Spin Geometry

&

Dirac operators

Questions:

- Clifford algebra as quantization of exterior algebra?
- why negative def. & not positive def?
  (does positive def only retrieve $O(n)$, not $PIN(n)$?) Sign convention?
- Clifford algebra philosophy: to study $O(n)$, things who preserve bilinear form Q, consider algebra w/ multiplication using Q
  Can we apply this to other things, e.g symplectic grp?
- why is spin a universal cover, while $spin^c$ isn't? is it a coincidence that $O(?) = \pi_1(SO(?))$
- what are the non-invertible elements of C_Q? just 0?
  no, many idempotents

$\mathbb{R}[x]/x^2=1 \cong \mathbb{R}\oplus\mathbb{R}$ as a ring
$\frac{1+x}{2}$ $\frac{1-x}{2}$

$\mathbb{R}[x]/x^2-1 \cong \mathbb{C}$ field

$$C(-Q_1)\hat{\otimes} C(-Q_2) = C\begin{pmatrix} -1 & \\ & -1\end{pmatrix} = \mathbb{H} = \langle 1, x_1, x_2, x_1x_2\rangle$$

(annotations: i over x_1, j over x_2, $-k$ over x_1x_2; "- definite" over the matrix)

$$C\begin{pmatrix} 1 & \\ & 1\end{pmatrix} = M_{2\times 2}(\mathbb{R})$$

$\mathbb{R}[x_1,x_2]/x_1^2=x_2^2=1,\ x_1x_2=-x_2x_1$

$\frac{1+x_1}{2} \otimes 1 \Leftrightarrow \begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}$

$x_1 \Leftrightarrow \begin{pmatrix} 1 & \\ & -1\end{pmatrix}$ $x_1^2 = \begin{pmatrix} 1 & \\ & 1\end{pmatrix}$

$x_2 \Leftrightarrow \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$ $x_2^2 = \begin{pmatrix} 1 & \\ & 1\end{pmatrix}$ $x_1x_2 = -x_2x_1$ ✓

pin^+: comes from $\begin{pmatrix} 1 & \\ & 1\end{pmatrix}$ pos def

pin^-: comes from $\begin{pmatrix} -1 & \\ & -1\end{pmatrix}$ neg def

$spin^+ \cong spin^-$
but $pin^+ \not\cong pin^-$
($\mathbb{Z}_2$ action different prob)

Q: why $\mathbb{K}$ & $\mathbb{K}(n)$ have same category of reps?
every rep of $A(n)$ on V^n comes from a rep of A on V

why is brauer group of $\mathbb{R}$ just $\mathbb{Z}_2 = [\mathbb{R}]$ & $[\mathbb{H}]$

tensor anything w/ $\mathbb{C}$ & get $\mathbb{C}$, and $\mathbb{C}$ is field?

Real super-division algebras

$C_K \otimes C_J \overset{\text{morita}}{\cong} C_{K+J \,\%\, 8}$?

Convert category of reps of C_K to loop space?

in 2x2 case,
spin acts by rotation
matrix squared
$\Rightarrow$ double cover

Clifford Algebras

Consider vector space $V^{\dim = n}$ (over field k) w/ Quadradic form Q

tensor algebra $T(V) = \bigoplus_{i=0}^{\infty} \underbrace{V^i}_{V \otimes \dots \otimes V \text{ } i \text{ times}} = \otimes^* V$ has ideal I_Q generated by $x \otimes x - Q(x)$

$$\begin{tikzcd} V \arrow[r, hook, "i", "\text{natural inc.}"'] \arrow[rr, bend right, "i_Q"'] & \oplus V^i \arrow[r, two heads, "/I(Q)", "\text{Quotient}"'] & C_Q \end{tikzcd}$$

Define <u>Clifford algebra</u> $C_Q = T(V)/I_Q$

Essentially, define product via Quadradic form Q.

<u>Universal prop. of Clifford algebras</u>: for any linear map $\phi : V \to \overset{\text{algebra}}{A}$, such that $\phi(x)^2 = Q(x) \cdot 1$, there is a unique lift $\tilde{\phi} : C_Q \to A$ s.t. $\phi = \tilde{\phi} \circ i$

$$\begin{tikzcd} V \arrow[r, "i_Q"] \arrow[dr, "\phi"'] & C_Q \arrow[d, dashed, "\tilde{\phi}"] \\ & A \end{tikzcd}$$

clifford algebra is the <u>universal</u> thing w/ products $x \cdot x = Q(x) \cdot 1$

i_Q is an injection:
suppose $i_Q(x_1) = i_Q(x_2)$
then $x_1 = x_2 + \underset{I_Q}{\underset{\Uparrow}{v}}$
but, I_Q has no elements purely of grade 1, so this implies $v = 0$ & $x_1 = x_2$

<u>Filtering</u>: C_Q has filtering induced from $\oplus V^i$ filtration not grading

Take field $= \mathbb{R}$:

choose a basis of E normalizing Q: $\overset{\text{orthonormal basis}}{\{\gamma_1, \dots, \gamma_n\}}$, $Q(\gamma_1) = \dots = Q(\gamma_p) = 1$, $Q(\gamma_{p+1}) = \dots = Q(\gamma_n) = -1$

(denote V w/ this form as $\mathbb{R}_{p, n-p}$) denote $\sigma_i = \begin{cases} 0 & 0 \leq i < p \\ 1 & p \leq i \leq n \end{cases}$

C_Q is generated by $\{\gamma_i\}$:

$Q(\gamma_i + \gamma_j) = Q(\gamma_i) + Q(\gamma_j) = (-1)^{\sigma_i} + (-1)^{\sigma_j}$; $Q(\gamma_i + \gamma_j) = (\gamma_i + \gamma_j)^2 = \cancel{\gamma_i^2}^{(-1)^{\sigma_i}} + \gamma_i \gamma_j + \gamma_j \gamma_i + \cancel{\gamma_j^2}^{(-1)^{\sigma_j}} \Rightarrow \gamma_i \gamma_j = \gamma_j \gamma_i$

C_Q has relations $\boxed{\begin{cases} \gamma_i^2 = (-1)^{\sigma_i} \\ \gamma_i \gamma_j = -\gamma_j \gamma_i \end{cases}}$ algebra w/ n anticommuting generating elements so, C_Q similar to exterior algebra $\Lambda^* V$

$\Rightarrow C_Q$ has basis $\{\underbrace{\gamma_{i_1} \cdot \dots \cdot \gamma_{i_k}}_{\text{Ordered product}} ; i_1 < \dots < i_k, 0 < k < n\}$ $\dim(C_Q) = 2^n$

The grading induced from $\otimes^* V$ is $\operatorname{rank}(\gamma_{i_1} \cdot \dots \cdot \gamma_{i_k}) = k$ Counts # of basis elements
denote this decomposition as $C_Q = \oplus C_Q^k$

<u>Super algebra structure</u>: like exterior algebra, splits into even and odd parts

$C_Q = C_Q^e \oplus C_Q^o$ $\begin{cases} C_Q^e = \oplus C_Q^{2k} \\ C_Q^o = \oplus C_Q^{2k+1} \end{cases}$

elements of C_Q^e commute, those of C_Q^o anticommute for $V = V_1 \oplus V_2$, $Q = Q_1 \oplus Q_2$ orthogonal decomp $\quad C_Q = C_{Q_1} \overset{\check{}}{\otimes} C_{Q_2}$ ($\mathbb{Z}_2$ graded tensor product)

in usual case w/ $Q = Q^-$ negative definite, this implies

$C_{Q^-} = C_1 \check{\otimes} \cdots \check{\otimes} C_1$ w/ C_1 = clifford algebra on $V = \mathbb{R}$ w/ $Q(\gamma_1) = -1$ i.e. $C_1 \cong \mathbb{R}[\gamma]/\gamma^2 = -1 \cong \mathbb{C}$

$C_{Q^-} = \mathbb{C} \check{\otimes} \cdots \check{\otimes} \mathbb{C}$

Operations on C_Q:

- $x \to x^t$ (transpose, $\otimes^* V$) $\quad x_1 \otimes \cdots \otimes x_n \longmapsto x_n \otimes \cdots \otimes x_1$: swap order of all elements

descends to C_Q, since preserves relation $x \otimes x = Q(x) \cdot 1$

$(xy)^t = y^t x^t \qquad (\gamma_1 \cdots \gamma_k)^t = \gamma_k \cdots \gamma_1 = (-1)^{k-1} \gamma_{k-1} \cdots \gamma_1 \gamma_k = (-1)^{(k-1)+(k-2)} \gamma_{k-2} \cdots \gamma_1 \gamma_k \gamma_{k-1}$

$= (-1)^{\sum_0^k k-i} \gamma_1 \cdots \gamma_k \qquad \sum k-i = k^2 - \sum i = k^2 - \frac{(k)(k-1)}{2} = \frac{k(k+1)}{2}$

$(-1)^{\sum k-i} = \begin{cases} 0 & k \bmod 4 = 0 \text{ or } 1 \\ 1 & k \bmod 4 = 2 \text{ or } 3 \end{cases}$

- $x \to \alpha(x) \qquad \gamma_{i_1} \cdots \gamma_{i_k} \mapsto (-\gamma_{i_1}) \cdots (-\gamma_{i_k})$: unique extension of $V \mapsto -V$ to C_Q

$\alpha|_{C_Q^e}: x \mapsto x \qquad \alpha|_{C_Q^o}: x \mapsto -x \qquad \alpha$ gives even/odd grading

- $x \to \bar{x} = \alpha(x^t) \qquad \gamma_{i_1} \cdots \gamma_{i_k} = (-1)^k \gamma_{i_k} \cdots \gamma_{i_1}$: $\quad \overline{xy} = \bar{y}\,\bar{x}$

- $\rho: C_Q^* \to \operatorname{Aut} C_Q \qquad \rho(x) y = \alpha(x) y x^{-1}$

(C_Q^*: elements of C_Q w/ multiplicative inverses (forms a group))

This is almost the adjoint rep of C_Q^*, but it's twisted by α (which modifies sign according to grading)

"Twisted Adjoint representation"

let $x^e \in C_Q^e \quad x^o \in C_Q^o$: $\quad \alpha(x^e) = x^e \quad \alpha(x^o) = -x^o$

say $\rho(x) y = y$: $\quad \rho(x^e) y = y \Rightarrow x^e y x^{e^{-1}} = y \Rightarrow x^e y = y x^e$

$\rho(x^o) y = y \Rightarrow -x^o y x^{o^{-1}} = y \Rightarrow x^o y = -y x^o$

"super" adjoint rep. ?

