Discrete Random Variables :

Let $(\Omega, \mathcal{S}, P)$ be a prob space and let $X : \Omega \to \mathbb{R}$ be a r.v. with induced prob space $(\mathbb{R}, \mathcal{B}, P_X)$ and d.f. F.

Defn: The r.v. X is said to be a discrete r.v. if there exists a countable set S (finite or infinite) such that

$$P_r(X=x) = F(x) - F(x-) > 0 \quad \forall\, x \in S.$$

and $P_r(X \in S) = 1$

The set S is called the support of r.v. X.

Remark: (i) If S is the support of a discrete r.v. then clearly $S = \{x \in \mathbb{R} : F(x) - F(x-) > 0\}$

= set of discontinuity points of F.

(ii) If x is a discontinuity point of d.f. F then

$$F(x) - F(x-) = \text{size of jump of } F \text{ at } x.$$

Thus a r.v. X is of discrete type

$\Leftrightarrow$ sum of jump ~~points~~ of F equal 1.

$$\left(P_r(X \in S) = \sum P_r(X=x) = \sum_{x \in S} [F(x) - F(x-)] = 1 \right.$$

Ex: In the previous example the set of discontinuity points of G is $D = \{1,2,3\}$

$$\sum_{x \in D} \left[F(x) - F(x-) \right] = \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = \frac{2}{3} < 1$$

$\Rightarrow$ X is not a discrete r.v.

Ex: In case of coin tossing example we have

$X \to$ no of heads &

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/8 & \text{if } 0 \le x < 1 \\ 1/2 & \text{if } 1 \le x < 2 \\ 7/8 & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$

Here the set of discontinuity points of F is $D = \{0,1,2,3\}$, & $\sum_{x \in D} \{F(x) - F(x-)\} = 1 \Rightarrow X$ is a discrete r.v. support of the distn is $S = D = \{0,1,2,3\}$.

Defn: Let X be a r.v. with c.d.f F_X and support S_X. Define the function $f_X : \mathbb{R} \to \mathbb{R}$ by

$$f_X(x) = \begin{cases} P_r(X = x) = F_X(x) - F_X(x-) & \text{if } x \in S_X \\ 0, & \text{o/w.} \end{cases}$$

The function f_X is called the prob. mass function (p.m.f) of r.v. X.

Whenever there is no ~~ab~~ ambiguity we will drop subscript X is F_X, S_X and f_X.

Remark (I) Let X be a discrete r.v with p.m.f f and d.f F. Then for any $A \subseteq \mathbb{R}$

$$P_r(X \in A) = P_r(X \in A \cap S) = \sum_{x \in A \cap S} f(x)$$

Where S is the support of X. Moreover

$$F(x) = \sum_{y \in S \cap (-\infty, x]} f(y)$$

Also for any $x \in S$, $f(x) = F(x) - F(x-)$.

(II) clearly a d.f determines the p.m.f uniquely and Vice-versa.

(III) Let X be a discrete r.v. with p.m.f f and support S then $f : \mathbb{R} \to \mathbb{R}$ satisfies

(I) $f(x) > 0 \quad \forall x \in S$.

(II) $\sum_{x \in S} f(x) = 1$.

conversly suppose that $g : \mathbb{R} \to \mathbb{R}$ is a function s.t for some countable set T

(I) $g(x) > 0$, $x \in T$

(II) $\sum_{x \in T} g(x) = 1$

Then $g(\cdot)$ is the p.m.f of some discrete r.v having support T.

Ex: Let X be a r.v. having d.f

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1/8, & \text{if } 0 \le x < 1 \\ 1/2, & \text{if } 1 \le x < 2 \\ 7/8, & \text{if } 2 \le x < 3 \\ 1, & \text{if } x \geq 3 \end{cases}$$

We have seen that X is a discrete r.v. with support $S = \{0, 1, 2, 3\}$. Then the p.m.f. of X is $f(x): \mathbb{R} \to \mathbb{R}$, where

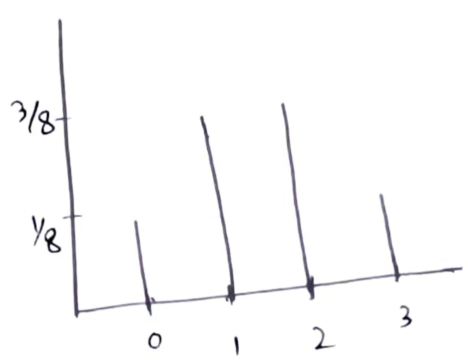
$f(0) = F(0) - F(0-) = \frac{1}{8}$, $f(1) = F(1) - F(1-) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

$f(2) = F(2) - F(2-) = 7/8 - 1/2 = 3/8$

$f(3) = F(3) - F(3-) = 1 - 7/8 = 1/8$

Thus the p.m.f of X is

$$f(x) = \begin{cases} \frac{1}{8}, & x = 0, 3 \\ 3/8, & x = 1, 2 \\ 0, & \text{o/w.} \end{cases}$$



Continuous Random Variable :⇒.

Defn: A random variable X is said to be a continuous r.v. if there exists a non-negative integrable function $f : \mathbb{R} \to [0, \infty)$ s.t. for any $x \in \mathbb{R}$

$$F(x) = Pr(X \le x) = \int_{-\infty}^{x} f(t)\, dt$$

The function $f(\cdot)$ is called the probability density function (p.d.f) of X. The support of a continuous r.v. X is the set $S = \{x \in \mathbb{R} : F(x+h) - F(x-h) > 0 \ \forall\, h > 0\}$. (or $S = \{x \in \mathbb{R} : f(x) > 0\}$.

