Limit of Functions

The intuitive idea of the function $f(x)$ having limit l at a point c is that the values of $f(x)$ are close to l when x is close to c.

To define the closeness in a technical way we use ϵ-δ defination.

The limit of a function f at a point c is meaningful, it is necessary that f should be defined at points near c. It need not be defined at the point c, but it should be defined at enough points close to c.

Def^n: Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a cluster point or limit point of A if for every $\delta > 0$ there exists at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

or

A point c is a cluster point or limit point of set A if every δ-neighborhood $V_\delta(c) = (c - \delta, c + \delta)$ of c contains at least one point of A distinct from c.

Note: The point c may or may not be in A. Even if $c \in A$, it is ignored when deciding it is a limit point of A or not.

We need $\quad V_\delta(c) \cap A \setminus \{c\} \neq \phi$.

Ex: $A = \{1, 2, 3\}$. The the point 1 is not a limit point. Take $\delta = 1/2$. Then $V_{1/2}(1) = (1 - 1/2, 1 + 1/2)$ does not contain any point of A other than 1.

i.e $A \cap V_{1/2}(1) \setminus \{1\} = \phi$.

Similarly for 2 & 3. So the set A has no limit point.

Theorem: A number $c \in \mathbb{R}$ is a limit point of a subset A of $\mathbb{R}$ iff there exists a sequence $\{x_n\}$ in A such that $\lim x_n = c$ and $x_n \neq c$ $\forall\, n \in \mathbb{N}$.

Example (i) $A_1 = (0,1)$. Every point of the closed interval $[0,1]$ is a limit point of A_1.

(ii) A finite set has no limit points.

(iii) The infinite set $\mathbb{N}$ has no limit points.

(iv) $A_4 = \{\frac{1}{n} : n \in \mathbb{N}\}$. The limit point $A_4 =$ is 0. None of the point in A_4 is a limit point.

(v) $I = [0,1]$, $A_5 = I \cap \mathbb{Q}$ consists all the rational numbers in I. From the density property every point in I is a limit point.

Limit: Let $A \subseteq \mathbb{R}$ and c be a limit point of A. For a function $f: A \to \mathbb{R}$ a real number l is said to be a limit of f at c if given $\epsilon > 0$ $\exists$ $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$ then $|f(x) - l| < \epsilon$.

Remark (I) Since the value of δ usually depends on ϵ we will some time write $\delta(\epsilon)$

(II) The inequality $0 < |x - c|$ is equivalent to $x \neq c$.

If l is a limit of f at c, then we write

$$l = \lim_{x \to c} f(x). \quad \text{or} \quad l = \lim_{x \to c} f$$

Symbolically

$$f(x) \longrightarrow l \quad \text{as } x \longrightarrow c.$$

Theorem: If $f: A \to \mathbb{R}$ and if c is a limit point of A. If the limit of f at c exists then f can have only one limit at c.

Proof: Suppose l_1 and l_2 be the two limits of $f(x)$ at c. Then for any $\epsilon > 0$ $\exists$ $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$ then $|f(x) - l_1| < \epsilon/2$

Also $\exists$ $\delta' \ni$ if $x \in A$ & $0 < |x - c| < \delta'$ then

$$|f(x) - l_2| < \epsilon/2$$

Now take $\delta = \min\{\delta, \delta'\}$. The if $x \in A$ & $0 < |x-c| < \delta$ implies that

$$|l_1 - l_2| = |l_1 - f(x) + f(x) - l_2| \leq |l_1 - f(x)| + |f(x) - l_2|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\epsilon > 0$ is arbitrary $\Rightarrow$ $l_1 - l_2 = 0 \Rightarrow l_1 = l_2$.

Example: Let $A = [0,\infty) \setminus \{9\}$ and define $f: A \to \mathbb{R}$ by

$$f(x) = \frac{x-9}{\sqrt{x}-3}.$$ Prove that $\lim_{x\to 9} f(x) = 6$.