"spinor norm"

- $N: C_Q \to C_Q \qquad N(x) = x\bar{x}$: $\quad$ note: have inclusion $V \cong C_Q^1 \hookrightarrow C_Q$, if $x \in C_Q^1$ say $x \in V$

$x \in V$: $\bar{x} = -x$, so $N(x) = -x \cdot x = -Q(x)$ is the norm of x, for Q negative definite.

$x = \gamma_{i_1} \cdots \gamma_{i_k}$: $N(x) = \gamma_{i_1} \cdots \gamma_{i_k} \cdot (-\gamma_{i_k}) \cdots (-\gamma_{i_1}) = \prod_{j=1}^{k} -Q(\gamma_{i_j}) = (-1)^{\sigma_{i_1}+1} \cdots (-1)^{\sigma_{i_k}+1}$

Clifford Group:

(or "Lipshitz group")

subgroup of elements of C_Q^* who preserve $V \subset C_Q$ under action of ρ

gives rep $\rho: \Gamma_k \to \operatorname{Aut}(\mathbb{R}^k)$

does this need Q negative def?

for $\gamma_i \in V$, $\rho(\gamma_i) \gamma_I = -\gamma_i \gamma_I \gamma_i^{-1}$, where $\gamma_i^{-1} = (-1)^{\sigma_i} \gamma_i$ (I: multi-index)

if $i \notin I$: $\gamma_i \gamma_I = (-1)^{|I|} \gamma_I \gamma_i$, so $\rho(\gamma_i) \gamma_I = (-1)^{|I|} \gamma_I$

if $i \in I$: suppose $i = I_j$. Then, $\gamma_i \gamma_I \gamma_i^{-1} = \gamma_i \gamma_{I_1} \cdots \gamma_{I_j} \cdots \gamma_{I_k} \gamma_i^{-1} = (-1)^{|I|-1} \gamma_{I_1} \cdots \gamma_i \gamma_{I_j} \gamma_i^{-1} \cdots \gamma_{I_k} = (-1)^{|I|-1} \gamma_I$: multiply by -1 when basis contains I

so, $\rho(\gamma_i)$ acts by reflection about $\gamma_i^\perp$ orthogonal hyperplane

$\ker \rho = \mathbb{R}^*$:

$x \in \ker \rho \Rightarrow \rho(x)y = y \Rightarrow \alpha(x)y = yx \Rightarrow \alpha(y)x = \alpha(x)\alpha(y)$

split $x = x^e + x^o$: this says $\begin{cases} x^e y = y x^e \\ x^o y = -y x^o \end{cases}$ or $\begin{matrix} x^e = y x^e y^{-1} = \rho(y) x^e \\ x^o = -y x^o y^{-1} = \rho(y) x^o \end{matrix}$

$\underset{C_Q^e}{\uparrow}$ $\underset{C_Q^o}{\uparrow}$

$\rho(\gamma_i)$ is reflection about $\gamma_i^\perp$, so $\rho(\gamma_i)x^e = x^e \Rightarrow x^e$ does not depend on γ_i, likewise x^o

this applies to all basis elements, so x must not have any factors of any γ^i: $x \in \mathbb{R}$

but, $x \in \Gamma_{P,Q} \Rightarrow x$ invertible, so $x \in \mathbb{R}^*$. every $x \in \mathbb{R}^*$ is in $\ker \rho$, so $\ker \rho = \mathbb{R}^*$ ∎

$x \in \Gamma_k \Rightarrow N(x) \in \mathbb{R}^*$: wts $N(x) \in \ker \rho$

$\rho(N(x))y = \alpha(x\bar{x})y(x\bar{x})^{-1} = \alpha(x)\alpha(\bar{x})\, y\, \bar{x}^{-1}x^{-1} = \alpha(x)x^t y \bar{x}^{-1} x^{-1}$, but, $\rho(N(x))y \in \mathbb{R}^k$, so $= (\rho(N(x))y)^t$

$\alpha(x)x^t y \cancel{\bar{x}^{-1}} x^{-1} = (\alpha(x)x^t y \bar{x}^{-1} x^{-1})^t = \bar{x}^{-t} x^{-1t} y\, x\, \alpha(x)^t = \alpha(x^{-1}) x^{-1t} y \cancel{x} x \Rightarrow y\alpha(x^t)x = x^t\alpha(x)y \Rightarrow \alpha(x^t)x \in \ker \rho$

but, $\alpha(x)x^t = N(x)^t$, so $\alpha(x)x^t \in \mathbb{R}^* \Rightarrow N(x) \in \mathbb{R}^*$ ∎

$N(x)$ acts like norm on Γ_k

Spin Group

$\mathrm{Pin}(k) := \ker N : \Gamma_k \to \mathbb{R}^*$ unit norm invertible elements preserving V

fits into SES $1 \to \mathbb{Z}_2 \to \mathrm{Pin}(k) \xrightarrow{\rho} O(k) \to 1$ double cover of $O(n)$

$\rho(\mathrm{Pin}(k))$ is generated by reflections across unit norm elements, which generate all $O(n)$, so ρ is onto

kernel of ρ consists of unit norm elements of $\mathbb{R}^*$, i.e $O(1) \cong \mathbb{Z}_2$

$\mathrm{Spin}(k) \subset \mathrm{Pin}(k) := \rho^{-1}(SO(k))$, so half of $\mathrm{Pin}(k)$

consists of orientation preserving maps, i.e generated by even # of reflections

$\mathrm{spin}(k)$ is <u>even</u> part of $\mathrm{Pin}(k)$

(usually, identity component)

$\mathrm{Spin}(k)$ is <u>non trivial</u> double cover, because it is connected.

so, $\mathrm{Spin}(k) \cong \widetilde{SO(k)}$ (for $k \geq 3$)

$\mathrm{Spin}^c(k)$: analagous for complex case $C_k \otimes \mathbb{C}$ w/ $\begin{matrix} \alpha(x \otimes z) = \alpha(x) \otimes z \\ (x \otimes z)^t = x \otimes \bar{z} \end{matrix} \Rightarrow N(x \otimes z) = N(x) \otimes z\bar{z}$

for complex, unit norm elements is $U(1)$, not $\mathbb{Z}_2$. So, $\mathrm{Spin}^c(k) \cong \mathrm{Spin}(k) \times U(1)$

<u>relation to complex structures</u> homomorphism $U(k) \to SO(2k)$ consists of forgetting complex structure: $\mathbb{C}^k \mapsto \mathbb{R}^{2k}$

does not lift: i.e

Choice of $U(k)$ frame bundle = Choice of almost complex structure

each vector represents a reflection, gives rep of V on itself
this uniquely extends to Clifford algebra by universal prop
(reflections represented by ρ
reflections across basis vectors generate Clifford algebra
Pin is all reflections, Spin is even # (orientation preserving)

the path $v \to -v$ induces a loop $\rho(v) \to \rho(-v)$ in $O(n)$ of reflections about $v^\perp$. This generates $\pi_1(O(n))$?

unit norm elements of C_k^1, along w/ ρ, give map $S^{k-1} \subset \mathbb{R}^k \subset C_k \longrightarrow O(k)$ (reflection)
generates $\pi_{k-1}(O(k))$?

Determination of Clifford Algebras

identities on classical algebras:

1) $K \otimes \mathbb{R}(n) = K(n)$ for $K \in \mathbb{R}, \mathbb{C}, \mathbb{H}$ (or any field)
$\mathbb{R}(n)$: $n \times n$ matrices in $\mathbb{R}$
obvious

2) $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$:

basis	$1 \otimes 1$	$i \otimes 1$	$1 \otimes i$	$i \otimes i$
square	$1 \otimes 1$	$-1 \otimes 1$	$-1 \otimes 1$	$1 \otimes 1$

$(a+bi) \otimes (c+di) = ac\ 1\otimes 1 + bc\ i \otimes 1 + ad\ 1 \otimes i + bd\ i \otimes i$

$(a+bi) \otimes (c+di) \cdot (a'+b'i) \otimes (c'+d'i)$
$= (ac\ 1\otimes 1 + bc\ i\otimes 1 + ad\ 1 \otimes i + bd\ i \otimes i) \cdot (a'c'\ 1 \otimes 1 + b'c'\ i \otimes 1 + a'd'\ 1 \otimes i + b'd'\ i \otimes i)$
$= (ac a'c' - bc b'c' - ad a'd' + bd b'd')\ 1 \otimes 1 + \; = (aa' - bb')(cc' - dd')\ 1 \otimes 1 + \ldots$
$ac b'c' + bc a'c' - b'd' ad - bd a'd'\ \ i \otimes 1 + \ldots$

$(a+bi, c+di)(a'+b'i, c'+d'i) = (aa' - bb' + (ab' + b'a)i,\ cc' - dd' + (cd' + c'd)i$
$= \text{above}$

so, as an algebra, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$

3) $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2)$:

$1 \otimes 1 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ $1 \otimes i = \begin{pmatrix} i & \\ & i \end{pmatrix}$
$\hat{i} \otimes 1 \mapsto \begin{pmatrix} i & \\ & -i \end{pmatrix}$ $\hat{i} \otimes i = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$
$\hat{j} \otimes 1 \mapsto \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ $\hat{j} \otimes i = \begin{pmatrix} & i \\ -i & \end{pmatrix}$
$\hat{k} \otimes 1 \mapsto \begin{pmatrix} & i \\ i & \end{pmatrix}$ $\hat{k} \otimes i = \begin{pmatrix} & -1 \\ -1 & \end{pmatrix}$

usual quaternion embedding:
$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \longleftrightarrow a\hat{1} + b\hat{i} + c\hat{j} + d\hat{k}$

$\rho: \mathbb{H} \to \mathbb{C}(2)$
$\rho': \mathbb{H} \otimes \mathbb{C} \to \mathbb{C}(2)$

$\rho'(h \otimes z) = z\rho'(h)$ $\ker \rho' = 0$
ρ' surjective
$\Rightarrow \rho'$ iso

4) $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4)$

Q: Geometric interpretation of this?

we have injective homo $\mathbb{H} \to \mathbb{C}(2)$ $\quad a + b\hat{i} + c\hat{j} + d\hat{k} \mapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} =$

