# Special Continuous Distribution :

Let $-\infty < \alpha < \beta < \infty$. An continuous type r.v. $X$ is said to have a uniform (or rectangulare) $dist^n$ over the interval $(\alpha, \beta)$ written as $X \sim U(\alpha, \beta)$ if its p.d.f is given by

$$f(x|\alpha,\beta) = \begin{cases} \dfrac{1}{\beta-\alpha}, & \alpha < x < \beta \\ 0, & o/w \end{cases}$$

$\{ U(\alpha,\beta) : -\infty < \alpha < \beta < \infty \}$ is a family of $dist^n$ with parameter $\alpha$ & $\beta$.

$$\mu_r' = E(X^r) = \int_\alpha^\beta \frac{x^\alpha}{\beta-\alpha}\,dx = \frac{\beta^{\alpha+1} - \alpha^{r+1}}{(r+1)(\beta-\alpha)}$$

$$E(X) = \frac{\alpha+\beta}{2} = \mu$$

$$\mu_r = E(X-\mu)^2 = \int_\alpha^\beta \left(x - \frac{\alpha+\beta}{2}\right)^2 \frac{1}{\beta-\alpha}\,dx$$

$$= \int_{-\frac{\beta-\alpha}{2}}^{\frac{\beta-\alpha}{2}} \frac{t^r}{\beta-\alpha}\,dt = \begin{cases} 0, & r = 1, 3, 5, \ldots \\ \dfrac{(\beta-\alpha)^r}{2^r(r+1)}, & r = 2, 4, 6, \ldots \end{cases}$$

$$\operatorname{Var}(X) = \frac{(\beta-\alpha)^2}{12}$$

c.d.f

$$F_X(x) = \begin{cases} 0, & x < \alpha \\ \dfrac{x-\alpha}{\beta-\alpha}, & \alpha \le x < \beta \\ 1, & x \geq \beta \end{cases}$$

m.g.f. $M_X(t) = E(e^{tX}) = \int_{\alpha}^{\beta} \frac{e^{tx}}{\beta-\alpha}\,dx$

$$= \begin{cases} \dfrac{e^{t\beta} - e^{t\alpha}}{t(\beta-\alpha)}, & t \neq 0 \\ 1 & t = 0. \end{cases}$$

## Exponential Distribution :

Consider a Poisson process $X(t)$ with rate $\lambda (>0)$.

Let $T$ be the time of the first occurence

we want prob. dist of $T$


$P(T > t) = P($upto to time $t$ no occurence is zero$)$

$$= P(X(t) = 0) = \begin{cases} e^{-\lambda t}, & t > 0 \\ 1, & t \le 0. \end{cases}$$

$$F_T(t) = P(T \le t) = 1 - P(T > t)$$
$$= \begin{cases} 0, & t \le 0 \\ 1 - e^{-\lambda t}, & t > 0 . \end{cases}$$

So the pdf of T is

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0, & o/w . \end{cases}$$

<u>Def^n</u>: A random T variable said to follow a exponential dist^n written as $T \sim Exp(\lambda)$ if T has pdf.

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0, & o/w . \end{cases}$$

The family of dist^n $\{Exp(\lambda) : \lambda > 0\}$ is a one parameter family of dist^n.

$$\mu_k' = E(T^k) = \int_0^\infty t^k e^{-\lambda t} dt = \lambda \frac{\Gamma(k+1)}{\lambda^{k+1}} = \frac{\lambda k!}{\lambda^{k+1}}$$
$$= \frac{k!}{\lambda^k}, \quad k = 1, 2, \ldots$$

$$\mu_1' = E(T) = \frac{1}{\lambda}, \quad \mu_2' = \frac{2}{\lambda^2}, \quad Var(T) = \frac{1}{\lambda^2}$$

$$\mu_3' = \frac{6}{\lambda^3}, \quad \mu_4' = \frac{24}{\lambda^4}, \quad \mu_3 = \frac{2}{\lambda^3}, \quad \mu_4 = \frac{9}{\lambda^4}$$

$$\beta_1 = \frac{2}{\lambda^3} \Big/ \frac{1}{\lambda^3} = 2 > 0$$ Positively skewed.

$$\beta_1 = \frac{9}{\lambda^4} \Big/ \frac{1}{\lambda^4} - 3 = 6 > 0$$ always have high peak.

Consider

$$P(T > a) = e^{-\lambda a}$$

$$P(\underbrace{T > a+b}_{E_1} \mid \underbrace{T > b}_{E_2}) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(T > a+b)}{P(T > b)}$$

$$= e^{-\lambda a} = P(T > a)$$

This is called the lack of memory property of exponential distn.

Let $T$ denote that waiting time for the failure of a system

$P(T > a+b \mid T > b)$ = P (given that the system not fail till time b, the system will not fail an additional time a)

$P(T > a) = P($ system will not fail till time $a)$.

