MATH 53

Parametize $y = f(x)$
$t \mapsto (t, f(t))$
Circle! $x = \cos t$, $y = \sin t$.

Area Δ
$\frac{1}{2} xy \sin\theta$.

Tangent line to parametric curve
$y = m(x - x_0) + y_0$; $y_0 = F(x_0)$
$m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
If tangent line horizontal ; $g'(t_0) = 0$, $f'(t_0) \neq 0$
vertical: $f'(t_0) = 0$, $g'(t_0) \neq 0$.

$\text{comp}_a b = \frac{a \cdot b}{|a|}$ > magnitude
$\text{proj}_a b = \frac{a \cdot b}{|a|} \cdot \frac{a}{|a|}$ ← direction.

Vector Equation of Line L
$\langle x, y, z \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$
$= (x_0, y_0, z_0) + t\langle a, b, c \rangle$
$x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$ } need point and vector $\langle a, b, c \rangle$ to define line

Symmetric Equation
$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$
Skew - not parallel and do not intersect.
Normal plane = plane defined by tangent vector — Normal

Area under curve
$y = F(x)$: $A = \int_a^b F(x)\,dx$
for parametric: $x = f(t)$, $y = g(t)$ $\alpha \le t \le \beta$
$A = \int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t)\,dt$

Arc Length
$y = F(x)$: $L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\,dx$.
Parametric: $L = \int_\alpha^\beta \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$.
C is traversed once as t increases from α to β
Surface Area: $S = \int_\alpha^\beta 2\pi y \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$.

Polar (r, θ)
$x = r\cos\theta$, $y = r\sin\theta$
$x^2 + y^2 = r^2$ $\tan\theta = \frac{y}{x}$
e.g. $r = 2\cos\theta$
$r = 2\frac{x}{r}$
$2x = r^2$
$2x = x^2 + y^2$

Properties of Polar Curves.
θ replaced w/ $-\theta$
⇒ symmetric about horizontal
θ replaced w/ $\pi - \theta$
⇒ symmetric about $\theta = \frac{\pi}{2}$
r replaced w/ $-r$ OR θ.. $\theta + \pi$
⇒ symmetric about diagonal.

Hyperboloid
1 sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
2: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Tangents to Polar!
$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

Area of Sector of Circle: $A = \int_a^b \frac{1}{2}(f(\theta))^2\,d\theta$
$= \int_a^b \frac{1}{2} r^2\,d\theta$.
Area of region inside $r = f(\theta)$, outside $r = g(\theta)$
- intersection: $f(\theta) = g(\theta) \to$ get θ_1, θ_2
$A = \frac{1}{2}\left[\int_{\theta_1}^{\theta_2} f(\theta) - \int_{\theta_1}^{\theta_2} g(\theta)\right] d\theta$
Arc Length: $L = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2}\,d\theta$.

Cross Product
$\vec{a} \times \vec{b}$ = vector ⊥ to both a and b.
$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ ad - bc
$|a \times b|$ = area parallelogram determined by a, b
If $a \times b = 0$, $a \parallel b$.
$|a \times b| = |a||b|\sin\theta$ — for Δ, $\frac{1}{2}$ area.
θ = angle between a and b.

Properties:
$a \times b = -b \times a$
$ca \times b = c(a \times b) = a \times cb$
$a \times (b + c) = a \times b + a \times c$
$a \cdot (b \times c) = (a \times b) \cdot c$
$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$

Plane specified by normal vector and point.
Point $P_0(x_0, y_0, z_0)$
$N \langle a, b, c \rangle$
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
↳ $ax + by + cz + d = 0$.
Two planes are parallel if their normal vectors are parallel.
θ between 2 planes:
normal vectors: n_1, n_2
$\frac{n_1 \cdot n_2}{|n_1||n_2|} = \cos\theta$.
Intersection between 2 planes:
find point on line L - find where line intersects $z = 0$.
$n_1 \times n_2$ = ⊥ to both normal vectors
= ∥ to L.
Distance from point (x_1, y_1, z_1) to plane $ax + by + cz + d = 0$.
$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$ } proj point onto normal.

3D
Distance: $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
Sphere w/ radius r centered at (h, k, l)
$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$.