& injective homo $\mathbb{C} \to \mathbb{R}(2)$ $\quad a+bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + b\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

so, $\mathbb{H} \hookrightarrow \mathbb{R}(4)$ $\quad a + b\hat{i} + c\hat{j} + d\hat{k} \mapsto a\begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix} + c\begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ -1 & 0 & & \\ 0 & -1 & & \end{pmatrix} + d\begin{pmatrix} & & 0 & 1 \\ & & -1 & 0 \\ 0 & 1 & & \\ -1 & 0 & & \end{pmatrix}$

$\mathbb{H} \otimes \mathbb{H}$ has 10? roots of 1

$1 \otimes 1$ & $\{\hat{i}, \hat{j}, \hat{k}\} \otimes \{\hat{i}, \hat{j}, \hat{k}\}$

Isomorphism: rep of $\mathbb{H} \otimes \mathbb{H}$ on $\mathbb{R}^4$

treat $\mathbb{R}^4$ as $\mathbb{H}$: $\quad (x \otimes y)\, z = x z \bar{y}$

bar to reverse multiplication order (negate $\hat{i}, \hat{j}, \hat{k}$)

this is surjective, thus isomorphism by dim counting

small clifford algebras

let C_k be the clifford algebra of the negative def. form on $\mathbb{R}^n$

let C_k' — " — positive — " —

$C_0 = \mathbb{R}$

$C_1 = \mathbb{C}$

$= \mathbb{R}[x] / \langle x^2 = -1 \rangle$

$C_2 = \mathbb{H}$

$= \mathbb{R}[x_1, x_2] / \langle x_1^2 = x_2^2 = -1,\ x_1 x_2 + x_2 x_1 = 0 \rangle$

$= \mathbb{C} \check{\otimes} \mathbb{C}$

basis elements $1 \quad i \quad j \quad k = ij$

$i^2 = -1 \quad j^2 = -1 \quad (k)^2 = ijij = -ijji = +ii = -1$

$ik + ki = i^2 j + iji = -j + i(-ij) = -j - i^2 j = 0$

likewise $jk + kj = 0$

so, spanned by 3 anticommuting roots of -1, or $\cong \mathbb{H}$

$C_0' = \mathbb{R}$

$C_1' = \mathbb{R} \oplus \mathbb{R}$

$= \mathbb{R}[x] / \langle x^2 = 1 \rangle$

$P_1 = \frac{1+x}{2} \qquad P_2 = \frac{1-x}{2}$

idempotent: corresponds to proj. onto 1st factor

$\left(\frac{1+x}{2}\right)^2 = \frac{1 + 2x + x^2}{4} = \frac{2+2x}{4} = \frac{1+x}{2}$

$\left(\frac{1-x}{2}\right)^2 = \frac{1 - 2x + x^2}{4} = \frac{1-x}{2} \qquad$ proj. onto 2nd factor

$\left(\frac{1+x}{2}\right)\left(\frac{1-x}{2}\right) = \frac{1-x^2}{2} = 0$

$(aP_1 + bP_2)(a'P_1 + b'P_2) = aa'P_1 + bb'P_2$

so, basis $\{P_1, P_2\}$ gives iso $C_1' = \mathbb{R}[x]/_{x^2=1} \cong \mathbb{R} \oplus \mathbb{R}$

$C_2' = \mathbb{R}(2)$

$= \mathbb{R}[x_1, x_2] / \langle x_1^2 = x_2^2 = 1,\ x_1 x_2 = -x_2 x_1 \rangle$

use same basis as above: $P_1^+ = \frac{1+x_1}{2} \qquad P_1^- = \frac{1-x_1}{2}$

$P_1^+ = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \qquad P_1^- = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \qquad x_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

other square root in $\mathbb{R}(2)$: $x_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$

so $x_1 x_2 = -x_2 x_1$ ✓

& $\mathbb{R}(2)$ spanned by $\underset{1}{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}}, \underset{x_1}{\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}, \underset{x_2}{\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}}, \underset{x_1 x_2}{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}$

k	C_k (neg. def)	C_k' (pos. def)	$C_k \otimes \mathbb{C}$
0	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C} \oplus \mathbb{C}$
2	$\mathbb{H}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{C}(4)$
5	$\mathbb{C}(4)$	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$
7	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \otimes \mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$
$\vdots$			

$\otimes \mathbb{H}$ $\quad$ $\otimes \mathbb{R}(2)$

$\vdots$

$\mathbb{K} \otimes_{\mathbb{R}} \mathbb{K}'(n) = \mathbb{K} \otimes_{\mathbb{R}} \mathbb{K}'(n)$

$\mathbb{K}(n) \otimes \mathbb{K}(m) = \mathbb{K}(nm)$

$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$

$\mathbb{H} \otimes \mathbb{C} = \mathbb{C}(2)$

$\mathbb{H} \otimes \mathbb{H} = \mathbb{R}(4)$

$$\begin{cases} C_{k+2} \cong C_k' \otimes C_2 \\ C_{k+2}' \cong C_k \otimes C_2' \end{cases}$$

$\Rightarrow$ $C_{k+4} = C_k \otimes C_2' \otimes C_2 = C_k \otimes \mathbb{H}(2)$

$C_{k+8} = C_k \otimes (C_2' \otimes C_2)^2 = C_k \otimes \mathbb{H}(2) \otimes \mathbb{H}(2) = C_k \otimes \mathbb{R}(16)$

Thm: $C_k \otimes_{\mathbb{R}} C_2' \cong C_{k+2}'$

$C_k' \otimes_{\mathbb{R}} C_2 \cong C_{k+2}$

define iso $\psi : C_{k+2} \to C_k' \otimes_{\mathbb{R}} C_2$

let $\mathbb{R}_k$ be the underlying vector space for C_k, w/ basis $\{\gamma_1, \dots, \gamma_k\}$ & $\mathbb{R}_k'$ of C_k' w/ basis $\{\gamma_1', \dots, \gamma_k'\}$

let $\mathbb{R}_{k+2}$ have basis $\{\gamma_1, \dots, \gamma_k, e_1, e_2\}$

define $\psi(\gamma_i) = \gamma_i' \otimes e_1 e_2$;
$\psi(e_i) = 1 \otimes e_i$

$\psi(\gamma_i)^2 = {\gamma_i'}^2 \otimes (e_1 e_2)^2 = 1 \otimes -1 = -1 = Q(\gamma_i)$

$\psi(e_i)^2 = 1 \otimes e_i^2 = -1 = Q(e_i)$

so, this satisfies universal prop. & extends to an algebra homomorphism of the clifford algebra C_k

$\dim C_{k+2} = 2^{k+2} = 4 \cdot 2^k = \dim(C_k' \otimes C_2) = 2^k \cdot 2^2 = 4 \cdot 2^k$

this map takes generating basis to generating basis, & dim is equal, so is iso.

replacing dashed & non-dashed gives $C_{k+2}' = C_k \otimes C_2'$

Classification of $\mathbb{R}$-central real super division algebras

$A = A^e \oplus A^o$ $\quad$ A^e is closed under multiplication so A^e is a real division algebra

thus, $A^e = \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ $\quad$ if $A^o \neq \{0\}$, then $A^o \cong A^e$

$Cl_0 \sim \mathbb{R} \oplus \{0\}_e$

$\mathbb{C}l_0 \sim \mathbb{C} \oplus \{0\}_e$

$Cl_4 \sim \mathbb{H} \oplus \{0\}_e$

$Cl_7 \sim \mathbb{R} \oplus \mathbb{R}_e$ $\quad e^2 = 1$ $\quad \cong \mathbb{R} \oplus \mathbb{R}$

$Cl_1 \sim \mathbb{R} \oplus \mathbb{R}_e$ $\quad e^2 = -1$ $\quad \cong \mathbb{C}$

$A^e \ni a \mapsto eae^{-1}$ is aut of $\mathbb{C}$, so either id or conjugation

$\mathbb{C}l_1 \sim \mathbb{C} \oplus \mathbb{C}_e$ $\quad ei = ie$ $\quad e^2 = -1$ $\quad \cong \mathbb{C} \oplus \mathbb{C}$

$Cl_6 \sim \mathbb{C} \oplus \mathbb{C}_e$ $\quad ei = ie$ $\quad e^2 = 1$ $\quad \cong \mathbb{R}[2]$

$Cl_2 \sim \mathbb{C} \oplus \mathbb{C}_e$ $\quad ei = -ie$ $\quad e^2 = -1$ $\quad \cong \mathbb{H}$

$Cl_3 \sim \mathbb{H} \oplus \mathbb{H}$ $\quad e$ commutes w/ A^e $\quad e^2 = 1$

$Cl_5 \sim \mathbb{H} \oplus \mathbb{H}$ $\quad e$ commutes w/ A^e $\quad e^2 = -1$ $\quad \cong \mathbb{C}[2]$

morita equivalence

morita equivlence of A & $A(n)$: ← algebra

Pf) $X = A^n$ is a module of both $A(n)$ & A ($A(n)$-A module) (acts on left; acts on right)

$X^* = A^n$ is A-$A(n)$ bimodule (row vectors)

V is A-module	$V \otimes X$ is $A(n)$	$V \otimes X \otimes X^* \cong V$
V' is $A(n)$	$V' \otimes X^*$ is A	$V' \otimes X^* \otimes X \cong V$

up to Morita equivlence, central simple algebras over field form a group under $\otimes$

Brauer grp

for $\mathbb{R}$, $Br(\mathbb{R}) = \{[\mathbb{R}], [\mathbb{H}]\} = \mathbb{Z}_2$

for any central simple A over F A^{op} (same F-space, but w/ left rather than right multiplication)

$[A] \cdot [A^{op}] = [F]$ $A^{op} = A^{-1}$ in brauer

$\mathbb{H} \otimes \mathbb{H}^{op} = M(4)$ $\mathbb{H}^{op} \xrightarrow{\sim} \mathbb{H}$

"cols" "rows" $x \longmapsto \bar{x}$

i.e $A \otimes_{F} A^{op} \cong End_F(A) \cong F(n)$

$(a \otimes b^{op})c = acb$

Assertion: sending graded C_k module to even part C_k^0 ($= C_{k-1}$) is equivlence of categorys

