treated equally but transmission media—wires or microwaves—introduce distortion in the amplitude and phase of the sinusoids that comprise the desired signal and this distortion must be removed. Filters to correct the distortion are called **equalizers**. Finally, the dynamic response of control systems requires modification in order for the complete system to have satisfactory dynamic response. We call the devices that make these changes **compensators**.

Whatever the name—filter, equalizer, or compensator—many fields have use for linear dynamic systems having a transfer function with specified characteristics of amplitude and phase. Increasingly the power and flexibility of digital processors makes it attractive to perform these functions by digital means. The design of continuous electronic filters is a well-established subject that includes not only very sophisticated techniques but also well-tested computer programs to carry out the designs [Van Valkenburg (1982)]. Consequently, an important approach to digital filter design is to start with a good analog design and construct a filter having a discrete frequency response that approximates that of the satisfactory design. For digital control systems we have much the same motivation: Continuous-control designs are well established and one can take advantage of a good continuous design by finding a discrete equivalent to the continuous compensator. This method of design is called **emulation**. Although much of our presentation in this book is oriented toward direct digital design and away from emulation of continuous designs with digital equivalents, it is important to understand the techniques of discrete equivalents both for purposes of comparison and because it is widely used by practicing engineers.

emulation

Chapter Overview

The specific problem of this chapter is to find a discrete transfer function that will have approximately the same characteristics over the frequency range of importance as a given transfer function, $H(s)$. Three approaches to this task are presented. The first method is based on *numerical integration* of the differential equations that describe the given design. While there are many techniques for numerical integration, only simple formulas based on rectangular and trapezoid rules are presented. The second approach is based on comparisons of the s and z domains. Note that the natural response of a continuous filter with a pole at some point $s = s_o$ will, when sampled with period T, represent the response of a discrete filter with a pole at $z = e^{s_o T}$. This formula can be used to map the poles and zeros of the given design into poles and zeros of an approximating discrete filter. This is called *pole and zero mapping*. The third and final approach is based on taking the samples of the input signal, extrapolating between samples to form an approximation to the signal, and passing this approximation through the given filter transfer function. This technique is called *hold equivalence*. The methods are compared with respect to the quality of the approximation in the frequency domain as well as the ease of computation of the designs.

6.1 Design of Discrete Equivalents via Numerical Integration

The topic of numerical integration of differential equations is quite complex, and only the most elementary techniques are presented here. For example, we only consider formulas of low complexity and fixed step-size. The fundamental concept is to represent the given filter transfer function $H(s)$ as a differential equation and to derive a difference equation whose solution is an approximation of the differential equation. For example, the system

$$\frac{U(s)}{E(s)} = H(s) = \frac{a}{s+a} \tag{6.1}$$

is equivalent to the differential equation

$$\dot{u} + au = ae. \tag{6.2}$$

Now, if we write Eq. (6.2) in integral form, we have a development much like that described in Chapter 4, except that the integral is more complex here

$$\begin{aligned} u(t) &= \int_0^t [-au(\tau) + ae(\tau)]d\tau, \\ u(kT) &= \int_0^{kT-T} [-au + ae]d\tau + \int_{kT-T}^{kT} [-au + ae]d\tau \\ &= u(kT-T) + \left\{ \begin{array}{l} \text{area of } -au + ae \\ \text{over } kT - T \le \tau < kT. \end{array} \right. \end{aligned} \tag{6.3}$$

Many rules have been developed based on how the incremental area term is approximated. Three possibilities are sketched in Fig. 6.2. The first approximation leads to the **forward rectangular rule**[2] wherein we approximate the area by the rectangle looking forward from $kT - T$ and take the amplitude of the rectangle to be the value of the integrand at $kT - T$. The width of the rectangle is T. The result is an equation in the first approximation, u_1

$$\begin{aligned} u_1(kT) &= u_1(kT-T) + T[-au_1(kT-T) + ae(kT-T)] \\ &= (1 - aT)u_1(kT-T) + aTe(kT-T). \end{aligned} \tag{6.4}$$

The transfer function corresponding to the forward rectangular rule in this case is

$$\begin{aligned} H_F(z) &= \frac{aTz^{-1}}{1-(1-aT)z^{-1}} \\ &= \frac{a}{(z-1)/T + a} \qquad \text{(forward rectangular rule).} \end{aligned} \tag{6.5}$$

