Task 3.5

Let a discreet stationary random process $x(\xi, t)$. The outcomes of the process are the values $x_1 = -1, x_2 = 0$ and $x_3 = 1$

The probabilities of the occurence of those outcomes are

$$P(\{x(\xi, t+\tau) = x_i\} | \{x(\xi, t) = x_j\}) =$$

$$\Rightarrow \begin{cases} \frac{1}{3}(1+2e^{-|\tau|}) & \text{for } i=j \\ \frac{1}{3}(1-e^{-|\tau|}) & \text{for } i \neq j \end{cases} \quad i,j = 1,2,3$$

a) Calculate the probabilities

$P(\{x(\xi, t) = x_i\})$ for $i = 1,2,3$

Solution

Let,

$\rightarrow P(\{x(e, t) = x_i\}) = P_i$ for all t,

$\rightarrow P(\{x(e, t+\tau) = x_i\}) | \{x(e, t) = x_j\}) =$

$$\Rightarrow \begin{cases} \frac{1}{3}(1+2e^{-|\tau|}) = a & \text{for } i=j \\ \frac{1}{3}(1-e^{-|\tau|}) = b & \text{for } i \neq j \end{cases} \quad i,j = 1,2,3$$

$\rightarrow$ we know, Total Probability theorem,

$$\boxed{P(A) = \sum_n P(A \cap B_n) = \sum_n P(A|B_n) P(B_n)}$$

For $i=1$,

$$P(\{x(e,t+\tau)=x_1\}) =$$

$$\sum_{j=1}^{3} P(\{x(e,t+\tau)=x_1\} \mid \{x(e,t)=x_j\}) * P(\{x(e,t)=x_j\})$$

$$\Rightarrow P(\{x(e,t+\tau)=x_1\} \mid \{x(e,t)=x_1\}) P(\{x(e,t)=x_1\}) +$$
$$P(\{x(e,t+\tau)=x_1\} \mid \{x(e,t)=x_2\}) P(\{x(e,t)=x_2\}) +$$
$$P(\{x(e,t+\tau)=x_1\} \mid \{x(e,t)=x_3\}) P(\{x(e,t)=x_3\})$$

$$\Rightarrow P_1 = aP_1 + bP_2 + bP_3 \quad \text{———} \quad (i)$$

Similarly for $i=2$ and $i=3$

$$P_2 = bP_1 + aP_2 + bP_3 \quad \text{———} \quad (ii)$$
$$P_3 = bP_1 + bP_2 + aP_3 \quad \text{———} \quad (iii)$$

$\Rightarrow$ (i) $-$ (ii), $\quad P_1 - P_2 = a(P_1 - P_2) + b(P_2 - P_1)$

$$\Rightarrow P_1 - aP_1 + bP_1 = P_2 - aP_2 + bP_2$$
$$\Rightarrow P_1(1-a+b) = P_2(1-a+b)$$
$$\Rightarrow P_1 = P_2 \quad \text{———} \quad (iv)$$

(ii) $-$ (iii),

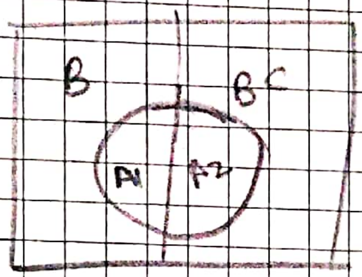
$$\Rightarrow P_2 - P_3 = a(P_2 - P_3) + b(P_3 - P_2)$$
$$\Rightarrow P_2 - aP_2 + bP_2 = P_3 - aP_3 + bP_3$$
$$\Rightarrow P_2(1-a+b) = P_3(1-a+b)$$
$$\Rightarrow P_2 = P_3 \quad \text{———} \quad (v)$$

From (iv) and (v),

$P_1 = P_2 = P_3$

Also, $P_1 + P_2 + P_3 = 1$,

$$\boxed{P_1 = P_2 = P_3 = \frac{1}{3}}$$



TOTAL PROBABILITY THEOREM

$P(A) = P(A \cap B) + P(A \cap B^c)$

By Bayes theorem

$P(A) = P(A|B)\,P(B) + P(A|B^c)\,P(B^c)$

For n sections of B

$P(A) = \sum_n P(A|B_n)\,P(B_n)$

b) Calculate the ACF $S_{xx}(\tau)$

$x_1 = -1$, $x_2 = 0$ and $x_3 = 1$

Pg 43 eq 4.12

$$\rightarrow S_{xx}(\tau) = E\{x(e,t+\tau)\,x(e,t)\}$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j\, P(\{x(e,t+\tau)=x_i\} \mid \{x(e,t)=x_j\}) \cdot P(\{x(e,t)=x_j\})$$

