Eigenvalue eigenvector

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Linear algebra- II (IC152)

Annihilating polynomial

- Let T be a linear operator on a finite dimensional vector space V over $\mathbb{F}$.
- A polynomial $p(x) \in P[x]$ is called a monic polynomial, if the co-efficient of the highest power of x in $p(x)$ is unity.
- For example, $p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \in P[x]$ is a monic polynomial.
- We say that T satisfies the monic polynomial $p(x)$ if

$$P(T) = T^n + a_1 T^{n-1} + \cdots + a_{n-1}T + a_n I = 0.$$

- A polynomial $p(x)$ such that $p(T) = 0$ is called an annihilating polynomial for T.

Definition

Let T be a linear operator on a finite-dimensional vector space V over the field $\mathbb{F}$. The monic polynomial $p_T(x)$ of least degree such that $p_T(T) = 0$, is called the minimal polynomial of T.

The minimal polynomial p for the linear operator T is uniquely determined by these three properties:

- p is a monic polynomial over the scalar field $\mathbb{F}$.
- $p(T) = 0$.
- No polynomial over $\mathbb{F}$ which annihilates T has smaller degree than p has.

Remark

Similarly, we can define the minimal polynomial for square matrix A.

Examples

- Let us compute the minimal polynomial $p(x)$ of each of the matrix operators

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- The characteristic polynomial $f(x)$ for each of these is simply $x^2(x-1)$.
- Since the minimal polynomial divides the characteristic polynomial and a root of the characteristic polynomial is necessarily the root of the minimal polynomial (Which I will prove later), the possibilities for the minimal polynomials of these matrices are only $x(x-1)$ and $x^2(x-1)$.
- Denoting these by $p_1(x)$ and $p_2(x)$, respectively.

- Note that

$$p_1(A) = A(A-I) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \mathbf{0}_{3\times 3},$$

$$p_1(B) = B(B-I) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}_{3\times 3}$$

and

$$p_1(C) = C(C-I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}_{3\times 3}.$$

- Hence the minimal polynomials of A, B and C are $x^2(x-1), x(x-1)$ and $x(x-1)$ respectively.

Theorem

Let T be a linear operator on a finite-dimensional vector space V over the field $\mathbb{F}$. Then the characteristic and minimal polynomials for T have the same roots, except for multiplicities.

Outline of the proof

- Let $p(x)$ be the minimal polynomial for T, so that $p(T) = \mathbf{0}$.
- let c be a root of $p(x)$, so that $p(c) = 0$.
- Since c be a root of $p(x)$, $(x - c)$ divides $p(x)$ in $P[x]$, that is, there exist some $q(x)$ in $P[x]$ such that

$$p(x) = (x - c)q(x), \tag{1}$$

  where $\deg q(x) < \deg p(x)$.
- Since $p(x)$ is the minimal polynomial for T and $\deg q(x) < \deg p(x)$, so $q(T) \neq 0$.
- This means that there exist a non-zero vector $v \in V$ such that $q(T)v \neq 0$.

- Let $x = q(T)v \neq 0$. From (1), we have

$$0 = p(T)v = (T - cI)q(T)v = (T - cI)x,$$

  which shows that c is a eigen value of T.
- Thus, c is a root of the characteristic polynomial for T.

Outline of the proof cont. converse part

- Conversely, let c be a root of the characteristic polynomial for T, that is, c is a eigen value of T.
- Then there exists some $0 \neq v \in V$ such that $Tv = cv$.
- Since $p(T)$ is a polynomial, we observe that $p(T)v = p(c)v$.
- Also, by hypothesis, we have $p(T) = 0$, which implies $p(c) = 0$.
- This shows that c is a root of the minimal polynomial for T.
- This completes the proof.

Theorem

Let T be a diagonalizable linear operator on a finite-dimensional vector space V over the field $\mathbb{F}$ and let $c_1, \ldots, c_k$ be the distinct characteristic values of T. Then the minimal polynomial for T is the polynomial

$$p(x) = (x - c_1) \cdots (x - c_k).$$

Outline of the proof

- We know that each characteristic value of T is a root of the minimal polynomial for T.
- Each of the polynomials $(x - c_1) \cdots (x - c_k)$ is a factor of the minimal polynomial for T.
- Let v be the eigen vector of T. Then

$$(T - c_i I)v = 0, \quad \text{for some } i, 1 \leq i \leq k.$$

- It follows that for each eigenvector v,

$$(T - c_1 I) \cdots (T - c_k I)v = 0. \tag{2}$$

- Since T is diagonalizable, there exists a basis β of V consisting of eigenvectors of T.

- Using (2) we see that for all $x \in V$

$$(T - c_1 I) \cdots (T - c_k I)x = 0. \tag{3}$$

- This shows that $p(T) = \mathbf{0}$.
- Hence $p(x) = (x - c_1) \cdots (x - c_k)$ is a minimal polynomial for T.

Remark

The above theorem tells that if T is a diagonalizable, then the minimal polynomial for T is a product of distinct linear factors.

Theorem

The minimal polynomial of a linear operator T divides its characteristic polynomial.