2 Also known as *Euler's rule*.

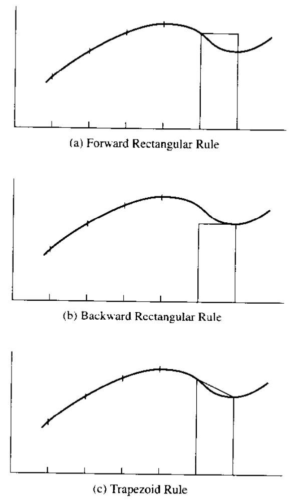


Figure 6.2
Sketches of three ways the area under the curve from kT to $kT + T$ can be approximated: (a) forward rectangular rule, (b) backward rectangular rule, (c) trapezoid rule

A second rule follows from taking the amplitude of the approximating rectangle to be the value looking *backward* from kT toward $kT - T$, namely, $-au(kT) + ae(kT)$. The equation for u_2, the second approximation,[3] is

$$\begin{aligned} u_2(kT) &= u_2(kT - T) + T[-au_2(kT) + ae(kT)] \\ &= \frac{u_2(kT - T)}{1 + aT} + \frac{aT}{1 + aT}e(kT). \end{aligned} \tag{6.6}$$

3 It is worth noting that in order to solve for Eq. (6.6) we had to eliminate $u(kT)$ from the right-hand side where it entered from the integrand. Had Eq. (6.2) been nonlinear, the result would have been an implicit equation requiring an iterative solution. This topic is the subject of predictor-corrector rules, which are beyond our scope of interest. A discussion is found in most books on numerical analysis. See Golub and Van Loan (1983).

Again we take the z-transform and compute the transfer function of the **backward rectangular rule**

$$\begin{aligned} H_B(z) &= \frac{aT}{1 + aT}\frac{1}{1 - z^{-1}/(1 + aT)} = \frac{aTz}{z(1 + aT) - 1} \\ &= \frac{a}{(z - 1)/Tz + a} \qquad \text{(backward rectangular rule).} \end{aligned} \tag{6.7}$$

Our final version of integration rules is the **trapezoid rule** found by taking the area approximated in Eq. (6.3) to be that of the trapezoid formed by the average of the previously selected rectangles. The approximating difference equation is

$$\begin{aligned} u_3(kT) &= u_3(kT - T) + \frac{T}{2}[-au_3(kT - T) \\ &\quad + ae(kT - T) - au_3(kT) + ae(kT)] \\ &= \frac{1 - (aT/2)}{1 + (aT/2)}u_3(kT - T) + \frac{aT/2}{1 + (aT/2)}[e_3(kT - T) + e_3(kT)]. \end{aligned} \tag{6.8}$$

The corresponding transfer function from the trapezoid rule is

$$\begin{aligned} H_T(z) &= \frac{aT(z + 1)}{(2 + aT)z + aT - 2} \\ &= \frac{a}{(2/T)[(z - 1)/(z + 1)] + a} \qquad \text{(trapezoid rule).} \end{aligned} \tag{6.9}$$

Suppose we tabulate our results obtained thus far.

$H(s)$	**Method**	**Transfer function**
$\frac{a}{s+a}$	Forward rule	$H_F = \frac{a}{(z-1)/T + a}$
$\frac{a}{s+a}$	Backward rule	$H_B = \frac{a}{(z-1)/Tz + a}$
$\frac{a}{s+a}$	Trapezoid rule	$H = \frac{a}{(2/T)\{(z-1)/(z+1)\} + a}$

(6.10)

From direct comparison of $H(s)$ with the three approximations in this tabulation, we can see that the effect of each of our methods is to present a discrete transfer function that can be obtained from the given Laplace transfer function

$H(s)$ by substitution of an approximation for the frequency variable as shown below

Method	**Approximation**
Forward rule	$s \leftarrow \dfrac{z-1}{T}$
Backward Rule	$s \leftarrow \dfrac{z-1}{Tz}$
Trapezoid Rule	$s \leftarrow \dfrac{2}{T}\dfrac{z-1}{z+1}$

(6.11)