$$= x_1x_1\, a\left(\frac{1}{3}\right) + x_1x_2\, b\left(\frac{1}{3}\right) + x_1x_3\, b\left(\frac{1}{3}\right) + x_2x_1\, b\left(\frac{1}{3}\right) + x_2x_2\, a\left(\frac{1}{3}\right) + x_2x_3\, b\left(\frac{1}{3}\right) + x_3x_1\, b\left(\frac{1}{3}\right) + x_3x_2\, b\left(\frac{1}{3}\right) + x_3x_3\, a\left(\frac{1}{3}\right)$$

$$= (-1)(-1)\,a\left(\frac{1}{3}\right) + (-1)(1)\,b\left(\frac{1}{3}\right) + (1)(-1)\,b\left(\frac{1}{3}\right) + (1)(1)\,a\left(\frac{1}{3}\right)$$

$$= \frac{a}{3} - \frac{b}{3} - \frac{b}{3} + \frac{a}{3} = \frac{2a}{3} - \frac{2b}{3}$$

$$= \frac{2}{3}(a-b) = \frac{2}{3}\left[\frac{1}{3}(1+2e^{-|\tau|}) - \frac{1}{3}(1-e^{-|\tau|})\right]$$

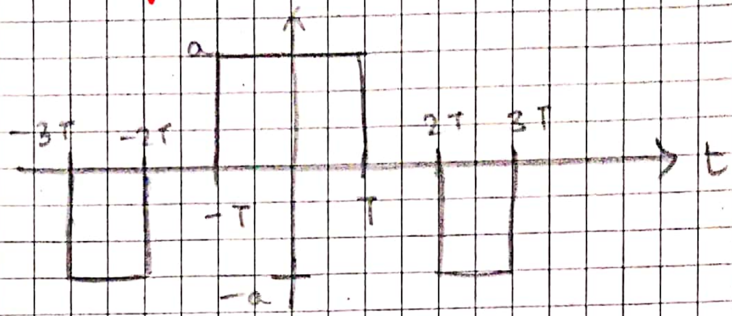
$$= \frac{2}{3}\left[\frac{1}{3}(3e^{-|\tau|})\right] = \frac{2}{3}e^{-|\tau|}$$

$$\boxed{\tilde{S}_{xx}(\tau) = \frac{2}{3}e^{-|\tau|}}$$

Scanned with CamScanner

Task 3.6

a) Given is signal $x(t)$, sketch ACF $\zeta_{xx}(\tau)$



Solution:

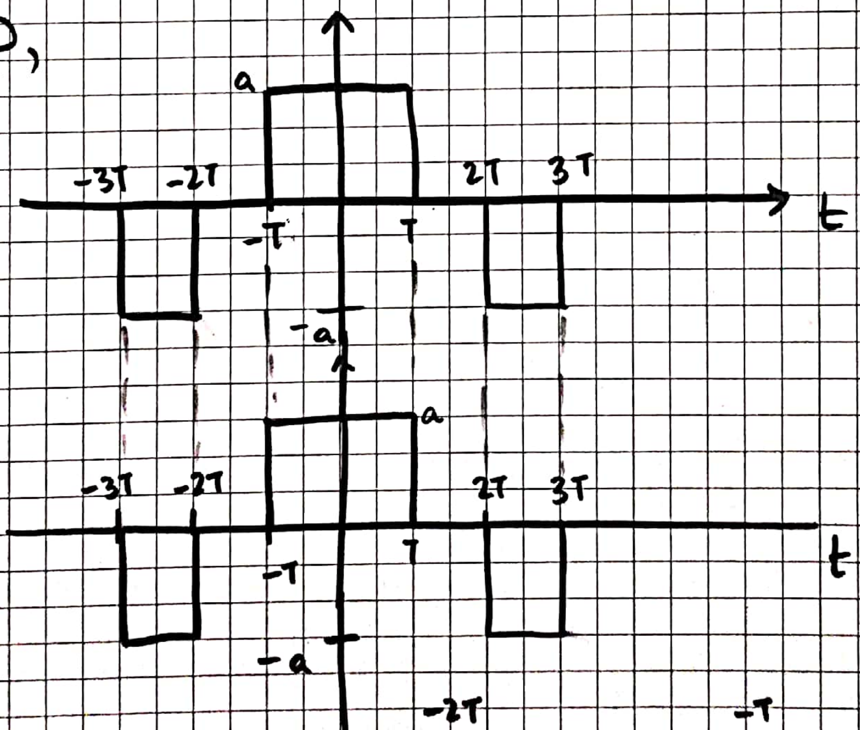
we know

$$\tilde{S}_{xx}(\tau) = E\{x(t)\,x(t+\tau)\}$$

$$= \int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,dt$$

For auto correlation, the amplitudes multiply at shifted instances by τ at every possible overlap.