C_k	k
$\mathbb{R}$	0
$\mathbb{C}$	1
$\mathbb{H}$	2

graded $\mathbb{C}$ ($= C_1$) - module V: as real vect space $V \cong_{\mathbb{R}} V^0 \oplus V^1$

s.t $i: V^0 \to V^1$, $V^1 \to V^0$

i is odd element of C_1, so multiplication by i is odd operator

i.e $V = \mathbb{C} \otimes_{\mathbb{R}} V^0$

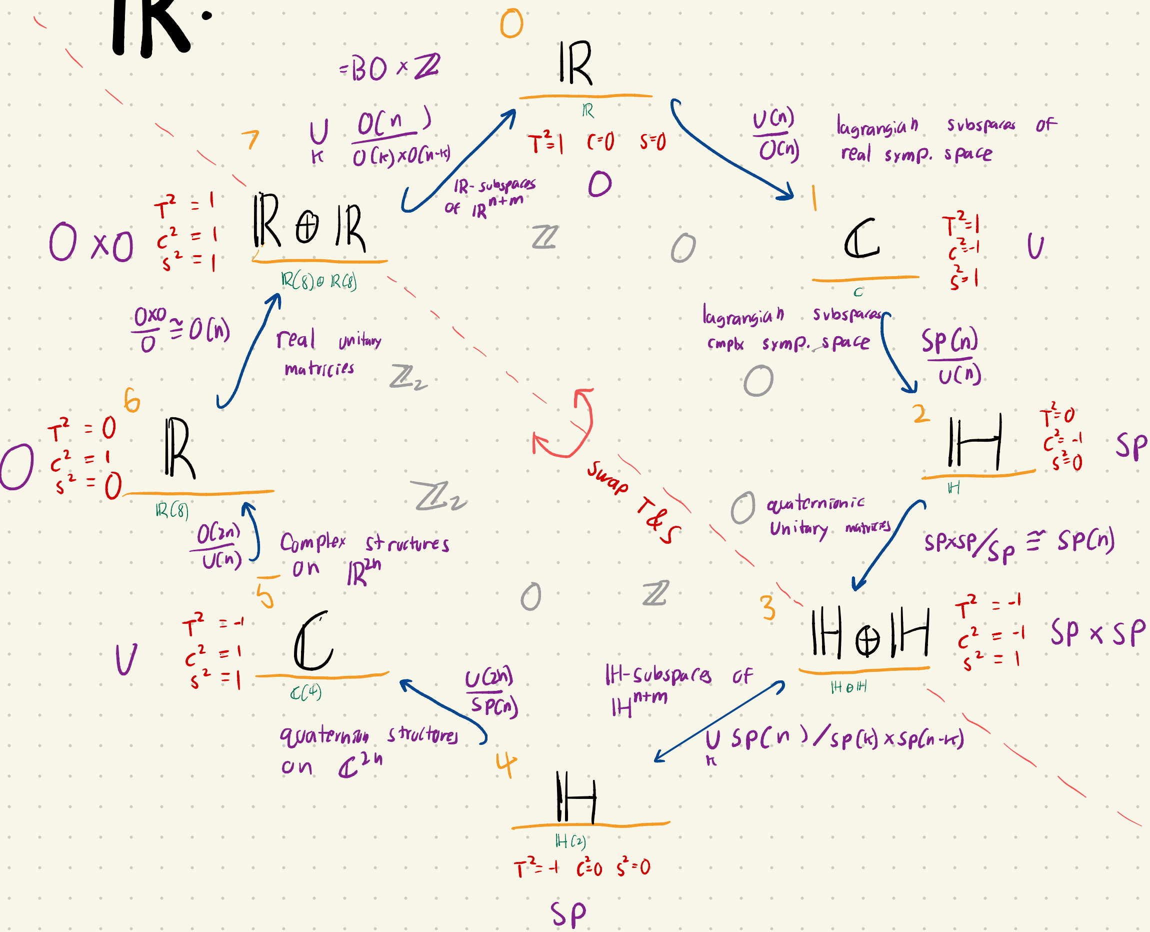
graded $\mathbb{C}$-module: complexification of real vector space

in general, $C_k = C_{k-1} \hat{\otimes} C_1$ so C_k is complexification of ungraded C_{k-1}

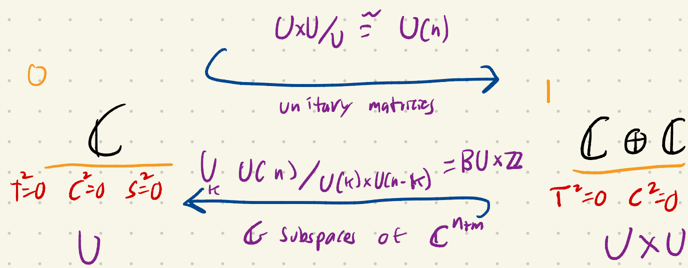
(C_{k-1} is even part; $C_1 = \mathbb{C}$)

$k=2$: ungraded $C_1 = \mathbb{C}$ vect space gives graded $\mathbb{H}$-vect space $V = V^0 \oplus V^1$ w/ $j \in \mathbb{H}$ sending $V^0 \to V^1$, $V^1 \to V^0$

$\mathbb{R}$:



$\mathbb{C}$:



Super division algebra: all $\mathbb{Z}_2$-homogenous elements invertible

super brauer group: invertible elements of Morita classes of reps of $\mathbb{Z}_2$ graded algebras

central algebra: those w/ k as its center

$$\underset{\text{induced by } \mathbb{R}_k \hookrightarrow \mathbb{R}_{k+1}}{C_k \xhookrightarrow{i} C_{k+1}}$$

: this is equivalent to splitting

$$C_{k+1} = C^e_{k+1} \check{\oplus} C^o_{k+1} \cong C_k \check{\otimes} \mathbb{C}$$

$C^e_{k+1} \overset{i}{\hookleftarrow} C_k$

add 1 odd root of -1 to C_k

Symmetric space

$\mathbb{R}^n - O(n)$

$\mathbb{C}^n - U(n)$

$\mathbb{H}^n - Sp(n)$

the clifford group of C_k (unit & preserves $\mathbb{R}^n$?) is compact

the inclusion $C_k \hookrightarrow C_{k+1}$ induces inclusion $\Gamma_k \hookrightarrow \Gamma_{k+1}$, s.t $\Gamma_k \triangleleft \Gamma_{k+1}$ closed subgrp thm

then Γ_{k+1}/Γ_k defines a symmetric space

these give rise to all infinite familys of sym spaces

Q: why?

Free fermion theories periodic table of topological invariants

10 classes, depends on which symmetrys are possessed:

time reversal symmetry $T^2 = 1$ or $T^2 = -1$

Charge conjugation symmetry $C^2 = 1$ or $C^2 = -1$

Chiral symmetry $S = TC$ $S^2 = 1$

Vector spaces w/ these operators are morita equivalent to clifford algebras

Condensed matter $\Leftrightarrow$ Symmetric spaces

only care abt deformation class of gap preserving hamiltonian, i.e π_0 of space of gapped hamiltonians w/ particular symmetry

for space w/ no symmetry ($T^2 = S^2 = C^2 = \emptyset$), hamiltonian can send to diagonal w/ only + & - 1s. Thus, for N states, equivalent to choice of positive subspace of $\mathbb{C}^N$, or $BU \times \mathbb{Z}$

each class can be reduced to a symmetric space

A free theory is a rep of a clifford algebra? & an interface btwn materials is classified by a clifford algebra extension?

$$H_D(\vec{k}) = \vec{k} \cdot \vec{\gamma} + m\gamma_0(r) \qquad \gamma_0(r) \in \Gamma_{k+1}/\Gamma_k \text{ is extention } C_k \hookrightarrow C_{k+1}$$

r describes spacial dependence of theory: @ phase bdry, $\gamma_0(r) = 0$, so $\gamma_0(r)$ fails to be continuous

only take things up to addition by trivial bands, i.e stable props)

Distinct topological phases classified by $\pi_0(\Gamma_{k+1}/\Gamma_k)$

or... starting w/ $O(16r)$, & demanding matrices preserve a set $\{J_k \mid \{J_i, J_j\} = g,\ J_i^2 = -1\}$ for incrementally added J_i gives a chain of symmetry breaking (some $r \in \mathbb{Z}_{\geq 0}$)

$$\dots O(16r) \supset U(8r) \supset Sp(4r) \supset Sp(2r) \times Sp(2r) \supset Sp(2r) \supset U(2r) \supset O(2r) \supset O(r) \times O(r) \supset O(r) \dots$$

dim $= 2$

in complex:
$$\dots \supset U(2r) \supset U(r) \times U(r) \supset U(r) \supset \dots$$

the even part of C_{k+1} is equal to C_k, the parts of C_{k+1} which preserve all $J_1, \dots, J_{k+1}$ are half the size of those w/in C_k of $J_1, \dots, J_k$

algebraically, for ring of modules M_k, we have $i^*: M_{k+1} \to M_k$ given by restriction

$M_k / i^*(M_{k+1})$ is modules of C_k that aren't restrictions of those in C_{k+1} $\quad M_k / i^*(M_{k+1}) \cong \pi_0(\Gamma_k / \Gamma_{k+1})$??

extentions $C_k \to C_{k+1}$ parametrized by sym. space: if J_{j+1} is an addition to $\{J_1, \dots, J_i\}$ then $g^{-1} J_{i+1} g$ is too (anticommutes + squares to -1), for g in clifford group Γ_k

but, $J_{j+1} = g^{-1} J_{i+1} g$ iff g & J_{i+1} commute, which happens iff g is part of grp Γ_{k-1}

so, parametrized by Γ_k / Γ_{k+1}!

