rows: how much variables affect 1 measurement
cols: how much one variable affects all measurements.

$\exists \vec{x}$ st $A\vec{x} = \vec{b} \Rightarrow$ span (cols of A) incl. $\vec{b}$.

Linear Dependence

$\{\vec{a}_1, \dots, \vec{a}_m\}$ are LD if $\exists\, \alpha_1, \dots, \alpha_m$ st $\alpha_1\vec{a}_1 + \dots + \alpha_m\vec{a}_m = \vec{0}$

Not all $\underline{\alpha_1, \dots, \alpha_m} = 0$.

2) $\{\vec{v}_1, \dots, \vec{v}_n\}$ is LD if $\exists\, \alpha_1, \dots, \alpha_n$ and index i st $\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$.

One vector is linear combination of others

---

transition matrix:
cols represent outflows,
rows represent inflows

Matrix Inversion is unique

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$Ax = b$ has unique soln
$\Leftrightarrow$ col of A are LI $\Leftrightarrow \det \neq 0$
$\Leftrightarrow$ A is invertible $\Leftrightarrow$ all $\lambda_i \neq 0$
$\Leftrightarrow$ every col in any REF has pivot
$\Leftrightarrow$ RREF of A leads to identity matrix

Subspace $\mathbb{U}$

3 Properties:
1) contains zero vector $\vec{0} \in \mathbb{U}$
2) closed under vector addition
$\vec{v}_1 \in \mathbb{U}$, $\vec{v}_2 \in \mathbb{U}$
$\vec{v}_1 + \vec{v}_2 \in \mathbb{U}$
3) closed under scalar multiplication
$\alpha \cdot \vec{v}_1 \in \mathbb{U}$

$\mathbb{V}$ is a vector space and $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subset \mathbb{V}$, span$\{\vec{v}_1, \dots, \vec{v}_n\}$ is ALWAYS a V.S. and subspace of $\mathbb{V}$.

Basis: min set of vectors to represent all vectors in V.S.
- vectors should be LI.
- span vector space and LI

Dimension: # of basis vectors.

col space: span (cols(A)) (range)
row space: span (rows(A))
Null Space: space you cannot reach $\boxed{A\vec{x} = 0}$
dim (col space (A)) = dim (range (A))
↳ # independent rows = RANK.
= # independent cols
= # pivots in REF

Null Space is a Vector Space.

Rank-Nullity Theorem:
rank(A) + dim(Nul(A)) = N
pivots   # cols w/out pivots   # cols (A).

rank(A) = dim (row space (A))
# LI cols = # LI rows.

Orthogonality

$(A\vec{x} = 0) \perp$ if:

$\begin{bmatrix} -a_1^T- \\ \vdots \\ -a_n^T- \end{bmatrix}\begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

. Null Space $\perp$ row space
- null space vector x will not affect result of multiplying A times b when x + b,
$Ab = A(b + x)$.

---

trace: sum of diagonal elements of A.

Steady state.
$A\vec{x} = \vec{x}$
eigvec. $\lambda = 1$.
$\exists$ steady state when $\exists!$ $\lambda = 1$, all other $\lambda < 1$.

Find λ:
$\det(A - \lambda I) = 0$. – NullSpace$(A-\lambda I)$ – $(a-\lambda)(d-\lambda)-bc$ ← char poly
$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Find eigvec:
Plug λ into $(A - \lambda I)\vec{x} = 0$.

Eigenspace:
vector space mapped out by eigvec.
↳ $\mathrm{Nul}(A - \lambda I)$

---

eigval = 0 $\Rightarrow$ not invertible

In Gaussian Elimination:
1) scale row by α:
scale det by α
2) Add multiple of one row to another:
shear – det stays same
3) swap rows – multiply det by –1.

$\lambda_1, \dots, \lambda_k$ are distinct
$\Rightarrow \vec{v}_1, \dots, \vec{v}_k$ are LI (eigvec)

Represent $\vec{u}$ in terms of $\vec{a}_1$ and $\vec{a}_2$ (Change of Basis)

$\vec{u} = u_{a1}\vec{a}_1 + u_{a2}\vec{a}_2$

$\begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}\begin{bmatrix} u_{a1} \\ u_{a2} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Find u_{a1}, u_{a2} st $\vec{a}_1, \vec{a}_2$ are expressed as $\vec{u}$

$\vec{u}_a = A^{-1}\vec{u}$

change $\vec{u}$ to basis of A.

