Graphs and Matrices

1 The Adjacency Matrix of a Graph

The ***adjacency matrix*** A of a graph is defined by numbering the vertices, say from 1 up to n, and then putting $a_{ij} = a_{ji} = 1$ if there is an edge from i to j, and $a_{ij} = 0$ otherwise. We can do the same for a digraph: putting $a_{ij} = 1$ if there is an arc from i to j, and $a_{ij} = 0$ otherwise. For example, here is an adjacency matrix of a directed cycle on 4 vertices:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

2 Powers of the Adjacency Matrix

The powers of the adjacency matrix counts things. In particular,

entry i, j in A^s gives the number of walks from i to j of length s.

The proof is by induction argument. For example, the number of walks of length 2 is the number of vertices k such that there is an arc from i to k and an arc from k to j. And this is exactly the i, j entry in A^2, by the definition of matrix multiplication.

(If we have weights on the edges, then the result is still valid. We redefine the adjacency matrix to have the weights as its entries, and define the weight of a walk as the product of the weights of the arcs. Then if want to know the total sum of weights of i, j paths of given length, that is the entry in the appropriate power.)

3 Eigenvalues of Graphs

We consider next undirected graphs without loops. The ***eigenvalues*** of a graph are the eigenvalues of its adjacency matrix. For example: the triangle K_3 has eigenvalues, $2, -1, -1$.

Observation: An r-regular graph has r as an eigenvalue. Proof: It has the all-1 eigenvector.

The complete bipartite graph $K_{m,n}$ has an adjacency matrix with rank 2. So it has eigenvalue 0 with multiplicity $m + n - 2$. It can be checked that the remaining two eigenvalues are $\pm\sqrt{mn}$.

4 Moore Graphs

Consider an r-regular simple graph of girth 5. That is, for every vertex v there is no edge between the neighbors of v, and none of the neighbors have a common neighbor apart from v. Then we must have at least $1 + r + r(r - 1) = r^2 + 1$ distinct vertices. A ***Moore graph*** is an r-regular graph with $r^2 + 1$ vertices with that property. For example, a 5-cycle is the Moore graph for $r = 2$ and the Petersen graph for $r = 3$.

Theorem 1 *There is a Moore graph if and only if* $r = 2, 3, 7$ *or possibly* 57*.*

Proof: Assume a Moore graph exists and let A be its adjacency matrix. Then if we consider any pair of vertices, they are either neighbors or they have a unique common neighbor, but not both. Consider what this means for the matrix A^2. Off the diagonal, it has a 1 wherever A has a 0 and vice versa. On the diagonal it has r (just because the graph is r-regular). This means that

$$A^2 + A - (r - 1)I = J,$$

where J is the all-1 matrix.

Now let λ be an eigenvalue of A with eigenvector v. There are two possibilities. First, assume the entries of v sum to something nonzero. Then we can normalize v

such that its entries sum to 1. Then

$$[\lambda^2 + \lambda - (r-1)]v = J,$$

where 1 is the all-1 vector. It follows that $v = \frac{1}{n}J$, where n is the number of vertices; and that $\lambda = r$ (with multiplicity 1).

The second possibility is that the entries of v sum to zero. Then

$$[\lambda^2 + \lambda - (r-1)]v = 0.$$

And so we have $\lambda^2 + \lambda - (r-1) = 0$. Hence

$$\lambda = \frac{-1 \pm \sqrt{1 + 4(r-1)}}{2}.$$

Now, suppose the $\lambda^+ = (-1 + \sqrt{4r-3})/2$ is an eigenvalue b times. It follows that $\lambda^- = (-1 - \sqrt{4r-3})/2$ is an eigenvalue $n - b - 1 = r^2 - b$ times.

And the trace of A is clearly 0. So we have that

$$b\lambda^+ + (r^2 - b)\lambda^- + r = 0.$$

And thus that

$$2b - r^2 = \frac{r(r-2)}{\sqrt{4r-3}}.$$

The right-hand side must therefore be an integer. If the numerator is 0, which happens when $r = 2$, then we are okay. Otherwise, we certainly need that $\sqrt{4r-3}$ be rational, which only happens if it is an integer, say m.

Substitute this into the RHS, and we get

$$\frac{m^4 - 2m^2 - 15}{16m}$$

Thus, m must be a divisor of 15, which forces $m \in \{1, 3, 5, 15\}$. Thus $r \in \{1, 3, 7, 57\}$. It can be checked that in each case of r, b is a whole number. [EndProof]

Moore graph have been constructed for $r = 2, 3, 7$, but the existence for $r = 57$ is unsolved.