Symmetric spaces $\Leftrightarrow$ loop spaces

A symmetric space is a homogenous G/H $H \trianglelefteq G$ s.t H is invariant grp. for some involution of G

consider geodesics connecting J_1 & $-J$ in Ω_k, the space of matrices preserving $J_1, \dots, J_k$ w/ $J_i = \rho(e_i)$ for some rep $\rho: C_k \to \mathbb{R}^n$

let $\gamma(t) = J_1 \exp(\pi t\, A)$ be such a geodesic: $A \in \mathfrak{o}(n)$ $\quad$ A in lie alg

$(J \exp A)^2 = -1 \Rightarrow J \exp A \underset{J^{-1}}{(J)} \exp A = 1 \Rightarrow \exp(JAJ^{-1}) \exp A = 1 \Rightarrow JAJ^{-1} = -A \Rightarrow JA = -AJ$

geodesic is path of complex structures iff A anticommutes w/ J_1 likewise $\{J_i \exp A, J_i\} = 0 \Rightarrow [A, J_i] = 0$

if J extends $J_1, \dots, J_k$ & anticommutes, then $J = J_k A$, where A commutes w/ all $J_{i<k}$ but anticommutes w/ J_k $\Rightarrow$ $\gamma(t) = J_k \exp(\pi t A)$ is geodesic of complex structures compatible w/ $(J_1, \dots, J_{k-1})$, so a geodesic in Ω_k connecting $+J_k$ & $-J_k$ (as $A^2 = -1$)

Γ_{k-1}/Γ_k parametrize minimal geodesics in Γ_k/Γ_{k+1} connecting J_i & $-J_i$

use morse theory to build up loopspace $\Omega(\Gamma_k/\Gamma_{k+1})$:

the loops based @ J_i have 1 conjugate pt, $-J_i$, w/ index

$\Rightarrow$ dim (geodesics $J_i \mapsto -J_i$) = dim $\Gamma_{k-1}/\Gamma_k \propto n$ w/ n dim of space

$\Rightarrow$ morse index is prop to n

$\Rightarrow$ for $n > 2n_0$, $\pi_{n_0} \Omega(\Gamma_k/\Gamma_{k+1}) = \pi_{n_0} \Gamma_{k-1}/\Gamma_k$ as $\Omega(\Gamma_{k+1}/\Gamma_k)$ only built from cells $< n$

non minimal geodesics have large index

$\Rightarrow$ $\boxed{\Omega(\lim_{n\to\infty} \Gamma_k/\Gamma_{k+1}) \simeq \lim_{n\to\infty} \Gamma_{k-1}/\Gamma_k}$

inductive limit homotopy equivalence

$\Omega(\Gamma_i/\Gamma_{i-1}) \sim \Gamma_{i+1}/\Gamma_i = R_{i+2}$

Conclusion: Classifying spaces Γ_{i+1}/Γ_i form a loop spectrum mod 8!

Bott periodicity corollary: $\pi_k(O) = \pi_k(R_0) = \pi_0(\Omega^k R_0) = \pi_0(\Omega^{k \bmod 8} R_0) = \pi_{k \bmod 8}(O)$ ✓

Symmetric spaces $\Rightarrow$ K-Theory

The classifying space for K-theory is fredholm operators: $\widetilde{KO}^0(X) = [X, \mathcal{F}^0]$

$\mathcal{F}^0$ = fredholm operators

$\mathcal{F}^1 \subset \mathcal{F}^0$ = skew-adjoint fredholm operators, K algebra of $O(\mathcal{H}^k_R)$ ← hilbert space

$\mathcal{F}^k$ = subset of $\mathcal{F}^1$ anticommuting w/ first k elements $\{J_1, \dots, J_k\}$

In fact, $\mathcal{F}^k \sim \Gamma_{k-1}/\Gamma_{k-2}$! or smth

$\mathcal{F}^{k+1} \sim \Omega \mathcal{F}^k$ by $\underset{\mathcal{F}^{k+1}}{A} \mapsto J_{k+1} \cos \pi t + A \sin \pi t$ $t \in [0, 2]$

generates geodesics

$$A_1 = \frac{\text{real vect spaces}}{\text{complex vect spaces ignoring } i} = \mathbb{Z}_2$$ as this is true only when dim is even

$$A_2 = \frac{\text{cmplx vect space}}{\text{quaternion, forgetting } j} = \mathbb{Z}_2$$ as dim has to be even

$$A_3 = \frac{\text{quaternion}}{\text{those that can become a pair}}$$ (preimage of $\mathbb{H} \oplus \mathbb{H}$ $\Rightarrow$

$$A_4 = \frac{\text{pairs of quaternions}}{\text{quaternions (those that can become a module of } M_2(\mathbb{H})\text{)}}$$ characterized by pairs of dims

extend from $\begin{pmatrix} /// & \\ & //// \end{pmatrix}$

it can extend only when dimensions are the same — as need to have isomorphism btwn 2 factors (factor induced by $*$)

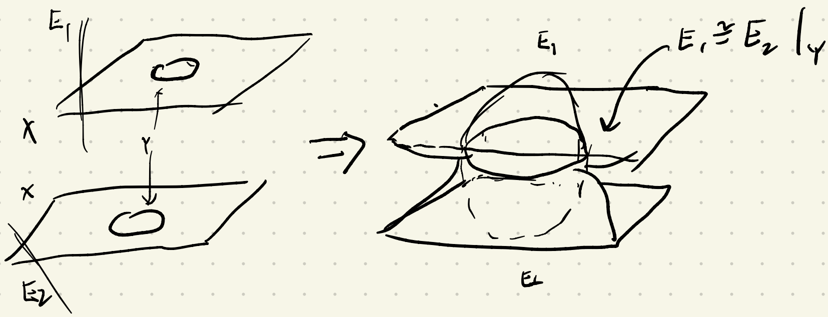
$$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

$$A_4 = \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z}} = \frac{\{(n,m)\}}{\{(n,n)\}} \cong \mathbb{Z}$$

$$0 \to E_1 \xrightarrow{\sigma} E_2 \to 0$$

Construction of euler charactastic $\chi : L(X, Y) \to K(X, Y)$

$\sigma|_Y$ is iso



Connection w/ Bott periodicity

consider V.B $V \to X$: V_p has an associated clifford algebra $C(V_p)$

form Clifford Bundle $C(V) = \coprod_{p \in X} C(V_p)$ $\quad$ E is clifford algebra based on V

choose Clifford Module $E = E^0 \oplus E^1$ of V : E is $C(V)$ module, $v(v(e)) = -\|v\|^2 e$

i.e $V \otimes E^0 \mapsto E^1$, $V \otimes E^1 \to E^0$, $\quad$ $E \to X$ is V.B

E^0 is $\mathrm{spin}(V)$ module, E is $\mathrm{Pin}(V)$ module, E has invariant metric w/ $E^{0\perp} = E^1$

E pulls back via projection to V.B over $\pi: B(V) \to X$ $\quad$ basically, consider action of V on E:

at each point in fiber, define $\sigma(E) : \pi^* E^0 \to \pi^* E^1$
$e \mapsto ve$

i.e
$$\begin{tikzcd} 0 \arrow[r] & \tilde{E}^0 \arrow[r, "\mu"] \arrow[d] & \tilde{E}^1 \arrow[r] \arrow[d] & 0 \\ & B(V) & B(V) & \end{tikzcd}$$
$\tilde{E} = \pi^* E$

$\mu(p, v, e) = (p, v, ve)$ $\quad$ Clifford multiplication

This is an iso when $v \neq 0$, in praticular on $\partial B(V) = S(V)$

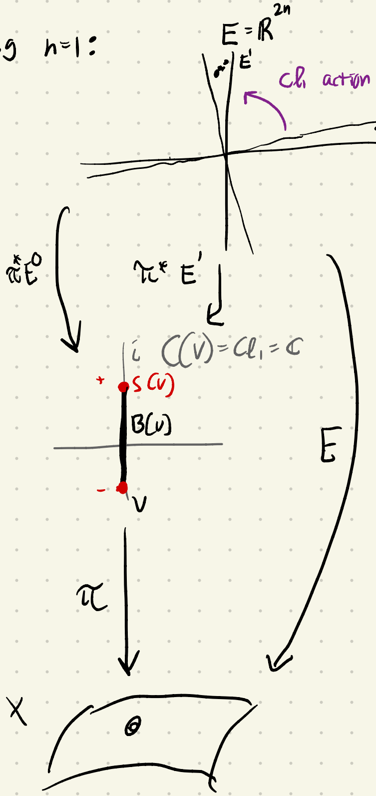
this is not an iso @ $\quad$ so, we get $\chi(E) \in KO(B(V), S(V)) = \tilde{KO}(X^V)$

Thom space

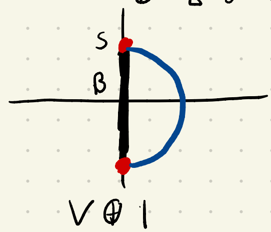
This gives homo $\varphi: M(V) \to KO(B(V), S(V))$

This is 0 iff we can extend iso to whole unit ball, iff it comes from $C_{n+1} \to C_n$ ...

e.g $n=1$:



Or... add another generator by adding trivial line bundle $\mathbb{C} = E^0$ of $Cl(2)$



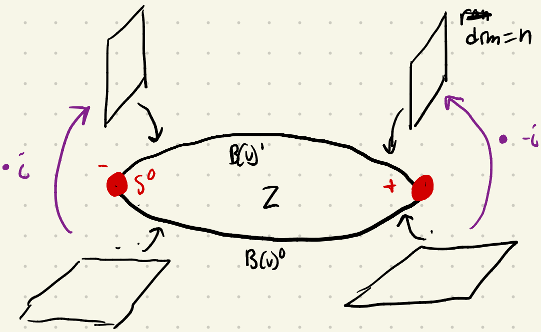
gives explicit extension of multi @ S to a disk. V.B extendable by trivial thing

$\Rightarrow$ V.B trivial as all V.Bs are trivial on disc

so, $\ker \chi : M(V) \to \tilde{KO}(B(V), S(V)) \supseteq \iota_* M(V \oplus 1)$

in fact $\boxed{M(V) / \iota_* M(V \oplus 1) \cong \tilde{KO}(X^V)}$

$\chi(E) \in K(B(V), S(V))$



dim n V.b on S^1 w/ twist multiplying all dimensions by -1

$\Rightarrow$ V.b trivial iff n is even

take base to be point

Class of V.B on $Z \Leftrightarrow$ homotopy class of map

$\varphi \in [S^n, \overset{GL(E^0, E')}{E^{0*} \otimes E'}]$

w/ $\varphi(v) = v \cdot \_$ is clifford multiplication

Can identify $E^0 \cong E'$ as real vector bundles w/ arbitrarily large rank

$\|\varphi(v)e\| = \|ve\| = \|v\|\|e\| \Rightarrow \varphi(v)$ is orthogonal, so this gives