→ **For $\tau = 0$,**



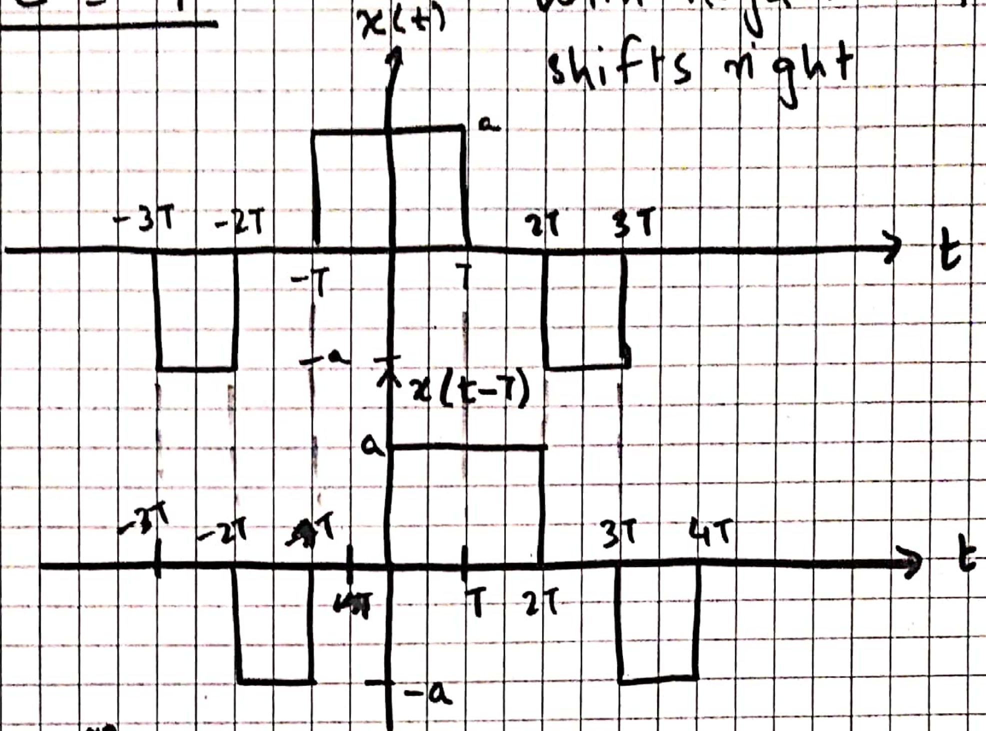
$$\tilde{S}_{xx}(0) = \int_{-\infty}^{\infty} x(t)\,x(t+0)\,dt \Rightarrow \int_{-3T}^{-2T}(-a)(-a)\,dt + \int_{-2T}^{-T} 0\,dt + \int_{-T}^{0}(a)(a)\,dt$$

$$+ \int_{0}^{T}(a)(a)\,dt + \int_{T}^{2T}(0)\,dt + \int_{2T}^{3T}(-a)(-a)$$

$$= a^2T + a^2T + a^2T + a^2T \Rightarrow \boxed{\tilde{S}_{xx}(0) = 4a^2T}$$

→ <u>For $\tau = -T$</u>

with negative τ, signal shifts right



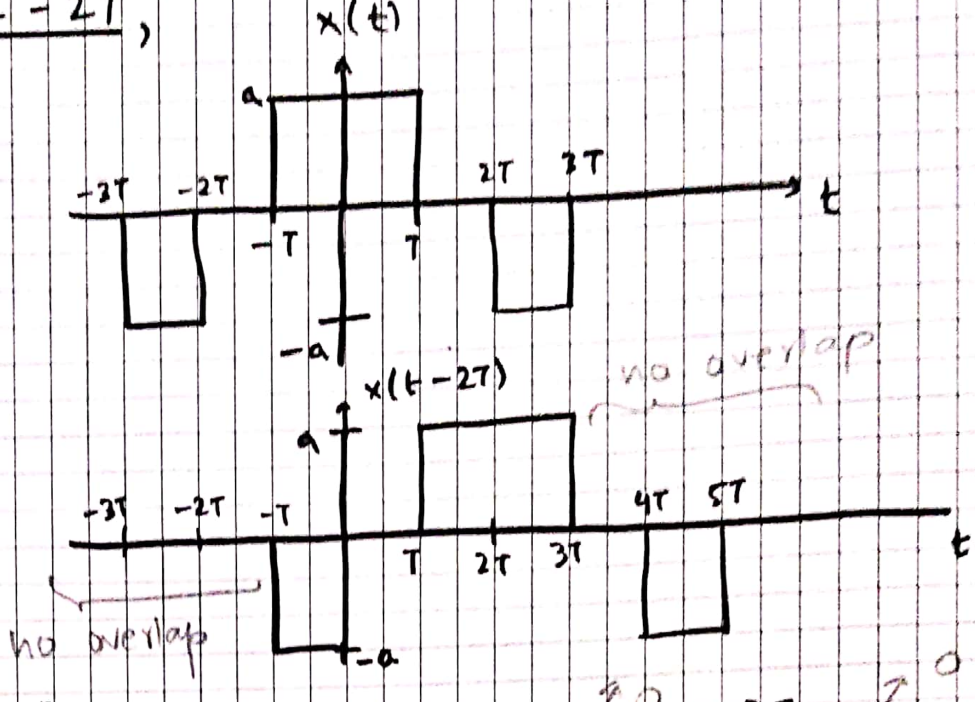
$$\vec{S}_{xx}(-T) = \int_{-\infty}^{\infty} x(t)\,x(t-T)\,dt$$