Vector given 2 points: $a = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$
Vector length: $\|a\| = \sqrt{a_1^2 + a_2^2}$; $a = \langle a_1, a_2 \rangle$
Dot Product: $\vec{a} \cdot \vec{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle$
$= a_1 b_1 + a_2 b_2 + a_3 b_3$.
$= |a||b|\cos\theta$.
If $\vec{a} \cdot \vec{b} = 0$, $\vec{a} \perp \vec{b}$
Unit Vector: $\frac{\vec{u}}{\|\vec{u}\|}$
↳ same direction as $\vec{u}$

Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Cone: $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
Elliptic Paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
Hyperbolic paraboloid: $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Direction Angle of $\vec{a}$: $\angle \alpha, \beta, \gamma$ that $\vec{a}$ makes with positive x, y, z axes.
$\cos\alpha = \frac{a \cdot i}{\|a\|\|i\|} = \frac{a_1}{\|a\|}$
$\cos\beta = \frac{a \cdot j}{\|a\|\|j\|} = \frac{a_2}{\|a\|}$
$\cos\gamma = \frac{a \cdot k}{\|a\|\|k\|} = \frac{a_3}{\|a\|}$
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
$a = \|a\|\langle \cos\alpha, \cos\beta, \cos\gamma \rangle$
Area parallelepiped
$\vec{u} \cdot (\vec{v} \times \vec{w})$

$r(t) = \langle f(t), g(t), h(t) \rangle$
$r'(t) = \langle f'(t), g'(t), h'(t) \rangle$
Props: u, v are vector fns; f is a real valued fn; c is a scalar.
$\frac{d}{dt}(u(t) + v(t)) = u'(t) + v'(t)$
$\frac{d}{dt}(cu(t)) = c\,u'(t)$
$\frac{d}{dt}(f(t)u(t)) = f'(t)u(t) + f(t)u'(t)$
$\frac{d}{dt}(u(t) \cdot v(t)) = u'(t) \cdot v(t) + u(t) \cdot v'(t)$
$\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$
$\frac{d}{dt}[u(f(t))] = u'(f(t))\,f'(t)$

cannot know if limit exists, unless Δ, ε
BUT
can prove DNE if approach (x_0, y_0) for $\lim_{(x,y) \to (x_0, y_0)}$ on different ways and lim is different.

Partial Differentiation
f_x, f_y ; $f_{xx} = \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}$
$f_{xy} = \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$
$f_{yx} = \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$
$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$

Implicit Differentiation.
$\frac{\partial}{\partial x}(z^3) = \frac{\partial(z^3)}{\partial z} \cdot \frac{\partial z}{\partial x}$
$= 3z^2 \cdot \frac{\partial z}{\partial x}$

Tangent Planes
Tangent plane to surface $z = f(x, y)$ at point $P(x_0, y_0, z_0)$
$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Linearization of f at (a, b):
Approximation
$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$
If partial derivatives f_x, f_y exist near (a, b) and are continuous at (a, b), f is differentiable at (a, b).

Total differential, dz.
$dz = f_x(x, y)\,dx + f_y(x, y)\,dy$
$= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$; $dx = \Delta x$, $dy = \Delta y$

Chain Rule.
$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$
$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$

For partial:
$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$

Implicit
$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y}$
$\frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ $\frac{\partial z}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$

$t \to x, y \to z$
$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$
$s, t \to x, y \to z$
$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t}$

In general:
u is a function with n vars, $x_1, x_2, \dots x_n$
Each x_j has m vars, $t_1, t_2, \dots, t_m$
$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$

Wave: $\frac{\partial^2 u}{\partial t^2} = a^2 \cdot \frac{\partial^2 u}{\partial x^2}$ u: displacement of vibrating string
t: time
x: point along string
Laplace: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ u: equilibrium concentration at (x, y)
Heat: $\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2}$ u: temp at time t at point x along a 1D rod, with heat conductivity σ.

Directional Derivative
Unit vector $u = \langle a, b \rangle$
$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b$
Gradient: $\nabla f = \langle f_x, f_y \rangle$
∴ $D_u f(x, y) = \nabla f(x, y) \cdot u$
$= |\nabla f||u|\cos\theta$ ∇f along
∇f gives dir of fastest increase of f
$\nabla f(x_0, y_0, z_0)$
To maximize directional derivative - u has same direction as ∇f.
Tangent plane to surface $F(x, y, z)$ at Point $P = (x_0, y_0, z_0)$:
$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$. $\nabla f \cdot u = 0$.
∇F is ⊥ to level surface S of F
Normal line to surface S at P
$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$

If eq = symmetric, can use polar.
If left with expression containing θ, lim DNE.
e.g. approach from $x = 0$, $y = 0$, $y = mx$.
Continuous: $\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$

MATH 53

$\sin^2 t = \frac{1}{2}(1-\cos 2t)$ $\langle 1+\cos t, 2+\sin t\rangle$ $=(x-1)^2+(y-2)^2=1$

local max/min:

$f_x(a,b)=0,\ f_y(a,b)=0$

$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$D>0$: $f_{xx}(a,b)>0 \Rightarrow f(a,b)$ is local min

$D>0$: $f_{xx}(a,b)<0 \Rightarrow$ local max

$D<0$: neither. Saddle point.

