# Solution of Tutorial-3

(1) (i) $a_n = \frac{n^2+1}{(n+3)(n+4)} \to 1 \neq 0$ So $\sum a_n$ not convergent

(ii) $a_n = (-1)^n \frac{n}{n+2}$, $a_{2n} = (-1)^{2n} \frac{2n}{2n+2} \to 1$

So $a_n \not\to 0 \Rightarrow \sum a_n$ not convergent

(iii) $-1 \leq \sin n \leq 1 \Rightarrow 0 \leq \sin n + 1$

So $0 \leq a_n = \frac{1+\sin n}{1+n^2} \leq \frac{2}{n^2}$. Now $\sum \frac{2}{n^2}$ convergent

$\Rightarrow \sum a_n$ convergent by comparision test.

(iv) $a_n = \frac{1}{2^n+n} < \frac{1}{2^n} = b_n$

$\sum b_n$ convergent $\Rightarrow \sum a_n$ convergent.

(v) $a_n = \frac{1}{\sqrt{n(n-1)}} > \frac{1}{n} > 0$. Sin $\sum \frac{1}{n}$ divergent so

$\sum \frac{1}{\sqrt{n(n-1)}}$ divergent

(vi) $a_n = \frac{n}{4n^3-2}$, take $b_n = \frac{1}{n^2}$

$\lim \frac{a_n}{b_n} = \frac{1}{4} \neq 0$ & $\sum b_n < \infty \Rightarrow \sum a_n < \infty$.

(vii) $a_n = \frac{n}{2^n}$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2} < 1$

$\Rightarrow \sum a_n$ convergent

(viii) $a_n = \frac{(n!)^n}{n^{n^2}}$, ~~$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim (n+1)$~~

Let $\{a_n\}$ be sequⁿ with $a_n > 0$ $\forall n$, s.t $\lim \frac{a_{n+1}}{a_n} = l$.

(i) $0 \le l < 1 \Rightarrow \lim x_n = 0$

(ii) If $l > 1$ then $\lim x_n = \infty$

We will use this result

$a_n > 0 \quad \forall n \in \mathbb{N}$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{n!}{n^n}$$

Now let $x_n = \frac{n!}{n^n}$, $\frac{x_{n+1}}{x_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$

$$= \frac{n+1}{(n+1)^{n+1}} \cdot n^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} < 1$$

$\Rightarrow \lim x_n = 0$ i.e $\lim_{n\to\infty} (a_n)^{1/n} = 0 < 1$

$\Rightarrow$ By root test $\sum a_n$ convergent.

(ix) $a_n = \frac{5^n}{3^n + 4^n} > 0 \quad \forall n \in \mathbb{N}$

$$\lim (a_n)^{1/n} = \lim \frac{5}{(3^n+4^n)^{1/n}} = \frac{5}{4} > 1$$

[We have seen in tutorial to $\lim (3^n+4^n)^{1/n} = 4$ Tutorial-2 (12(v))]

So by root test $\sum a_n$ not convergent.

(X) $a_n = (-1)^{n+1}\frac{1}{n^p}$

Now $\frac{1}{n^p} \not\to 0$ if $p \le 0$ $\Rightarrow \frac{(-1)^{n+1}}{n^p} \not\to 0$ as $n \to \infty$

$\Rightarrow \sum \frac{(-1)^{n+1}}{n^p}$ not convergent if $p \le 0$

If $p > 0$ then $\frac{1}{n^p}$ is decreasing and $\frac{1}{n^p} \to 0$

So by Leibniz's test $\sum \frac{(-1)^{n+1}}{n^p}$ convergent.

(XI) Apply Leibniz's test.

(2)(i) $a_n = \frac{\sqrt{n}}{n^2+5n-1}$, $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{n^2}{n^2+5n-1} = \lim \frac{1}{1+5n^{-1}-1/n^2} = 1 > 0$$

So $\sum b_n < \infty \Rightarrow \sum a_n < \infty$

(ii), (iii) similar.

(iv) $a_n = \frac{3^n+1}{7^n+4}$, Take $b_n = \left(\frac{3}{7}\right)^n$

Apply limit comparison test.

(v) Similare to (iv)

(vi) $a_n = \sin\frac{1}{n^2}$ , $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n^2}}{1/n^2} = 1 < \infty \quad \text{So } \sum b_n < \infty \Rightarrow \sum a_n < \infty .$$

(VII) Similare to (VI)

(ix) Take $b_n = \frac{1}{n}$, Find $\lim \frac{a_n}{b_n}$ apply limit comparison test.

