Tutorial 2. (Solutions)

Linear Algebra-(IC152)
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Answer 1

We have

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -5 & 2 \\ 6 & -7 & 3 \end{bmatrix}$$

Then,

$$A^2 = A.A = \begin{bmatrix} -3 & 4 & -2 \\ -4 & 7 & -4 \\ -4 & 8 & -5 \end{bmatrix}$$

Continuing in the same way,

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

and $A^{100} = (A^4)^{25} = I^{25}$, where I is the identity matrix.

Answer 2

We have A is 3×3 matrix having one eigenvalue 1. Let λ_1, λ_2 be other two eigenvalues of A. Since sum of eigenvalues is trace and product of eigenvalues is determinant of matrix. Therefore, $\lambda_1 + \lambda_2 = 2$ and $\lambda_1.\lambda_2 = 2$. On solving these two equations, we get $\lambda_1 = 1 + \iota$ and $\lambda_2 = 1 - \iota$. Now if λ is an eigenvalue of A then $p(\lambda)$ is an eigenvalue of $p(A)$. Therefore, eigenvalues of $A^2 - 2I$ are $-1, 2\iota - 2, -2\iota - 2$.

Answer 3

We know that if λ is an eigen value of A, then λ^n is an eigen value of A^n.
Claim If λ is eigen value of A, then $p(\lambda)$ is eigen value of polynomial $p(A)$.
suppose for some non zero vector v, we have $Av = \lambda v$. Write $p(x) = \sum_{i=0}^{k} a_i x^i$ be a polynomial, then

$$p(A)v = \left(\sum_{i=0}^{k} a_i A^i\right) v = \sum_{i=0}^{k} a_i A^i v.$$

Now using the fact that if $Av = \lambda v \implies A^i v = \lambda^i v$, $i = 1, 2, ..., n$, we have

$$= \sum_{i=0}^{k} a_i \lambda^i v = p(\lambda)v.$$

For this question, $p(x) = x^{100} + 1$ and $\lambda = 1, -1, 0$. So the eigen value of $p(A) = A^{100} + I$ are $p(1) = 2, p(-1) = 2, p(0) = 1$. Hence the determinant of $p(A) = A^{100} + I$ is $p(1) \times p(-1) \times p(0) = 1 = 4$.

Answer 4

Given that A is 2×2 matrix satisfying $(tr(A))^2 > 4\det(A)$. Note that for any 2×2 matrix A, the characteristic equation is

$$0 = \det(A - \lambda I) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A), \tag{1}$$

which is a quadratic equation in variable λ. So the roots of this equation are given by

$$\lambda = \frac{\mathrm{tr}(A) \pm \sqrt{(\mathrm{tr}(A)^2 - 4\det(A)}}{2}.$$

As $(\mathrm{tr}(A))^2 > 4\det(A)$, Therefore the equation (1) has two real and unequal roots, say λ_1 and λ_2. i.e. λ_1 and λ_2 are eigen values of A. Hence the matrix A is digonalizable. Because Corresponding to eigen value λ_i, there is an eigen vector v_i so that $P^{-1}AP = D$, where $P = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$, and $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

Answer 5

Suppose $\lambda \in \mathbb{R}$ is an eigen value of the operator T associated with the non zero eigen vector f such that

$$T(f(x)) = \int_0^x f(t)dt = \lambda f(x), \quad \forall x \in \mathbb{R}.$$

Note that from calculus, the function $F(x) = \int_0^x f(t)dt$ is differentiable and $F'(x) = f(x)$ with $F(0) = 0$. Thus we have the following initial value problem (ivp)

$$\lambda F'(x) = F(x), F(0) = 0.$$

If $\lambda = 0$ we get $f(x) = F'(x) = 0$ a contradiction.

If $\lambda \neq 0$, on solving ivp, we get $F(x) = ke^{x/\lambda}$ for some constant k. But using $F(0) = 0$, we have $\int_0^x f(t)dt = F(x) \equiv 0$, a contradiction. Thus this operator has no eigenvalue.

Answer 6

The given matrix is

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Characteristic equation of A is given by

$$\begin{aligned} \det(Ix - A) &= 0 \\ \implies (x-3)^3 &= 0 \end{aligned}$$

with $A - 3I \neq 0$ and $(A - 3I)^2 = 0$. Therefore, $(x - 3)^2$ is the minimal polynomial of A.