## Shifted Expenential Distn

Defn: A random variable X of continuous type is said to follow a shifted expenential distn written as $X \sim Exp(\mu, \sigma)$ if its probability density function is given by

$$f_X(x) = \begin{cases} \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}} & , x > \mu \\ 0 & , o/w \end{cases} \qquad \begin{matrix} \sigma > 0 \\ \mu \in \mathbb{R} \end{matrix}$$

$$E(X-\mu)^k = \int_{\mu}^{\infty} (x-\mu)^k \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}} dx$$

$$= \int_0^{\infty} \sigma^k t^k e^{-t} dt = \Gamma(k+1)\sigma^k = k!\,\sigma^k$$

$$E(X-\mu) = \sigma \Rightarrow E(X) = (\mu + \sigma)$$

$$E(X-\mu)^2 = 2\sigma^2 \Rightarrow E(X^2) = 2\sigma^2 + 2\mu\sigma + \mu^2$$

$$Var(X) = E(X^2) - (E(X))^2 = \sigma^2.$$

m.g.f

$$M_X(t) = E(e^{tX}) = \int_{\mu}^{\infty} \frac{1}{\sigma} e^{tx} e^{-\frac{x-\mu}{\sigma}} dx$$

take $\frac{x-\mu}{\sigma} = y$

$$M_X(t) = \int_0^{\infty} e^{t(\sigma y+\mu)} e^{-y} dy = \int_0^{\infty} e^{t\mu} e^{y(\sigma t-1)} dy$$

$$= e^{\mu t} \left. \frac{e^{y(\sigma t-1)}}{(\sigma t-1)} \right|_0^{\infty} \qquad \sigma t < 1 = t < \frac{1}{\sigma}$$

$$= \frac{e^{\mu t}}{1-\sigma t}$$

$$M_X(t) = \frac{e^{\mu t}}{1-\sigma t}. \qquad \text{Take } \mu = 0, \quad M_X(t) = \frac{1}{1-\sigma t}$$

Take $\frac{1}{\sigma} = \lambda \Rightarrow \sigma = \frac{1}{\lambda}$.

$$M_X(t) = \frac{\lambda}{\lambda - t} \leftarrow \text{m.g.f of exponential dist}^n.$$

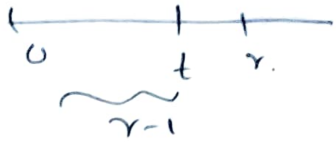
Result: $X \sim Exp(\mu, \sigma)$, $Y = aX+b \sim Exp(a\mu+b, a\sigma)$

Proof: H.W. use m.g.f of X find m.g.f of Y.

Consider a a Poisson process $X(t)$ with rate $\lambda$.

Let $T_r$ denotes the time of $r^{th}$ occurrence

$$P(T_r > t) = P(X(t) \le r-1), \quad t > 0$$




[ $P(T_r > t) = P($ $r^{th}$ occurrence not take place till time $t)$ ]

$$P(T_r > t) = \begin{cases} P(X(t) \le r-1), & t > 0 \\ 1, & t \le 0. \end{cases}$$

$$= \begin{cases} \sum_{j=0}^{r-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!}, & t > 0 \\ 1 & t \le 0. \end{cases}$$

$$F_{T_r}(t) = 1 - P(T > t) = \begin{cases} 0, & t \le 0 \\ 1 - \sum_{j=0}^{r-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!}, & t > 0. \end{cases}$$

So the pdf $T_r$ is

$$f_{T_r}(t) = \begin{cases} \dfrac{\lambda^r t^{r-1} e^{-\lambda t}}{\Gamma(r)}, & t>0, \quad \lambda>0 \\ 0 & o/w. \end{cases}$$

This gives the pdf of the waiting time for the $r^{th}$ occurrence in a Poisson process.

Def$^n$: An continuous type r.v. $X$ is said to follow a gamma distribution as written as $X \sim \text{Gamma}(r, \lambda)$ if its pdf is given as

$$f_X(x) = \begin{cases} \dfrac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, & x>0, \quad \lambda>0, \ r>0. \\ 0 & o/w \end{cases}$$

$$\mu_k' = E(X^k) = \int_0^\infty \frac{x^k \lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)} dx$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{k+r-1} e^{-\lambda x} dx = \frac{\lambda^r}{\Gamma(r)} \frac{\Gamma(k+r)}{\lambda^{k+r}}$$

$$= \frac{\Gamma(k+r)}{\Gamma(r)} \frac{1}{\lambda^k}.$$

$$E(X) = \frac{\Gamma(r+1)}{\Gamma(r)} \cdot \frac{1}{\lambda} = \frac{r}{\lambda}$$

$$E(X^2) = \mu_2' = \frac{r(r+1)}{\lambda^2}, \quad Var(X) = \frac{r(r+1)}{\lambda^2} - \frac{r^2}{\lambda^2} = \frac{r}{\lambda^2}$$

$$M_X(t) = \int_0^\infty e^{tx} \frac{\lambda^r e^{-\lambda x} x^{r-1}}{\Gamma(r)} dx$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty e^{-x(\lambda-t)} x^{r-1} dx$$

$$= \frac{\cancel{\Gamma(r)}}{(\lambda-t)^r} \cdot \frac{\lambda^r}{\cancel{\Gamma(r)}}, \quad \lambda > t.$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^r, \quad \lambda > t.$$

Note: $\{G(r,\lambda) : r>0, \lambda>0\}$ is a family of distn.

Note: If we take $r=1$, $X \sim exp(\lambda)$.

Beta Distribution: Let $X$ be a r.v. of continuous type and let $a>0$ & $b>0$ be constants. The r.v. $X$ is said to follow the distn with parameter $(a, b)$ written as $X \sim Be(a,b)$ if its prob. density fun is given by