Eigenvalue eigenvector

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Linear algebra- II (IC152)

Example

- Let $W = \{(x, y, z, w) \in \mathbb{R}^4 : x = y, z = w\}$ be a subspace of W. Then an orthonormal ordered basis of W is

$$\Big(\frac{1}{\sqrt{2}}(1, 1, 0, 0), \frac{1}{\sqrt{2}}(0, 0, 1, 1)\Big),$$

and that of $W^{\perp}$ is

$$\Big(\frac{1}{\sqrt{2}}(1, -1, 0, 0), \frac{1}{\sqrt{2}}(0, 0, 1, -1)\Big).$$

- Therefore, if $P_W : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is an orthogonal projection of $\mathbb{R}^4$ onto W along $W^{\perp}$, then the corresponding matrix A is given by

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

- Hence, the matrix of the orthogonal projection P_W in the ordered basis

$$\mathcal{B} = \big(\frac{1}{\sqrt{2}}(1,1,0,0), \frac{1}{\sqrt{2}}(0,0,1,1), \frac{1}{\sqrt{2}}(1,-1,0,0), \frac{1}{\sqrt{2}}(0,0,1,-1)\big)$$

is

$$P_W[\mathcal{B},\mathcal{B}] = AA^t = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

- It is easy to see that the matrix $P_W[\mathcal{B},\mathcal{B}]$ is symmetric, $P_W[\mathcal{B},\mathcal{B}]^2 = P_W[\mathcal{B},\mathcal{B}]$ and

$$\big(I_4 - P_W[\mathcal{B},\mathcal{B}]\big)P_W[\mathcal{B},\mathcal{B}] = \mathbf{0} = P_W[\mathcal{B},\mathcal{B}]\big(I_4 - P_W[\mathcal{B},\mathcal{B}]\big).$$

Example cont.

- Also, for any $(x, y, z, w) \in \mathbb{R}^4$, we have $[(x, y, z, w)]_{\mathcal{B}} = \left(\frac{x+y}{\sqrt{2}}, \frac{z+w}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}, \frac{z-w}{\sqrt{2}}\right)^t$.
- Thus, $P_W\big((x, y, z, w)\big) = \frac{x+y}{2}(1, 1, 0, 0) + \frac{z+w}{2}(0, 0, 1, 1)$ is the closest vector to the subspace W for any vector $(x, y, z, w) \in \mathbb{R}^4$.

Definition (Special Matrices)

$A^* = (\overline{a_{ji}})$, is called the conjugate transpose of the matrix A. Note that $A^* = \overline{A^t} = \overline{A}^t$. A square matrix A with complex entries is called

1. a Hermitian matrix if $A^* = A$.
2. a unitary matrix if $A\,A^* = A^*A = I_n$.
3. a skew-Hermitian matrix if $A^* = -A$.

Definition (Special Matrices)

A square matrix A with real entries is called

1. a symmetric matrix if $A^t = A$.
2. an orthogonal matrix if $A\,A^t = A^tA = I_n$.
3. a skew-symmetric matrix if $A^t = -A$.

Remark

Note that a symmetric matrix is always Hermitian, a skew-symmetric matrix is always skew-Hermitian and an orthogonal matrix is always unitary. Each of these matrices are normal. If A is a unitary matrix then $A^* = A^{-1}$.

Example

- Let $B = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$. Then B is skew-Hermitian.
- Let $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.
- Then A is a unitary matrix and B is a normal matrix.
- Note that $\sqrt{2}A$ is also a normal matrix.

Definition (Unitary Equivalence)

Let A and B be two $n \times n$ matrices. They are called unitarily equivalent if there exists a unitary matrix U such that $A = U^*BU$.
Note that $U^* = U^{-1}$ as U is a unitary matrix. So, A is unitarily similar to the matrix B .

Theorem

Let A be an $n \times n$ Hermitian matrix. Then all the eigenvalues of A are real.

