Post-Gödel Logic and Unanswered Questions

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#blairlogicmath

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Arithmetic

Recap: Gödel's theorems

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    - Gödel-number the formulas of arithmetic
    - Define $BEW(x)$ as "formula with Gödel-number x is provable"
    - Apply BEW to its own negation to get a formula G with Gödel-number g meaning "The formula with Gödel-number g is not provable"
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   - A simple statement of number theory cannot be proven or disproven!
3. **Second incompleteness theorem**: Any sufficiently powerful theory of arithmetic A cannot prove its own consistency.

$$A \nvdash \forall x(WFF(x) \to \neg(BEW(x) \land BEW(\neg x)))$$

Primitive recursive arithmetic

- There are other important theories of arithmetic besides Peano
- Many philosophers and mathematicians accept **primitive recursive arithmetic** as an inherently consistent starting point for arithmetic
- Constructive and finite, so it pleases intuitionists
- Quantifier-free arithmetization of arithmetic; no quantification over infinite domains means no absolute infinity
- Infinite number of variables $x, y, z, \ldots$ and a symbol for each **primitive recursive function**
  - Basically a recursively defined function
  - You know how long many steps it will take to compute beforehand (for but not while loops)
  - Each primitive recursive function's recursive definition is included as an axiom
  - For example, $ADD(x, 0) = x$, $ADD(x, S(y)) = S(ADD(x, y))$, $MUL(x, 0) = 0$, $MUL(x, S(y)) = ADD(MUL(x, y), y)$ are all axioms
- Only relies on extra axioms $S(x) \neq 0$, $S(x) = S(y) \rightarrow x = y$, and $\{\phi(0), \phi(x) \rightarrow \phi(S(x))\} \vdash \phi(y)$

Gentzen's consistency proof

- Gentzen proved Peano arithmetic consistent relative to primitive recursive arithmetic when we allow induction up to ϵ_0
- Gentzen's system and Peano's system are **incomparable**; neither is strictly stronger
  - Gentzen's system can prove that Peano's system is consistent, whereas Peano's can't (second incompleteness theorem)
  - Peano's system can compute certain functions (e.g. Ackermann function) that aren't primitive recursive
- Many people see Gentzen's proof as "as close as we can get" to an absolute proof of the consistency of PA

Logic

Löwenheim-Skolem theorem

- **Löwenheim-Skolem theorem**: If a set of first-order axioms has an infinite model, then it has a model of cardinality κ for all infinite cardinals κ
- First-order theories cannot prevent arbitrarily large models
- All first-order axiomatizations with infinite models are non-categorical
  - Gödel's theorems also imply this for sufficiently strong theories
  - *Axioms are supposed to completely categorize the objects they define, but if they're not categorical, they just can't*
- Leads to **Skolem's paradox**
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- Leads to **Skolem's paradox**
  - Löwenheim-Skolem says set theory has countable models
  - Do we know that there are uncountable objects in set theory?
  - Yes, by Cantor's theorem, so what?!??!
- Resolution: externally countable sets can appear uncountable inside of countable models because no bijection with $\mathbb{N}$ exists inside the model but one exists outside the model
- People started realizing that even first-order logic is *really hard*

Kinds of logic

- Distinguish simple syntax, **semantics** (meaning), and **proof theory** (what proofs exist)
- **Propositional logic**: Everything is either true or false
  - Connectives combine propositions to create new propositions
  - Semantics: truth tables
- **First-order logic**: Quantify over a domain
  - Same as propositional but variables, predicates, constants, functions, and quantifiers are added
  - Used for all our axiomatizations (Peano, ZFC, etc.)
  - Non-categorical
  - Semantics: model theory
- **Second-order logic**: Quantify over predicates/relations
  - Can go even higher order
  - More expressive but also much more difficult to work with
  - Incompleteness, etc. still apply, but we can bypass some Löwenheim-Skolem issues
- Note that, for example, Peano arithmetic in FOL requires infinite axioms but only five in SOL

