**Cauchy Condensation Test**: suppose $\{a_n\}_{n\geq 1}$ is a decreasing sequence of nonnegative terms. Then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 2^n a_{2^n}$ are either both convergent or both divergent.

Ex: Show that the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges

$a_n = \frac{1}{n\ln(n)}$, $a_n > 0 \quad \forall\, n \geq 2$ $\quad a_n$ is decreasing

$$a_{2^n} = \frac{1}{2^n \ln(2^n)} = \frac{1}{2^n n \ln(2)}$$

Thus

$$\sum 2^n a_{2^n} = \sum 2^n \frac{1}{2^n n \log(2)} = \frac{1}{\ln 2} \sum \frac{1}{n}$$

$\because \sum \frac{1}{n}$ divergent so $\sum 2^n a_{2^n}$ is divergent

$\Rightarrow \sum a_n$ divergent.

Ex: check convergence or divergence of $\sum_{n=2}^{\infty} \frac{1}{n\ln(n^3)}$

## Absolute convergent:

**Def$^n$**: Let $\sum a_n$ be any infinite series. Then we say $\sum a_n$ is absolute convergent if the series $\sum |a_n|$ is convergent.

**Conditional convergent**: If $\sum |a_n| = \infty$ but $\sum a_n$ convergent then $\sum a_n$ is called conditionally convergent.

**Remark**: It is trivial that a series of positive terms is absolutely convergent iff it is convergent.

**Theorem**: If a series is absolutely convergent then it is convergent.

**Proof**: Given $\sum |a_n|$ is convergent. So by Cauchy's criterian for a given $\epsilon > 0$ $\exists N \in \mathbb{N} \ni \forall m, n \geq N$ $(n > m)$

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon$$

Now $|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| < \epsilon$

$\Rightarrow \sum a_n$ is convergent.

Remark : If a series have arbitrary term i.e. not all positive or negative (some terms are positive and some negative) in this case if we show that the series is absolutely convergent then the series converges.

Now we consider a special case of the series with arbitrary terms.

Alternating series : Let $\{a_n\}$ be sequence of nonzero real numbers is said to be alternating if the terms $(-1)^{n+1} a_n$, $n \in \mathbb{N}$ are all positive (or all negative). If the sequence $\{a_n\}$ is alternating real numbers. If the sequence $\{a_n\}$ is an alternating we say that the series $\sum a_n$ is an alternating series.

Alternating series test (Leibniz's test)

If $\{a_n\}$ be a decreasing seq$^n$ of positive real numbers with $\lim a_n = 0$. Then the alternating series $\sum (-1)^{n+1} a_n$ is convergent.

Ex show that the series $\sum_{n=1}^{\infty} (-1)^{n+1} n^{-1/2}$ is convergent. Examine its absolute convergence.

Solu$^n$: Here $a_n = \frac{1}{\sqrt{n}} > 0 \quad \forall n$

and $\lim a_n = 0$.

Also $a_n$ is decreasing. So by Leibniz's test the series $\sum (-1)^{n+1} n^{-1/2}$ is convergent

$$\left|(-1)^{n+1}\frac{1}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}}. \text{ so } \sum \frac{1}{\sqrt{n}} \text{ not convergent } \because \tfrac{1}{2} \leq 1$$

$$\left[\sum \frac{1}{n^p} = \infty, \; p \leq 1\right]$$

So the series is not absolutely convergent.

Note: The above example shows that if $\sum a_n < \infty$ $\not\Rightarrow$ $\sum |a_n| < \infty$.

Ex: $\sum (-1)^n \frac{1}{n}$. $a_n = \frac{1}{n}$ decreasing to 0.

so $\sum (-1)^n \frac{1}{n}$ convergent.

$$\sum \left|(-1)^n \frac{1}{n}\right| = \sum \frac{1}{n} \text{ divergent.}$$

**Theorem**: The series $\sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ converges for $p > 1$ and diverges for $p \le 1$.

**Proof**:$\Rightarrow$ Case-1 $p > 1$, $\sum a_n = \sum \frac{1}{n^p}$

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

$$= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \cdots + \frac{1}{15^p}\right)$$

$$+ \cdots$$

Define $b_1 = 1$

$$b_2 = \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{1}{2^{p-1}}$$

$$b_3 = \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{4}{4^p} = \left(\frac{1}{2^{p-1}}\right)^2$$

$$b_4 = \frac{1}{8^p} + \cdots + \frac{1}{15^p} < \frac{8}{8^p} = \left(\frac{1}{2^{p-1}}\right)^3$$

$\vdots$

Define $v_n = \left(\frac{1}{2^{p-1}}\right)^{n-1}$. Then $b_n < v_n$

$\forall\ n \geq 2$

$\sum v_n = \sum \left(\frac{1}{2^{p-1}}\right)^{n-1}$ which is geometric series with common ratio $\frac{1}{2^{p-1}}$

Since $p>1$. so $0 < \frac{1}{2^{p-1}} < 1$

$\Rightarrow$ $\sum v_n$ convergent

Therefor by comparison test $\sum b_n = \sum a_n$ convergent

Case-2: $p=1$. In this case

$$\sum a_n = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

Let $S_n = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}$.