Basis $\vec{a}_1, \dots, \vec{a}_n$ $\underset{A^{-1}}{\overset{A}{\rightleftarrows}}$ Std basis

Std basis $\underset{B}{\overset{B^{-1}}{\rightleftarrows}}$ Basis $\vec{b}_1, \dots, \vec{b}_n$

$$\begin{tikzcd} \vec{u} \arrow[r, "T"] \arrow[d, shift left, "A^{-1}"] & \vec{v} \arrow[d, shift left, "A^{-1}"] \\ \vec{u}_a \arrow[r, "T_a"] \arrow[u, shift left, "A"] & \vec{v}_a \arrow[u, shift left, "A"] \end{tikzcd}$$

Diagonalizable $T_{n \times n}$ if it has n LI eigvec

$T = ADA^{-1}$

$A = [\vec{a}_1, \dots, \vec{a}_n]$ (eigvec)

$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

determinant of upper triangular 3x3 matrix is product of diagonal entries.

Matrix Props.
$(c+d)A = cA + dA$
$c(A+B) = cA + cB$
$AB(C) = A(BC)$
$A(B+C) = AB + AC$
$(B+C)A = BA + CA$

Page Rank: Normalize importance scores

16A

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & 0 \end{bmatrix}$ $x_2 = 0$: $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ span$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$\begin{bmatrix} -8 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 = 0$ $\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ span$\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$A\vec{x} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \dots & a_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
$= x_1\begin{bmatrix} | \\ a_1 \\ | \end{bmatrix} + x_2\begin{bmatrix} | \\ a_2 \\ | \end{bmatrix} + \dots + x_n\begin{bmatrix} | \\ a_n \\ | \end{bmatrix}$

trace(A) = sum of n eigvals of A.
product of n eigvals is same as determinant of A
$\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I)$ ☆

If λ is eigval of matrix A, eigval of matrix A^T
- conservative state transition matrices have columns that sum to 1.

---

a matrix B st set of valid net traffic flows is nullspace of B
→ incidence matrix of traffic network
row represents intersection
col represents road between two intersections
→ null space of traffic flow

Inverse
$[M \mid I_n] \rightarrow [I_n \mid M^{-1}]$

If you have symmetric transition matrix, $A^T = A$, both rows and cols sum to one.
All uniform vectors are equilibrium state.

Imaging: $\vec{s} = H\vec{i} + \vec{w}$
If $A \rightarrow \lambda$, $A^{-1} \rightarrow \frac{1}{\lambda}$
$\therefore$ large eigval of H $\Rightarrow$ lower noise
small eigval of H $\Rightarrow$ higher noise
S: sensor reading
H: matrix of image masks.
$\vec{i}$: image
$\vec{w}$: noise

Col Space
- reduce matrix to echelon form
- columns w/ pivots are columns of original matrix that make up col space

---

- Use nullspace to prove distinct vectors operated only m x n, m < n, matrix can give same measurements
→ non trivial nullspace
→ let i be $\in$ Nul, j $\in$ Col
$Mj = b$, $Mi = 0$
$Mj + Mi = M(j + i) = b + 0 = b$.

$m \times n$
$(AB)^T = B^T A^T$

$n \times n$
$Av = \lambda v$

Full set eigvec -
$A = S \Lambda S^{-1}$
$(AB)^{-1} = B^{-1} A^{-1}$
$(A^T)^{-1} = (A^{-1})^T$
$A = S\Lambda S^{-1}$ $A^{-1} = S \Lambda^{-1} S^{-1}$
(same eigvec, inverse eigval)

Matrix Transformations
- rotation
- scaling
- reflecting

rotation
$T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

rotation ccw by angle θ

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$

Scale x by a, y by b

reflection

x-axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

y-axis $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Origin $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$y = x$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

See transformation

$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ 2a \end{bmatrix}$

e.g. $= \begin{bmatrix} x+2y \\ 2x+4y \end{bmatrix}$
$= \begin{bmatrix} a_1 \\ 2a_1 \end{bmatrix}$

---

same characteristic polynomial $\Longrightarrow$ same eigval.

