Tutorial-3(Solution)

Linear Algebra-(IC152)
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Answer 1

(a) $\langle Ax, Ay\rangle = \langle A^tAx, y\rangle = \langle Ix, y\rangle = \langle x, y\rangle$.

(b) Let for some non-zero x, we have $Ax = \lambda x$, so

$$\begin{aligned} x^t x &= x^t A^t A x = (Ax)^t A x = \lambda x^t \lambda x = \lambda^2 x^t x \\ \implies \lambda^2 &= 1 \implies \lambda = 1 \text{ or } -1. \end{aligned}$$

(c) Since all the eigenvalues are non zero so A is an invertible matrix. Hence set of column vectors of A form a basis of $\mathbb{R}^n$ and $A^tA = AA^t = I$ implies that the columns are orthonormal vectors.

Answer 2

Here $W = span\left\{ u = \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} \right\}$. Then orthonormal basis for W is $\{\frac{u}{||u||}\}$ where $||u|| = \sqrt{2}$. Take $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in W^\perp$, then $u.x = 0 =$ implies that $x = \begin{pmatrix} a \\ b \\ -ai \end{pmatrix}$

Therefore $W^\perp = span\left\{ v = \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}, w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. Thus $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis of $W^\perp$. Clearly S is orthogonal set. Hence the orthonormal basis is $S = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

Answer 3

Definition: Let V be a vector space over the field F. An **inner product** on V is a function that assigns, to every ordered pair of vectors x and y in V, a scalar in F, denoted $\langle \cdot, \cdot \rangle$ such that for all x, y and z in V and $c \in F$ the following hold:

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(ii) $\langle cx, y \rangle =$ c$\langle x, y \rangle$

(iii) $\overline{\langle x, y \rangle} = \langle y, x \rangle$

(iv) $\langle x, x \rangle > 0$ if $x \neq 0$.

(a) Take $0 \neq (a, b) = (c, d) = (1, 1) \in \mathbb{R}^2$ then $\langle (a, b), (c, d) \rangle = ac - bd = 1 - 1 = 0$. Thus $\langle (a, b), (c, d) \rangle = ac - bd$ is not defines an inner product on $\mathbb{R}^2$ as it does not satisfy condition (iv).

(b) Take $A = B = I_2$, Then

$$\langle 2A, B \rangle = 3 \neq 2\langle A, B \rangle.$$

Therefore, (ii) not holds good.

(c) Take $f(x) = g(x) = 1$ then

$$\langle f, f \rangle = \int_{-1}^{1} f f' dx = \int_{-1}^{1} 1 \cdot 0 dx = 0.$$

Thus, (iv) not holds good as $f(x) \neq 0$.

Answer 4

Here $V = C([-1, 1])$. Let $f \in W_e$ and $g \in W_o$. Then

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, \mathrm{d}x = 0,$$

since product of odd function with even function is again an odd function. Hence $W_o \subset W_e^{\perp}$. Now, take $\phi \in V$ and $h \in W_e^{\perp}$, then

$$\begin{aligned}\int_{-1}^{1} \phi(x)(h(x) + h(-x)) \, \mathrm{d}x &= \int_{-1}^{1} \phi(x) h(x) \, \mathrm{d}x + \int_{-1}^{1} \phi(x) h(-x) \, \mathrm{d}x \\ &= \int_{-1}^{1} (\phi(x) + \phi(-x)) h(x) = 0\end{aligned}$$

since $\phi(x) + \phi(-x)$ is an even function. So, $h(x) + h(-x) = 0$. Hence $W_e^{\perp} \subset W_o$.

Answer 5

Let $\beta_1 = (1, -1, 1, 1)$, $\beta_2 = (1, 0, 1, 0)$ and $\beta_3 = (0, 1, 0, 1)$. By Gram-Schmidt process, we get

$$
\begin{aligned}
u_1 &= \beta_1 = (1, -1, 1, 1) \\
u_2 &= \beta_2 - \frac{\langle \beta_2, u_1 \rangle}{||u_1||^2} u_1 = (1, 0, 1, 0) - \frac{(1, -1, 1, 1)}{2} = (1/2, 1/2, 1/2, -1/2) \\
u_3 &= \beta_3 - \frac{\langle \beta_3, u_1 \rangle}{||u_1||^2} u_1 - \frac{\langle \beta_3, u_2 \rangle}{||u_2||^2} u_2 \\
&= (0, 1, 0, 1) - 0 - 0 = (0, 1, 0, 1).
\end{aligned}
$$

Then consider $v_1 = \frac{u_1}{||u_1||}$, $v_2 = \frac{u_2}{||u_2||}$ and $v_3 = \frac{u_3}{||u_3||}$