$$= \int_{-3T}^{-2T} \cancel{(-a)(0)}\,dt + \int_{-2T}^{-T} \cancel{(0)(-a)}\,dt + \int_{-T}^{0} \cancel{(a)(0)}\,dt + \int_{0}^{T}(a)(a)\,dt$$

$$+ \int_{T}^{2T} \cancel{(0)(a)}\,dt + \int_{2T}^{3T} \cancel{(-a)(0)}\,dt + \int_{3T}^{4t} \cancel{(0)(-a)}\,dt$$

$$= \boxed{\vec{S}_{xx}(-T) = a^2 T}$$

→ For $\tau = -2T$,

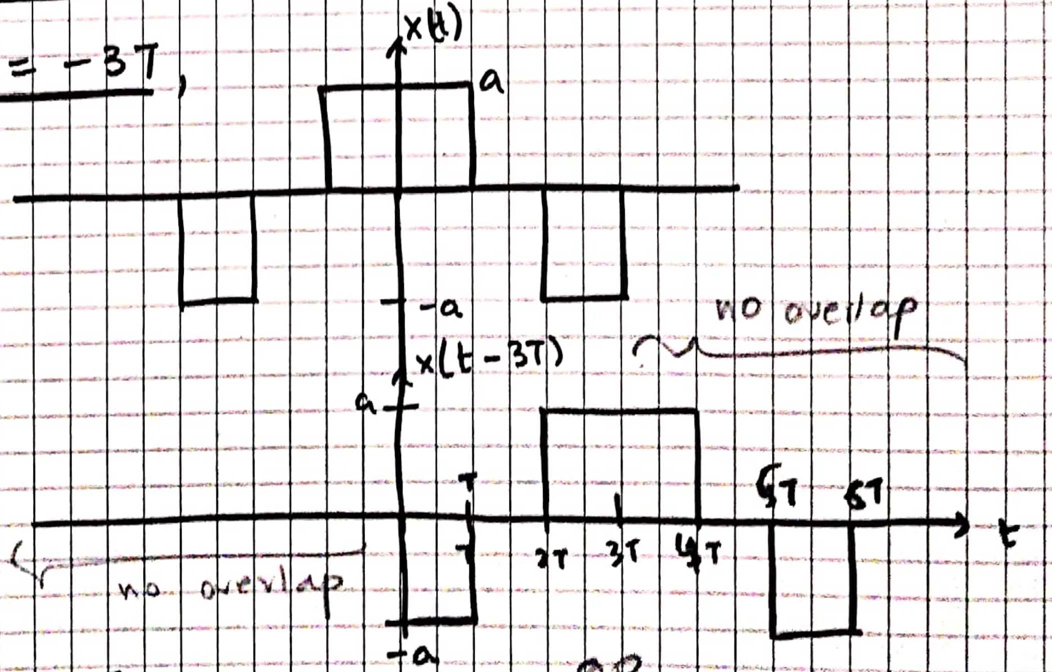


$$\tilde{S}_{xx}(-2T) = \int_{-T}^{0} (a)(-a)\,dt + \cancel{\int_{0}^{T} (a)(0)\,dt}^{\,0} + \cancel{\int_{T}^{2T} (0)(a)\,dt}^{\,0}$$

