The next result is to compute inverse of a matrix.

Theorem: Suppose $[A|I]$ is row equivalent to $[R|B]$ and R is RRE matrix. Then A is invertible iff $R=I$ and in this case $B=A^{-1}$

Proof:- Let $E_1, E_2, \ldots E_k$ be elementary matrices such that

$$R = E_1 E_2 \cdots E_k A$$

Let us denote $E = E_1 E_2 \cdots E_k$, then

$$R = EA \text{ and } EI = B$$

As R is RRE matrix ~~and~~ equivalent to A implies (by previous theorem) $R=I$ for invertibility of A.

Hence $I = EA$ and $EI = B$ which implies $A^{-1} = B$

The following examples help to understand the above result.

Example: Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Compute the inverse using GJE.

Consider

$$[A|I] = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

and try to convert into $[R|B]$ where R is RRE matrix.

We perform the following steps.

Step 1 Apply $R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

Step 2: Apply $R_1 \to R_1 - R$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

Step 3 Apply $R_2 \to R_2 - R_3$.

$$[R|B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

As $R = I$, we have A to be an invertible matrix and

$$A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Now we see an application of RRE matrices in finding the rank of a matrix.

Let us first define, what is the meaning of rank of a matrix $A \in M_{m \times n}(\mathbb{R})$.

Definition: Let $A \in M_{m \times n}(\mathbb{R})$, then an integer $0 \leq r \leq \min\{m,n\}$ is called the rank of A if there exists an invertible submatrix (or block) of A of size $r \times r$ and there is no invertible submatrix of A of size $(r+1) \times (r+1)$.

If rank of $A = 0$, then A is a zero matrix.

The following theorem gives a complete characterization of rank of a matrix via RRE form.

Theorem :

i) Row equivalent matrices have same rank

ii) If A is an RRE matrix then rank of A is no of nonzero rows of A

iii) The rank of a matrix is the number of nonzero rows in the RRE form of the matrix.

Let us see the following example.

Example : Take

$$A = \begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then $RRE(A) = \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Hence $Rank(A) = 3$

Verify the same !!

Let us now see the application of RRE matrices in solving system of equations.

Let us have $Ax = b$, then we have the following theorem

Theorem:- The system of linear equation $Ax = b$ has a solution if and only if rank(A) = rank$([A+b])$

This system has unique solution iff rank(A) is equal to the no. of unknowns.

First see the following few examples.

Example. Consider the system

$$x + y + z = 3$$
$$x + 2y + 3z = 6$$
$$y + 2z = 1$$

The augmented matrix is

$$[A \; b] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Apply elementary row operations and convert A in RRE form.

$R_2 \to R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$
$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$R_3 \to -\frac{1}{2} R_3$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_3$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ which gives $$\left.\begin{aligned} x - z &= 0 \\ y + 2z &= 0 \\ 0 &= 1 \end{aligned}\right\}$$

Note that $0 = 1$ is absurd, Hence No solution (rank$(A) = 2 <$ rank$[A \; b] = 3$)

Example: Consider

$$x+y+z=3$$
$$x+2y+4z=7$$
$$x-y+z=1$$

The augmented matrix of the system is

$$[A\ b] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

Apply elementary row operations

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & 0 & -2 \end{bmatrix}$$

$R_1 \to R_1 - R_2$
$R_3 \to R_3 + 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$R_3 \to \frac{1}{6} R_3$

$$\sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Next we apply $R_2 \to R_2 - 3R_3$ & $R_1 \to R_1 + 2R_3$ to get,

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

As rank$(A) = 3 =$ rank $([A\ b])$
$=$ no of unknowns.

Hence system has unique solution $(1,1,1)^T$.

Example:- Consider

$$\left.\begin{aligned} x+y+z&=3\\ x-y+2z&=4\\ 2x+3z&=7 \end{aligned}\right\}$$

The augmented matrix of this system is

$$[A\ b] = \begin{bmatrix} 1 & 1 & 1 & 3\\ 1 & -1 & 2 & 4\\ 2 & 0 & 3 & 7 \end{bmatrix}$$

Apply

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3\\ 0 & -2 & 1 & 1\\ 0 & -2 & 1 & 1 \end{bmatrix}$$

Next, $R_2 \to -\frac{1}{2} R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3\\ 0 & 1 & -1/2 & -1/2\\ 0 & -2 & 1 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$, $R_3 \to R_3 + 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 3/2 & 7/2\\ 0 & 1 & -1/2 & -1/2\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Observe that $\text{rank}(A) = 2 = \text{rank}([A\ b])$
$<$ no of variables

Hence there are infinitly many solutions. In fact $\left(\frac{7-3a}{2}, \frac{a-1}{2}, a\right)$, $a \in \mathbb{R}$ solve the system.

Now we write a proof of the previous theorem.

Proof:- We have the system $Ax=b$. As the row equivalent systems have same solution set, we can assume (without loss) that A is RRE matrix.

Let us assume that i^{th} row of A is zero (of course all the rows below i^{th} row will also be zero) and i^{th} row of $[A\ b]$ is non zero ($b_i \neq 0$) then the system has it's ith equation $0 = b_i$ which is absurd for $b_i \neq 0$. Hence there will be No solution. Thus we have proved that if at all rank of $A <$ rank$[A\ b]$ the system will have no solution.

Next we assume that rank $(A) =$ rank $[A\ b]$

Then there are two cases

Either rank $(A) =$ rank $([A\ b]) =$ no of unknowns (in this case there will be no. free unknown. Hence the solution of the given system will be unique.

The second situation will be if rank$(A) =$ rank$([A\ b]) <$ no of unknowns then there will be at least one free unknown, and therefore by choosing different values of this free variable, you can have infinitly many solutions.

Remark:
When the system is homogeneous and A is a square matrix then the unique solution is the trivial (zero) solution.