The partition function shouldn't depend on $\hbar$ since the theory is topological. With $2n$ zero modes of η, every Feynman diagram that goes as $\hbar^s$ with $s < 2n$ dies. It can be shown that in fact $s = 2n$. This reduces the lagrangian to one with only two vertices (both of fourth order).

2. After (1) has been established, we can systematically analyze the 1-loop contributions *completely*.

2 Atiyah-Hitchin Space

For $G = \mathrm{SU}(2)$, the Coulomb branch of 3d $\mathcal{N} = 4$ pure gauge theory arises by dimensionally reducing the F^2 auxiliary field term. Note again that we begin with a gauge field A and Weyl fermions ψ in the adjoint representation of G under an $\mathrm{SU}(2)_R$ symmetry in 6 dimensions with $\mathcal{N} = 1$. Reduction to 3d gives 3 new scalars $\varphi_i = A^{i+3}$. The scalars transform as $(\mathbf{1},\mathbf{3},\mathbf{1})$ under $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_N \times \mathrm{SU}(2)_E$. We would find that the potential is given by the equation

$$V = \frac{1}{4e^2}\sum_{i<j} \mathrm{tr}[\varphi_i, \varphi_j]^2,$$

where e is the gauge coupling. Classical solutions of $V = 0$ are clearly those φ_i which lie in the Cartan of $\mathfrak{g}$. For SU(2), which is of rank $r = 1$, the φ_i break the G-symmetry to a $U(1)$-symmetry; thus, there is additionally one massless photon, or gauge field. In 3d, these all sit in the same vector multiplet; hence, there is a total of 4 massless scalars. It is possible to introduce masses for them, but we won't consider that possibility (the Coulomb branch won't arise). We can also include FI terms, but those don't arise for semisimple G.

We can also add hypermultiplets to the theory; these would arise by including N_f pairs of chiral superfields. The most basic object is four real scalars that transform under $(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})$ under $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_N \times \mathrm{SU}(2)_E \times G$ along with fermions transforming as $(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2})$.

The physics is free in the IR and can be described by a conventional low-energy EFT. The most general one for 4 massless scalars in 3d $\mathcal{N} = 4$ SUSY is a sigma model with target space a hyperkähler manifold of quaternionic dimension 1, as Che explained a couple of weeks ago. It follows that the Coulomb branch is a hyperkähler manifold, which greatly restricts the possibilities.

Suppose $\|\varphi\|$ is large, so that we are in the semiclassical regime. Then $V = 0$ is just a copy of $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$, where we include the S^1 which parametrizes a fourth scalar σ dual to the gauge boson and the $\mathbb{Z}_2$ is the action of the Weyl group. There is a classical hyperkähler metric on the moduli space

$$\frac{1}{e^2}\,\mathrm{d}\varphi \otimes \mathrm{d}\varphi + e^2\,\mathrm{d}\sigma \otimes \mathrm{d}\sigma.$$

Infinity in $\mathbb{R}^3$ is homotopic to S^2, so that topologically we get $S^2 \times S^1$ at infinity classically. The S^2 has a "radius" proportional to $\|\varphi\|$, while the S^1 has radius $e/2\pi$. In the quantum theory, instead of $S^2 \times S^1$ at infinity, we might have an S^1 fiber bundle over S^2, with a metric given by

$$\frac{1}{e^2}\sum \left[\mathrm{d}\varphi_i \otimes \mathrm{d}\varphi_i + e^2(\mathrm{d}\sigma - sB_i(\varphi)\,\mathrm{d}\varphi_i) \otimes (\mathrm{d}\sigma - sB_i(\varphi)\,\mathrm{d}\varphi_i)\right],$$

where $s \in \mathbb{Z}$ and B is the Dirac monopole U(1) gauge field over S^2.

It can be shown that the monopole and 1-loop corrections force two things. First, N_f must be 0 or 1, in which case $s = -4 + 2N_f$ must be either -4 or -2. Second, at infinity we end up with a Lens space $L_s/\mathbb{Z}_2$ with $s \in -4, -2$.

