13. Directional Derivatives and the Gradient Vector

Given a function $f(x_1, \ldots, x_m)$, the directional derivative is the rate of increase of the function as the point in the domain moves in some direction. For example, if that direction were along the y-axis (with y increasing), then the directional derivative would be the partial derivative of f with respect to y. To find a directional derivative, we must specify a starting point in the domain and a direction as a vector from this point.

If $\mathbf{x} \in \mathbf{R}^m$ is the point and $\mathbf{u} \in \mathbf{R}^m$ is the direction, the definition of the directional derivative is

$$D_{\mathbf{u}} f(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{|t\mathbf{u}|}.$$

The gradient is an m-dimensional vector ∇f that takes different values at different points of the domain of f according to the equation

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_m} \right\rangle.$$

The gradient has two fundamental geometric properties:

1. At every point of the domain of f, it points in the direction of maximal increase of f.

2. At every point of the domain of f, it is perpendicular to the level surface of f passing through that point.

Given any surface in $\mathbf{R}^3$, in order to find the tangent plane at a point, simply write the surface as a level surface of a function $f(x, y, z)$; then ∇f at the point is perpendicular to the surface, and hence is the normal vector to the tangent plane. Then the point and the normal vector determine the tangent plane. For example, if the surface is the graph of a function $z = g(x, y)$, then it is the level surface $f(x, y, z) = z - g(x, y) = 0$.

The gradient is used to compute directional derivatives via the following formula

$$D_{\mathbf{u}} f(\mathbf{x}) = \nabla f \cdot \frac{\mathbf{u}}{|\mathbf{u}|}.$$

Questions

1. True or False? The gradient of the function $f(x, y) = x^2 + y^2$ at the point $(1, 1)$ is $2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

2. Suppose that for $f(x, y)$ the level curve through a certain point forms an "$\times$". How can ∇f be perpendicular to all of the level curve segments at this point?

3. A dye is used to detect the presence of a certain chemical — higher concentrations of the chemical make the dye grow darker. For each slide treated with the dye seen in Figure 11 below:

(a) Draw in curves of constant concentration.

(b) Draw in gradient vectors pointing in the direction of increasing concentration.

(c) Indicate points where the gradient is zero.

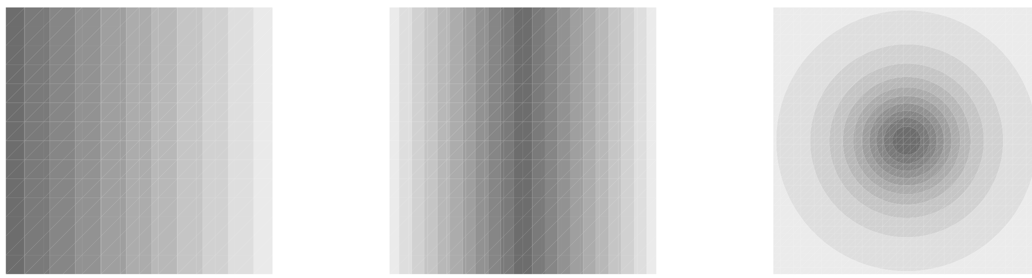

Figure 11: Slides treated with a special dye.

Problems

1. Let $f(x,y) = x^2 + y^2 - xy$.

   (a) Compute $\nabla f(1,3)$.

   (b) Compute the directional derivative of f at $(1,3)$ in the direction of $\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$.

2. The sphere $x^2 + y^2 + z^2 = 6$ and the ellipsoid $x^2 + 3y^2 + 2z^2 = 9$ intersect at the point $(2,1,1)$. Find the angle between their tangent planes at this point.

3. Suppose the function $f\colon \mathbf{R}^m \to \mathbf{R}$ is *even*, i.e. $f(\mathbf{x}) = f(-\mathbf{x})$ for every $\mathbf{x} \in \mathbf{R}^m$, and suppose f is differentiable. Find ∇f at the origin.