The trapezoid-rule substitution is also known, especially in digital and sampled-data control circles, as **Tustin's method** [Tustin (1947)] after the British engineer whose work on nonlinear circuits stimulated a great deal of interest in this approach. The transformation is also called the **bilinear transformation** from consideration of its mathematical form. The design method can be summarized by stating the rule: Given a continuous transfer function (filter), $H(s)$, a discrete equivalent can be found by the substitution

$$H_T(z) = H(s)|_{s=\frac{2}{T}\frac{z-1}{z+1}}. \tag{6.12}$$

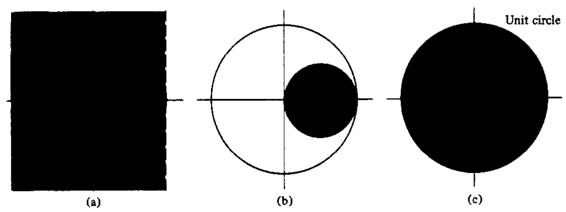
Each of the approximations given in Eq. (6.11) can be viewed as a map from the s-plane to the z-plane. A further understanding of the maps can be obtained by considering them graphically. For example, because the $(s = j\omega)$-axis is the boundary between poles of stable systems and poles of unstable systems, it would be interesting to know how the $j\omega$-axis is mapped by the three rules and where the left (stable) half of the s-plane appears in the z-plane. For this purpose we must solve the relations in Eq. (6.11) for z in terms of s. We find

$$\begin{aligned}
&\text{i)} \quad z = 1 + Ts, && \text{(forward rectangular rule)},\\
&\text{ii)} \quad z = \frac{1}{1-Ts}, && \text{(backward rectangular rule)},\\
&\text{iii)} \quad z = \frac{1+Ts/2}{1-Ts/2} && \text{(bilinear rule)}.
\end{aligned} \tag{6.13}$$

If we let $s = j\omega$ in these equations, we obtain the boundaries of the regions in the z-plane which originate from the stable portion of the s-plane. The shaded areas sketched in the z-plane in Fig. 6.3 are these stable regions for each case. To show that rule (ii) results in a circle, $\frac{1}{2}$ is added to and subtracted from the right-hand side to yield

$$\begin{aligned}
z &= \frac{1}{2} + \left\{\frac{1}{1-Ts} - \frac{1}{2}\right\}\\
&= \frac{1}{2} - \frac{1}{2}\frac{1+Ts}{1-Ts}.
\end{aligned} \tag{6.14}$$

Figure 6.3
Maps of the left-half of the s-plane by the integration rules of Eq. (6.10) into the z-plane. Stable s-plane poles map into the shaded regions in the z-plane. The unit circle is shown for reference. (a) Forward rectangular rule. (b) Backward rectangular rule. (c) Trapezoid or bilinear rule



Now it is easy to see that with $s = j\omega$, the magnitude of $z - \frac{1}{2}$ is constant

$$|z - \frac{1}{2}| = \frac{1}{2},$$

and the curve is thus a circle as drawn in Fig. 6.3(b). Because the unit circle is the stability boundary in the z-plane, it is apparent from Fig. 6.3 that the forward rectangular rule could cause a stable continuous filter to be mapped into an unstable digital filter.

It is especially interesting to notice that the bilinear rule maps the stable region of the s-plane exactly into the stable region of the z-plane although the entire $j\omega$-axis of the s-plane is compressed into the 2π-length of the unit circle! Obviously a great deal of distortion takes place in the mapping in spite of the congruence of the stability regions. As our final rule deriving from numerical integration ideas, we discuss a formula that extends Tustin's rule one step in an attempt to correct for the inevitable distortion of real frequencies mapped by the rule. We begin with our elementary transfer function Eq. (6.1) and consider the bilinear rule approximation

$$H_T(z) = \frac{a}{(2/T)[(z-1)/(z+1)] + a}.$$

The original $H(s)$ had a pole at $s = -a$, and for real frequencies, $s = j\omega$, the magnitude of $H(j\omega)$ is given by

$$\begin{aligned}
|H(j\omega)|^2 &= \frac{a^2}{\omega^2 + a^2}\\
&= \frac{1}{\omega^2/a^2 + 1}.
\end{aligned}$$