This Lens space is by definition $L_s = S^3/\mathbb{Z}_s$, where $\mathbb{Z}_s$ is generated by $u_\alpha \mapsto e^{2\pi i/s} u_\alpha$. It arises as the quotient $(S^3 \times S^1)/\,\mathrm{U}(1)$ where the U(1) action is given by

$$u_\alpha \mapsto e^{i\theta} u_\alpha$$
$$\psi \mapsto \psi + s\theta.$$

We can gauge out the ψ to obtain the desired $\mathbb{Z}_s$ action.

After the smoke clears, we're left with the Atiyah-Hitchin manifold, or AH manifold, which looks like a 2-plane bundle over $\mathbb{RP}^2$ with structure at infinity that looks like $L_{-4}/\mathbb{Z}_2$. Its simply connected double cover is a 2-plane bundle over S^2 with $s = -2$. In both cases, $N_f \in \{0,1\}$, we're left with a complete, nonsingular metric on the Coulomb branch.

3 Reduction of RW

We want to evaluate $Z_X(M)$ for RW theory with target X and spacetime M. We'll investigate two cases: $M = S^2 \times S^1$ and $M = T^3$; other cases can be obtained from examining cutting-and-gluing relations and the other Betti numbers, 0 and 2.

There is only one minimal Feynman graph which contributes to $Z_X(M)$, and it has prefactor

$$\frac{1}{2} b_\theta(X) |H_1(M,\mathbb{Z})|',$$

where

$$b_\theta(X) = \frac{1}{8\pi^2} \operatorname{tr} \int_X R \wedge R$$

and $|H_1(M,\mathbb{Z})|'$ is the number of torsion elements in $H_1(M,\mathbb{Z})$. Now, nothing depends on X in this formula except $b_\theta(X)$ (the integrals arising from the interaction of the Feynman diagram don't depend on X either). It follows that if we take X to be 4-dimensional, then if we can compute $Z_X(M)$ for arbitrary M, we can also compute it for $Z_{X'}(M)$, where X' is any other 4-dimensional target. In particular, we can take X' to be the Atiyah-Hitchin space of the previous section, and take X to be say a K3 surface. It will then follow that

$$Z_{\mathrm{AH}}(M) = \frac{b_\theta(\mathrm{AH})}{b_\theta(\mathrm{K3})} Z_{\mathrm{K3}}(M).$$

Let M be a 3-manifold with boundary $\partial M = \Sigma$, where Σ is a Riemann surface. For example, we can take $M = S^1 \times \Sigma$. We'd like to evaluate the partition function $Z_X(M)$. The path integral will be taken over the fields $\phi^i, \eta^I, \chi^I_\mu$ which satisfy certain boundary conditions on Σ, so $Z_X(M)$ becomes a function of the boundary values. Q-invariant boundary functions form a Hilbert space $\mathcal{H}_\Sigma$; we say $Z_X(\Sigma) = \mathcal{H}_\Sigma$. It follows then that $Z_X(M) = \operatorname{sdim} \mathcal{H}_\Sigma$.

If we can find M_1, M_2 with $\partial M_1 = \partial M_2 = \Sigma$, then we can glue them along Σ to obtain a new manifold M. Then,

$$Z_X(M) = \langle M_1^* | M_2 \rangle = \langle M_2^* | M_1 \rangle \in \mathcal{H}_\Sigma.$$

We could also act on ∂M_1 by an element U of the mapping class group of Σ prior to gluing. This group is represented in $\mathcal{H}_\Sigma$, so that we get

$$Z_X(M^U) = \langle M_2^* | U | M_1 \rangle.$$

When we reduce on S^1, we'll get a space of classical solutions to the equations of motion which can be expanded in a basis of modes of fixed frequency. In the leading approximation, there is a space of zero modes tensored with a space of non-zero modes; however, Q is acyclic in the space of non-zero modes away from the ground state, so we can only quantize the zero modes!

Thus, the 1-form field χ^I now becomes $\chi^I = \chi^I_\Sigma + \chi^I_0 \,\mathrm{d}x^0$, where χ^I_Σ is a pull-back of a 1-form ρ^I on Σ, and χ^I_0 becomes just a 0-form on Σ. We also have 0-forms ϕ^i and η^I.

Remark. You can check that the BRST transformations of the constant modes now become the same as that of the B-model with target X. They become

$$\begin{aligned} \delta\phi^I &= 0 & \delta\phi^{\bar I} &= \eta^{\bar I} \\ \delta\eta^{\bar I} &= 0 & \delta\theta_I &= 0 \\ \delta\rho^I &= \mathrm{d}\phi^I, \end{aligned}$$

where $\theta_I = \Omega_{IJ}\chi^J_0$.