Determinant of matrix and that of its transpose are equal.

A and A^T have same eigvals

Distinct eigval $\Longleftrightarrow$ All LI eigvec $\Longleftrightarrow$ diagonalizable

---

$\vec{v} = \begin{bmatrix} 2m \\ m+3 \\ 0 \end{bmatrix} \in \text{span}(S) \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

if $\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2m \\ m+3 \\ 0 \end{bmatrix}$
$= m \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

Find eigval $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$\det\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right)$
$= \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix} \right) = 0.$

$(1-\lambda)(2-\lambda) - 2 = \lambda^2 - 3\lambda = \lambda(\lambda - 3) = 0.$

$\lambda = 0.$

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right]$

$x_1 + x_2 = 0$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2$
$x_1 = -x_2$
$\Rightarrow \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$\lambda = 3.$

$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right]$

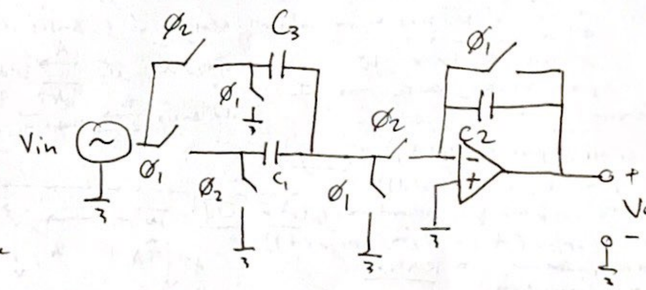
$-2x_1 + x_2 = 0$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1$
$x_2 = 2x_1$
$\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Null Space:
$A\vec{x} = 0.$

Col Space:
row reduce
cols in OG corresponding to cols w/ pivots in RR form span col space.

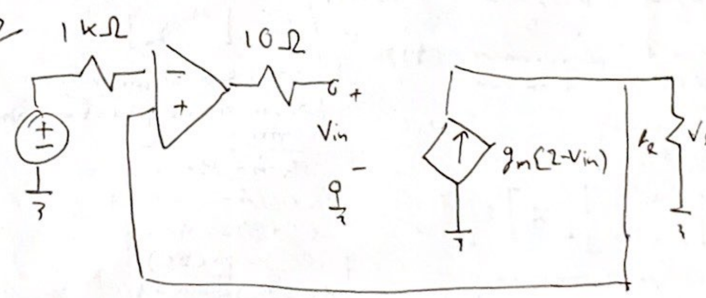
Finding eigvec
$\begin{bmatrix} 1-b & a \\ 0 & 0 \end{bmatrix}$
$(1-b)x_1 = -a x_2$
$(b+1)x_1 = a x_2$
$\vec{v} = \begin{bmatrix} a \\ b-1 \end{bmatrix}$

AB or BA has eigval λ, BA has same
$AB\vec{x} = \lambda \vec{x}$
$BAB\vec{x} = B\lambda\vec{x}$
$BA(B\vec{x}) = \lambda(B\vec{x})$



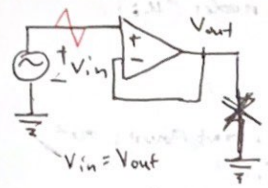
Ø1: $V_{out} = 0$
Ø2: $V_{out} = \frac{C_1 - C_3}{C_2} V_{in}$

NFB



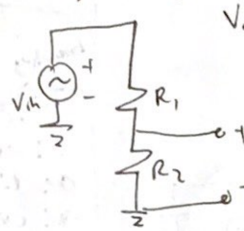
- increase Vin, dependent current src ↓
- current must go thru RL
$\Rightarrow$ V_{load} ↑, V^+ ↓, V_{in} ↓