$$+ \int_{2T}^{3T} (-a)(a)\,dt + \cancel{\int_{3T}^{4T}}$$

$$\boxed{\tilde{S}_{xx}(-2T) = -2a^2T}$$

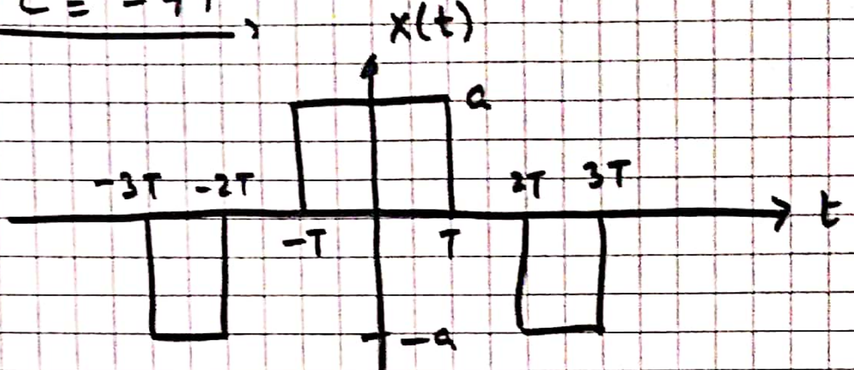
→ For $\tau = -3T$,

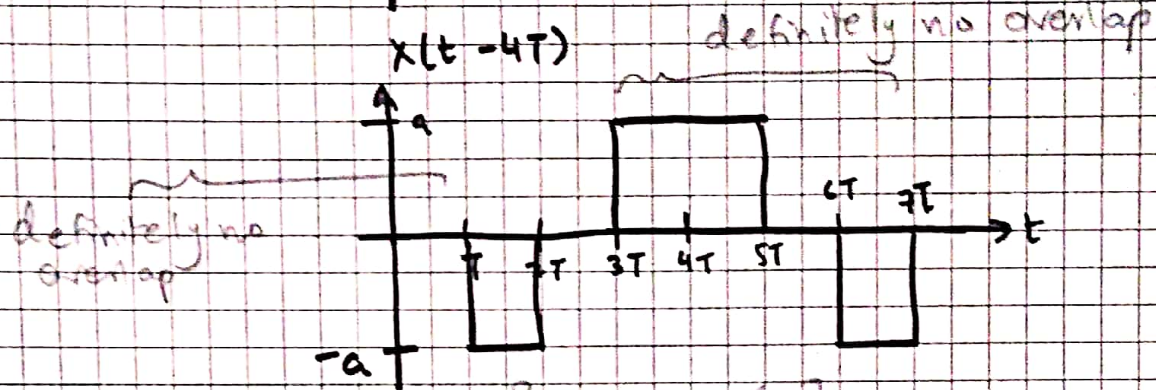


$$\tilde{S}_{xx}(-3T) = \int_{0}^{T} (a)(-a)\,dt + \cancel{\int_{T}^{2T} (0)(0)\,dt}^{\,0} + \int_{2T}^{3T} (-a)(a)\,dt$$

$$\cancel{\int_{3T}^{4T}} \qquad \boxed{\tilde{S}_{xx}(-3T) = -2a^2T}$$

→ **For $\tau = -4T$,**

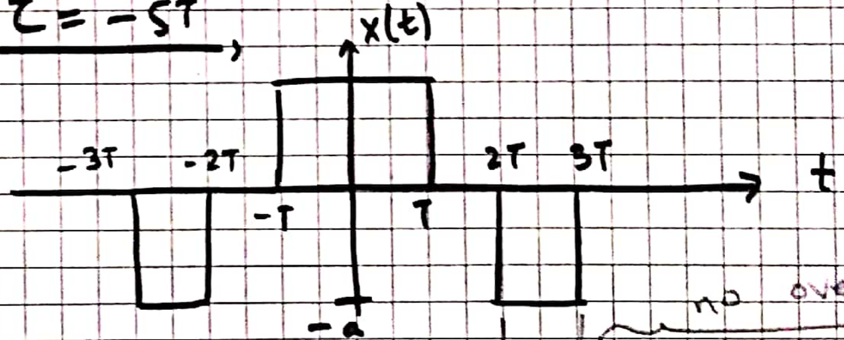


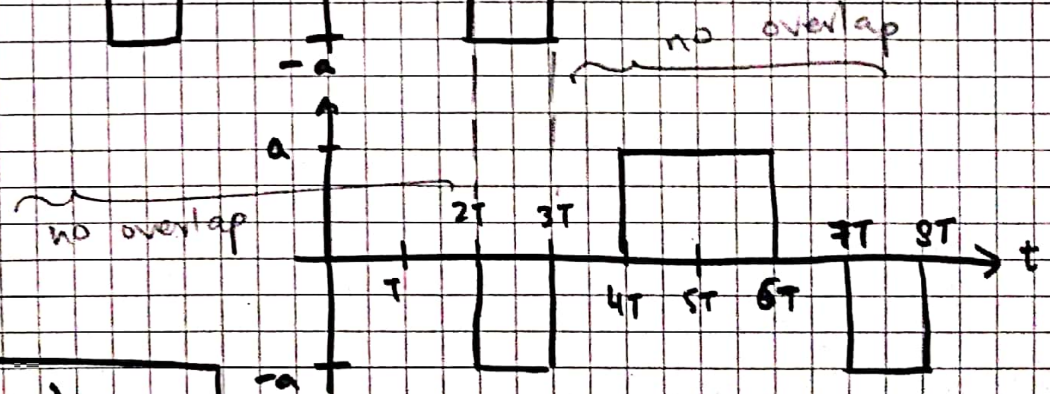


$$\tilde{S}_{xx}(-4T) = \int_{T}^{2T} (0)(-a)\,dt + \int_{2T}^{3T} (-a)(0)\,dt$$

$$\boxed{\tilde{S}_{xx}(-4T) = 0}$$

→ **For $\tau = -5T$,**





$$\boxed{\tilde{S}_{xx}(-5T) = a^2T}$$

For further right shifts, signals dont overlap

$\tilde{S}_{xx}(-6T) = 0$

As autocorrelation function is an even function, $\tilde{S}_{xx}(\tau) = \tilde{S}_{xx}(-\tau)$

