Tutorial 5: Calculus I (IC153)

Indian Institute of Technology Bhilai

---

1. Is the function $f(x) = \begin{cases} 1 & \text{if} \quad x \neq 1 \\ 0 & \text{if} \quad x = 1 \end{cases}$ integrable over the interval $[0, 2]$.

2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If there is a partition P of $[a, b]$ such that $L(f, P) = U(f, P)$, then show that f is a constant function.

3. Let $0 < a < b$ and $f(x) = \begin{cases} 0 & \text{if} \quad x \in [a, b] \cap \mathbb{Q} \\ x & \text{if} \quad x \in [a, b] \cap \mathbb{Q} \end{cases}$. Find the upper and lower Durboux integrals of $f(x)$ over $[a, b]$.

4. Evaluate the limit: $\lim\limits_{n\to\infty} \frac{1}{n^2} \sum\limits_{k=1}^{n} \sqrt{n^2 - k^2}$

5. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that there exists $c \in (a, b)$ such that $\frac{1}{b-a} \int_a^b f(x) = (c)$. (This result is called the mean value theorem of Riemann integrals.)

6. If $f : [-1, 1] \to \mathbb{R}$ is continuously differentiable, then evaluate $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} f'\left(\frac{k}{3n}\right)$

7. Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous and let $g(x) \geq 0$ for all $x \in [a, b]$. Show that there exists $c \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$. (This result is called the generalized mean value theorem of Riemann integrals.)

8. Prove the following inequality (a) $\frac{\pi^2}{9} \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx \leq \frac{2\pi^2}{9}$ (b) $\frac{\sqrt{3}}{8} \leq \int_{\pi/4}^{\pi/3} \frac{\sin x}{x} dx \leq \frac{\sqrt{2}}{6}$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $g(x) = \int_0^x (x - t)f(t)dt$ for all $x \in \mathbb{R}$. Show that $g'' = f(x)$ for all $x \in \mathbb{R}$.

10. Given $\phi(x) = \int_{x^2}^{x^3} \frac{1}{1+t^2} dt,\ x \geq 1$. Find $\phi'(x)$.