Unity Gain Buffer



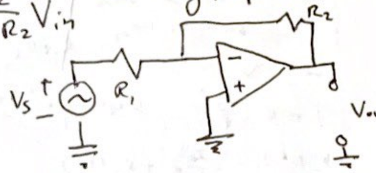
$V_{in} = V_{out}$

Voltage Divider



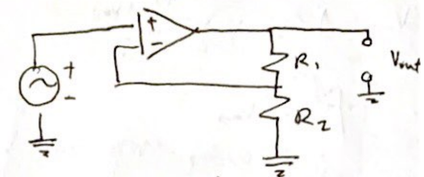
$$V_{out} = \frac{R_2}{R_1+R_2} V_{in}$$

Inverting Amplifier



$$V_{out} = \frac{-R_2}{R_1} V_{in}$$

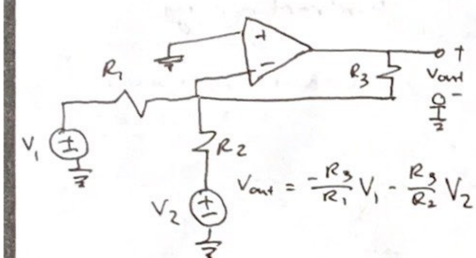
Non-Inverting Amplifier



$$V_{out} = \left(1+\frac{R_1}{R_2}\right) V_{in}$$

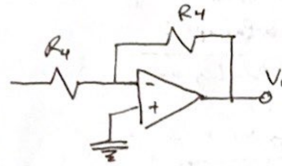
positive gain $\beta = 1+\frac{R_1}{R_2}$

Inverting Summer - negative gain β



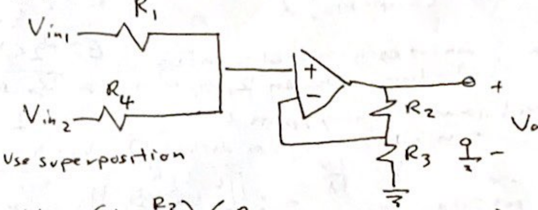
$$V_{out} = \frac{-R_3}{R_1} V_1 - \frac{R_3}{R_2} V_2$$

Add inverter



$$V_{out} = \frac{R_3}{R_1} V_1 + \frac{R_3}{R_2} V_2$$

Summer

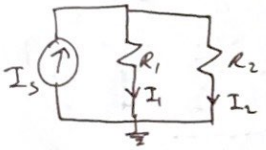


Use superposition

$$V_{out} = \left(1+\frac{R_2}{R_3}\right)\left(\frac{R_4}{R_1+R_4} V_{in1} + \frac{R_1}{R_1+R_4} V_{in2}\right)$$

e.g. if $R_1 = R_4 (=1)$, $R_2 = R_3 (=1)$; $V_{out} = V_{in1} + V_{in2}$

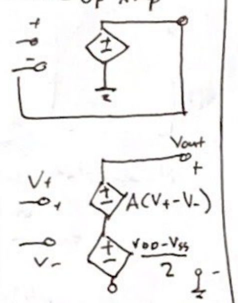
Current Divider



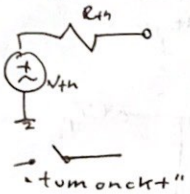
$$I_1 = \frac{R_2}{R_1+R_2} I_S$$

$$I_2 = \frac{R_1}{R_1+R_2} I_S$$

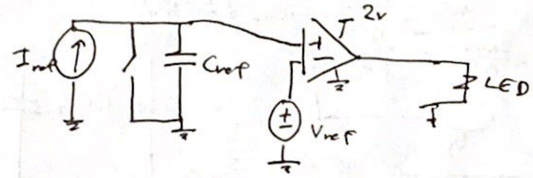
Inside Op Amp



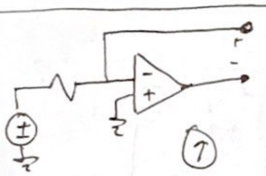
Sensor



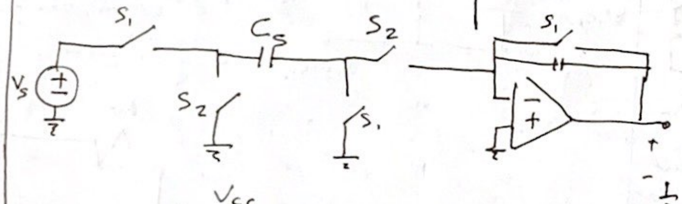
"turn on ckt"



