An Introduction to Ordinals

D. Salgado, N. Singer

#blairlogicmath

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Introduction

History

- Bolzano defined a **set** as "an aggregate so conceived that it is indifferent to the arrangement of its members" (1883)
  - Cantor defined set membership, subsets, powersets, unions, intersections, complements, etc.
- Given two sets A and B, Cantor says they have the same **cardinality** iff there exists a **bijection** between them
  - Denoted $|A| = |B|$
  - More generally, $|A| \leq |B|$ iff there is an **injection** from A to B
- Every set has a **cardinal number**, which represents its cardinality
  - Infinite sets have cardinalities $\aleph_0, 2^{\aleph_0}, \ldots$
  - Arithmetic can be defined, e.g.

$$|A| + |B| = |(A \times \{0\}) \cup (B \times \{1\})|$$

Intuition

- Cantor viewed sets as *fundamentally structured*
- When we view the natural numbers, why does it make sense to think of them as just an "infinite bag of numbers"?
- The natural numbers is constructed in an **inherently structured** way: through successors

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- The natural numbers is constructed in an **inherently structured** way: through successors
- That structure is derived from the **order** of the set
- The notions of cardinality only apply to *unstructured* sets and are determined by bijections
- Every set also has an **order type** determined by **order-preserving bijections**: $f : A \to B$ is a bijection and

$$\forall x, y \in A\,[(x < y) \to (f(x) < f(y))]$$

Cardinals and ordinals

- The cardinality of a set corresponds to its cardinal number; the order type of a set corresponds to its **ordinal number**
- Two sets can have the same cardinality but different order types
- The cardinality of $\mathbb{N}$, $|\mathbb{N}|$, is $\aleph_0$
- The order type of $\mathbb{N}$, $\operatorname{Ord}\mathbb{N}$, is ω
- Under order-preserving bijections, ordered sets are *equivalent up to the labeling of the elements*

Naturals and primes

Consider the naturals $0, 1, 2, 3, \ldots$ and the primes $2, 3, 5, 7, 11, \ldots$.

- Can they be placed in a bijection?

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Naturals	0	1	2	3	...
Primes	2	3	5	7	...

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- Can they be placed in an order-preserving bijection?
- Why not?

What is ω?

- ω is a mathematical object that we can explicitly define (we'll do it later)
- It represents the *structure* of *any set* when it's ordered like

$$0 < 1 < 2 < 3 < \cdots$$

- An ordered set has order type ω when the set has a first element, a second element, and so on forever
- $\mathbb{N}$ has order type ω, but so does $\{x : x \text{ is prime}\}$, $\{2^x : x \in \mathbb{N}\}$, and $\{-4, -3, -2, -1, 0, 1, 2, 3, ...\}$

Order types

Let's look at some structured sets and figure out their order types!
Consider the sets $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\{0, 1, 2, 3, \ldots, a\}$.

- Do they have the same cardinality?

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Consider the sets $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\{0, 1, 2, 3, \ldots, a\}$.

- Do they have the same cardinality?
- Do they have the same order type?
- Define the order type of the new set to be $\omega + 1$
- What about $\{0, 1, 2, 3, \ldots, a, b, c\}$?

Ordinals and hyperjumps

- There are only two kinds of ordinals: **successor ordinals** and **limit ordinals**
- Successor ordinals come "after" any given ordinal
- Limit ordinals come after you take a jump into "hyperspace"
  - Also the *supremum* of an infinitely increasing set of ordinals

Figure: The original USS Enterprise (NCC-1701) making the jump into "warp".

Order types

Consider the set $\{0, 1, 2, 3, \ldots, 0', 1', 2', 3', \ldots\}$.

- Does this have the same cardinality as $\mathbb{N}$?

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- The answer is $\omega \cdot 2 = \omega + \omega$
  - Two sequential copies of ω
  - What's the order type of $\{0, 0', 1, 1', 2, 2', 3, 3', \ldots\}$?

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- What's the order type of $\{0, 1, 2, 3, \ldots, 0', 1', 2', 3', \ldots, 0'', 1'', 2'', 3'', 4''\}$?

Order types

Consider the set
$\{0_0, 1_0, 2_0, 3_0, \ldots, 0_1, 1_1, 2_1, 3_1, \ldots, 0_2, 1_2, 2_2, 3_2, \ldots, \ldots\}$.

- Does this have the same cardinality as $\mathbb{N}$?
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Order types

Consider the set
$\{0_0, 1_0, 2_0, 3_0, \ldots, 0_1, 1_1, 2_1, 3_1, \ldots, 0_2, 1_2, 2_2, 3_2, \ldots, \ldots\}$.

- Does this have the same cardinality as $\mathbb{N}$?
- What do you think its order type is?
- The answer is ω^2
- We could also think of it as ordered pairs

Ordinals and hyperjumps

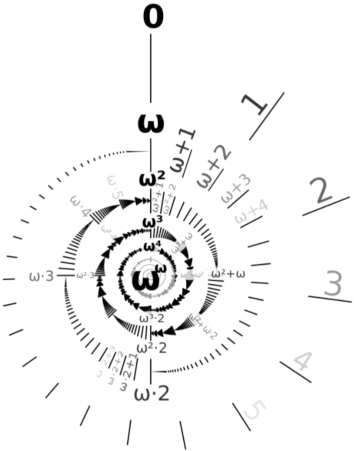

Figure: All the ordinals from 0 to ω^ω. Each turn of the spiral represents another power of ω.

Von Neumann Ordinals

Peano axioms

Now, we've reduced everything in math down to arithmetic or set theory. But what if we could reduce arithmetic to set theory too? Here are the Peano axioms:

1. 0 is a number
2. $S(M)$ is a number
3. $S(M) = S(N) \to M = N$
4. $S(M) \neq 0$
5. Induction:

$$[\phi(0) \wedge \forall x[\phi(x) \to \phi(S(x))]] \to \forall x \phi(x)$$

The von Neumann construction as a model of arithmetic

- Peano arithmetic has one operation, *successor* (S)
- Set theory has one relation, *membership* ($\in$)
- So we must somehow model the successor function using the only thing we have - set membership.
- Let's define every natural number as the set of all lesser natural numbers, e.g.

$$4 = \{0, 1, 2, 3\}$$

- What is 0?
- What is 1?
- What is 2?
- What is 3?

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- Let's define every natural number as the set of all lesser natural numbers, e.g.
$$4 = \{0, 1, 2, 3\}$$
- What is 0?
- What is 1?
- What is 2?
- What is 3?
- In general, $S(x) = x \cup \{x\}$
- Cantor naturally extended this thinking to the infinite ordinals

Order types, again

$$
\begin{aligned}
&0 := \emptyset \\
&1 := \{\emptyset\} \\
&2 := \{\emptyset, \{\emptyset\}\} \\
&3 := \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
&\omega := \{0, 1, 2, 3, \ldots\} \\
&\omega + 1 := \{0, 1, 2, 3, \ldots; \omega\} \\
&\omega \cdot 2 := \omega + \omega = \{0, 1, 2, 3, \ldots; \omega, \omega + 1, \omega + 2, \omega + 3, \ldots\} \\
&\omega^2 := \{0, \ldots; \omega, \ldots; \omega \cdot 2, \ldots; \ldots\}
\end{aligned}
$$

Transfinite induction

Induction can be applied to any well-ordering. When applied to infinite sets, induction is often called **transfinite induction**. In general, we split the proof into three cases:

1. $\alpha = 0$
2. α is a successor ordinal
3. α is a limit ordinal

Transfinite example

Theorem

Every ordinal can be written as the sum of a limit ordinal and a finite ordinal.

Proof.

1. *Zero case*: Zero is the sum of a limit ordinal (0) and a finite ordinal (0).
2. *Limit case*: Suppose α is a limit ordinal. Then α is the sum of a limit ordinal (α) and a finite ordinal (0).
3. *Successor case*: Suppose α is a successor ordinal, so $\alpha = S(\beta)$. By the inductive hypothesis, $\beta = \gamma + f$, for some finite ordinal f and limit ordinal γ, and so $\alpha = \gamma + S(f)$.

$\square$

Rigor

Goals

- So far, we've been talking about sets as if they were structured
- From a modern perspective, sets are inherently unstructured, so we need to extrinsically define orderings on them!

We need to rigorize the following definitions:

- Ordering
- Order-preserving bijection
- Order type
- Ordinal

Relations

Definition (Binary relation)

A binary relation R between the sets S and T is a set satisfying the property:

$$R \subseteq S \times T$$

If the relation is between the set S and itself, then it is said to be "on" S.

1. What is the value of $\{1, 4\} \times \{3, 6, 7\}$?
2. Which of the following are relations between the sets $\{a, b, c\}$ and $\{d, e\}$?
    1. $\{(d, b), (a, e)\}$
    2. $\{(b, e), (c, d), (a, d), (a, e)\}$
    3. $\{\}$

Total orderings

Definition (Total ordering)

A total ordering $(<, S)$ is a binary relation $<$ on S satisfying two properties:

1. $\forall x \in S \; \neg[x < x]$ (**irreflexivity**)
2. $\forall x, y, z \in S \; [(x < y \wedge y < z) \to (x < z)]$ (transitivity)
3. $\forall x, y \in S \; [x < y \vee x = y \vee y < x]$ (**trichotomy**)

where $a < b$ means that $(a, b) \in <$.

Which of the following is a total ordering on $\{1, 2, 3\}$?

1. $\{(2,3), (3,1), (1,3)\}$
2. $\{(1,3), (3,2), (1,2), (2,2)\}$
3. $\{(3,2), (2,3), (1,3), (1,2)\}$
4. $\{(1,2), (2,3), (1,3)\}$

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are totally ordered. $\mathbb{C}$ is not.

Well-orderings

Definition (Well-ordering)

A well-ordering $<$ on a set S is a total ordering where every non-empty subset has a least element:

$$\forall T \subseteq S \, [(T \neq \emptyset) \rightarrow (\exists y \in T \, \forall x \in T \, [y < x \vee y = x])]$$

Which of the following is well-ordered under the conventional definition of less-than?

- $\mathbb{N}$
- $\mathbb{Z}$
- $\mathbb{Q}$
- $\mathbb{R}$

Well-orderings can be equivalently defined as total orderings that have no infinitely decreasing subsets.

Order isomorphisms

Definition (Order isomorphism)

For two total orders $(S, <_S)$ and $(T, <_T)$, a bijection $f : S \to T$ is an order isomorphism between the sets iff it preserves ordering, i.e.

$$\forall x, y \in S\ [(x <_S y) \to (f(x) <_T f(y))]$$

If there exists an order isomorphism between two well-orderings $(S, <_S)$ and $(T, <_T)$, they are said to be **order isomorphic**, and we write $S \cong T$. Order isomorphism splits all the well-ordered sets up into a bunch of classes called **order types**.

The problem

- Order types are bags of sets that are all order isomorphic to each other
- Right now, ordinals and order types are the same
- Turns out order types are so big that they aren't really well-defined
- We need to pick **canonical representatives** of the order types, and we'll call them **ordinals**
- The ordinals should be "model citizens" of the order types that have a lot of nice properties that are happy and useful

The solution

Each order type has a **canonical representation**.

Definition

Von Neumann construction Every ordinal is precisely the set of all smaller ordinals.

- For any ordinal α, $S(\alpha) = \alpha \cup \{\alpha\}$
- As a consequence, the ordinals are well-ordered both by set membership ($\in$) and subset ($\subset$)

More definitions

Definition (Limit ordinal)

An ordinal α is a limit ordinal iff it has no maximal element, i.e.

$$\forall\beta < \alpha \, \exists\gamma \, [\beta < \gamma < \alpha]$$

Definition (Successor ordinal)

An ordinal α is a successor ordinal iff it has no maximal element, or equivalently, if it is not a limit ordinal.

Often, zero is considered by itself as a "zero ordinal", and "limit ordinal" is interpreted as "non-zero limit ordinal".

Addition

Definition (Ordinal addition)

For two ordinals α and β,

$$\alpha + \beta = \mathrm{Ord}((\alpha \times \{0\}) \cup (\beta \times \{1\}), <_+)$$

where the ordering $<_+$ is defined as

$$\{(a, b) : (a \in \alpha \wedge b \in \alpha \wedge a < b) \vee (a \in \alpha \wedge b \in \beta) \vee (a \in \beta \wedge b \in \beta \wedge a < b)\}$$

Multiplication

Definition (Ordinal multiplication)

For two ordinals α and β,

$$\alpha \cdot \beta = \mathrm{Ord}(\alpha \times \beta, <_*)$$

where the **lexicographic ordering** $<_*$ is defined as

$$\{((a_0, b_0), (a_1, b_1)) : (a_0 < a_1) \vee (a_0 = a_1 \wedge b_0 < b_1)\}$$

Exponentiation

Burali-Forti paradox

Suppose O is the set of all ordinals. Then, since O is a well-ordered and complete set of ordinals, O is itself an ordinal. Then $O \in O$, but this means that $O < O$, which is a contradiction.

More ordinals

- The limit of $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ is ϵ_0
  - First **epsilon number** to solve $\omega^\epsilon = \epsilon$
- When we write down (**notate**) the ordinals, how many symbols in our alphabet are there?