Solu^n: Let $\epsilon > 0$ be given. Consider

$$|f(x) - 6| = \left|\frac{x-9}{\sqrt{x}-3} - 6\right| = |\sqrt{x} - 3| = \left|\frac{x-9}{\sqrt{x}+3}\right|$$

$$\leq \frac{1}{3}|x-9|$$

Take $\delta = 3\epsilon$. Then $|f(x) - 6| < \epsilon$ whenever $0 < |x-9| < \delta$.

$$\Rightarrow \lim_{x\to 9} f(x) = 6.$$

Sequential criterian of limits:

Theorem: Let $f: A \to \mathbb{R}$ and let c be a limit point of A. Then the following are equivalent

(1) $\lim_{x\to c} f(x) = l$

(ii) For every sequence $\{x_n\}$ in A that converges to c such that $x_n \neq c$ $\forall n \in \mathbb{N}$, the sequence $f(x_n)$ converges to l.

The following result is very useful to show (i) that a certain number is not a limit of a function at a point or (ii) that the function does not hava limit at a point.

Theorem: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ ba a limit point of A.

(a) If $l \in \mathbb{R}$ then $f(x)$ does not have limit l at c iff there exists a sequence $\{x_n\}$ in A with $x_n \neq c$ $\forall n \in \mathbb{N}$ such that the sequence $\{x_n\}$ converges to c but the sequn $\{f(x_n)\}$ does not converge to l.

(b) The function $f(x)$ does not hava limit at c iff there exists a sequence $\{x_n\}$ in A with $x_n \neq c$ $\forall n \in \mathbb{N}$ such that $\{x_n\}$ converges to c but the sequn $\{f(x_n)\}$ does not converge in $\mathbb{R}$.

Ex $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined as $f(x) = \sin \frac{1}{x}$

Prove that $\lim_{x \to 0} f(x)$ does not exist.

Soln: $f(x) = \sin \frac{1}{x}$, $x \neq 0$.

Take $x_n = \frac{1}{2\pi n}$. we have $x_n \to 0$ as $n \to \infty$

$f(x_n) = \sin(2n\pi) = 0 \quad \forall n \in \mathbb{N}$

$\Rightarrow \lim_{n\to\infty} f(x_n) = 0$

Again take $y_n = \dfrac{1}{2n\pi + \pi/2}$. So $y_n \to 0$ as $n \to \infty$

$f(y_n) = \sin(2n\pi + \pi/2) = 1 \quad \forall n \in \mathbb{N}$

$\Rightarrow \lim_{n\to\infty} f(y_n) = 1$

Two values are different $\Rightarrow \lim_{x\to 0} \sin\frac{1}{x}$ does not exists.

<u>Ex</u>: Prove $\lim_{x\to 0} \frac{1}{x}$ does not exist

<u>Solu^n</u>. Take $x_n = \frac{1}{n}$. Then $x_n \to 0$ as $n \to \infty$.

But $f(x_n) = \dfrac{1}{1/n} = n$.

So $\{f(x_n)\}$ is a divergent sequence. Hence $\lim_{x\to 0} \frac{1}{x}$ does not exist.

<u>Properties of Limit</u>: (1) Suppose $f, g : A \to \mathbb{R}$ and c is a limit point of A. If $f(x) \leq g(x) \ \forall x \in A$ and $\lim_{x\to c} f(x)$ & $\lim_{x\to c} g(x)$ exist then

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

Result (2) $f, g, h : A \to \mathbb{R}$ and c is a limit point of A

If $f(x) \le g(x) \le h(x) \ \forall x \in A$ & $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = l$

Then $\lim_{x\to c} g(x)$ exists & $\lim_{x\to c} g(x) = l$.