$\varphi \in [S^n, O(\infty)] = \pi_n(O(\infty))$

So, $\boxed{\tilde{KO}^{-n}(pt) = \pi_n(O(\infty))}$ $\mu : M(V) \to \pi_n(O)$

π_{n+k}

$\varphi = 0 \Leftrightarrow \varphi(v) = v \cdot \_$ extends to solid disc

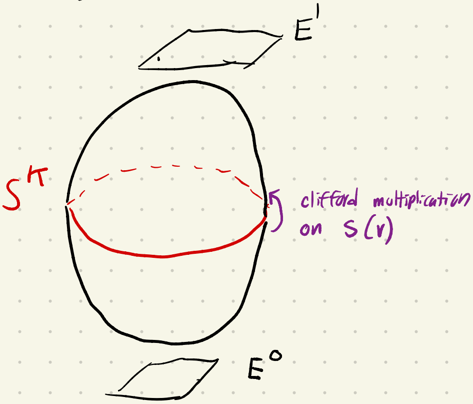
So modules that extend to $M(V \oplus 1)$ are in $\ker \mu$

Take base to be X:

glue together copies of $B(V)$ along $S(V)$

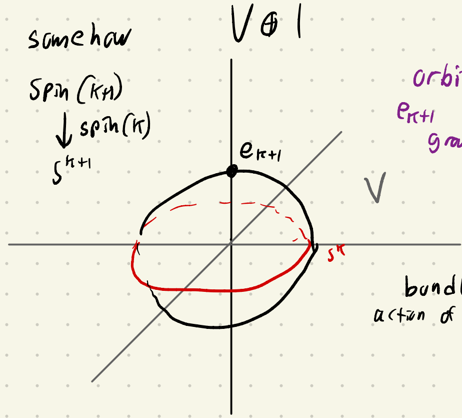
modules are $\mathbb{R}^n$:

in general:



somehow

$Spin(k+1)$
$\downarrow spin(k)$
S^{k+1}



orbit of e_{k+1} under clifford group

bundle given by action of S^k on $E^0 \to E^1$

Choice of spin(k) bundle over S^{k+1} gives different $KO(S^{k+1})$ elements

correspond to grp of modules?

trivial extention of $Cl_n \to Cl_{n+1} \Rightarrow$ trivial bundle

hence, $KO \cong M_k / i^* M_{k+1}$

Atiyah '67

actually proves periodicity

"Bott periodicity & index of ellipitic operators"

Bott periodicity pf (w/ elliptic operators)

K-theory w/ compact supports: X space, X^+ its 1 pt compactification, ∞ pt @ ∞

then $K(X) := \ker(K(X^+) \xrightarrow{i^*} K(\infty))$ i.e. v.bs who are trivial @ ∞ (outside any compact set)

define $K^{-n}(X) = K(X \times \mathbb{R}^n)$ shouldn't this be n-fold suspension?

Thm homo: for X cmpt, $\beta: K(X) \xrightarrow[\alpha^*]{} K(X \times \mathbb{C}) \cong K(X \times S^2) \cong K^{-2}(X)$

explicitly, β represented by multiplication w/ class in $K(\mathbb{C})$: given by

$$0 \to \Lambda^0(\mathbb{C}) \xrightarrow{\wedge v} \Lambda^1(\mathbb{C}) \xrightarrow{\wedge v} \Lambda^2(\mathbb{C}) \to 0$$ acyclic @ ∞ gives element of $K(\mathbb{C})$

$\Lambda^0(\mathbb{C}) = \mathbb{C}$, $\Lambda^2(\mathbb{C}) = \mathbb{C}$

this yeild class $\lambda_{\mathbb{C}} \in K(\mathbb{C})$; $\lambda_{\mathbb{C}} = \Lambda^1(\mathbb{C}) - \mathbb{C} \oplus \mathbb{C}$ define bott class $b = \lambda_{\mathbb{C}}^*$

bott periodicity: $K(X) \xrightarrow{\cdot b} K^{-2}(X)$ is an isomorphism

explicitly, $K(S^2) \cong \mathbb{Z}[H]/(H-1)^2$ w/ H tautological line bundle over $\mathbb{P}\mathbb{C}^1 \cong S^2$

formal trick: wish to construct $\alpha: K^{-2}(X) \to K(X)$ s.t. $\alpha\beta = \mathrm{id}_{K^{-2}(X)}$, $\beta\alpha = \mathrm{id}_{K(X)}$

suppose α functorial, α $K(X)$-module homo, $\alpha\beta = 1$

then $\beta\alpha = 1$; i.e. only need to show 1 direction (!!)

Pf) first such an α extends to $K^{-q-2}(X) \to K^{-q}(X)$

1) extent α to locally complex X, as is iso for X^+ & ∞
2) apply to $X \times \mathbb{R}^q$

α is $K(X)$-module homo, so sends $K^{-2}(X) \otimes Y$ to $K(X) \otimes Y$

$$\begin{tikzcd} K^{-p-2}(X) \otimes K^{-q}(Y) \arrow[r] \arrow[d, "\alpha \otimes 1"] & K^{-p-q-2}(X \times Y) \arrow[d, "\alpha"] \\ K^{-p}(X) \otimes K^{-q}(Y) \arrow[r] & K^{-p-q}(X \times Y) \end{tikzcd}$$

$\Rightarrow \alpha(xy) = \alpha(x)y$ on all $K^{-n}(X)$

crux: $K(X) \underset{\alpha}{\overset{\beta}{\rightleftarrows}} K^{-2}(X) \underset{\alpha}{\overset{\beta}{\rightleftarrows}} K^{-4}(X) \cong K(X \times \mathbb{R}^2 \times \mathbb{R}^2)$

swapping factors of $\mathbb{R}^2$ in $K^{-4}(X)$ swaps products $x, y \in K^{-2}(X)$, $xy \mapsto yx$

$\beta\alpha(y) = \alpha(\underbrace{y}_{\text{any element of } K(X)})b = \alpha(yb) = \alpha(by) = \alpha(b)y = y$ so $\beta\alpha = 1$

$y \in K^{-2}(X)$

swap factors of $\mathbb{R}^2 \times \mathbb{R}^2$

Define homomorphism

$\text{index}_d : K(M \times X) \to K(X)$ by

d is family of elliptic operators $d_x : \Gamma(E) \to \Gamma(F)$ paramertized by X

ker d_x, coker d_x are smoothly varying vector spaces over X

(add trivial bundles to these s.t everything is well defined dimension)

then subtract them in K-group?)

That is, consider E as v.b $E \to M \times X$

main homo: $Q : d \to d_Q \quad d_Q : \Gamma(E \oplus Q) \to \Gamma(F \oplus Q) \quad \sigma(d_Q) = \sigma(d) \oplus id_Q$

e.g for line bundle Q, $\bar\partial_Q : \Omega^{0,0}(Q) \to \Omega^{0,1}(Q)$ Q-valued 0,1 forms

$\ker \bar\partial_Q = H^0(Q) \quad \text{coker } \bar\partial_Q = H^1(Q)$

this applies to any family of v.Bs $Q \to M$ paramatized by X, i.e

a $\bar{Q} \to M \times X$

then, $\text{index } d\_ : K(M \times X) \to K(X)$

$Q \longmapsto \text{index } d_Q$

becomes tensor product

as Q_0 comes from X, & $M \times X$

$Q = Q_0$ is M trivial: $Q_0 \xrightarrow{\text{pull back bundle}} M \times X \xrightarrow{\pi} X$

then, $\ker d_{Q_0} = \ker d \otimes Q_0$

$\text{coker } d_{Q_0} = \text{coker } d \otimes Q_0$

$\text{index } d_{Q_0} = \text{index } d \otimes Q_0$

$\Rightarrow \text{index}_d(\underset{\in K(M\times X)}{a} \; \underset{\in K(X)}{b}) = \text{index}_d(a)\, b$

is a K-module homomorphism

Bott periodicity: take $M = S^2 = \mathbb{CP}^1$

$b = L - H^1$ bott class

wish to construct K-module homo $\alpha : K(S^2 \times X) \to K(X)$ s.t $\alpha(b) = 1$

Take $\alpha = \text{index}_{\bar\partial}$ for $\bar\partial$ elliptic operator on $\mathbb{CP}^1$

$\text{index } \bar\partial_Q = \dim(H^0(Q)) - \dim(H^1(Q)) = c_1(Q) + 1 - g^{=0}$

Thus $\text{index } \bar\partial_Q = 1 + d$ for a degree d bundle anonical bundle is $\mathcal{O}(-1)$ has $c_1 = -1$, index $d_{\mathcal{O}(-1)} = 0$

Proof of Bott periodicity

(real case)

$V \cong \mathbb{R}^8$

Goal: prove there is isomorphism $K(X) \cong K(X \times V)$ $\quad \dim(V) = 8$

bott class of V induses hom, $K(X) \to K(X \times V)$, just need to show it is reversible,

Construct eliptic complex $E \xrightarrow{D} F$ over V exact @ ∞

or, $E \xrightarrow{P} F$ over some compactification $\bar{V}$ exact @ $\bar{V} - V$

$\text{index}_D : KO(\bar{V}) \to KO(pt)$

Find class $u \in KO(\bar{V})$ s.t. $\underline{\text{index}_D\, u = 1}$

Construction: take compactification $V^+ \simeq S^{8n}$

S^{8n} has natural spinor bundle structure: $S^{8n} \cong \text{spin}(8n+1)/\text{spin}(8n)$

$$\begin{array}{ccc} \text{Spin}(8n+1) & \hookleftarrow & \text{Spin}(8n) \\ \downarrow & & \\ S^{8n} & & \end{array}$$

call this S: $S = \underset{\text{even}}{S^0} \oplus \underset{\text{odd}}{S^1}$

Dirac operator $\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{g_{ij}} \Gamma(TM \otimes S) \xrightarrow{\text{clifford}} \Gamma(S)$

$D = \not{\nabla} = \gamma_i \nabla^i$ Differential operator w/ principle symbol clifford multiplication

D muliplies by order 1 element $\Rightarrow$ sends $\begin{array}{l} S^+ \to S^- \\ S^- \to S^+ \end{array}$ $\quad D^\pm : S^\pm \to S^\mp$

$D = \begin{pmatrix} 0 & D^+ \\ D^- & \end{pmatrix}$ $\quad$ D self adjoint $\Rightarrow$ $D^\pm = D^{\mp *}$

take differential operator $S^+ \xrightarrow{D^+} S^-$

D is homogenous operator on $S^{8n} = \text{spin}(8n+1)/\text{spin}(8n)$

Consider spinor bundle w/ spin $\pm 1/2$ over S^{8n} associated to Spin bundle

call these Δ^+ & Δ^- $\quad$ D acts on these?