- Let $(\lambda, \mathbf{x})$ be an eigenpair. Then $A\mathbf{x} = \lambda\mathbf{x}$ and $A = A^*$ implies

$$\mathbf{x}^* A = \mathbf{x}^* A^* = (A\mathbf{x})^* = (\lambda\mathbf{x})^* = \overline{\lambda}\mathbf{x}^*.$$

- Hence

$$\lambda\mathbf{x}^*\mathbf{x} = \mathbf{x}^*(\lambda\mathbf{x}) = \mathbf{x}^*(A\mathbf{x}) = (\mathbf{x}^*A)\mathbf{x} = (\overline{\lambda}\mathbf{x}^*)\mathbf{x} = \overline{\lambda}\mathbf{x}^*\mathbf{x}.$$

- But $\mathbf{x}$ is an eigenvector and hence $\mathbf{x} \neq \mathbf{0}$ and so the real number $\|\mathbf{x}\|^2 = \mathbf{x}^*\mathbf{x}$ is non-zero as well.
- Thus $\lambda = \overline{\lambda}$. That is, λ is a real number.

Theorem

Let A be an $n \times n$ Hermitian matrix. Then A is unitarily diagonalisable. That is, there exists a unitary matrix U such that $U^*AU = D$; where D is a diagonal matrix with the eigenvalues of A as the diagonal entries. In other words, the eigenvectors of A form an orthonormal basis of $\mathbb{C}^n$.

Outline of the proof

- We will prove the result by induction on the size of the matrix.
- The result is clearly true if $n = 1$. Let the result be true for $n = k - 1$. we will prove the result in case $n = k$.
- So, let A be a $k \times k$ matrix and let $(\lambda_1, \mathbf{x})$ be an eigenpair of A with $\|\mathbf{x}\| = 1$.
- We now extend the linearly independent set $\{\mathbf{x}\}$ to form an orthonormal basis $\{\mathbf{x}, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}$ (using Gram-Schmidt Orthogonalisation) of $\mathbb{C}^k$.
- As $\{\mathbf{x}, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}$ is an orthonormal set,

$$\mathbf{u}_i^*\mathbf{x} = 0 \quad \text{for all} \quad i = 2, 3, \ldots, k.$$

- Therefore, observe that for all $i,\ 2 \leq i \leq k$,

$$(A\mathbf{u}_i)^*\mathbf{x} = (\mathbf{u}_i^* A^*)\mathbf{x} = \mathbf{u}_i^*(A^*\mathbf{x}) = \mathbf{u}_i^*(A\mathbf{x}) = \mathbf{u}_i^*(\lambda_1\mathbf{x}) = \lambda_1(\mathbf{u}_i^*\mathbf{x}) = 0.$$

- Hence, we also have $\mathbf{x}^*(A\mathbf{u}_i) = 0$ for $2 \leq i \leq k$.

Outline of the proof cont.

- Now, define $U_1 = [\mathbf{x},\ \mathbf{u}_2,\ \cdots, \mathbf{u}_k]$ (with $\mathbf{x}, \mathbf{u}_2, \ldots, \mathbf{u}_k$ as columns of U_1).
- Then the matrix U_1 is a unitary matrix and

$$
\begin{aligned}
U_1^* A U_1 &= U_1^* [A\mathbf{x}\ A\mathbf{u}_2\ \cdots A\mathbf{u}_k] \\
&= \begin{bmatrix} \mathbf{x}^* \\ \mathbf{u}_2^* \\ \vdots \\ \mathbf{u}_k^* \end{bmatrix} \begin{bmatrix} \lambda_1 \mathbf{x}) & \cdots & \mathbf{u}_k^*(A\mathbf{u}_k) \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 \mathbf{x}^* \mathbf{x} & \cdots & \mathbf{x}^*(A\mathbf{u}_k) \\ \vdots & \cdots & \vdots \\ u_k^*(\lambda_1 \mathbf{x} & \cdots & u_k^*(Au_k) \end{bmatrix} \\
&= \left(\begin{array}{c|c} \lambda_1 & \mathbf{0} \\ \hline \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} & B \end{array} \right)
\end{aligned}
$$

where B is a $(k-1) \times (k-1)$ matrix.

Outline of the proof cont.

- As $A^* = A$,we get $(U_1^* A U_1)^* = U_1^* A U_1$.
- This condition, together with the fact that λ_1 is a real number, implies that $B^* = B$. That is, B is also a Hermitian matrix.
- Therefore, by induction hypothesis there exists a $(k-1) \times (k-1)$ unitary matrix U_2 such that

$$U_2^* B U_2 = D_2 = \text{diag}(\lambda_2, \ldots, \lambda_k).$$

- Recall that , the entries λ_i, for $2 \leq i \leq k$ are the eigenvalues of the matrix B.
- We also know that two similar matrices have the same set of eigenvalues. Hence, the eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_k$.
- Define $U = U_1 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{bmatrix}$.

