# Integration

Let $f(x)$ be a bounded function on a closed interval $[a,b]$.

A Partion $P = \{x_0, x_1, \ldots, x_n\}$ of an interval $[a,b]$ is a finite set of points arranged in such a way that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The partition P defines n closed subintervals

$$[x_0, x_1],\ [x_1, x_2],\ \ldots\ [x_{k-1}, x_k],\ \ldots\ [x_{n-1}, x_n]$$

of $[a,b]$

The typical closed subinterval $[x_{k-1}, x_k]$ is called $k^{th}$ subinterval of the partition P. The length of the $k^{th}$ subinterval is

$$\Delta x_k = x_k - x_{k-1}, \quad k = 1, 2, \ldots n.$$

The largest of the lengths of these subintervals is called the norm (some-times called the mesh or width) of the partition P and is denoted by $\|P\|$

that is,

$$\|P\| = \max_{k=1,2,\ldots n} \Delta x_k = \max_{k=1,2,\ldots n} (x_k - x_{k-1}).$$

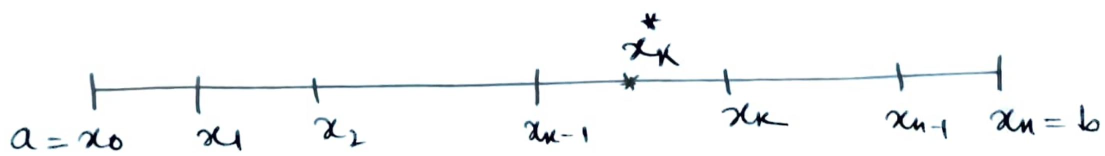
The family of all partitions of $[a,b]$ will be denoted by $\mathcal{P}[a,b]$ or $\mathcal{P}$ when the interval under ~~area~~ discussion is clear.

Note : A standard partition or equally spaced partion is a partion all of whose subintervals are of equal length.

Now we have a partion

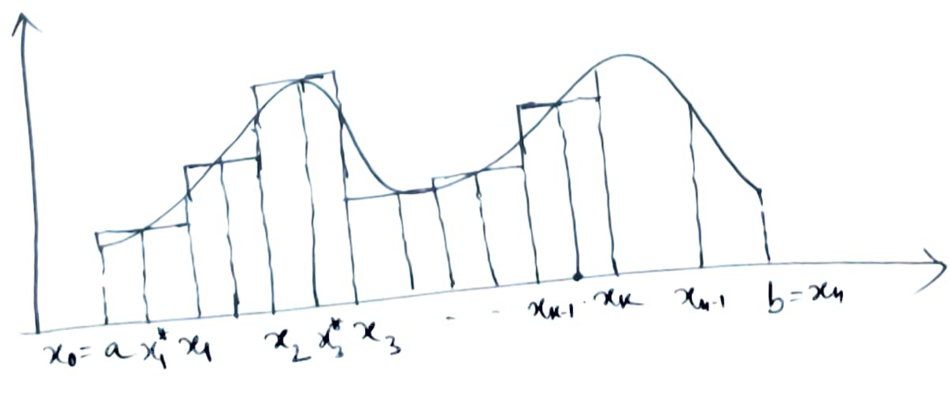
$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}.$$

For each $k = 1, 2, \ldots, n$ choose an arbitrary point $x_k^* \in [x_{k-1}, x_k]$




Define $A_k = f(x_k^*)\,\Delta x_k$ and $S_n = \sum_{k=1}^{n} A_k$

This sum, which depends on the partion P and the choice of points $x_1^*, x_2^*, \ldots, x_n^*$ is called the Riemann sum of integral sum also called with respect to P and f over the interval $[a, b]$ points $x_k^* \in [x_{k-1}, x_k]$, $k = 1, 2, \ldots n$.



Also define

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x) \quad , \quad M_k = \sup_{x \in [x_{k-1}, x_k]} f(x)$$

Then each partition determines two sums that correspond to overestimates and underestimates of the possible area

$$\overline{S_n} = \sum_{k=1}^{n} M_k \Delta x_k \quad , \quad \underline{S_n} = \sum_{k=1}^{n} m_k \Delta x_k .$$

Here $\overline{S_n}$ and $\underline{S_n}$ aree reffered as an upper sum and lower sum of $f$ on $[a, b]$ respectevely.