Zermelo-Frankel set theory

- Foundational system still in use today
- We know that ZF is incomplete and cannot demonstrate consistency internally by Gödel's theorems, since we can construct a model of Peano arithmetic
- We actually know two statements that are independent of ZF (true in some models, false in others)
  - **Axiom of choice** (AC): can choose an element from each set in a (possibly very large) collection of sets
  - **Continuum hypothesis** (CH): $\aleph_1 = 2^{\aleph_0}$
- Gödel showed that $ZF + AC$ and $ZF + CH$ are both consistent relative to ZF
- Cohen introduced **forcing** to show that $ZF + \neg AC$ and $ZF + \neg CH$ are also consistent relative to ZF
- Thus, AC and CH are both independent of ZF–a concrete example of how ZF is incomplete

Conclusions

Difficulties

- Hilbert's program didn't really work
- Hilbert's second problem–prove arithmetic is consistent–cannot be solved outright (Gödel's incompleteness theorem)
- Hilbert's Entscheidungsproblem–find a decision procedure for formulas of FOL–cannot be solved outright (halting problem)
- Hilbert's tenth problem–find a decision procedure for Diophantine equations (polynomial equations with integer coefficients and solutions)–cannot be solved outright (Matiyasevich reduced it to the halting problem)
- Arithmetic is incomplete, undecidable, non-categorical, and possibly inconsistent
- Since our theories like ZFC and Peano arithmetic are incomplete, Platonism suffers a huge setback

Current work

- Keep exploring other branches of math that were never really bothered by foundational issues
- ZFC is basically the universally accepted foundation of math
  - We don't think we can prove that it's absolutely consistent, but nobody has found a contradiction so far
- Explore other foundational areas of math
  - Category theory
- Do computer science
- Keep studying logic and do really hard things, possibly hoping to discover errors or new methods that will bypass current difficulties
  - New axiomatizations of arithmetic
  - New set theories
  - New logical systems
- Incorporate philosophical arguments... some Platonists still hold out hope!

Our heroes

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- Kurt Gödel
  - Arguably the most pivotal logician (along with Aristotle and Frege)
  - Massive contributions demolished Hilbert, Russell, and Brouwer
  - Targeted by Nazis for mathematical collaborations with Jews and conscripted; fled to America in 1940
  - Became besties with Einstein
    - Einstein had to coach him on how not to derail his citizenship ceremony by pointing out logical flaws in the Constitution
    - Lowkey showed that time travel is possible in Einstein's theories as a "birthday present" and caused Einstein to doubt his entire career
  - Obsessive fear of poisoning and would only eat wife's cooking; starved to death in 1974 when she was hospitalized

Our heroes, continued

- Bertrand Russell
  - Started working outside of mathematics after the *Principia*
  - Huge pacifist and anti-war activist
    - Arrested for speaking out in WWI
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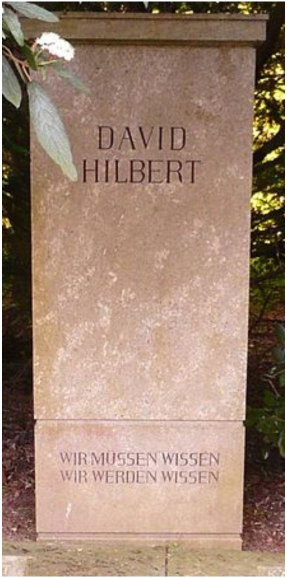
- David Hilbert
  - Nazis purged the University of Göttingen in 1933, most promiment mathematicians were Jewish or had Jewish family (Weyl, Noether, Landau, etc.)
  - Hilbert remained and died in obscurity 1943; funeral attended by ¡10 people

Spirit



Unanswered questions

In mathematics:

1. What does it mean for an object to exist?
2. How do we interpret mathematical results?
3. Why is math useful in the real world? Should it be?
4. Should we even bother doing math?
5. What is truth? How can we verify it?