Outline of the proof cont.

- Then U is a unitary matrix and

$$
\begin{aligned}
U^*AU &= \left(U_1 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{bmatrix}\right)^* A \left(U_1 \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U_2^* \end{bmatrix}\right) (U_1^* A U_1) \left(\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U_2^* \end{bmatrix} \begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & U_2^* B U_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & D_2 \end{bmatrix}.
\end{aligned}
$$

- Thus, U^*AU is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_k$, the eigenvalues of A.
- Hence, the result follows.

Corollary

Let A be an $n \times n$ real symmetric matrix. Then

- *the eigenvalues of A are all real,*
- *the corresponding eigenvectors can be chosen to have real entries, and*
- *the eigenvectors also form an orthonormal basis of $\mathbb{R}^n$.*

Lemma (Schur's Lemma)

Every $n \times n$ complex matrix is unitarily similar to an upper triangular matrix.

Definition (Bilinear Form)

Let A be a $n \times n$ matrix with real entries. A bilinear form in $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t,\ \mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ is an expression of the type

$$Q(\mathbf{x}, \mathbf{y}) = \mathbf{x}^t A \mathbf{y} = \sum_{i,j=1}^{n} a_{ij} x_i y_j.$$

- Observe that if $A = I$ (the identity matrix) then the bilinear form reduces to the standard real inner product.
- Also, if we want it to be symmetric in $\mathbf{x}$ and $\mathbf{y}$ then it is necessary and sufficient that $a_{ij} = a_{ji}$ for all $i, j = 1, 2, \ldots, n$.
- Hence, any symmetric bilinear form is naturally associated with a real symmetric matrix.

Definition (Sesquilinear Form)

Let A be a $n \times n$ matrix with complex entries. A sesquilinear form in $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t, \ \mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ is given by

$$H(\mathbf{x}, \mathbf{y}) = \sum_{i,j=1}^{n} a_{ij} x_i \overline{y_j}.$$

- Note that if $A = I$ (the identity matrix) then the sesquilinear form reduces to the standard complex inner product.
- Also, it can be easily seen that this form is ‘linear’ in the first component and ‘conjugate linear’ in the second component.
- Also, if we want $H(\mathbf{x}, \mathbf{y}) = \overline{H(\mathbf{y}, \mathbf{x})}$ then the matrix A need to be an Hermitian matrix.

Observation

- Note that if $a_{ij} \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then the sesquilinear form reduces to a bilinear form.
- The expression $Q(\mathbf{x}, \mathbf{x})$ is called the quadratic form and $H(\mathbf{x}, \mathbf{x})$ the Hermitian form.
- We generally write $Q(\mathbf{x})$ and $H(\mathbf{x})$ in place of $Q(\mathbf{x}, \mathbf{x})$ and $H(\mathbf{x}, \mathbf{x})$, respectively.
- It can be easily shown that for any choice of $\mathbf{x}$, the Hermitian form $H(\mathbf{x})$ is a real number.
- Therefore, in matrix notation, for a Hermitian matrix A , the Hermitian form can be rewritten as

$$H(\mathbf{x}) = \mathbf{x}^t A \mathbf{x}, \qquad \text{where } \mathbf{x} = (x_1, x_2, \ldots, x_n)^t, \text{ and } A = [a_{ij}].$$

Example

- Let $A = \begin{bmatrix} 1 & 2-i \\ 2+i & 2 \end{bmatrix}$.
- Check that A is an Hermitian matrix
- For $\mathbf{x} = (x_1, x_2)^t$, the Hermitian form

$$\begin{aligned} H(\mathbf{x}) &= \mathbf{x}^* A \mathbf{x} \\ &= (\bar{x}_1, \bar{x}_2) \begin{bmatrix} 1 & 2-i \\ 2+i & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \bar{x}_1 x_1 + 2\bar{x}_2 x_2 + (2-i)\bar{x}_1 x_2 + (2+i)\bar{x}_2 x_1 \\ &= |x_1|^2 + 2|x_2|^2 + 2\text{Re}[(2-i)\bar{x}_1 x_2] \end{aligned}$$

where 'Re' denotes the real part of a complex number.

- This shows that for every choice of $\mathbf{x}$ the Hermitian form is always real

Thank You