<u>Darboux Integral</u>: Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is a paretition and $x_k^* \in [x_{k-1}, x_k]$ $(k = 1, 2, \ldots, n)$ is arbibarey. Then

$\overline{S_n} : \Rightarrow$ upper Darboux sum or upper integral sum

$\underline{S_n} : \Rightarrow$ lower Darboux sum or lower integral sum

$S_n : \Rightarrow$ Riemann sum,

of the function $f$ associated with partion $P$.

These aree usually denoted by

$$\overline{S_n} = U(P, f) \;,\; \underline{S_n} = L(P, f) \quad \&$$

$$S_n = \sigma(P, f, x^*) .$$

For a bounded fu$^n$ $f$ on $[a,b]$, we define

$$m = \inf_{x\in[a,b]} f(x) \quad , \quad M = \sup_{x\in[a,b]} f(x)$$

Then

$$m \le m_k \le f(x_k^*) \le M_k \le M$$

$$m\,\Delta x_k \le m_k\,\Delta x_k \le f(x_k^*)\,\Delta x_k \le M_k\,\Delta x_k \le M\,\Delta x_k$$

Taking sum

$$m(b-a) \le L(P,f) \le \sigma(P,f,x^*) \le U(P,f) \le M(b-a)$$

holds for every partition $P$. In other words

$$\{U(P,f) : P\in \mathcal{P}[a,b]\} \ \& \ \{L(P,f) : P\in\mathcal{P}[a,b]\}$$

form bounded sets.

<u>Def$^n$ (Darboux Integral)</u>: The upper (Darboux) integral of $f$ on $[a,b]$ is defined by

$$U(f) = \overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) : P\in\mathcal{P}[a,b]\}$$

and the lower (Darboux) integral of $f$ on $[a,b]$ defined by

$$L(f) = \underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) : P\in\mathcal{P}[a,b]\}.$$

A bounded function $f$ defined on $[a,b]$ is said to be integrable or Darboux integrable if

$$U(f) = L(f).$$

The common value is called the integral of $f$ on $[a,b]$ or the definite integral of $f$ from $a$ to $b$ and is denoted by $\int_a^b f(x)\,dx$.

If $U(f) > L(f)$, then we say that $f$ is not Darboux integrable.

<u>Remark</u> (i) If $\int_a^b f(x)dx$ exists then

$$L(P,f) \leq \int_a^b f(x)\,dx \leq U(Q,f) \quad \forall\ P, Q \in \mathcal{P}[a,b]$$

(ii) We follow the convention that whenever an interval $[a,b]$ is employed, we assume $a<b$ and therefore $\int_a^b f(x)\,dx$ for $a<b$ only. If $a=b$ we set

$$\int_a^b f(x)\,dx = 0.$$

(iii) The function $f$ that is being integrated is called the integrand, the interval $[a,b]$ is the interval of integration and the endpoints $a$ and $b$ are called lower limit & upper limit respectively.

(IV) Some time people call the lowe & upper Dareboux integral the lower and upper Riemann integral reespetively. Also $\int_a^b f(x)\,dx$ is often reeferred to as the Riemann integral of $f$ on $[a,b]$. This is because Riemann's definition of integrability is slightly different. However will see that these two definations arce actully equivalent.

Ex

$$f(x) = c, \text{ constant} \quad \forall\, x \in [a,b]$$

$$U(P,f) = \sum_{k=1}^{n} M_k \Delta x_k = c(b-a)$$

$$L(P,f) = \sum_{k=1}^{n} m_k \Delta x_k = c(b-a)$$

So $$\sup_{P\in\mathcal{P}} L(P,f) = \inf_{P\in\mathcal{P}} U(P,f) = c(b-a)$$

$$\Rightarrow \int_a^b f(x)\,dx = c(b-a).$$

Ex:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$$

$$U(P,f) = 1, \quad L(P,f) = 0, \quad \forall\, P \in \mathcal{P}[0,1]$$

Since: $P = \{x_0, x_1, \dots, x_n\}$, be any partion on $[0,1]$. Since every interval $[x_{k-1}, x_k]$ contains both rational and irrational point, so we have

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x) = 0, \quad M_k = \sup_{x \in [x_{k-1}, x_k]} f(x) = 1$$

$$U(P,f) = \sum_{k=1}^{n} 1 \cdot (x_k - x_{k-1}) = (x_n - x_0) \cdot 1 = 1$$

$$L(P,f) = \sum_{k=1}^{n} 0 \cdot (x_k - x_{k-1}) = 0.$$

So $\overline{\int_0^1} f(x)\,dx = 1, \neq \underline{\int_0^1} f(x) = 0.$

$\Rightarrow f$ is not integrable on $[0,1]$

Criteria for Integrability :