Now $\{S_n\}$ is the sequence of partial sum

Already we proved that the sequ^n^ $\{S_n\}$ is not cauchy (see sequ^n^ Note Page-22)

So $\{S_n\}$ not convergent $\Rightarrow$ $\sum a_n$ divergent.

[ $\because$ $\{S_n\}$ is a sequ^n^ which is increasing so it is divergent to infinity. so $\sum a_n$ divergent to $\infty$ ].

Case-3: $0<p<1$. Then we have $\frac{1}{2^p} > \frac{1}{2}$, $\frac{1}{3^p} > \frac{1}{3}, \ldots$

Therefor $\frac{1}{n^p} > \frac{1}{n}$ $\forall$ $n \geq 2$.

Since $\sum \frac{1}{n}$ is divergent so by comparison test

$\sum \frac{1}{n^p}$ is divergent.

Case - 4 $p \leq 0$ then $\lim \frac{1}{n^p} \neq 0 \Rightarrow \sum \frac{1}{n^p}$ is not convergent

( By comparison test we say that $\sum \frac{1}{n^p} \to \infty$ when $p \leq 0$ )

Finding sum of the $p$ series is not an easy task.

The Riemann $\zeta$ - function defined for $1 < p < \infty$ by

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

In particular $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$.

$\zeta(8) = \frac{\pi^8}{9450}$, $\zeta(10) = \frac{\pi^{10}}{93555}$.

On the other hand, the values of $\zeta$- function at odd natural numbers are harder to study.

(The proof the above result of $\zeta(p)$ beyond the syllabus.).

The irrationality of e: We have defined

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n.$$

First we obtain an alternative expression for e as the sum of a series.

$\sum \frac{1}{n!}$

$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$

$\lim \frac{a_{n+1}}{a_n} = 0 < 1$

so $\sum \frac{1}{n!}$ converges

Result: $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

Proof: Let $x_n = \left(1+\frac{1}{n}\right)^n \longrightarrow e$

Define $S_n = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$

$S_n \longrightarrow S$. Now we will prove $S = e$.

$$x_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

$$= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots$$

$$+ \frac{n(n-1)(n-2)\cdots 2\cdot 1}{n!}\frac{1}{n^n}.$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots$$

$$+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\frac{2}{n}\,\frac{1}{n}.$$

$$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \quad \left[\because \left(1-\frac{k}{n}\right)\leq 1 \ \forall\ 1\leq k\leq n.\right]$$

$= S_n \quad \forall n.$

Taking limit $n \to \infty$ we get $e \leq s$.

Again we have

$$x_n = \left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots$$

$$\cdots+\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)+\cdots$$

$$\cdots\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\frac{2}{n}\cdot\frac{1}{n}$$

$$\geq 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots\frac{1}{k!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$(\because\ 2 \leq k < n).$$

Now fix $k$ and take $n \to \infty$

$$e \geq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{k!} = S_k \quad \forall k.$$

Take limit $k \to \infty$ we have $e \geq S$.

$$\Rightarrow e = S$$

Note: $S_n \to e \Rightarrow$ that for large value of $n$, $S_n$ approximate $e$. Now we will estimate $e - S_n$

We know that $S_n \leq e \Rightarrow 0 \leq e - S_n$

So $$e - S_n = \frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots$$

$$= \frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+\cdots\right)$$

$\because \quad (n+1) < (n+2) \Rightarrow \frac{1}{n+1} > \frac{1}{n+2}$

$(n+3) > (n+1) \Rightarrow \frac{1}{(n+1)} > \frac{1}{(n+3)}$

So $\quad \frac{1}{(n+1)^2} > \frac{1}{(n+2)(n+3)} \cdots\cdots$

So $\quad e - S_n \le \frac{1}{(n+1)!}\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\cdots\right]$

$$= \frac{1}{(n+1)!} \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{(n+1)!} \frac{n+1}{n} = \frac{1}{n(n!)}$$

$\Rightarrow \quad 0 < e - S_n \le \frac{1}{n!\,n} \quad \forall n.$

**Theorem**: The number is irrational.

**Proof**: suppose $e$ is rational, then $e = \frac{p}{q}$, $p, q \in \mathbb{N}$. $[\because e > 0]$

Now we have $\quad 0 < e - S_n \le \frac{1}{n!\,n}$

$$\Rightarrow 0 < n!\,(e - S_n) \le \frac{1}{n}$$

Take $n = q$ then

$$0 < q!\left(\frac{p}{q} - S_q\right) \le \frac{1}{q}$$

Now $q! \times \frac{p}{q}$ is an integer, $q!\,S_q$ is also integer

Since $\quad S_q = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}$.

So $q!\left(\frac{p}{q} - S_q\right)$ also an integer which lies strictly between 0 & $\frac{1}{q}$ which is a contradiction.

So our assumption e is ratational is not correct. Hence e is irrational.