<u>Hence for positive τ :</u>

→ $\tilde{S}_{xx}(0) = 4a^2T$

→ $\tilde{S}_{xx}(-T) = \tilde{S}_{xx}(T) = a^2T$

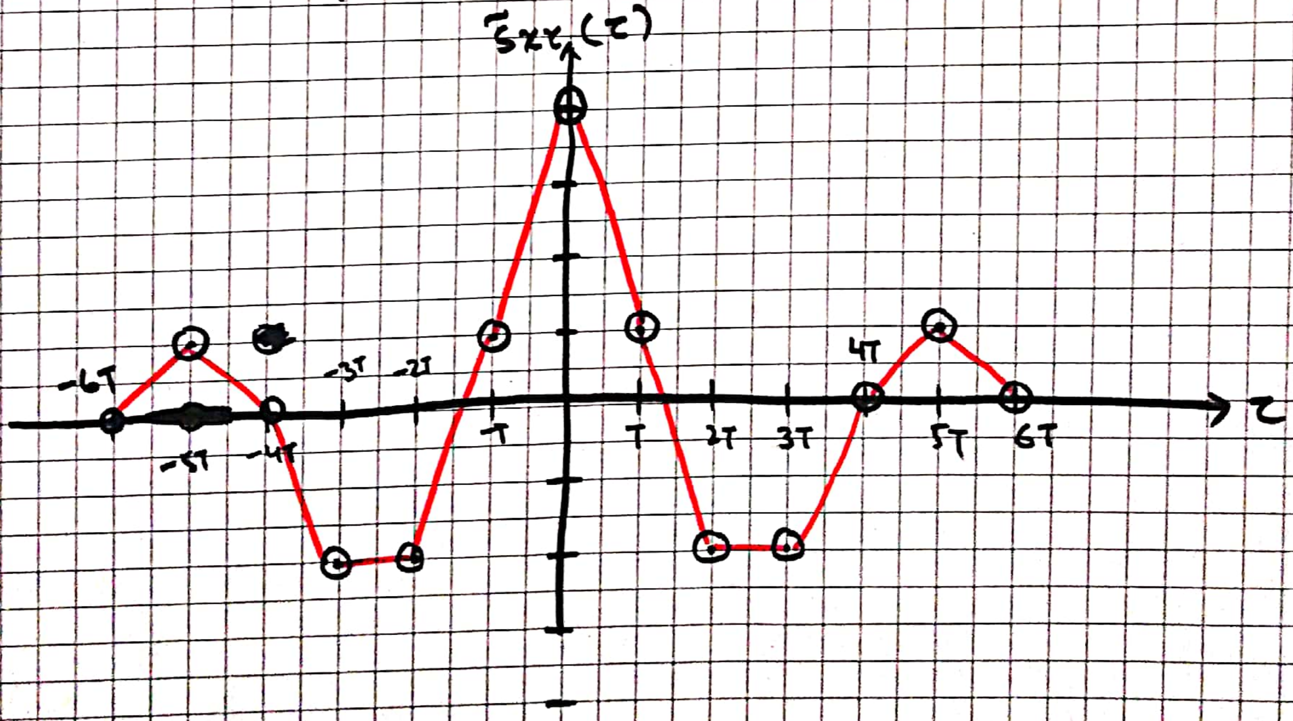
→ $\tilde{S}_{xx}(-2T) = \tilde{S}_{xx}(+2T) = -2a^2T$

→ $\tilde{S}_{xx}(-3T) = \tilde{S}_{xx}(+3T) = -2a^2T$

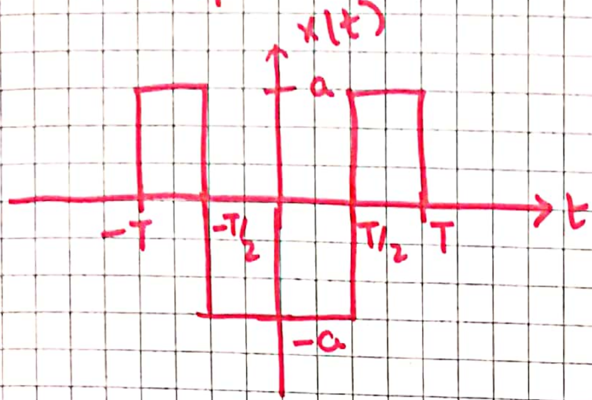
→ $\tilde{S}_{xx}(-4T) = \tilde{S}_{xx}(4T) = 0$

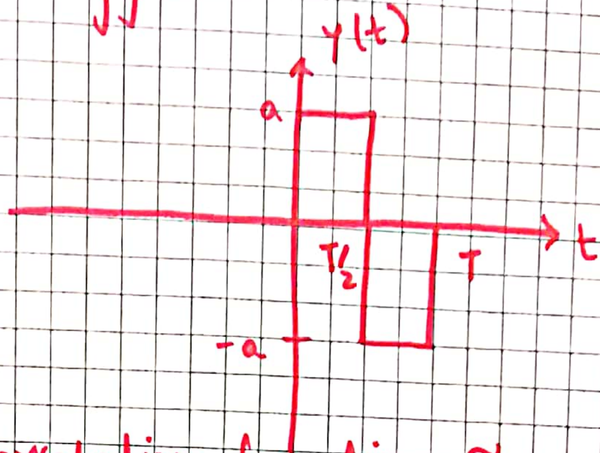
→ $\tilde{S}_{xx}(-5T) = \tilde{S}_{xx}(5t) = a^2T$