Riemann's criterion for integrability: Let $f$ is a bounded fun$^n$ on $[a,b]$, then $f$ is integrable on $[a,b]$ if and only if for each $\epsilon > 0$ there is a partition $P$ of $[a,b]$ s.t

$$U(P,f) - L(P,f) < \epsilon$$

Theorem: If $f$ is bounded function on $[a,b]$, then $f$ is integrable iff for each $\epsilon > 0$ $\exists$ $\delta > 0$ s.t

$$U(P,f) - L(P,f) < \epsilon$$

for all partitions $P$ of $[a,b]$ for which $\|P\| < \delta$.

**Defn (Riemann Integrability):** A bounded function $f$ defined on $[a,b]$ is said to be Riemann integrable on $[a,b]$ if there exist a number $I$ with following property: For every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|\sigma(P, f, x^*) - I| < \epsilon$$

for every Riemann sum $\sigma(P, f, x^*)$ of $f$ associated with a partition $P$ of $[a,b]$ for which $\|P\| < \delta$. In this case we write

$$\lim_{\|P\| \to 0} \sigma(P, f, x^*) = I.$$

Formally the quantity $I$ is the definite integral of $f$ on $[a,b]$

## Equivalence of the defination of Reemann and Darboux

If $f$ is a bounded function on $[a,b]$, then $f$ is Riemann integrable iff $f$ is Darboux integrable.

**Note:** Standard partition or equally spaced partition $[a,b]$ $P = \{x_0, x_1, \ldots, x_n\}$, where

$$x_k = a + \frac{k}{n}(b-a), \quad k = 0, 1, 2, \ldots n.$$

## Some examples

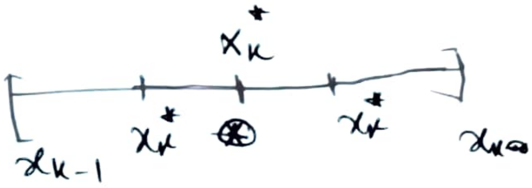
**Example (1)** Let $f(x) = x$, on $[a, b]$

Then the Riemann sum takes the form

$$S_n = \sigma(P, f, x^*) = \sum_{k=1}^{n} x_k^* (x_k - x_{k-1})$$

where $f(x_k^*) = x_k^*$ and $x_k^* \in [x_{k-1}, x_k]$ is arbitrary.

Note that $x_k^*$ is either the midpoint of the interval $[x_{k-1}, x_k]$ or to the left of it or to the right of it. Consequently we write




$$x_k^* = \frac{x_{k-1} + x_k}{2} + \delta_k ,$$

where

$$|\delta_k| \le \frac{x_k - x_{k-1}}{2} \le \frac{\|P\|}{2} \quad \text{for } k = 1, 2, \dots n.$$

So we have

$$S_n = \sum_{k=1}^{n} \frac{x_{k-1} + x_k}{2} (x_k - x_{k-1}) + \sum_{k=1}^{n} \delta_k (x_k - x_{k-1})$$

$$= \frac{1}{2} \sum_{k=1}^{n} \left(x_k^2 - x_{k-1}^2\right) + E_n = \frac{b^2 - a^2}{2} + E_n.$$

$$E_n = \sum_{k=1}^{n} \delta_k (x_k - x_{k-1})$$

Now

$$|E_n| \leq \sum_{k=1}^{n} |\delta_k| (x_k - x_{k-1}) \leq \frac{\|P\|}{2} \sum_{k=1}^{n} (x_k - x_{k-1})$$

$$= \frac{\|P\|}{2} (b-a) \longrightarrow 0$$

as $\|P\| \to 0$. Hence $\int_a^b x\,dx = \frac{b^2 - a^2}{2}$ as

$$\lim_{\|P\| \to 0} \sigma(P, f, x^*) = \frac{b^2 - a^2}{2}.$$

The next example gives us a method of evaluating the definite integral of an integrable function as the limit of a sequence.

