Let us see the application of rank nullity theorem in the following result.

Let us consider the system

$$Ax = y \quad , A \in M_{m\times n}(\mathbb{F})$$

In case of homogeneous system $y=0$, we get $Ax=0$

Let us think of a linear transformation $T : \mathbb{F}^{n\times 1} \to \mathbb{F}^{m\times 1}$ defined as $Tx = Ax$, then the solution space of $Ax=0$ is nothing but the null space of the linear transformation T.

In case of non-homogeneous system, $y \in \text{Range}(T)$ if the system $Ax=y$ has a solution.

Let us recall that the ~~so~~ column rank of a matrix is the dimension of subspace (of $\mathbb{F}^{n\times 1}$) spanned by columns of the matrix.

Let $A_1, A_2, \ldots A_n$ are the columns of the matrix, then

$$Tx = Ax = x_1A_1 + x_2A_2 + \cdots x_nA_n$$

where $x = (x_1, x_2, \ldots x_n)^T$

It is clear that range of T is the subspace spanned by the columns of matrix A and hence the dimension of range space of T is the column rank of A

i.e. $\text{rank}(T) =$ column rank of A

Let us consider that $S =$ solution space of homogeneous system $Ax=0$ which is the null space of T. Hence $\dim(S) =$ nullity of T.

Now apply rank nullity theorem to get

nullity of T + rank T = dim of $F^{n\times 1} = n$

$\Rightarrow$ dim S + column rank of $A = n$

Now look back the theory of system of equations, as

if RRE of A has r non zero rows then row rank of $A = r$ and solution of A consists of vectors from $F^{n\times 1}$ with $n-r$ free variables.

or the basis of solution space of $Ax=0$ consists $n-r$ vectors hence $\dim S = n-r$

Therefore column rank $A = n - \dim S$
$= r =$ row rank of A

Thus for a matrix row rank of A and column rank of A are equal. We call this no as rank of the matrix.

<u>Remark</u>: what will be the dimension of solution space of $Ax=0$ if A is invertible, where $A \in M_{n\times n}(\mathbb{R})$.

Observe that the set of all linear transformations from V into W forms a vector space under the following vector addition and scalar multiplication defined as

Let $T, U : V \to W$ be linear transformation

then $(T+U)(\alpha) = T\alpha + U\alpha$

$(cT)(\alpha) = cT(\alpha)$

The vector space such obtained is denoted as $L(V, W)$.

It is left as an exercise to the students to verify all the properties of vector addition and scalar multiplications defined above.

<u>Remark</u>: We know that the set of functions $f : V \to W$ (where V and W need not be vector spaces) forms a vector space under pointwise addition and scalar multiplication defined above, then what is $L(V, W)$?

It is nothing but the subspace of the vector space of functions from V into W, when V & W are vector spaces.

If $T : V \to W$ & $U : W \to Z$ are two linear transformations, where V, W & Z are vector spaces over a field $\mathbb{F}$, then their composition $U \circ T = UT : V \to Z$ is also a linear transformation from V into Z

and is defined as

$$(U \circ T)(\alpha) = U(T(\alpha)) \quad \forall \alpha \in V$$

It is easy to check. Infact,

$$(U \circ T)(c\alpha+\beta) = U(T(c\alpha+\beta))$$
$$= U(cT\alpha + T\beta)$$
$$= U(cT\alpha) + U(T\beta)$$
$$= cU(T\alpha) + U(T\beta)$$
$$= c(U \circ T)(\alpha) + (U \circ T)\beta$$

We denote this composition as UT.

Remark 1

In case of $V=W=Z$, & $T=U$, one can define $T^2 = T \circ T$ in a similar way. Also $T^n = T.T. \ldots T$ (n copies)

Remark:

In general $U \circ T \neq T \circ U$!!

Take T and D defined on space of polynomials over F as "multiplication" by x and differentiation respectively, then it is easy to check

$$TD \neq DT.$$