Department of Mathematics
Indian Institute of Technology Bhilai
IC104: Linear Algebra-I
Hints of Tutorial Sheet 1: Systems of Equations

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1. (a) The augmented matrix of the system, $[A\ b]$ can be written as $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & -3 \\ 2 & 2 & -2 \end{bmatrix}$.

   Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1$, we get $\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -5 \\ 0 & -4 & -4 \end{bmatrix}$. Applying $R_2 \to (-1/5)R_2, R_3 \to R_3 + 4R_2 \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   Therefore, we get $y = 1, x + 3y = 1 \Rightarrow y = 1, x = -2$

   (b) Here augmented matrix $[A\ b]$ can be given as $\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 4 \end{bmatrix}$. Applying $R_3 \to R_3 - R_1 \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{bmatrix}$. Applying $R_3 \to R_3 + 2R_2 \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which implies $x+2y = 4$ and $y-z = 0$. Thus if $z = k$, then $y = k$ and $x = 4-2k$. Therefore a typical solution will be of the type $(4 - 2k, k, k)$, $k \in \mathbb{R}$.

2. The augmented matrix of the given system is $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & \alpha \end{bmatrix}$. Applying $R_2 \to R_2 - 3R_1$ we get $\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & \alpha - 3 \end{bmatrix}$. Now if $\alpha = 3$, then $rank\ (A) = rank\ ([A\ b]) < 2$. Hence infinitely many solutions. If $\alpha \neq 3$, then $rank\ (A) < rank\ ([A\ b])$. Hence no solution.

3. The matrices B and C are elementary matrices. The matrix B can be achieved by applying $R_1 \to R_1 + R_2$ on identity matrix of size 3×3 and matrix C can be achieved by applying $R_2 \to R_2 - 5R_1$ on identity matrix of size 3×3.

4. Using $e(A) = e(I).A$, we get $E = e(I)$.

5. If RRE (A) = RRE (B) then A will be row equivalent to matrix B. To change matrix A into RRE form we use the following set of elementary operations,

   (i) $R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 5R_1$

   (ii) $R_3 \to R_3 - R_2$

(iii) $R_2 \to -R_2$

we get $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$.

Similarly changing matrix B into RRE form using $R_2 \leftrightarrow (1/2)R_2, R_3 \leftrightarrow R_3 - R_2$ we get the same RRE as above. Hence A and B are row equivalent.

6. (a) $$\left[\begin{array}{cccc|cccc} 0 & 1 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 1 & 0 & 0 \\ 1 & 5 & 1 & 5 & 0 & 0 & 1 & 0 \\ 2 & 3 & 7 & 9 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_4 \to R_4 - 2R_3} \left[\begin{array}{cccc|cccc} 0 & 1 & 3 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 5 & 6 & 0 & 1 & 0 & 0 \\ 1 & 5 & 1 & 5 & 0 & 0 & 1 & 0 \\ 0 & -7 & 5 & -1 & 0 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{cccc|cccc} 1 & 5 & 1 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 6 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 & 0 & 1 \\ 0 & -7 & 5 & -1 & 0 & 0 & -2 & 1 \end{array}\right] \xrightarrow[R_1 \to R_1 - 5R_3]{R_4 \to R_4 + 7R_3}$$
$$\left[\begin{array}{cccc|cccc} 1 & 0 & -14 & -5 & -5 & 0 & 1 & -5 \\ 0 & 0 & 5 & 6 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 26 & 13 & 7 & 0 & -2 & 8 \end{array}\right]$$

$$\xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & -14 & -5 & -5 & 0 & 1 & -5 \\ 0 & 1 & 3 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 5 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 26 & 13 & 7 & 0 & -2 & 8 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{5}R_3}$$
$$\left[\begin{array}{cccc|cccc} 1 & 0 & -14 & -5 & -5 & 0 & 1 & -5 \\ 0 & 1 & 3 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{6}{5} & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 26 & 13 & 7 & 0 & -2 & 8 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - 26R_3,\ R_1 \to R_1 + 14R_3]{R_2 \to R_2 - 3R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{59}{5} & -5 & \frac{14}{5} & 1 & -5 \\ 0 & 1 & 0 & -\frac{8}{5} & 1 & -\frac{3}{5} & 0 & 1 \\ 0 & 0 & 1 & \frac{6}{5} & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & -\frac{91}{5} & 7 & -\frac{26}{5} & -2 & 8 \end{array}\right] \xrightarrow{R_4 \to -\frac{5}{91}R_4}$$
$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{59}{5} & -5 & \frac{14}{5} & 1 & -5 \\ 0 & 1 & 0 & -\frac{8}{5} & 1 & -\frac{3}{5} & 0 & 1 \\ 0 & 0 & 1 & \frac{6}{5} & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{5}{13} & \frac{2}{7} & \frac{10}{91} & -\frac{40}{91} \end{array}\right]$$