Minimize distance - find abs min of distance equation.

$d^2x,\ d^2y = 0.$

Lagrange Multiples

To find max, min of $f(x,y,z)$: subject to $g(x,y,z)=k$

Find all x,y,z,λ s.t.

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$,

$g(x,y,z)=k$

Evaluate $f\ \forall (x,y,z)$

largest is max

smallest is min

$f_x = \lambda g_x$ | $f_x = \lambda g_x + \mu h_x$

$f_y = \lambda g_y$ | $f_y = \lambda g_y + \mu h_y$

$f_z = \lambda g_z$ | $f_z = \lambda g_z + \mu h_z$

$g(x,y,z)=k$

Cylindrical Coordinates (r,θ,z)

$x = r\cos\theta,\ y = r\sin\theta,\ z=z$

$r^2 = x^2+y^2 \quad \tan\theta = \frac{y}{x}$

$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta)\, r\,dz\,dr\,d\theta$

Spherical (ρ,θ,ϕ)

ρ: distance from origin to P

θ: angle from positive x-axis

ϕ: angle from positive z-axis

-use on problems w/ symmetry around a

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$

$\rho^2 = x^2+y^2+z^2$

$\iiint_E f(x,y,z)\,dV = \int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

Jacobian

$x = g(u,v),\ y = h(u,v)$

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$

$\Delta A = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \Delta u\,\Delta v$

$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

3D

$x = g(u,v,w);\ y = h(u,v,w);\ z = k(u,v,w)$

$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$

$\Rightarrow \iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$

Directional Derivative: $\nabla f\cdot\frac{v}{\|v\|}$

$u,v \to x,y,z$

$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$

$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial v}$

$f(x,y)=c$, c is max/min of $f(x,y)$ subject to constraint $g(x,y)=k$ is tangent to

Constant curve $g(x,y)=k$ at pt of intersection

tangent vector to curve $= \nabla f\times\nabla g$

Vector Field - continous iff P, Q, R continous

$F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$

Gradient Field

$\nabla f(x,y) = f_x(x,y)i + f_y(x,y)j$

$\nabla f(x,y,z) = f_x(x,y,z)i + f_y(x,y,z)j + f_z(x,y,z)k$

Line Integrals.

$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

☆ $r(t) = (1-t)r_0 + t r_1$

$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$

$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$ } wrt arc length

$\Rightarrow \int_C P(x,y)\,dx + Q(x,y)\,dy$

Space: $\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$

Length of Curve: $L = \int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

Line Integral of Vector Fields

$W = \int_C F(x,y,z)\cdot T(x,y,z)\,ds = \int_C F\cdot T\,ds$

force field F; T = unit tangent vector at (x,y,z)

F: continous vector field on smooth curve C given by vector fn $r(t)$, $a\le t\le b$

Line Integral of F along C:

$\int_C F\cdot dr = \int_a^b F(r(t))\cdot r'(t)\,dt = \int_C F\cdot T\,ds$

$\hookrightarrow F(x(t), y(t), z(t))$

$F = Pi + Qj + Rk$

FT

C: smooth curve given by vector fn $r(t)$

$\int_C \nabla f\cdot dr = f(r(b)) - f(r(a))$

$\to$ line integral of ∇f is net change in f

Line integral of a conservative vector field depends only on initial and terminal point of a curve

$\int_C F\cdot dr$ is independent of path if $\int_{C_1}F\cdot dr = \int_{C_2}F\cdot dr$ for any two paths C_1 and C_2 that have same initial = terminal pt. terminal points.

$\int_C F\cdot dr$ is IOP iff $\int_C F\cdot dr = 0\ \forall$ closed path C in D.

If $\int_C F\cdot dr$ is IOP, F is a conservative vector field on D, i.e. $\exists$ function f st $\nabla f = F$.

If $F(x,y) = P(x,y)i + Q(x,y)j$ is conservative,

$\frac{\partial P}{\partial y} = P_y = Q_x = \frac{\partial Q}{\partial x}$

$\hookrightarrow$ If $F = Pi + Qj$ is a vector field on open s.c. region D, + ↑, F is conservative.