→ $\tilde{S}_{xx}(-6T) = \tilde{S}_{xx}(6T) = 0$



b) Let two deterministic signals $x(t)$ and $y(t)$ of finite energy





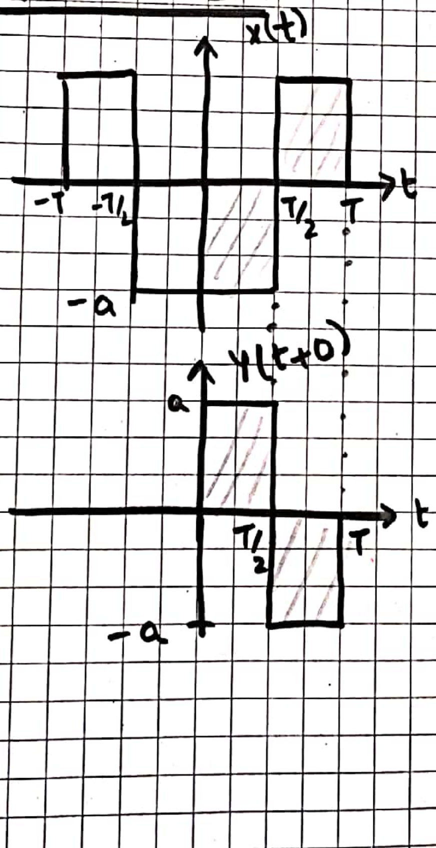
sketch the cross-correlation function $\tilde{S}_{xx}(\tau)$

$$\tilde{S}_{xy}(\tau) = E\{x(t)\,y(t+\tau)\}$$

$$= \int_{-\infty}^{\infty} x(t)\,y(t+\tau)\,dt$$

We have the choice to shift any signal with the lag. We set signal $x(t)$ and shift $y(t)$ across to calculate correlation. until no overlap occurs.

→ For $\tau = 0$



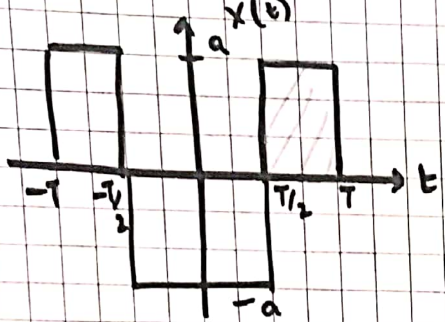
$$\tilde{S}_{xy}(0) = \int_{-\infty}^{\infty} x(t)\,y(t+\tau)\,dt$$

$$= \int_{0}^{T/2} (-a)(a)\,dt + \int_{T/2}^{T} (a)(-a)\,dt$$

$$= -a^2\left(\frac{T}{2}\right) - a^2\left(\frac{T}{2}\right)$$

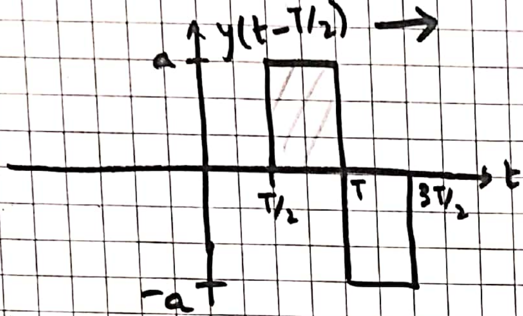
$$\boxed{\tilde{S}_{xy}(0) = -a^2 T}$$

→ For $\tau = -T/2$,



$$\tilde{S}_{xy}(-T/2) = \int_{T/2}^{T} (a)(a)\,dt$$

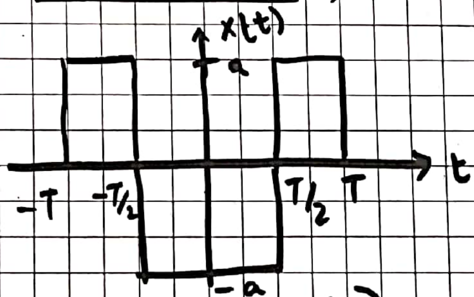
$$\boxed{\tilde{S}_{xy}(-T/2) = a^2(T/2)}$$



→ For $\tau = -T$ and so on the signals dont overlay hence $\tilde{S}_{xy}(-T) = 0$

$\tilde{S}_{xy}$ unlike $\tilde{S}_{xx}$ is not an even function. For that matter it has to be calculated with right lags as well.