(3) (i) $a_n = \dfrac{5^{n+1}\, 7^{n-1}}{n!}$ . Now $\dfrac{a_{n+1}}{a_n} = \dfrac{5^{n+2}\, 7^{n}}{(n+1)!} \cdot \dfrac{n!}{5^{n+1}\, 7^{n-1}}$

$$= \frac{5\cdot 7}{n+1} \to 0 < 1$$

So by ratio test $\sum a_n < \infty$ .

(II), (iii) Similare to (i)

(iv) $a_n = \dfrac{a^n\, n!}{n^n}$ . $\dfrac{a_{n+1}}{a_n} = \dfrac{a^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \dfrac{n^n}{a^n\, n!}$

$$= \frac{a.\, n^n}{(n+1)^n} = \frac{a}{\left(1+\frac{1}{n}\right)^n} \to \frac{a}{e}$$

So if $\frac{a}{e} < 1$ then $\sum a_n$ convergent.

(v) is similare to (i) $\rightarrow$ in this case series diverge

(vi) 

$$a_n = \frac{2\cdot 4 \cdots 2n}{5\cdot 8 \cdots (3n+2)} \quad \text{so} \quad \frac{a_{n+1}}{a_n} = \frac{2n+2}{3n+5} \to 2/3 < 1$$

$\Rightarrow$ converges

(viii) $\sum \frac{n^n}{n!}$, $\quad \frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \to e > 1$

$\Rightarrow$ So the series diverges.

④ (i) $a_n = \left(\frac{n+1}{2n+3}\right)^n$, $\quad \sqrt[n]{a_n} = \frac{n+1}{2n+3} \to \frac{1}{2} < 1$

So $\sum a_n$ convergent

(ii) $a_n = \frac{1}{(\ln n)^n}$, $\quad$ & $\sqrt[n]{a_n} = \frac{1}{\ln n} \to 0$

So $\sum a_n$ convergent

(iii) $a_n = \frac{5^n}{n^{n+1}}$

So $\sqrt[n]{a_n} = \frac{1}{\sqrt[n]{n}} \frac{5}{n} \to 1\cdot 0 = 0 \quad \left[\because \lim n^{1/n} = 1\right.$

$\Rightarrow \sum a_n$ convergent by root test

⑤ (i) The seires is $\sum \frac{1}{n^p}$

This is a series of decreasing, positive terms.

$\frac{1}{n^p}$ if $p > 0$

Now $\sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$ is a geometric series

and converges iff $\frac{1}{2^{p-1}} < 1$ i.e iff $p > 1$

So By cauchy's condensation test $\sum \frac{1}{n^p}$ converges iff $p > 1$

(ii) The series has decreasing positive terms.

Now $\sum \frac{2^n}{2^n (\log 2^n)^p} = \sum \frac{1}{(\log 2)^p} \frac{1}{n^p} = \frac{1}{(\log 2)^p} \sum \frac{1}{n^p}$

converges if $p > 1$

So by Cauchy's condensation test the series is convergent

(III) By cauchy's condensation test we know that for a series $\sum a_n$ with positive decreasing terms

$$\sum a_n = \infty \Leftrightarrow \sum 2^n a_{2^n} = \infty$$

The given series has decreasing positive terms. So by condensation test it diverges iff the following series diverges

$$\sum_{n=3}^{\infty} \frac{1\times 2^n}{2^n (\log 2^n)(\log\log 2^n)} = \sum \frac{1}{n\log 2 \, \log(n\log 2)}$$

$$= \frac{1}{\log 2} \sum \frac{1}{n(\log n + \log\log 2)}$$

The series has again decreasing positive terms. By condensation test it diverges iff the following series diverges.

$$\sum_{n=3}^{\infty} \frac{2^n}{2^n(\log 2^n + \log\log 2)} = \sum_{n=3}^{\infty} \frac{1}{n\log 2 + \log\log 2}$$

Which diverges iff by condensation test the following series diverges:

$$\sum \frac{2^n}{2^n \log 2 + \log\log 2}$$ which is diverges. Because

$$a_n = \frac{2^n}{2^n\log 2 + \log\log 2} \rightarrow \frac{1}{\log 2} \neq 0.$$

So the result.

(iv) By apply Leibniz's test the series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n(\log n)^{1/3}}$ converges. But by cauchy's condensation test $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{1/3}}$ diverges. So the series is conditionally convergent.