Green's

$\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$

Area of D: $A = \oint_C x\,dy = -\oint_C y\,dx$

$= \frac{1}{2}\oint_C x\,dy - y\,dx$

$\text{curl}\,F = \nabla\times F\ (Pi+Qj+Rk)$

$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k$

$\text{Curl}(\nabla f) = 0$ if f is fn of 3 vars w/ continous 2nd order PD,

F is conservative $\Rightarrow$ curl $F = 0$.

curl $F = 0$, F_x, F_y, F_z are continous $\Rightarrow$ conservative F

$\text{div}\,F = \nabla\cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

If $F = Pi + Qj + Rk$ on $\mathbb{R}^3$, P, Q, R continous,

div curl $F = 0$.

$\text{div}(\nabla f) = \nabla\cdot\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$\oint_C F\cdot dr = \iint_D (\text{curl}\,F)\cdot k\,dA$

$\oint_C F\cdot n\,ds = \iint_D \text{div}\,F(x,y)\,dA$

Smooth Curve - tangent line at all points

$\Rightarrow \nabla f\times\nabla g \ne 0$

$|a\times b|$ = area of parallelogram

Law of Gravitation

$|F| = \frac{mMG}{r^2}$ $X = \langle x,y,z\rangle$

$F(x) = \frac{-mMG}{|x|^3}X$

Conservative: $\exists$ f st $F = \nabla f$

e.g. $F(x,y) = (3+2xy)i + (x^2-3y^2)j$

$\nabla f = \langle 3+2xy, x^2-3y^2\rangle$

$f_x = 3+2xy,\ f(x,y) = 3x + x^2y + g(y)$

$f_y = x^2 + g'(y)$

$\therefore x^2 - 3y^2$

$g'(y) = -3y^2 \Rightarrow g(y) = -y^3 + K$

$\therefore f(x,y) = 3x + x^2y - y^3 + K$.

Parametric Surfaces

$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$

Revolution: $x = x,\ y = f(x)\cos\theta,\ z = f(x)\sin\theta$

Tangent Plane to Vector

$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$

$r_v = \frac{\partial x}{\partial v}(u_0,v_0)i + \frac{\partial y}{\partial v}(u_0,v_0)j + \frac{\partial z}{\partial v}(u_0,v_0)k$

$r_u = \frac{\partial x}{\partial u}(u_0,v_0)i + \frac{\partial y}{\partial u}(u_0,v_0)j + \frac{\partial z}{\partial u}(u_0,v_0)k$

Smooth surface $\Rightarrow r_u\times r_v$ is normal vector to tangent plane

Surface Area: $A(S) = \iint_D |r_u\times r_v|\,dA$

S has equation $z = f(x,y) \to x = x,\ y = y,\ z = f(x,y)$

$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$

Surface Integral

$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$

$\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))\,|r_u\times r_v|\,dA$

w/ surface $z = g(x,y)$:

$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA$

Vector Field

$\iint_S \rho v\cdot n\,dS = \iint_S \rho(x,y,z)\,v(x,y,z)\cdot n(x,y,z)\,dS$

on surfaces w/ normal vector n

$\iint_S F\cdot dS = \iint_S F\cdot n\,dS$

$\hookrightarrow$ Flux of F across S

Reduce vector field to 2 variables.

$S = r(u,v) \Rightarrow n = \frac{r_u\times r_v}{|r_u\times r_v|}$

$\iint_S F\cdot dS = \iint_D F\cdot(r_u\times r_v)\,dA$

OR

If S given by $z = g(x,y)$

$F\cdot(r_x\times r_y) = (Pi + Qj + Rk)\cdot\left(-\frac{\partial g}{\partial x}i - \frac{\partial g}{\partial y}j + k\right)$

$\Rightarrow \iint_S F\cdot dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)dA$

$\iint_S F\cdot dS = \iint_{S_1} F\cdot dS + \iint_{S_2} F\cdot dS$

Stokes

$\int_C F\cdot dr = \iint_S \text{curl}\,F\cdot dS = \iint_S \nabla\times F\cdot dS$

$= \int F(r(t))\cdot r'(t)\,dt$

e.g. $\int_C F\cdot dr,\ 0\le t\le 2\pi$

$= \int_0^{2\pi} F(r(t))\cdot r'(t)\,dt$

Divergence

closed surface oriented outwards

$\iint_S F\cdot dS = \iiint_E \text{div}\,F\,dV$

$\iiint_E \text{div}\,F\,dV = \iint_S F\cdot dS = \iint_S F\cdot n\,dS$

closed, bounded region $\to$ max/min on D

$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$

Normal vector: $r_u\times r_v$

gradient is tangent plane to curve

$\int_C \nabla f\cdot dr = f(r(b)) - f(r(a))$

$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \int_C P\,dx + Q\,dy$

$\iint_S \text{curl}\,F\cdot dS = \int_C F\cdot dr$

$\iiint_E \text{div}\,F\,dV = \iint_S F\cdot dS$