→ For $\tau = T/2$,

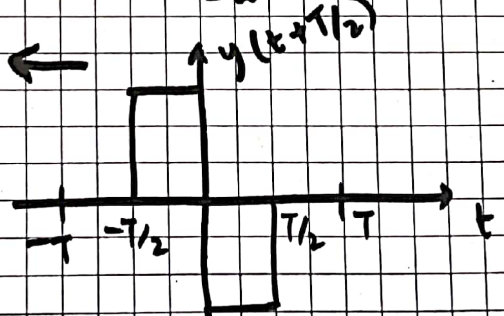


$$\tilde{S}_{xy}(T/2) = \int_{-\infty}^{\infty} x(t)\,y(t+T/2)\,dt$$

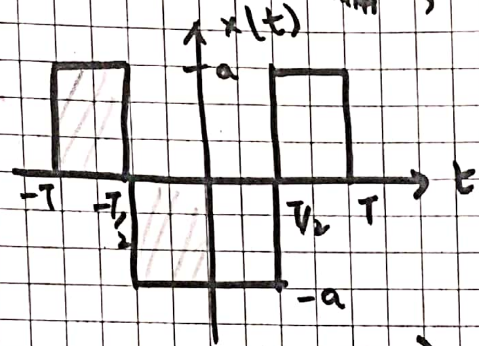
$$\Rightarrow \int_{-T/2}^{0} (a)(a)\,dt + \int_{0}^{T/2} (-a)(-a)\,dt + \int_{T/2}^{T} (a)(0)\,dt$$

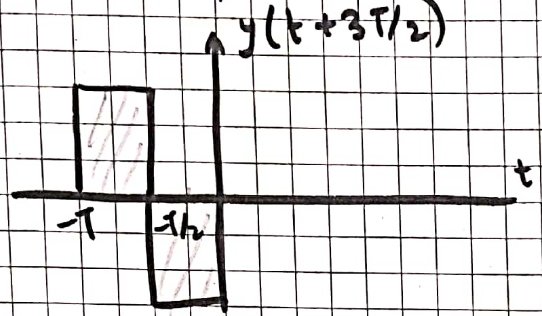
$$\Rightarrow a^2\left(\frac{T}{2}\right) + a^2\left(\frac{T}{2}\right) = a^2 T$$

$$\boxed{\tilde{S}_{xy}(T/2) = a^2 T}$$



→ **For $\tau = T$,**





$$\tilde{S}_{xy}(T) = \int_{-\infty}^{\infty} x(t)\,y(t+T)\,dt$$

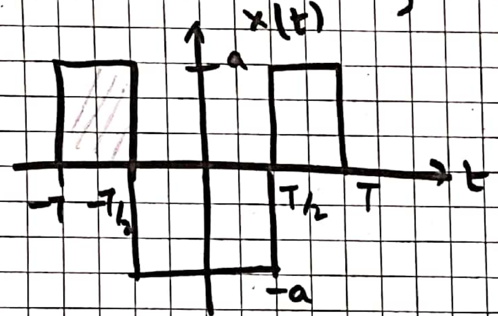
$$\tilde{S}_{xy}(T) = \int_{-T}^{-T/2} (a)(a)\,dt$$

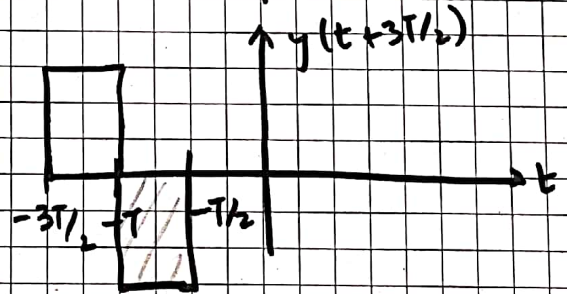
$$+\int_{-T/2}^{0} (-a)(-a)\,dt$$

$$\tilde{S}_{xy}(T) = a^2\frac{T}{2} + a^2\frac{T}{2}$$

$$\boxed{\tilde{S}_{xy}(T) = a^2 T}$$

→ **For $\tau = 3T/2$,**





$$\tilde{S}_{xy}(3T/2) = \int_{-\infty}^{\infty} x(t)\,y\left(t+\frac{3T}{2}\right)dt$$

$$\tilde{S}_{xy}(3T/2) = \int_{-T}^{-T/2} (a)(-a)\,dt$$

$$\Rightarrow \boxed{\tilde{S}_{xy}\left(\frac{3T}{2}\right) = -a^2 T/2}$$

As signals dont overlap beyond this left shift the $\tilde{S}_{xy} = 0$ for lags further left.

$\tilde{S}_{xy}(2T) = 0$

The following $\tilde{S}_{xy}$ are:

$\tilde{S}_{xy}(0) = -a^2 T$

$\tilde{S}_{xy}(-T/_2) = a^2 T/_2$

$\tilde{S}_{xy}(T/_2) = 0$

$\tilde{S}_{xy}(-T) = 0$

$\tilde{S}_{xy}(T) = a^2 T$

$\tilde{S}_{xy}(3T/_2) = -a^2 T/_2$

$\tilde{S}_{xy}(2T) = 0$

The $\tilde{S}_{xy}$ sketch: