Invertibility of a linear transformation

Let $T: V \to W$ be given where V & W are vector spaces over field F. Then under what conditions we can obtain the inverse of T, i.e. existence of $U: W \to V$ (a linear transf) such that $TU = UT = I$

Recall that $TU = T \circ U$, is the composition of linear transformations, T & U.

We have the following formal definition.

Definition: Let $T: V \to W$ be a linear transformation. Then T is called invertible if there exists a linear transformation $U: W \to V$ such that $UT = TU = I$, where I is the identity transformation on V.

Remark: The linear transformation U is called inverse of T and is unique.

The trivial transformation, $I: V \to V$ is always invertible and the inverse is I.

Q. Is zero linear transformation invertible ??

Similar to the functions in calculus, $T: V \to W$ will be invertible if

1) T is one to one
2) T is onto i.e. Range of $T = W$.

We see that, if T is invertible then the inverse of T is also a linear transformation. For, let $\alpha, \beta \in W$ and $c \in \mathbb{F}$, then

$T^{-1}(c\alpha + \beta)$ should be equal to $cT^{-1}\alpha + T^{-1}\beta$ if T^{-1} is invertible.

As Range of $T = W \Rightarrow \exists\, \alpha_1 \in V$ and $\beta_1 \in V$ such that $T\alpha_1 = \alpha \Rightarrow \alpha_1 = T^{-1}\alpha$

$T\beta_1 = \beta \Rightarrow \beta_1 = T^{-1}\beta$, which implies for any $c \in \mathbb{F}$

$$cT(\alpha_1) + T\beta_1 = c\alpha + \beta$$

$$T(c\alpha_1 + \beta_1) = c\alpha + \beta$$

which implies

$$c\alpha_1 + \beta_1 = T^{-1}(c\alpha + \beta)$$

$$\Rightarrow cT^{-1}\alpha + T^{-1}\beta = T^{-1}(c\alpha + \beta).$$

Note that, by one to one of a linear transformation we mean by

$$T\alpha = T\beta \Rightarrow \alpha = \beta$$

$$\text{or } T\alpha - T\beta = 0 \Rightarrow T(\alpha - \beta) = 0$$

$$\Rightarrow \alpha - \beta = 0$$

Thus T is one to one if $T\alpha = 0 \Rightarrow \alpha = 0$

In other words null space of T must be singleton $\{0\}$.

The next observation is from the following theorem

Theorem:- Let $T : V \to W$ be a linear transformation then T is one to one iff T carries each linearly independent subset to a linearly independent subset.

Proof: Let $\{\alpha_1, \alpha_2, \cdots \alpha_k\}$ be a linearly independent subset of V. We aim to show first that $\{T\alpha_1, T\alpha_2, \cdots T\alpha_k\}$ is a linearly independent subset of W.

Consider

$$c_1 T\alpha_1 + c_2 T\alpha_2 + \cdots c_k T\alpha_k = 0$$

$$\Rightarrow T(c_1\alpha_1 + c_2\alpha_2 + \cdots c_k\alpha_k) = 0$$

As T is one to one

$$c_1\alpha_1 + c_2\alpha_2 + \cdots c_k\alpha_k = 0$$

$$\Rightarrow c_i = 0 \ \forall \ i = 1, 2, \cdots k.$$

Conversely assume that T sends every linearly independent subset of V to a linearly independent subset of W. Assume if possible, $0 \neq \alpha \in Null(T)$ then by assumption $\{T\alpha\}$ is a linearly independent subset of W but $T\alpha = 0$ leads us to absurd as $\{0\}$ can not be linearly independent.

Let us see the following example

$T: F^2 \to F^2$ be defined as

$T(x,y) = (x+y, x)$

Then we claim that T is 1-1

Let $T(x_1, y_1) = T(x_2, y_2)$

$\Rightarrow (x_1+y_1, x_1) = (x_2+y_2, x_2)$

$\Rightarrow x_1 = x_2,\ x_1+y_1 = x_2+y_2 \Rightarrow y_1 = y_2$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$

Hence T is one to one.

Note that using linearity of T, it is equivalent to show that

$T(x,y) = (0,0) \Rightarrow (x,y) = (0,0)$

Or null space of T is $\{0\}$.

