is inside the unit circle, the corresponding pulse response decays with time geometrically and is stable. Thus, if all poles are inside the unit circle, the system with rational transfer function is stable; if at least one pole is on or outside the unit circle, the corresponding system is not BIBO stable. With modern computer programs available, finding the poles of a particular transfer function is no big deal. Sometimes, however, we wish to test for stability of an entire class of systems; or, as in an adaptive control system, the potential poles are constantly changing and we wish to have a quick test for stability in terms of the literal polynomial coefficients. In the continuous case, such a test was provided by Routh; in the discrete case, the most convenient such test was worked out by Jury and Blanchard(1961).[9]

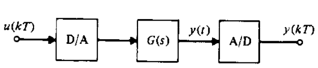
4.3 Discrete Models of Sampled-Data Systems

The systems and signals we have studied thus far have been defined in discrete time only. Most of the dynamic systems to be controlled, however, are continuous systems and, if linear, are described by continuous transfer functions in the Laplace variable s. The interface between the continuous and discrete domains are the A/D and the D/A converters as shown in Fig. 1.1. In this section we develop the analysis needed to compute the discrete transfer function between the samples that come from the digital computer to the D/A converter and the samples that are picked up by the A/D converter.[10] The situation is drawn in Fig. 4.12.

4.3.1 Using the z-Transform

We wish to find the discrete transfer function from the input samples $u(kT)$ (which probably come from a computer of some kind) to the output samples $y(kT)$ picked up by the A/D converter. Although it is possibly confusing at first, we follow convention and call the discrete transfer function $G(z)$ when the continuous transfer function is $G(s)$. Although $G(z)$ and $G(s)$ are entirely different functions, they do describe the *same* plant, and the use of s for the continuous transform and z for the discrete transform is always maintained. To

Figure 4.12
The prototype sampled-data system



9 See Franklin, Powell, and Workman, 2nd edition, 1990, for a discussion of the Jury test.

10 In Chapter 5, a comprehensive frequency analysis of sampled data systems is presented. Here we undertake only the special problem of finding the sample-to-sample discrete transfer function of a continuous system between a D/A and an A/D.

find $G(z)$ we need only observe that the $y(kT)$ are samples of the plant output when the input is from the D/A converter. As for the D/A converter, we assume that this device, commonly called a zero-order hold or ZOH, accepts a sample $u(kT)$ at $t = kT$ and holds its output constant at this value until the next sample is sent at $t = kT + T$. The piecewise constant output of the D/A is the signal, $u(t)$, that is applied to the plant.

ZOH

Our problem is now really quite simple because we have just seen that the discrete transfer function is the z-transform of the samples of the output when the input samples are the unit pulse at $k = 0$. If $u(kT) = 1$ for $k = 0$ and $u(kT) = 0$ for $k \neq 0$, the output of the D/A converter is a pulse of width T seconds and height 1, as sketched in Fig. 4.13. Mathematically, this pulse is given by $1(t) - 1(t - T)$. Let us call the particular output in response to the pulse shown in Fig. 4.13 $y_1(t)$. This response is the difference between the step response [to $1(t)$] and the delayed step response [to $1(t - T)$]. The Laplace transform of the step response is $G(s)/s$. Thus in the transform domain the unit pulse response of the plant is

$$Y_1(s) = (1 - e^{-Ts})\frac{G(s)}{s}, \tag{4.40}$$

and the required transfer function is the z-transform of the samples of the inverse of $Y_1(s)$, which can be expressed as

$$\begin{aligned} G(z) &= \mathcal{Z}\{Y_1(kT)\} \\ &= \mathcal{Z}\{\mathcal{L}^{-1}\{Y_1(s)\}\} \triangleq \mathcal{Z}\{Y_1(s)\} \\ &= \mathcal{Z}\{(1 - e^{-Ts})\frac{G(s)}{s}\}. \end{aligned}$$

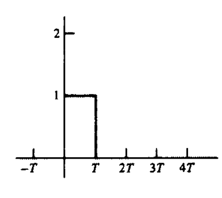
This is the sum of two parts. The first part is $\mathcal{Z}\{\frac{G(s)}{s}\}$, and the second is

$$\mathcal{Z}\{e^{-Ts}\frac{G(s)}{s}\} = z^{-1}\mathcal{Z}\{\frac{G(s)}{s}\}$$

because e^{-Ts} is exactly a delay of one period. Thus the transfer function is

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}. \tag{4.41}$$

Figure 4.13
D/A output for unit-pulse input



◆ **Example 4.5** *Discrete Transfer Function of 1st-Order System*

What is the discrete transfer function of

$$G(s) = a/(s+a)$$

preceded by a ZOH?

Solution. We will apply the formula (4.41)

$$\frac{G(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a},$$

and the corresponding time function is

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} = 1(t) - e^{-at}1(t).$$

The samples of this signal are $1(kT) - e^{-akT}1(kT)$, and the z-transform of these samples is

$$\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{z}{z-1} - \frac{z}{z-e^{-aT}}$$
$$= \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}.$$

We could have gone to the tables in Appendix B and found this result directly as Entry 12. Now we can compute the desired transform as

$$G(z) = \frac{z-1}{z}\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$
$$= \frac{1-e^{-aT}}{z-e^{-aT}}. \tag{4.42}$$

◆

◆ **Example 4.6** *Discrete Transfer Function of a $1/s^2$ Plant*

What is the discrete transfer function of

$$G(s) = \frac{1}{s^2}$$

preceded by a ZOH?

Solution. We have

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s^3}\right\}.$$

This time we refer to the tables in Appendix B and find that the z-transform associated with $1/s^3$ is

$$\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3}.$$

and therefore Eq. (4.41) shows that

$$G(z) = \frac{T^2(z+1)}{2(z-1)^2}. \tag{4.43}$$

◆

The MATLAB function, c2d.m computes Eq. (4.41) (the ZOH method is the default) as well as other discrete equivalents discussed in Chapter 6. It is able to accept the system in any of the forms.

◆ **Example 4.7** *Discrete Transfer Function of a $1/s^2$ Plant Using MATLAB*

Use MATLAB to find the discrete transfer function of

$$G(s) = \frac{1}{s^2}$$

preceded by a ZOH, assuming the sample period is $T = 1$ sec.

Solution. The MATLAB script

```
T = 1
numC = 1,   denC = [1   0   0]
sysC = tf(numC,denC)
sysD = c2d(sysC,T)
[numD,denD,T] = tfdata(sysD)
```

produces the result that

```
numD = [0   0.5   0.5]     and     denD = [1   -2   1]
```

which means that

$$G(z) = \frac{0z^2 + 0.5z + 0.5}{z^2 - 2z + 1} = 0.5\frac{z+1}{(z-1)^2}$$

which is the same as Eq. (4.43) with $T = 1$.

◆

4.3.2 *Continuous Time Delay

We now consider computing the discrete transfer function of a continuous system preceded by a ZOH with pure time delay. The responses of many chemical process-control plants exhibit pure time delay because there is a finite time of transport of fluids or materials between the process and the controls and/or the