$$\xrightarrow[R_{\rightarrow} R_1-\frac{59}{5}R_4]{R_2\rightarrow R_2+\frac{8}{5}R_4,\ R_3\rightarrow R_3-\frac{6}{5}R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{6}{13} & -\frac{4}{7} & -\frac{27}{91} & \frac{59}{91} \\ 0 & 1 & 0 & 0 & \frac{5}{13} & -\frac{1}{7} & \frac{16}{91} & -\frac{8}{91} \\ 0 & 0 & 1 & 0 & \frac{6}{13} & -\frac{1}{7} & -\frac{12}{91} & \frac{6}{91} \\ 0 & 0 & 0 & 1 & -\frac{5}{13} & \frac{2}{7} & \frac{10}{91} & -\frac{5}{91} \end{array}\right]$$

Here inverse of the matrix A is

$$\begin{bmatrix} -\frac{6}{13} & -\frac{4}{7} & -\frac{27}{91} & \frac{59}{91} \\ \frac{5}{13} & -\frac{1}{7} & \frac{16}{91} & -\frac{8}{91} \\ \frac{6}{13} & -\frac{1}{7} & -\frac{12}{91} & \frac{6}{91} \\ -\frac{5}{13} & \frac{2}{7} & \frac{10}{91} & -\frac{5}{91} \end{bmatrix}$$

(b) By similiar technique for the matrix B, we get that

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{13}{11} & \frac{9}{11} & \frac{3}{11} \\ 0 & 1 & 0 & \frac{10}{11} & -\frac{1}{11} & -\frac{4}{11} \\ 0 & 0 & 1 & -\frac{2}{11} & -\frac{2}{11} & \frac{3}{11} \end{array}\right]$$

Therefore the RRE of B is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Again the inverse of B is $\begin{bmatrix} -\frac{13}{11} & \frac{9}{11} & \frac{3}{11} \\ \frac{10}{11} & -\frac{1}{11} & -\frac{4}{11} \\ -\frac{2}{11} & -\frac{2}{11} & \frac{3}{11} \end{bmatrix}$.

7. (a) Applying row operation on the matrix A we have, $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \xrightarrow{R_2\rightarrow R_2-2R_1} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

Hence rank of the matrix is 1.

(b) The RRE form of the matrix is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$, Rank is 2.

(c) The RRE form of the matrix is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, Rank is 2.

(d) The RRE form of the matrix is $\begin{bmatrix} 1 & 0 & \frac{1}{8} & \frac{1}{2} \\ 0 & 1 & \frac{5}{4} & 0 \end{bmatrix}$, Rank is 2.

8. (a) Here the system of equation is

$$x+y=0; \quad 2x+y+3z=3; \quad x+2y+z=3$$

Therefore we can write these equations as $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$ Then the augmented matrix is

$$[A\,b] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 3 \\ 1 & 2 & 1 & 3 \end{bmatrix}.$$

By Gauss Jordon Elimination on augmented matrix we have,

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}.$$

Therefore $x = -\frac{3}{2}; \quad y = \frac{3}{2}; \quad z = \frac{3}{2}$.

(b) By Gauss Jordon Elimination on augmented matrix we have,

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & 1 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Then $w = -1$, $y - \frac{1}{4}z = 0$, and $x + \frac{1}{2}z = 1$, that is $x = 1 - \frac{1}{2}z$, $y = \frac{1}{4}z$. Then the solution set is $\{(1, 0, 0, -1)^t + k(-\frac{1}{2}, \frac{1}{4}, 1, 0)^t : k \in \mathbb{R}\}$.

(c) By Gauss Jordon Elimination on augmented matrix we have,

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Then $x = 1; \quad y = 1; \quad z = 2$.

9. As $f(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$ and $(2, 3)$, then $2 = a + b + c$; $6 = a - b + c$; $3 = 4a + 2b + c$.

Therefore we can write these equations as $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$ Then the augmented matrix is

$$[A\,b] = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 6 \\ 4 & 2 & 1 & 3 \end{bmatrix}.$$

Now apply the row operation on $[A\,b]$, we get that

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 6 \\ 4 & 2 & 1 & 3 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - 4R_1} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 4 \\ 0 & -2 & -3 & 5 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & -3 & -9 \end{bmatrix}$$

Then we have $-3c = -9$, which implies $c = 3$. Again $-2b = 4$ implies $b = -2$ and $a + b + c = 2$ implies $a = 1$.