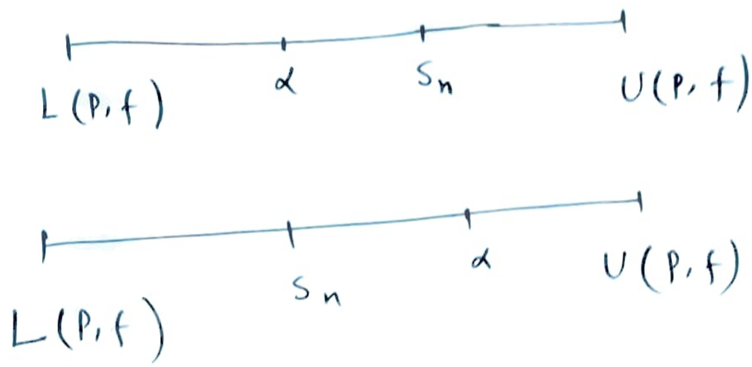
**Example**: Suppose that $f$ is integrable on $[a,b]$. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n, \quad S_n = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right)$$

**Solⁿ**: Suppose that $f$ is integrable on $[a,b]$. Then $L(f) = U(f) = \int_a^b f(x)\,dx =: \alpha$. Choose the standard partition $P = \{x_0, x_1, \dots, x_n\}$ and let $h = \frac{b-a}{n}$. Consider $x_k = a + kh$ for $k = 1, 2, \dots, n$ as points of division of $[a,b]$ into $n$ equal parts of length $\Delta x_k = \Delta x = h$

By definition of $L(P,f)$, $U(P,f)$, $L(f)$, $U(f)$ and $\sigma(P,f,x^*)$, it follows that

$$L(P,f) \le \sigma(P,f,x^*) \le U(P,f) \quad \& \quad L(P,f) \le \alpha \le U(P,f)$$




so we get $\quad |S_n - \alpha| \le U(P,f) - L(P,f)$

Since $f$ is integrable then for given $\epsilon > 0$ $\exists\, \delta > 0$

s.t. $\quad U(Q,f) - L(Q,f) < \epsilon$ for all partitions

$Q$ of $[a,b]$ for which $\|Q\| < \delta$.

Now we have for our partition $\|P\| = h = \dfrac{b-a}{n}$

Let $n \geq N$ then $\quad \|P\| \le \dfrac{b-a}{N}$.

If we choose $N$ s.t. $\dfrac{b-a}{N} < \delta$ then we are done

Now by Archemedian property $\exists\, N \in \mathbb{N} \ni$

$$N\delta > b-a$$

choose this $N$. then

~~$\forall n \geq N$~~ $\forall n \geq N$ we have $\|P\| \leq \frac{b-a}{N} < \delta$.

Consequently given $\epsilon > 0$ $\exists N$ s.t $\forall n \geq N$

$$|S_n - \alpha| \leq U(P,f) - L(P,f) < \epsilon,$$

$$\Rightarrow \lim_{n \to \infty} S_n = \alpha.$$

## Some Important Results

**Result-1**: Every monotone function on $[a,b]$ is integrable on $[a,b]$. The converse is false

**Result-2**: Every continuous fun $f$ on $[a,b]$ is integrable. The converse is false.

**Example**: $f(x) = x^2$ on $[a,b]$

$$S_n = \sum_{k=1}^{n} f(x_k^*)\Delta x \text{, which simplifies to}$$

$$S_n = \sum_{k=1}^{n} \left[a + \frac{k(b-a)}{n}\right]^2 \frac{b-a}{n}$$

$$= \frac{b-a}{n}\left[\sum_{k=1}^{n} a^2 + \frac{2a(b-a)}{n}\sum_{k=1}^{n} k + \frac{(b-a)^2}{n^2}\sum_{k=1}^{n} k^2\right]$$

$$S_n = \frac{b-a}{n}\left[a^2 n + 2a(b-a)\frac{n(n+1)}{2} + \frac{(b-a)^2}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

Taking limit $n \to \infty$ we conclude that

$$\int_a^b x^2\,dx = (b-a)\left[a^2 + a(b-a) + \frac{(b-a)^2}{3}\right] = \frac{b^3-a^3}{3}.$$

## Properties of integral :

Suppose that $f$ and $g$ are integrable on $[a,b]$. We have the following

(I) $c_1 f + c_2 g$ is integrable for constants $c_1$ & $c_2$

$$\int_a^b (c_1 f + c_2 g)\,dx = c_1\int_a^b f(x)\,dx + c_2\int_a^b g(x)\,dx$$

(II) $f(x) \le g(x)$ on $[a,b]$ then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

(III) If $m \le f(x) \le M$ for $x \in [a,b]$.

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

(IV) Let $c \in (a,b)$, then $f$ is integrable on $[a,c]$ & on $[c,b]$

Also $\int_a^b f\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$