Since homogenous space, only need to find index for Δ^+ & Δ^- w/ any operator

$$\text{index}_D S^+ + \text{index}_D S^- = \chi(V^+) = 2$$
$$\text{index}_D S^+ + \text{index}_D S^- = \text{sig}(V^+) = 0$$
$$\Rightarrow \text{index}_D S^+ = 1 \ !!!$$

Then subtract off trivial bundle 1 @ S^∞ to get proper behavior $\text{index}_D 1 = 0$

Qs: - Dirac operator as "the" eliptic operator

- $\text{index}_D S^+ = 1$ from fixed point thm?
- why $8n$??
- Dirac operator resolutions of harmonic spinors?

u is KO element from clifford module $\Delta = \Delta^+ \oplus \Delta^-$

Dirac operators are 'universal' classes represented by eliptic operators

Generate K-homology K_X ← ind_D : acts as 'dual' to K-theory

every $K_*(X)$ element generated by Dirac operator over mfld M

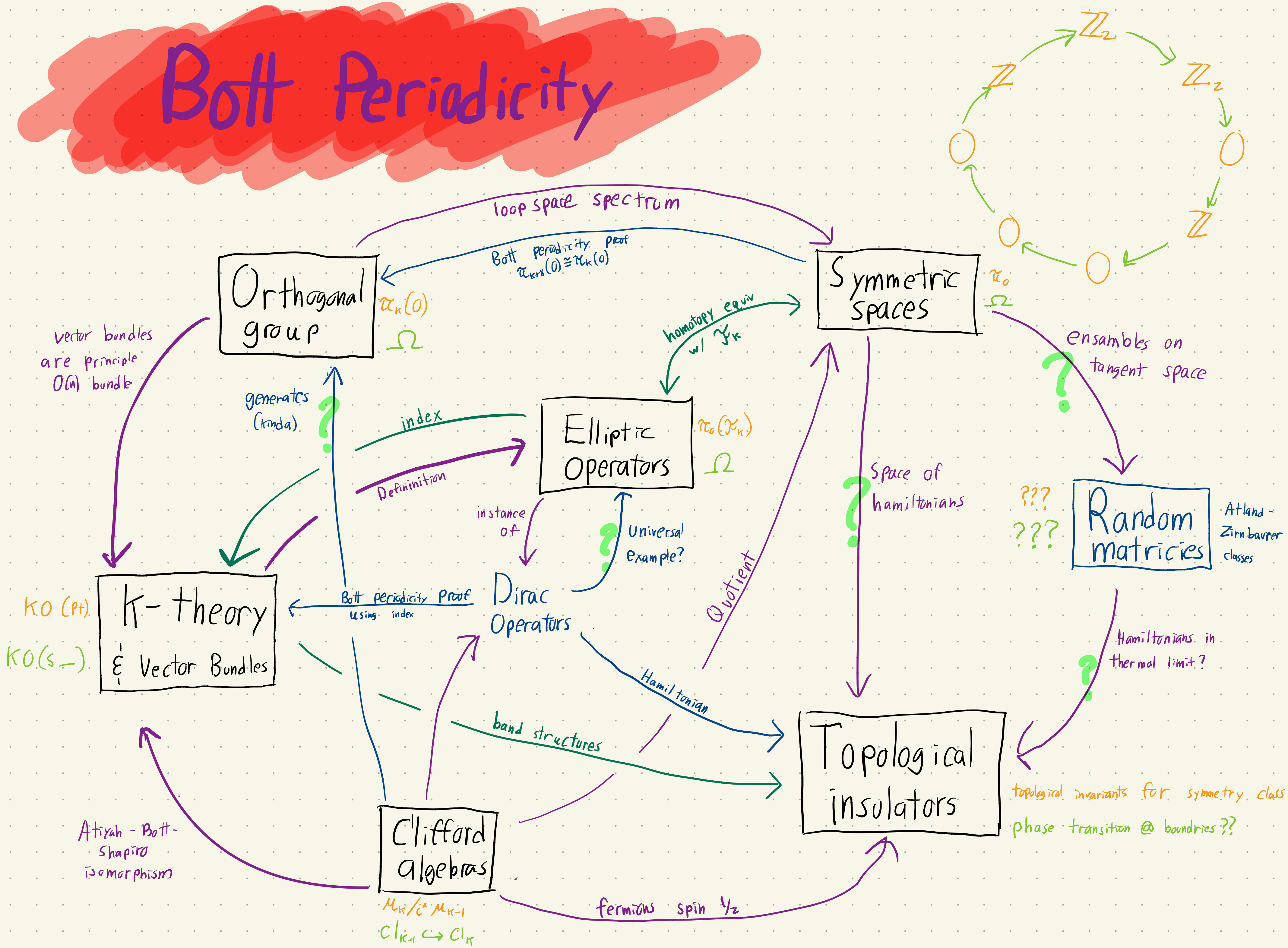
$$\begin{tikzcd} M \arrow[r] & X \\ D \arrow[u] & \end{tikzcd}$$

& map from M into X!!

so, only index calculations we need are Dirac operators!!

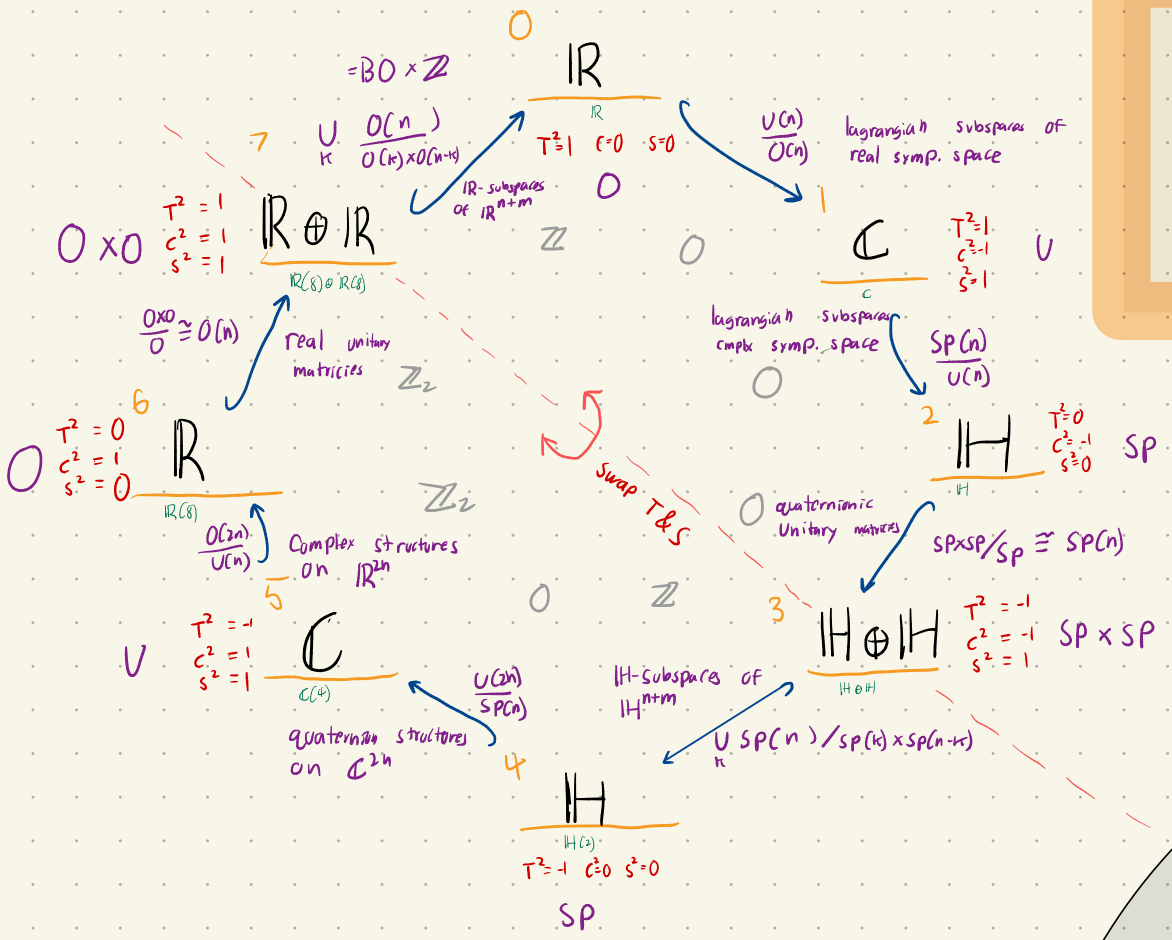
Other papers Librovich harmonic spinors French