10. Let

$$\begin{aligned} ax + by &= c \quad (1) \\ ax + dy &= e \quad (2) \end{aligned}$$

Case:-1 Let $a = 0$, then from the equation (1) we get that $y = \frac{c}{b}$ and from the equation (2) we obtain that $y = \frac{e}{d}$. For same set of solution of both these equation $\frac{c}{b} = \frac{e}{d}$.

Case:-2 Let $a \neq 0$, then from the equation (1), the solution set is $\{(x, y) : x = \frac{c-by}{a}\}$ and the solution set of the equation (2) is $\{(x, y) : x = \frac{e-dy}{a}\}$. Then for the same solution of these two equations we have $\frac{c-by}{a} = \frac{e-dy}{a}$, that is $c = e$ and $b = d$.

Moreover, if we think in geometrically, the equations (1) and (2) have same set of solution only when they are overlapping.

11. Let $A \in M_{m\times n}(\mathbb{R})$. Assume for the system $Ax = b$ has exactly 2 solution i.e p and q. Then $Ap = b$, $Aq = b$. Now, for $\frac{p+2q}{3}$ we have

$$A(\frac{p+2q}{3}) = \frac{1}{3}A(p+2q) = \frac{1}{3}A(p) + \frac{2}{3}A(q) = \frac{b}{3} + \frac{2b}{3} = b.$$

This implies that $\frac{p+2q}{3}$ is also the solution of A which is contrary to our assumption. Note that we can get the same contradiction if we assume that a system has exactly n solutions. In this case we will assume that $x_1, ..., x_n$ are the n solutions. There is a catch for choosing the n coefficients $a_1, ..., a_n$ such that $a_1 + ... + a_n = 1$.

12. In this question we will observe the pattern for $m = 1, 2, 3, ..$

I. When $m = 1$ then

$$ax + cy = b. \quad (3)$$

Clearly, in this case we have the matrix $[A\ b] = [a\ c\ b]$. Then possible choice for RRE form in this case is 1.

II. When $m = 2$ then

$$\begin{aligned} ax + by &= c \quad (4) \\ dx + ey &= f \quad (5) \end{aligned}$$

$$\implies \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}, \text{ then } [A\ b] = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

Now fixing $a_{11} = 1$ we have 3 choices

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$$

Again fixing $a_{11} = 0$ and $a_{12} = 1$, we have 2 choices

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$$

Again fixing $a_{11} = 0$, $a_{12} = 0$ and $a_{13} = 1$ we have 1 choice

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence we have in all 3+2+1=6 choices.

III. When $m = 3$ then

$$ax + by = c$$
$$dx + ey = f$$
$$gx + hy = i$$

$$\Longrightarrow \begin{bmatrix} a & b \\ d & e \\ g & h \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \\ i \end{bmatrix}, \text{ then } [A\ b] = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Now fixing $a_{11} = 1$ we have 4 choices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Again fixing $a_{11} = 0$ and $a_{12} = 1$, we have 2 choices

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Again fixing $a_{11} = 0$, $a_{12} = 0$ and $a_{13} = 1$ we have 1 choice

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, here also we have observed that there are in all 4+2+1=7 choices. Proceeding in the same way we will get that for $m = 4, 5, ..$ we also have 7 choices.

13. Let $A = (a_{ij})_{n\times n}$. We can assume that $a_{11} \neq 0$ (if $a_{11} = 0$, then choose $S = I$ and we are done). Let a_{1i} be the non zero element in the first row except a_{11}. Then S can be taken as the elementary matrix obtained by the row elementary operation

$e : R_i \to R_i + \frac{a_{11}}{a_{1i}} R_1$ applied on identity matrix I. If no such a_{1i} exists then search for the nonzero element in first column (except a_{11}), let it be a_{j1}. Then S can be taken as the elementary matrix obtained by the row elementary operation $e : R_1 \to R_1 - \frac{a_{11}}{a_{j1}} R_j$ applied on identity matrix I.

14. (a) False, beacuse consider the system

$$\begin{aligned} x + 2y + z - 3w &= 1 \\ 2x + 4y + 3z + w &= 3 \\ 3x + 6y + 4z - 2w &= 5 \end{aligned}$$

which has 4 unknowns and 3 equations. But this system is inconsistent.

(b) False, beacause consider the system

$$\begin{aligned} x + y &= 3 \\ x - y &= 1 \\ 5x - 7y &= 3 \end{aligned}$$

which has 2 unknowns and 3 equations. But this system has unique solution.

(b) False, beacuse consider the system

$$\begin{aligned} x &= 1 \\ x + y &= 2 \\ x + 2y &= 4 \end{aligned}$$

which has 2 unknowns and 3 equations. But this system has no solution.