Remark: (1) From ~~the~~ a result of calculus we know that

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

is a continuous function on $\mathbb{R}$. Thus the d.f. F of any continuous r.v. X is continuous everywhere in $\mathbb{R}$.

In particular

$$Pr(X = x) = F(x) - F(x-) = 0 \quad \forall\, x \in \mathbb{R}.$$

Generally if A is any countable subset of $\mathbb{R}$ then for any continuous r.v. X

$$Pr(X \in A) = \sum_{x \in S} Pr(X = x) = 0.$$

(ii) If X is a continuous r.v. then

(a) $Pr(X < x) = P(X \leq x) = F(x) \ \forall x \in \mathbb{R}$.

(b) $Pr(X \geq x) = 1 - Pr(X < x) = 1 - F(x) \quad \forall x \in \mathbb{R}$

(c) For any $a, b \in \mathbb{R}$, $-\infty < a < b < \infty$

$$Pr(a < X < b) = P(a \leq X \leq b) = P(a < X \leq b)$$
$$= P(a \leq X < b) = F(b) - F(a)$$
$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt = \int_{a}^{b} f(t)\,dt.$$

(iii) The p.d.f of a continuous r.v. is not unique.

(iv) There are random variables that are neither discrete nor continuous (see example page-11). Such random variables will not be studied here.

<u>Theorem</u>: Let X be a r.v. with d.f. F. Suppose that F is differentiable everywhere expect (possibly) on a countable set E. Further suppose that $\int_{-\infty}^{\infty} F'(t)\,dt = 1$

Then X is a continuous r.v. with p.d.f.

$$f(x) = \begin{cases} F'(x) & \text{if } x \in E^c \\ 0 & \text{if } x \in E. \end{cases}$$

Remark (1) Let X be a continuous r.v. with pdf $f(x)$. Then (i) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

(ii) $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Conversely suppose that $g: \mathbb{R} \to \mathbb{R}$ is a function s.t.

(i) $g(x) \geq 0 \quad \forall x \in \mathbb{R}$

(ii) $$\int_{-\infty}^{\infty} g(t)\,dt = 1$$

Then $g(\cdot)$ is a pdf of some continuous r.v. having support $T = \left\{ x \in \mathbb{R} : \int_{x-h}^{x+h} g(t)\,dt > 0 \quad \forall h > 0 \right\}$.

Ex: Let X be a r.v. with d.f.

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ x^2/2 & \text{if } 0 \leq x < 1 \\ x/2 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

Show that X is continuous r.v. Find the pdf of X and support of X.

Solun: clearly $F(x)$ is continuous everywhere.

More over F is differentiable everywhere except at three points $0, 1$ & 2 and

$$F'(x) = \begin{cases} 0, & x < 0 \\ x & 0 < x < 1 \\ 1/2 & 1 < x < 2 \\ 0, & x \geq 2 \end{cases}$$

Also $\int_{-\infty}^{\infty} F'(x) = \int_0^1 x\,dx + \int_1^2 \frac{1}{2}\,dx = 1$

$\Rightarrow X$ is a continuous r.v with pdf

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 1/2 & \text{if } 0 < x < 2 \\ 0 & o/w \end{cases}$$

The support of X is $[0, 2]$.

Ex: Let X be a continuous r.v. with p.d.f

$$f(x) = \begin{cases} x^2 & \text{if } 0 < x \leq 1 \\ ce^{-x} & \text{if } x \geq 1 \\ 0 & o/w \end{cases}, \quad \text{where } c \geq 0 \text{ const.}$$

(a) Find the value of c (b) Find $P(1/2 \leq X \leq 2)$
(c) Find the support of X (d) Find the d.f. of X.

Solu^n (a) We have $\int_a^b f(x)\,dx = 1$

$$\Rightarrow \int_0^1 x^2\,dx + \int_1^\infty ce^{-x}\,dx = 1 \Rightarrow \frac{1}{3} + ce^{-1} = 1 \Rightarrow c = \frac{2e}{3}$$

(b) $$P\left(\frac{1}{2} \le X \le 2\right) = \int_{1/2}^2 f(x)\,dx + \int_{1/2}^1 x^2\,dx + c\int_1^2 e^{-x}\,dx$$

$$= \frac{1}{3}\left(1-\frac{1}{8}\right) + c\left(e^{-1} - e^{-2}\right)$$

$$= \frac{7}{24} + \frac{2}{3}\left(1-e^{-1}\right)$$

(c) The support of X is $[0, \infty)$

(d) The d.f. of X is

$$F(x) = \int_{-\infty}^x f(t)\,dt$$

For $x < 0$, $F(x) = 0$.

For $0 \le x < 1$, $F(x) = \int_0^x t^2\,dt = \frac{x^3}{3}$

For $x \geq 1$, $F(x) = \int_0^1 t^2\,dt + c\int_1^x e^{-t}\,dt = \frac{1}{3} + \frac{2}{3}\left(1-e^{-(x-1)}\right)$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^3}{3}, & 0 \le x < 1 \\ \frac{1}{3} + \frac{2}{3}\left(1-e^{-(x-1)}\right), & x \geq 1. \end{cases}$$