Switch is on before button: connected |
→ resets capacitor



Current Source

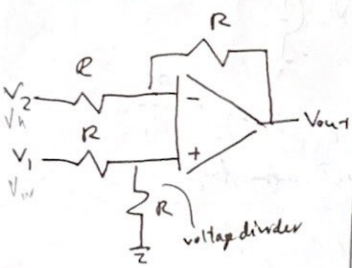


$Q_1 = Q_2$

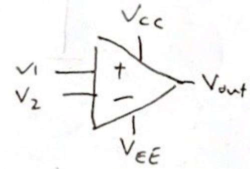
$V_{in}C_1 = V_{out}C_2$

$$V_{out_2} = \frac{C_S}{C_{ref}} \cdot V_S$$

Difference

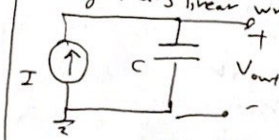


$V_{out} = V_1 - V_2$



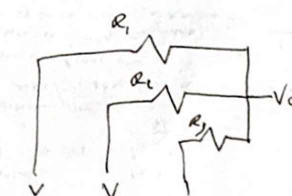
$$V_{out} = \begin{cases} V_{CC}, & V_1 > V_2 \\ V_{EE}, & V_1 < V_2 \end{cases}$$

Voltage that's linear wrt time



$$V_{out} = \frac{I}{C} t$$

Resistive Adder

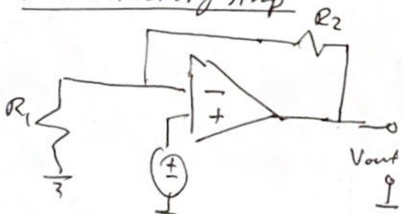


$$V_{out} = \frac{\frac{V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_3}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

If $R_1 = R_2 = R_3$

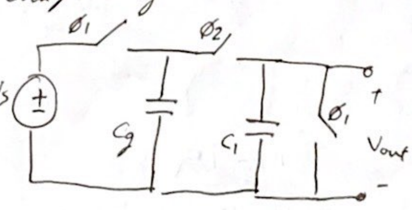
$$V_{out} = \frac{V_1+V_2+V_3}{3}$$

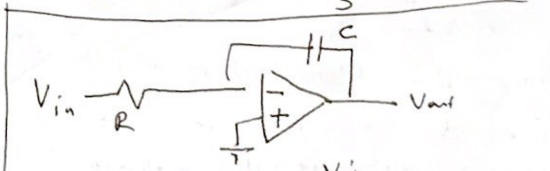
Non Inverting Amp



$$V_{out} = \frac{R_1+R_2}{R_1} \cdot V_+$$

Charge sharing

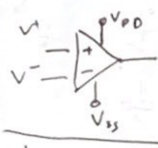




$$V_{out} = \frac{V_{in}}{R \cdot C} t$$

Op Amp

$V_{out} = A(V_+ - V_-) + \frac{V_{SS} + V_{DD}}{2}$



OpAmp w/ NFB: $V_{SS} = -V_{DD}$

charge: Q coulombs (C) carried by electrons
current: I measures movement of charge
$I = \frac{dQ}{dt}$ (amperes) A

Voltage: amount of energy needed to move unit charge between 2 points V

Resistance: amount of energy spent to move electrons thru conductor Ω

$V = IR$ $1\,\Omega = \frac{1V}{1A}$

$R = \rho\frac{L}{A}$ length / cross-sectional area $\rightarrow$ $L = h$, $A = \pi(\frac{1}{2}d)^2$

Open circuit: zero current | short circuit (wire): 0 V

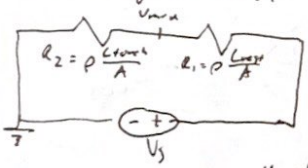
$P = \frac{dE}{dt} = V\frac{dQ}{dt} = VI$

