Mean Value Theorem (Lagrange's) (MVT)

Let $f(x)$ be a continuous function on $[a,b]$ and $f(x)$ differentiable on (a,b). Then there exists (at least one) x in (a,b) such that

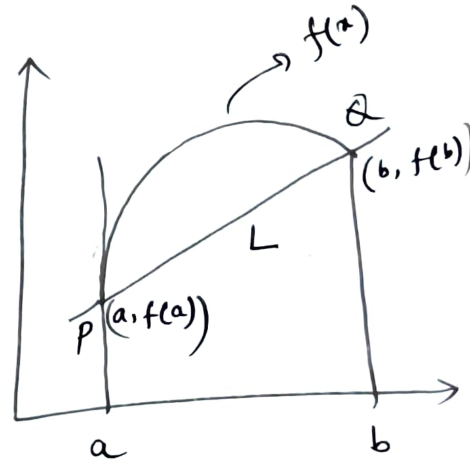
$$f'(x) = \frac{f(b)-f(a)}{b-a}$$

Proof: Let $g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right]$

[The equ^n of the line L is

$$\frac{y-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$



$g(x)$ is the difference of $f(x)$ and L i.e.

$$g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right].$$]

Now i) $g(x)$ is continuous on $[a,b]$

ii) $g'(x)$ exists on (a,b)

(iii) $g(a) = 0 = g(b)$

Then by Rolle's theorem we have $\exists\ x \in (a,b) \ni$

$$g'(x) = 0 \Rightarrow f'(x) = \frac{f(b)-f(a)}{b-a}.$$

Ex: Let $f(x)$ be differentiable on (a,b) s.t $f'(x)=0\ \forall x \in (a,b)$. Then $f(x)$ is a const function.

Solu^n: Let $x_1, x_2 \in (a,b)$. Then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Then by MVT $\exists\, x \in (x_1, x_2)$ s.t.
$$f'(x) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

But $f'(x)=0 \Rightarrow f(x_2)=f(x_1)$

$\because$ x_2 & x_1 are arbitrary points in (a,b)

$\Rightarrow f$ is const on (a,b).

Ex: Prove that between any two real roots of $e^x \sin x = 1$ there exists at least one real root of $e^x \cos x + 1 = 0$.

Solu^n: Let $g(x) = e^x \sin x - 1$. Let a, b be any two roots of $g(x)$ & $a<b$. Then $g(a) = g(b) = 0$.

Define $f(x) = e^{-x} g(x) = \sin x - e^{-x}$, &

$f(a) = f(b) = 0$.

Also $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). So by Rolle's th^m $\exists\, c \in (a,b)$, $f'(c)=0$

$\Rightarrow f'(c) = \cos c + e^{-c} = 0$

$\Rightarrow e^{c}\cos c + 1 = 0$

This proves the result.

Cauchy's mean value theorem:

If $f, g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differen on (a,b) then $\exists$ $x \in (a,b)$ such that

$$f'(x)\left(g(b)-g(a)\right) = g'(x)\left(f(b)-f(a)\right).$$

Proof: Define $h(x) = \left(f(x)-f(a)\right)\left(g(b)-g(a)\right) - \left(g(x)-g(a)\right)$

$$\times \left(f(b)-f(a)\right)$$

Apply Rolle's theorem on $h(x)$.

Def^n: Let I be an interval and $f : I \to \mathbb{R}$ a given function (i) we say that f is monotonically increasing on I if $f(x) \le f(y)$ $\forall$ $x, y \in I$ with $x \leq y$.

(ii) monotonically decreasing on I if $f(x) \geq f(y)$ for all $x, y \in I$ with $x < y$.

(iii) strictly increasing if $f(x) < f(y)$ $\forall$ $x, y \in I$ with $x < y$.

(iv) strictly decreasing if $f(x) > f(y)$ $\forall$ $x, y \in I$ with $x < y$.

Result: Let f be a differentiable fun on an interval (a, b). Then we have the following

(a) If $f'(x) > 0$ on (a, b) then f is strictly increasing on (a, b)

(b) If $f'(x) < 0$ on (a, b) then f is strictly decreasing on (a, b)

(c) If $f'(x) \geq 0$ (≤ 0) on (a, b) then f is increasing (decreasing) on (a, b).

Proof: (a) Let $a < x_1 < x_2 < b$. Then by MVT $\exists\, x \in (x_1, x_2) \ni$

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

Sin $x_2 - x_1 > 0$ so ~~$f(x_2) x_1)$~~ $f(x_2) - f(x) > 0$

$\Rightarrow f(x_2) > f(x_1)$ This proves $f(x)$ is strictly increasing on (a, b).

(b), (c) $\|^{ly}$ we can prove.

Again we consider some impotent result and remark on local maxima and minima. For the sake of completeness we will state again the local extremum theorem.

Theorem: Suppose $f : [a, b] \to \mathbb{R}$ and suppose f has either a local maximum or a local minimum at $x_0 \in (a, b)$ and if f is differentiable at x_0 then $f'(x_0) = 0$

Proof: Suppose f has a local maximum at x_0. Then $\exists\, \delta > 0$ such that

$$f(x) \leq f(x_0) \quad \forall\, x \in (x_0 - \delta, x_0 + \delta) \subset I.$$

First, considering the points to the left of x_0, we have

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0 \quad \text{for} \quad x_0 - \delta < x < x_0$$

$$\Rightarrow \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = Lf'(x_0) \geq 0$$

Next considering points to the right of x_0, we have

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0 \quad \text{for} \quad x_0 < x < x_0 + \delta$$

$$\Rightarrow \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = Rf'(x_0) \leq 0.$$

Since $f'(x)$ exists at x_0. So $Rf'(x_0) = Lf'(x_0) = f'(x_0)$

$\Rightarrow f'(x_0) \geq 0$ & $f'(x_0) \leq 0 \Rightarrow f'(x_0) = 0$.

Remark: The fu^n $f(x) = |x|$ has a local minimum at 0, although $f(x)$ is not differentiable at 0. This shows that a fu^n may have local extremum at a point without being diff^n at the point.

Remark: Consider $f(x) = x^3$, $x \in [-1, 1]$, does not have a local minimum or maximum at 0, although $f'(0) = 0$. It is notice that the above theorem does not assert that a point x_0 where $f'(x_0) = 0$ is necessarily a local extemum. That is the converse of the theorem is not true.

Def^n: (i) A point x_0 is called a critical point of f if either f is not differentiable at x_0 or if it is $f'(x_0) = 0$.

(ii) A critical point that is not a local extremum is called a sddle point.

Second order derivative test for relative extrema

Let f be a function defined in an open interval containing x_0 such that $f'(x_0) = 0$. Then we have the following

(i) If $f''(x_0) > 0$ then $f(x_0)$ is a local minimum for f

(ii) If $f''(x_0) < 0$ then $f(x_0)$ is a local maximum for f

(iii) If $f''(x_0) = 0$ then the test is inconclusive (a maximum or a minimum or neither may occur)

Example:

$$f(x) = 2x^3 + 3x^2 + 1 \quad \text{then}$$

$$f'(x) = 6x(x+1), \qquad f''(x) = 6(2x+1)$$

$f'(x) = 0 \Rightarrow x = 0,\ x = 1$ are the critical values of f.

$$f''(0) = 6 > 0, \qquad f''(-1) = -6 < 0$$

So -1, is a point of local maxima and 0 is a point of local minima.