Outline of the proof

- Let $p(x)$ be the minimal polynomial for T, that is, $p(T) = \mathbf{0}$.
- Let $f(x)$ be the characteristic polynomial of T. By Cayley Hamilton theorem, we have $f(T) = \mathbf{0}$.
- Thus, by division theorem there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)p(x) + r(x)$, where $\deg r(x) < \deg p(x)$.
- However, we can clearly see that $f(T) = \mathbf{0}$ implies $r(T) = \mathbf{0}$.
- Because $r(x)$ has degree strictly less than $p(x)$, this violates the minimality of the degree of $p(x)$ unless $r(x) = 0$.
- Thus, $p(x)$ divides $f(x)$.

Example 1

- Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
- The characteristic polynomial of T is

$$\det(A - xI_{2\times 2}) = x^2 + 1.$$

- The eigenvalues of A are $\pm i$.
- These roots also satisfy the minimal polynomial $p(x)$ for T and so $p(x)$ is divisible by $x^2 + 1$.
- Hence $p(x) = x^2 + 1$ is the minimal polynomial for T.
- It is easy to verify that $A^2 + I = \mathbf{0_{2\times 2}}$.

Example 2

- Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & -2 & 3 \end{bmatrix}$.
- The characteristic polynomial of T is

$$f(x) = \det(A - xI_{3\times 3}) = (x-5)(x-3)(x+3).$$

- The eigenvalues of A are $5, 3$ and -3.
- By previous Theorem, the minimal polynomial for T is $p(x) = (x-5)(x-3)(x+3)$.
- Hence $f(x) = p(x)$.
- It is easy to verify that

$$(A - 5I_{3\times 3})(A - 3I_{3\times 3})(A + 3I_{3\times 3}) = \mathbf{0_{3\times 3}}.$$

Example 3

- Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & -2 & 3 \end{bmatrix}$.
- The characteristic polynomial of T is

$$f(x) = \det(A - xI_{3\times 3}) = (x-2)^2(3-x).$$

- We know that minimal polynomial for T divides its characteristic polynomial.
- Thus the possible minimal polynomials for T can be either $p(x) = (x-2)(3-x)$ or $p(x) = (x-2)^2(3-x)$.
- Let us take $p(x) = (x-2)(3-x)$.

Example 3 cont.

- We have

$$
\begin{aligned}
p(A) &= (3I_{3\times3} - A)(A - 2I_{3\times3}) \\
&= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \mathbf{0}_{3\times3}.
\end{aligned}
$$

- This shows that $p(x) = (x-2)(3-x)$ is not the minimal polynomial for T.
- Hence the minimal polynomial for T is $p(x) = (x-2)^2(3-x)$ which is same as the characteristic polynomial of T.
- Note that T is not diagonalizable.

Example 4

- Let V be the finite-dimensional vector space over the field $\mathbb{F}$. We will find the minimal polynomial for Identity operator I and zero operator $\mathbf{0}$ on V.
- We have $I - 1.I = \mathbf{0}$, that is $p(I) = \mathbf{0}$, where $p(x) = x - 1$ is the lowest degree such that $p(I) = \mathbf{0}$.
- Hence $x - 1$ is the minimal polynomial for the identity operator.
- Similarly, $p(x) = x$ is is the lowest degree such that is the lowest degree such that $p(\mathbf{0}) = \mathbf{0}$.
- Hence x is the minimal polynomial for the zero operator.

Let V be the vector space of $n \times n$ matrices over the field $\mathbb{F}$. Let A be a fixed $n \times n$ matrix. Let T be the linear operator on V defined by

$$T(B) = AB \text{ for all } B \in V.$$

Then we will prove that minimal polynomial for T is the minimal polynomial for A.

Proof.

1. Let $p(x) = x^n + a_1x^{n-1} + ... + a_n \in \mathbb{F}[x]$ be the minimal polynomial for T of degree n and $q(x) = x^m + b_1x^{m-1} + ... + b_m \in \mathbb{F}[x]$ be the minimal polynomial for A of degree m. Then by C-H theorem $p(T) = O$ and $q(A) = O$.
2. We see that

$$O = p(T)I = (T^n + a_1T^{n-1} + ... + a_nI)I.$$

$$= A^n + a_1A^{n-1} + ... + a_nI = p(A).$$

3. Claim: $p(x) = q(x)$. We will first show $q(x)$ divides $p(x)$ and then show that $p(x)$ divides $q(x)$, which completes our proof.

Proof.

1. Let c is the root of $p(x)$, we can write $p(x) = (x - c)q(x) + r(x)$, where $r(x) = 0$ or $\deg r(x) < \deg q(x)$. But we have $p(A) = O$ and $q(A) = O$ therefore $r(A) = O$.
2. If $r(x) \neq 0$ then $\deg r(x) < \deg q(x)$ and $r(A) = O$ forces us to choose $r(x) = 0$ (use contradiction). Hence $p(x) = (x - c)q(x)$ implies $q(x)$ divides $p(x)$.
3. Finally we will show that $p(x)$ divides $q(x)$. We have

$$O = q(A)B = (A^n + b_1 A^{n-1} + \ldots + b_n I)B.$$

$$= [T^n(I) + b_1 T^{n-1}(I) + \ldots + b_n I]B.$$

$$= (T^n + b_1 T^{n-1} + \ldots + b_n I)B = q(T)B, \text{ for all } B \in V.$$

   This implies that $q(T) = O$.
4. Since $p(x)$ is the minimal polynomial for T and $q(T) = O$, so $p(x)$ divides $q(x)$. Hence $p(x) = q(x)$.

$\square$