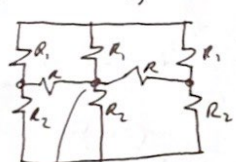
+: power dissipated; −: power supplied

measuring device has no influence $\Leftrightarrow$ no power dissipated

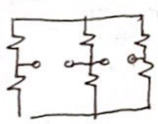
power: rate of change of energy

$u_{mid} = \frac{L_{touch}}{L} \cdot V_s$





$u_1 = u_2 = u_3$
$\Rightarrow R$ is useless;
I, V are 0



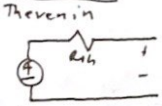
$u_2 = \frac{L_{touch}}{L} V_s$, $L_{touch} = L_{touch,\ vertical}$

Superposition: replace voltage sources with wires, current sources w/ open circuit

shorts: 0 V, 0 R / open ckt: 0 I, ∞ R

Equivalent: I-V relationships are same

Thevenin

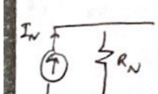


1. measure voltage across output terminals by connecting OC
2. zero out independent srcs V: wire, SC I: OC
apply test current into terminal and measure voltage
OR
apply test voltage and measure current

$V_{oc} = V_{th}$

$R_{th} = \frac{V_{test}}{I_{test}}$

Norton



1. connect short ckt across output terminals and measure current
$I_{sc} = I_N$
2. 0 out ind. srcs. test.
$R_{th} = \frac{V_{test}}{I_{test}}$.

$R_N = R_{th}$ V: SC, wire ; I: OC

V_{th} and I_N are on same line. $I_{sc} = \frac{V_{th}}{R_{th}}$

$R_{th} = \frac{V_{th}}{I_N}$

V_{test} if ckt elements in parallel

I_{test} if ckt elements in series

elements that share same start/end node are in parallel.

Resistors in series: $R_{eq} = R_1 + R_2$

parallel: $R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$

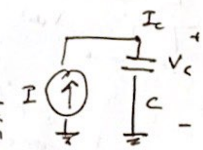
$Q = CV$

$I = C\frac{dV}{dt}$ Capacitor: Farads

$V_c(t) = \frac{I_c}{C}t + V_c(0)$

Parallel capacitors: $C_{eq} = C_1 + C_2$

Series: $C_{eq} = \frac{C_1 C_2}{C_1 + C_2} = \frac{C_2}{1 + \frac{C_2}{C_1}}$



$C = \varepsilon\frac{A}{d}$ ε = permittivity, how far apart

$E = \frac{1}{2}CV_s^2$

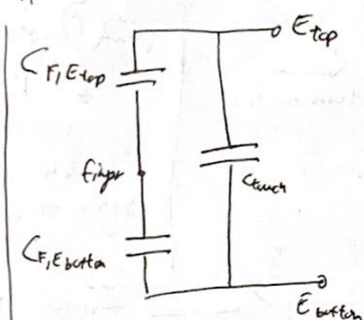
Golden Rules (Ideal Op Amp)

$I_+ = I_-$

$V_+ = V_-$, when NFB

$\boxed{V_{time} = \frac{I_{ref}}{C_{ref}} \cdot t}$

- capacitors in series have equal charge
- capacitors in parallel have same voltage



Rails of op-amp control output max/min

Incidence Matrix - KCL

nodes (U_i)

branches (I_i)

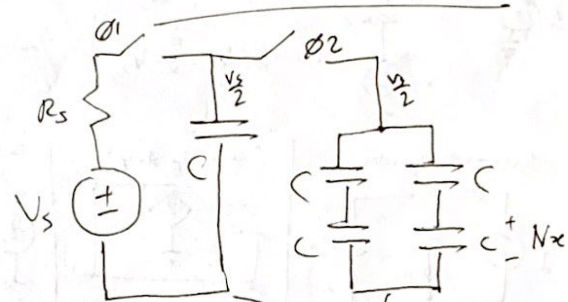
1: current flowing out

0: N/A

−1: current flowing in

KVL: arrow of loop goes into "+" of element, subtract voltage

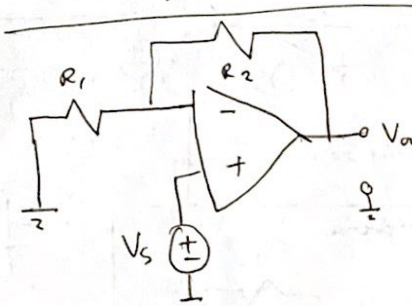
Find matrix - nullspace = soln space



steady state: capacitor charged

$V_x = \frac{V_s}{4}$

Voltage splits in parallel capacitor, flows through in series.