Let us think of situations when $T: V \to W$ ($\dim V = \dim W = n$) is one to one and not onto?

Recall the rank nullity theorem.

$\text{rank}(T) + \text{nullity}(T) = \dim V$

As T is one to one, hence nullity is 0

Thus $\dim V = \dim V = \text{rank}(T)$ and hence range space of T is nothing but W. Thus it can not be a case, that T is one to one and not onto provided $\dim V = \dim W$.

Similarly, if T is onto, i.e. $\text{rank} = \dim W = n$ then $\text{nullity} = \dim V - \text{rank}\, T = 0$, Hence T will be 1-1.

Observe that if T is one to one & $\dim V = \dim W$ then if $B = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ is a basis of V then $B' = \{T\alpha_1, T\alpha_2, \dots T\alpha_n\}$ is a basis of W. For,

$$\sum_{i=1}^{n} c_i T\alpha_i = 0 \Rightarrow T\left(\sum_{i=1}^{n} c_i \alpha_i\right) = 0$$

which implies (by 1-1 of T)

$\sum_{i=1}^{n} c_i \alpha_i = 0$. As $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ is linearly independent, $c_i = 0 \;\forall\; i = 1, 2, \dots n$.

Thus we have a linearly independent subset of W ($\dim W = n$) containing n elements, hence it is a basis for W.

Matrix Representation of a linear Transformation

Let us see the following example

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$T(x, y) = (0, y)$$

Then take $\mathcal{B} = \{(1,0), (0,1)\}$ a standard ordered basis of $\mathbb{R}^2$ and consider

$$T(1,0) = (0,0) = 0.(1,0) + 0.(0,1)$$

$$T(0,1) = (0,1) = 0.(1,0) + 1(0,1)$$

$$[T\alpha]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot [\alpha]_{\mathcal{B}}$$

for any $\alpha \in \mathbb{R}^2$.

In general we have the following result.

Theorem: Let V and W be n-dim & m-dim vector spaces over F respectively having $\mathcal{B}$ and $\mathcal{B}'$ as ordered bases. Then for any $T \in L(V, W)$ there exists a $A \in M_{m \times n}(F)$ such that

$$[T\alpha]_{\mathcal{B}'} = A[\alpha]_{\mathcal{B}}$$

for any $\alpha \in V$. Moreover

$T \mapsto A$ is a one to one correspondance between $L(V, W)$ and $M_{m \times n}(F)$.

The matrix A is called "matrix of T relative to ordered bases $\mathcal{B}$, $\mathcal{B}'$".

The column of A (say A_j) is given by $A_j = [T\alpha_j]_{\mathcal{B}'}$, where $\mathcal{B} = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ is an ordered basis of V.

Let us see the following example

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be

$T(x, y) = (x, 0)$

Let $\mathcal{B} = \{(1,0)\ (0,1)\}$ and

$\mathcal{B}' = \{(1,1), (2,1)\}$ be

two ordered bases of $\mathbb{R}^2$

Then $[T]_{\mathcal{B}}^{\mathcal{B}'} :=$ matrix of T relative to ordered basis $\mathcal{B}, \mathcal{B}'$

Then $A = [T]_{\mathcal{B}}^{\mathcal{B}'}$ will be a matrix of size 2×2. and the columns of A will be

$$A_1 = [T\alpha_1]_{\mathcal{B}'} \text{ and } A_2 = [T\alpha_2]_{\mathcal{B}'}$$

Note that

$$T\alpha_1 = T(1,0) = (1,0) = c_1(1,1) + c_2(2,1)$$

$$\Rightarrow c_1 + 2c_2 = 1, \quad c_1 + c_2 = 0$$

$$\Rightarrow c_2 = 1, \quad c_1 = -1$$

Thus $[T\alpha_1]_{\mathcal{B}'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Similarly

$$T\alpha_2 = T(0,1) = (0,0) = 0\cdot(1,1) + 0\,(2,1)$$

thus $[T\alpha_2]_{\mathcal{B}'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Therefore

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$$

You can verify

$$[T\alpha]_{\mathcal{B}'} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} [\alpha]_{\mathcal{B}}$$

for any $\alpha \in \mathbb{R}^2$