Romanian IMO TST 2005

Day I - *March 31, 2005*

Problem 1. Solve the equation $3^x = 2^x y + 1$ in positive integers.

Solution.

In the following, the notation $p^n || t$ will mean that $p^n | t$ and that p^{n+1} does not divide t. Note that if p is a prime, $p^i || a$ and $p^j || b$ then $p^{i+j} || ab$. This is trivial by looking at the prime factorization of a and b.
We will use with two simple lemmas:

LEMMA 1. For any $n \in \mathbb{N}$ we have $2^{n+2} || 3^{2^n} - 1$.

LEMMA 2. For any natural number $n \geq 3$ we have $2^n > n + 2$.

Proof of Lemma 1.
We set as the induction basis the case $n = 1$ (it is clear that $2^3 || 8$ holds).
Now let's suppose that LEMMA 1 is true for some $n \geq 1$ and we'll prove that it is true for $n + 1$. We notice that we have

$$3^{2^{n+1}} - 1 = \left(3^{2^n}\right)^2 - 1^2 = \left(3^{2^n} - 1\right)\left(3^{2^n} + 1\right).$$

From the induction hypothesis we know that $2^{n+2} || 3^{2^n} - 1$. But $n + 2 \geq 3$, so $4 | 2^{n+2}$. Therefore $4 | 3^{2^n} - 1$. As a consequence we get that 4 does not divide $3^{2^n} + 1$, so, by combining this with the fact that $3^{2^n} + 1$ is even, we have $2 || 3^{2^n} + 1$. Using the property mentioned at the beginning, we get $2^{n+3} || \left(3^{2^n} - 1\right)\left(3^{2^n} + 1\right) = 3^{2^{n+1}} - 1$, which completes the induction step, so LEMMA 1 is fully proven.

Proof of Lemma 2.
Again, the proof will be by induction over n. The base case, $n = 3$, is trivial, since $8 > 3$.
Now let's suppose that LEMMA 2 is true for some $n \geq 3$ and we'll prove that it is true for $n + 1$. We have $2^{n+1} = 2 \cdot 2^n > 2 \cdot (n + 2) = 2n + 4 > n + 3$, which completes the induction step and the short proof of LEMMA 2.

We rewrite the equation as

$$\begin{aligned} 2^x y &= (3-1)(3^{x-1} + 3^{x-2} + \ldots + 3 + 1) \\ &= 2(3^{x-1} + 3^{x-2} + \ldots + 3 + 1). \end{aligned}$$

Now let $n, \alpha \in \mathbb{N} \cup \{0\}$ such that $x = 2^n \alpha$, with α odd.

Case 1. $n = 0$, i.e. x is odd
Since $S = 3^{x-1} + 3^{x-2} + \ldots + 3 + 1$ is a sum of x odd terms and x is also odd, it follows that S is odd. Therefore $2 || 2^x y$, so x is at most 1. But we know that x is positive (≥ 1), so $x = 1$, which implies $y = 1$. Hence $(x, y) = (1, 1)$ is a solution of the equation.

Case 2. $n \geq 1$, i.e. x is even
In this case the equation becomes

$$2^{2^n \alpha} y = \left(3^{2^n}\right)^{\alpha} - 1^{\alpha} = \left(3^{2^n} - 1\right)\left(\left(3^{2^n}\right)^{\alpha-1} + \left(3^{2^n}\right)^{\alpha-2} + \ldots + \left(3^{2^n}\right)^{1} + \left(3^{2^n}\right)^{0}\right).$$

Since $T = \left(3^{2^n}\right)^{\alpha-1} + \left(3^{2^n}\right)^{\alpha-2} + \ldots + \left(3^{2^n}\right)^{1} + \left(3^{2^n}\right)^{0}$ is a sum of α odd terms and α is also odd, it follows that T is odd. Combine this with what we know from LEMMA 1, i.e. $2^{n+2} || 3^{2^n} - 1$, to obtain $2^{n+2} || RHS$. Let $t \in \mathbb{N}$ s.t. $2^t || 2^{2^n \alpha} y$. For equality to take place we must have $t = n + 2$. Furthermore, it is obvious that $t \geq 2^n \alpha$.
If $n \geq 3$, using LEMMA 2, we get $t \geq 2^n \alpha \geq 2^n > n + 2$, so in this case we can't have equality.
If $n = 2$ we must have $t = 4$. But $t \geq 4\alpha \geq 4$, so it is imperative to have $\alpha = 1$. Therefore $x = 2^n \alpha = 4$, which implies $y = 5$.
If $n = 1$ we must have $t = 3$. But $t \geq 2\alpha$, so any $\alpha > 1$ won't do. Therefore $\alpha = 1$ and $x = 2^n \alpha = 2$, which implies $y = 2$.

In conclusion, the solutions of the equation are $(1, 1)$, $(2, 2)$ and $(4, 5)$.