Harmonic Spinors — Hitchen,

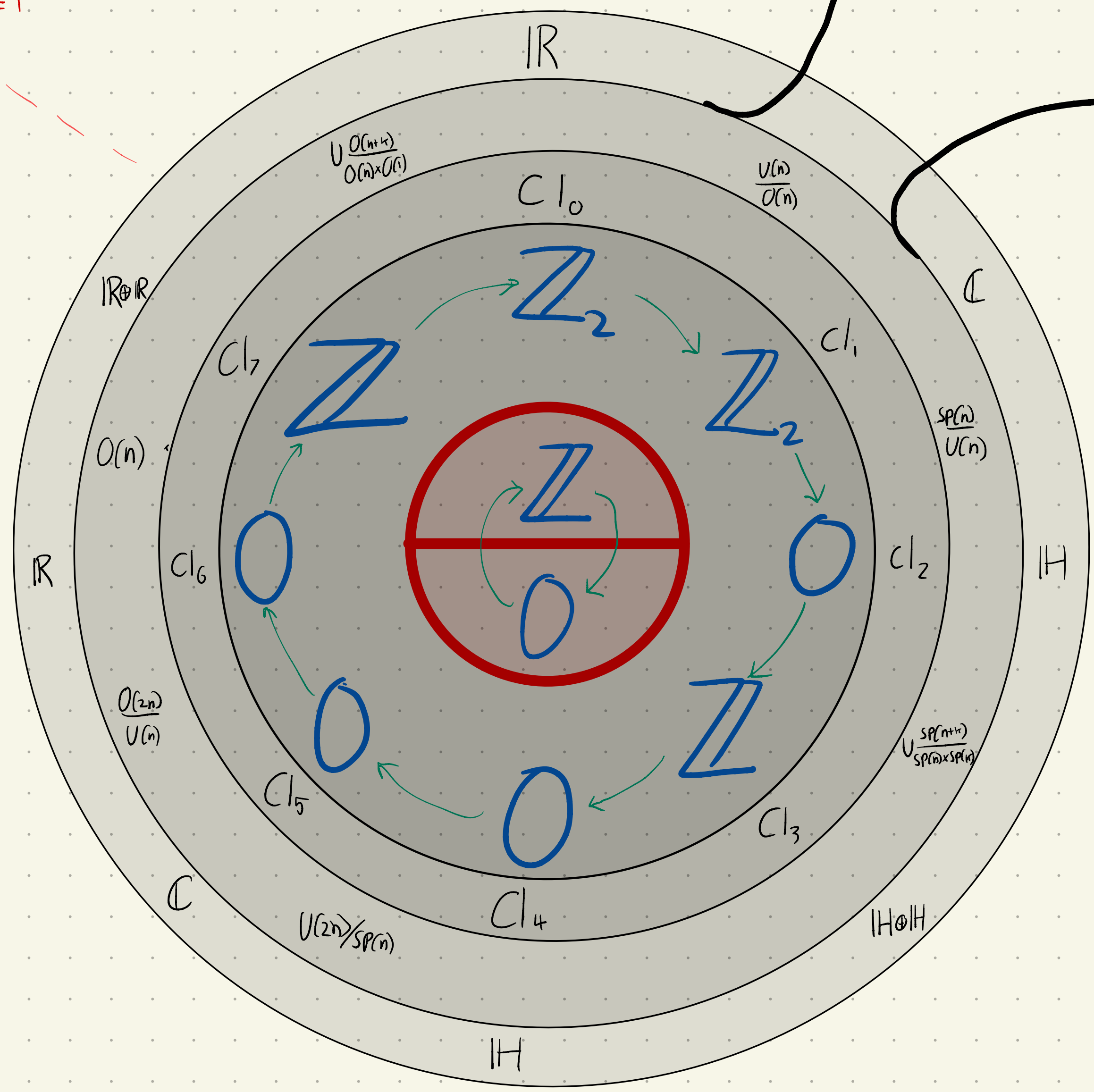


Bott Periodicity
Z
Z2
Z2
O
Z
O
O
O
Orthogonal group
$\pi_k(O)$
Ω
Symmetric spaces
π_0
Ω
loop space spectrum
Bott periodicity proof
$\pi_{k+8}(O) \cong \pi_k(O)$
vector bundles are principle O(n) bundle
generates (kinda)
?
index
Definition
Elliptic Operators
$\pi_0(\mathcal{F}_k)$
Ω
homotopy equiv w/ $\mathcal{F}_k$
ensambles on tangent space
?
Random matrices
Atland-Zirnbauer classes
???
???
Space of hamiltonians
?
instance of
Universal example?
?
Dirac Operators
Quotient
K-theory & Vector Bundles
KO (pt)
KO(s_)
Bott periodicity proof using index
Hamiltonian
band structures
Hamiltonians in thermal limit?
?
Topological insulators
topological invariants for symmetry class
phase transition @ boundries??
Atiyah-Bott-Shapiro isomorphism
Clifford algebras
$M_k / i^* M_{k-1}$
$Cl_{k-1} \hookrightarrow Cl_k$
fermions spin 1/2

Bott Periodicity



SYMMETRIC spaces



Dirac operators & Riemanian geometry

$D : \Gamma(S) \to \Gamma(S)$ $D\sigma = \gamma_i \nabla^i \sigma$ clifford multiplication

D self-adjoint $\Rightarrow$ $\operatorname{Ker} D = \operatorname{Ker} D^2$ (as $\operatorname{spec}(D^2) = \operatorname{Spec}(D)^2$

Lichnerowicz formula: ψ spinor (section of spinor bundle Δ^+

$$D^2\psi = \underbrace{\nabla^*\nabla\psi}_{\geq 0} + \frac{1}{4} K$$

K denotes scalar curvature

$$D^2 = \gamma_i \nabla^i \gamma_j \nabla^j = \gamma_i \gamma_j \nabla^i \nabla^j = \gamma_i \gamma_j \left(-\nabla^i \nabla^j + \nabla_{[\partial_i, \partial_j]}\right)$$

$\Rightarrow$ positively curved mfld has no harmonic spinors
like Bochner / other vanishing thms

Cor: on compact spin mfld w/ $K=0$, $D^2\psi = 0 \Leftrightarrow \nabla^*\nabla\psi = 0 \Longleftrightarrow \nabla\psi = 0$

i.e, on mfld w/ flat scalar curvature, harmonic $\Leftrightarrow$ parallel

Thm: X compact simply connected w/ parallel spinor $\psi \Rightarrow$ X is calabi-Yau (Kahler + ricci flat

Pf ψ parallel $\Rightarrow$ holonomy $\Phi \subset SO$ leaves ψ fixed (under lift $SO \hookrightarrow Spin$)
holonomy reduced to isotropy subgroup $G \subset Spin(2m)$ of ψ

Spin statistics Thm: Parity of spin $\longleftrightarrow$ sign under interchange

spin bundle $S \to X$, $\dim X = 3,1$ $\dim \Delta_k^{\pm} = 2k+1$

Consider spinor bundle $\Delta_k^{\pm} \to X$ w/ Δ_k^+ = rep w/ highest weight k

consider $\operatorname{Sym}^2 X = X \times X / a,b \sim b,a$: $\pi_1(\operatorname{Sym}^2 X) = \mathbb{Z}_2$ statistics = action of π_1 on Δ_k^+

Spin statistics Thm : π_1 can only act nontrivially on Δ_k^+ for k = half integer

Standard proof uses PT symmetry (action of $\times -1$ in $SO(3,1)$) to conclude that fields must be 0

lemma: G isotropy subgroup of $\psi \Rightarrow G_0 \cong SU(m)$
argue via weights & weyl groups

Q: all C-Y mflds have paralel spinor

Finally, holonomy reduces to $SU(n) \Rightarrow X$ calabi-Yau ∎

Dirac operator canonically associated to conformal structure
conformal structure is reduction $GL(n,\mathbb{R}) \to SO(n) \times \mathbb{R}^+$
$\Rightarrow$ each conformal structure has a trivial line bundle L

consider $\tilde{g} = e^{2\sigma} g$, & the associated dirac operators $P, \tilde{P}$
then $\tilde{P}\psi = -a\, P(a\psi)$ for some $a \in \Gamma(S)$
proof by expanding in local coords
Thus, $\tilde{P}\psi = 0 \Leftrightarrow P(a\psi) = 0$, so dimension of this space is conformally invariant
implies, in 2D, dim H harmonic is only dependent on complex structure

on Kahler manifolds

Thm: spin structures $\longleftrightarrow$ classes of holomorphic L w/ $L^2 \cong K$ $(= \Lambda^{n} TX)$ canonical line bundle
or spin structure $\Leftrightarrow$ choice of square root

let P be the principle tangent $SO(n)$ bundle of X. Choice of spin structure = double cover of P
given by double cover $spin(n) \to SO(n)$
fibration $SO(n) \xhookrightarrow{i} P \xrightarrow{j} X$ has L.E.S
this is labeled by $g \in H^1(SO(n), \mathbb{Z}_2)$

$$H^1(X, \mathbb{Z}_2) \xrightarrow{j_*} H^1(P, \mathbb{Z}_2) \xrightarrow{i_*} H^1(SO(n), \mathbb{Z}_2) \xrightarrow{\delta} H^2(X, \mathbb{Z}_2) \to \cdots$$

as we care abt double covers · $\ni g$

a double cover (indicated by) $H^1(P, \mathbb{Z}_2)$ is a spin structure if it sends to g under fiber restriction
by L.E.S, such a thing exists iff $\delta(g) = 0$ $\delta(g)$ is Stiefel-whitney class $w_2(X)$
if $\delta(g) = 0$, spin structures $\Leftrightarrow$ elements $q \in H^1(P, \mathbb{Z}_2)$ w/ $i_* q = g = q_0 + \underset{H^1(P,\mathbb{Z}_2)}{K}$ w/ $i_*(K) = 0$
$= q_0 + j_*(H^1(X, \mathbb{Z}_2))$ spin structures (if exist) $\Leftrightarrow$ elements of $H^1(X, \mathbb{Z}_2)$

Kähler case: $SO(2n)$ reduces to $U(n)$
then, $H^1(SO(2n), \mathbb{Z}_2) \xrightarrow{\text{inc}} H^1(U(n), \mathbb{Z}_2) \xrightarrow{\det} H^1(U(1), \mathbb{Z}_2)$ need $\mathbb{Z}_2$ coeffs are all isomorphism, so a lift to the double cover $SO(2n) \to Spin(2n)$ is equivalently a lift to the double cover $U(1) \to U(1)$
for reduction $SO(2n) \to U(n)$, choice double cover $\Leftrightarrow$ choice double cover $U(1) \cong \det(U(n))$ K canonical line bundle
$\Leftrightarrow$ square root of K (as for circles $L \to K$ double cover
$\cdot z \mapsto \cdot \sqrt{z}$ square of overlaps gives K)

specifically, need choice of holomorphic line bundle

analog of $w_2 = 0$ condition is $c_1(K) = 0$ mod 2

for Kahler mflds, w/ canonical spin^c structure, $\overset{\text{dirac}}{\not{D}} = \sqrt{2}(\bar\partial + \bar\partial^*)$

???

pf: have the same symbol: $\bar\partial + \bar\partial^*$ has $d(\pi_{\xi,1}\alpha) - \delta(\pi_{\xi,1}\alpha)$
different constant for some reason

w/ this, $\bar\partial + \bar\partial^*$ exchanges even & odd forms, so $S^+ = H^{even}(X, \mathcal{O})$
$S^- = H^{odd}(X, \mathcal{O})$

$H^+ = H^{even}(X, \mathcal{O}(\sqrt{K})),\quad H^- = H^{even}(X, \mathcal{O}(\sqrt{K}))$
bundle of spinors iso to $\Lambda^* T^{0,1} \otimes L$