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- When we write down (**notate**) the ordinals, how many symbols in our alphabet are there?
- We can only explicitly write down a countable number of ordinals!
- The first "un-notatable" ordinal is the **Church-Kleene ordinal** ω_{CK}^1
  - Cannot be *computed* or defined *recursively*

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- When we write down (**notate**) the ordinals, how many symbols in our alphabet are there?
- We can only explicitly write down a countable number of ordinals!
- The first "un-notatable" ordinal is the **Church-Kleene ordinal** ω^1_{CK}
  - Cannot be *computed* or defined *recursively*
- After *all* the countable ordinals comes the first uncountable ordinal ω_1
- *Question*: How many ordinals are there?

More ordinals

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- After *all* the countable ordinals comes the first uncountable ordinal ω_1
- *Question*: How many ordinals are there?
- *Answer*: SO MANY.

Bonus: RIGORRRR!

Induction

Here's a proof that transfinite induction works.

Theorem (Induction)

Given any well-ordering $(S, <)$ *and property* ϕ*,*

$$[\forall x \in S\ ([\forall y < x\ \phi(y)] \to \phi(x))] \to \forall x \phi(x)$$

Proof.

Assume $\forall x \in S\ ([\forall y < x\ \phi(y)] \to \phi(x))$.
Suppose ϕ does not hold for all $x \in S$. Let $T = \{x : x \in S \wedge \neg\phi(x)\}$.
Since $T \subseteq S$, T is non-empty, and $<$ well-orders S, T must have a least element x_0.
We know that $\neg\phi(x_0)$, so by our assumption, there is some $y < x_0$ such that $\neg\phi(y)$. Thus, $y \in T$. But x_0 is the minimal element of T!
Thus, we have a contradiction, and $\forall x \phi(x)$. □

Order isomorphisms

Theorem

Order isomorphism is an equivalence relation.

Proof.

1. Reflexivity: The identity function on $(S, <_S)$ is an order isomorphism.
2. Symmetry: Let f be an order isomorphism from $(S, <_S)$ to $(T, <_T)$. Then f^{-1} is an order isomorphism from $(T, <_T)$ to $(S, <_S)$.
3. Transitivity: Let f be an order isomorphism from $(S, <_S)$ to $(T, <_T)$ and g be an order isomorphism from $(T, <_T)$ to $(U, <_U)$. Then $g \circ f$ is an order isomorphism from $(S, <_S)$ to $(U, <_U)$.

$\square$

Comparison of well-orderings

Definition (Initial segment)

Let $(S, <)$ be a well-ordering. Any $x \in S$ generates a (proper) **initial segment** $(S, <)/x = \{y : y \in S \wedge y < x\}$.

What are the initial segments of the well-ordering $\omega^2 + \omega$?

Definition (Ordinal comparison)

For any two well-orderings $(S, <_S)$ and $(T, <_T)$, $(S, <_S) \lhd (T, <_T)$ iff $(S, <_S)$ is order isomorphic to some initial segment of T, that is,

$$\exists x \in T \ (S, <_S) \cong (T, <_T)/x$$

Properties of ordinal comparison

Let ON be the class of all ordinals.

Theorem

Ordinal comparison well-orders the class of all ordinals.

Proof.

To show that $(\mathrm{ON}, \lhd)$ is a partial order, let $\alpha, \beta, \gamma \in \mathrm{ON}$ and consider:

1. α cannot be order isomorphic to a proper initial segment of itself, so $\lhd$ is irreflexive.
2. If $\alpha \lhd \beta$, then it cannot be the case that $\beta \lhd \alpha$, because by composing the two resulting isomorphisms, α would be isomoprhic to a proper initial segment of itself. Thus, $\lhd$ is antisymmetric.
3. If $\alpha \lhd \beta \lhd \gamma$, compose the orderings. The new ordering is clearly an order isomorphism, and its image in γ must be an initial segment (why)?. Thus, $\lhd$ is transitive.

Properties of ordinal comparison

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Proof.

To prove totality, (in progress).
To see why $\lhd$ is well-founded, let A be a set of ordinals. Choose some $\alpha \in A$. If α is minimal, we are done. Otherwise, let B be the set of all ordinals less than α. Since we have trichotomy, each of these ordinals must be isomorphic to some initial segment of α. Let X be the set of x's which correspond to the initial segments in α. Since $X \subseteq \alpha$, choose the minimal such X, and the minimal element of B must be $(\alpha, \lhd)/x$. □

Properties of ordinal comparison

Theorem

Any well-ordering $(S, <_S)$ is order isomorphic to the set of its initial segments under ordinal comparison:

$$(S, <_S) \cong (\{(S, <_S)/x : x \in S\}, \lhd)$$

This motivates us to *define an ordinal as the set of all lesser ordinals*.