$V_{out} = V_s + \frac{R_2}{R_1}V_s$

NonInverting Amp.

Nodal Analysis
- set currents =,
use ohms

$1F = 1\,C/V$

$R = \frac{V}{A} = \Omega$ | steady state - current thru capacitors = 0

16A

Inner Product

$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = \sum_{i=1}^{n} x_i y_i$

$\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

$\langle c\vec{x}, \vec{y} \rangle = c \langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, c\vec{y} \rangle$

scalar

$\langle \vec{x}+\vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

$= \langle \vec{x}, \vec{y}+\vec{z} \rangle$

Inner Product is 0 $\Leftrightarrow$ orthogonal

Euclidean Norm:

$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\langle \vec{x}, \vec{x} \rangle}$

$\Rightarrow \|x\|_2^2 = \langle \vec{x}, \vec{x} \rangle$

magnitude, length of vector

$\|\alpha x\| = |\alpha| \|x\|$

$\|x+y\| \leq \|x\| + \|y\|$

$\|x\| \geq 0$

$\|x\| = 0$ only if $x = 0$

$\langle \vec{x}, \vec{y} \rangle = \|x\| \|y\| \cos\theta$

lengths multiplied by angle between them

Cauchy-Schwartz:

$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \|\vec{y}\|$

$Proj_x y = \|\vec{y}\| \cos\theta \cdot \frac{\vec{x}}{\|x\|}$

$= \langle \vec{y}, \frac{\vec{x}}{\|x\|} \rangle \frac{\vec{x}}{\|x\|} = \frac{\vec{x}^T \vec{y}}{\|x\|^2} \cdot \vec{x}$

Circulant Matrix

$$C_{\vec{x}} = \begin{bmatrix} x[0] & x[N-1] & \dots & x[1] \\ x[1] & x[0] & \dots & x[2] \\ \vdots & \vdots & \ddots & \vdots \\ x[N-1] & x[N-2] & \dots & x[0] \end{bmatrix}$$

Cross Correlation - measurement of similarity between two vectors, $\vec{y}$ and $\vec{u}$

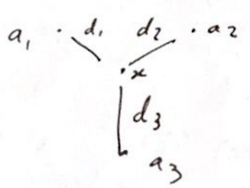
$$S_{\vec{u}\vec{y}}[j] = \frac{1}{\|u\|^2} \sum_{n=0}^{N-1} \vec{y}[n] \vec{u}[(n-j)_N]$$

j^{th} entry of cross correlation

- largest cross-correlations deemed to correspond to time-shifts of $\vec{u}$

Autocorrelation - cross correlation of signal w/ itself

- use delays to get distances of beacons



3 circles uniquely define a point in 2D

$d = v t$

velocity of sound.

delay

$\|\vec{x} - \vec{a}_1\|^2 = d_1^2 \Rightarrow \vec{x}^T\vec{x} - 2a_1^T\vec{x} + \|\vec{a}_1\|^2 = d_1^2$

$\|\vec{x} - \vec{a}_2\|^2 = d_2^2 \Rightarrow \vec{x}^T x - 2a_2^T\vec{x} + \|a_2\|^2 = d_2^2$

} subtract to get linear eqn.

$\|\vec{x} - \vec{a}_3\|^2 = d_3^2$

Least Squares

- minimize error using overdetermined set of eqns

n points, $(a_1, b_1) \dots (a_n, b_n)$ in $\mathbb{R}^2$

satisfy $b_i = x_1 a_i + x_2$

w/ error: $b_i + e_i = x_1 a_i + x_2$

- vertical distance between best fit line and points
- minimize this squared vertical distance.