Now, fix a complex structure on X. We claim that

$$\mathcal{H}_{\Sigma_g} = \sum_{q=0}^{\dim_{\mathbb{C}} X} H^q_{\bar\partial}(X, (\wedge^\bullet V)^{\otimes g}).$$

3.1 $S^2 \times S^1$

Now, since we've set $\Sigma = S^2$, $\dim H^1(S^2) = 0$, so we only need to deal with the zero modes η^I and χ^I_0. They satisfy the single anti-commutation relation

$$\{\eta^I, \chi^J_0\} = \varepsilon^{IJ}.$$

This space will contain the single vacuum state $|0\rangle_\eta$ which is annihilated by χ^I_0. States are produced by the action of operators η^I. If we also include the bosons, a general state is given by

$$|\psi\rangle = \psi_{I_1\cdots I_l}(\phi)\eta^{I_1}\cdots\eta^{I_l}|0\rangle_\eta.$$

Note that it can be interpreted as a section of $\wedge^l V$, where V is the $\mathrm{Sp}(n)$ bundle we met previously. The ineer product then becomes a scalar product between sections of $\wedge^l V$ and $\wedge^{2n-l} V$:

$$\langle\psi_1|\psi_2\rangle = \int_X \psi_1 \wedge \psi_2.$$

The action of the Q_A operators is then just obtained from the transformation laws; moreover, since $\wedge^l V$ can be identified with the space of $(0,l)$-forms on X, one of the Q's becomes the $\bar\partial$ operator. We thus get the finite-dimensional space

$$\mathcal{H}_{S^2} = \bigoplus_{l=0}^{2n} H^{0,l}(X).$$

Thus,

$$Z_X(S^2 \times S^1) = \operatorname{sdim}\mathcal{H}_{S^2} = \sum_{l=0}^{2n}(-1)^{1+l}\dim H^{0,l}(X).$$

3.2 $\Sigma = T^2$

This case is in some sense simpler. Let ω_i be a basis for $H^1(M,\mathbb{Z})$ in $H^1(M,\mathbb{R})$. Then the integral for the only Feynman diagram is just

$$I(M) = \left(\int_M \omega_1 \wedge \omega_2 \wedge \omega_3\right)^2.$$

This is an obvious classical invariant of M with $b_1(M) = 3$. The weight function is the same as before,

$$\begin{aligned} Z_X(M) =& \frac{1}{2}|H_1(M,\mathbb{Z})|' b_\theta(X) I(M) \\ =& \frac{1}{2} b_\theta(X). \\ Z_{\mathrm{AH}}(M) =& -|H_1(M,\mathbb{Z})|' I(M) \\ =& -1. \end{aligned}$$

The Hilbert space is again formed by a vacuum $|0\rangle_{\chi_2}$ annihilated by χ_1^I, together with states produced by the operators χ_2^I. Here, the fermion fields turn into operators, where $\omega^{(\alpha)}$ are harmonic 1-forms on Σ, $\chi_\mu^I = \chi_\alpha^I \omega_\mu^{(\alpha)}$, $\alpha \in \{1,\dots,\dim H^1(\Sigma)\}$, $\mu \in 1,2$.

Everything proceeds as before, except now a general state is given by

$$|\psi\rangle = \psi_{I_1\cdots I_l,J_1\dots J_m}(\phi)\chi_2^{I_1}\dots\chi_2^{I_l}\eta^{J_1}\cdots\eta^{J_m}|0\rangle_{\eta\chi}.$$

States are thus sections of $\wedge^\bullet V \otimes \wedge^\bullet V$. Thus,

$$\mathcal{H}_{T^2}(X) = \bigoplus_{l,m=1}^{2n} H^{l,m}(X).$$

Remark 1. Note that if X is not compact, then there is an obstruction to a reduction to a finite-dimensional space of physical states. Nonetheless, we would like to find a way to consider non-compact X.

Remark 2. We can actually calculate $Z_{\mathrm{K3}}(T^3)$ via the Hodge diamond of a K3 surface: It is just 24. This lets us work backwards and conjecture that $Z_{\mathrm{AH}}(T^3)$ should be -1.