$\vec{e} = A\vec{x} - \vec{b} = \begin{bmatrix} a_1 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$\min_{\vec{x}} \|e\|^2 = \min_{\vec{x}} (\|A\vec{x} - \vec{b}\|^2)$

$\vec{x} = (A^T A)^{-1} A^T \vec{b}$

Minimum Norm when eqn is underdetermined

$\vec{x} = A^T (A A^T)^{-1} b$

Beacon receiver

$\vec{r} = a_0 \overrightarrow{S_0^{(N_0)}} + \dots + a_n \overrightarrow{S_n^{(N_n)}}$

- a_i is a message that scales each song
- subscript is index of song
- superscript is ~~[illegible]~~ index of shift

---

cross correlation of received signal and S_i will have a peak at N_i (delay)

Problem: users transmitting w/ different messages (amplitudes) may be overshadowed by users w/ bigger messages.

OMP

Given: - received signal $\vec{r}$, $\mathbb{R}^n$

- set of m songs, each of length n: $S = \{\vec{S}_0, \dots, \vec{S}_{m-1}\}$
- sparsity level k
- threshold th

Initialize: $\vec{y} = \vec{r}$, $j = 1$, $k = 10$, $A = []$,

$F = \{\emptyset\}$

while $j \leq k$ and $\|r\| \geq th$:

1) cross correlate $\vec{y}$ w/ shifted versions of all songs. Find song index i, and the shifted version of song, $\vec{S_i}^{N_i}$ w/ which received signal has highest correlation value

2) Add i to set of song indices, F.

3) $A = [A \mid \vec{S_i}^{(N_i)}]$ column concat

4) $\vec{x} = (A^T A)^{-1} A^T \vec{r}$

5) $\vec{y} = \vec{r} - A\vec{x}$

6) $j = j + 1$

each col of A represents effect of song on $\vec{x}$

If columns of A_j are mutually orthogonal, projection of $\vec{r}$ on span(A_j) is sum of projection of $\vec{r}$ on each of columns of A_j.

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orthonormal - mutually orthogonal, of unit length

Gram-Schmidt

- takes set of LI vectors $\{\vec{S}_1, \dots, \vec{S}_n\}$ and generates set of orthonormal vectors $\{\vec{q}_1, \dots, \vec{q}_n\}$

1) $\vec{q}_1 = \frac{\vec{S}_1}{\|\vec{S}_1\|}$

2) $\vec{e}_2 = \vec{S}_2 - (\vec{S}_2^T \vec{q}_1) \vec{q}_1$

projection of S_2 on q_1

$\vec{q}_2 = \frac{\vec{e}_2}{\|\vec{e}_2\|}$

3) $\vec{e}_3 = \vec{S}_3 - (S_3^T q_2) \vec{q}_2 - (S_3^T \vec{q}_1) \vec{q}_1$

$\vec{q}_3 = \frac{\vec{e}_3}{\|\vec{e}_3\|}$

---

received signal $= \vec{x}$

Gram-Schmidt on OMP; initialize $Q = []$; $\vec{b}_0 = 0$

1) cross correlate $\vec{r}$ ...

After step 2: run Gram Schmidt on Q, $V_j = S_i^{N_i}$

3) $e_j = \vec{V}_j - proj_Q \vec{V}_j$

$\vec{q}_j = \frac{\vec{e}_j}{\|\vec{e}_j\|}$

$Q = [Q \mid \vec{q}_j]$

4) $\vec{b}_j = \vec{b}_{j-1} + (x^T q_j) \vec{q}_j$

5) $\vec{r} = \vec{x} - \vec{b}_j$

---

row space and nullspace of matrix are orthogonal

$$A\vec{v} = \begin{bmatrix} \langle \vec{a}_1, \vec{v} \rangle \\ \langle \vec{a}_2, \vec{v} \rangle \\ \vdots \\ \langle \vec{a}_m, \vec{v} \rangle \end{bmatrix}$$

Projection onto subspace = closest vector to that subspace (least squares)

For matrix w/ orthonormal cols, Q, $Q^T Q = I$

Diagonalizable $\Leftrightarrow$ n distinct eigval in $\mathbb{R}^n$ $\Leftrightarrow$ eigvecs form eigbasis

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$A\vec{x} = b + \vec{e}$

$RI = v + \vec{e}$

$\begin{bmatrix} \frac{1}{3} \\ \frac{3}{5} \end{bmatrix} [I] = \begin{bmatrix} \frac{r}{1} \\ \frac{r}{k} \end{bmatrix} + \vec{e}$