Algebric Properties: Let $f, g : A \to \mathbb{R}$, c is a limit point of A and $\lim_{x\to c} f(x) = l_1$, $\lim_{x\to c} g(x) = l_2$

(1) (a) $\lim_{x\to c} (f(x) \pm g(x)) = l_1 \pm l_2$, (b) $\lim (f(x)\, g(x)) = l_1 l_2$

(c) $\lim_{x\to c} (kf(x)) = kl_1$

(ii) If $g(x) \neq 0 \ \forall x \in A$ and $l_2 \neq 0$ then

$$\lim_{x\to c} \left(\frac{f}{g}\right) = l_1/l_2$$

One Sided Limits: Let $A \subset \mathbb{R}$ and let $f(x) : A \to \mathbb{R}$.

Right hand limit: If $c \in \mathbb{R}$ is a limit point of the set $A \cap (c, \infty) = \{x \in A : x > c\}$, then we say that $l \in \mathbb{R}$ is a right hand limit of $f(x)$ at c if given $\epsilon > 0$ there exists $\delta > 0$ s.t. for all $x \in A$ with $0 < x - c < \delta$

then $|f(x) - l| < \epsilon$.

We write $\lim_{x\to c^+} f = l$.

Left hand limit: If $c \in \mathbb{R}$ is a limit point of the set $A \cap (-\infty, c) = \{x \in A : x < c\}$, then we say that $l \in \mathbb{R}$ is a left-hand limit of f at c if given any $\epsilon > 0$ there exists $\delta > 0$ s.t $\forall x \in A$ with $0 < c - x < \delta$ or $(-\delta < x - c < 0)$ then $|f(x) - l| < \epsilon$.

We write $\lim_{x \to c^-} f(x) = l$.

Some examples:

(1) $f(x) = a$ (constant). $\lim_{x \to c} f(x) = b$.

Here we have $|f(x) - b| = |b - b| = 0 < \epsilon$

Since $\epsilon > 0$. Take $\delta = 1/2$. So we have

$|f(x) - b| < \epsilon$ whenever $|x - c| < \delta$

$\Rightarrow \lim_{x \to c} f(x) = b$

[Note: Any strictly positive δ will work].

(2) $\lim_{x \to c} x = c$. Let $\epsilon > 0$

$|f(x) - c| = |x - c| < \epsilon$

Take $\delta = \epsilon$ then $|x - c| < \delta \Rightarrow |f(x) - c| < \epsilon$

$\Rightarrow \lim_{x \to c} x = c$

(3) Show that $\lim_{x \to 4} (2x-5) = 3$

Solu^n: Let $\epsilon > 0$. Consider

$$|2x-5-3| = |2(x-4)| < \epsilon$$

$$\Rightarrow \quad |x-4| < \epsilon/2$$

Take $\delta = \epsilon/2$. Then for given $\epsilon > 0$,

$|f(x) - 3| < \epsilon$ whenever $0 < |x-4| < \delta$.

(4) Prove that $\lim_{x \to c} x^2 = c^2$, $f(x) = x^2$

Solu^n: $|f(x) - c^2| = |x^2 - c^2|$. We want to make $|x^2 - c^2|$ less than a preassigned $\epsilon > 0$ by taking x is sufficiently close to c.

If $|x-c| < 1$ then $|x| < 1 + |c|$, so

$$|x+c| \le |x| + |c| < 2|c| + 1$$

$-1 < x - c < 1$

$\Rightarrow c - 1 < x < c + 1$

So $|x| < |c+1|$
$< |c| + 1$

Therefore if $|x-c| < 1$, we have

$$|x^2 - c^2| = |x+c|\,|x-c| < (2|c|+1)|x-c| < \epsilon$$

if $|x-c| < \dfrac{\epsilon}{2|c|+1}$

Now if we tenee $\delta = \min\left\{1, \frac{\epsilon}{2|c|+1}\right\}$

then if $0<|x-c|<\delta \Rightarrow |x-c|<1$ so that

$|x^2-c^2| < 2(|c|+1)|x-c|$ holds true.

and thereefore $|x-c| < \frac{\epsilon}{2(|c|+1)}$ we have

$$|x^2-c^2| = 2(|c|+1)|x-c| < \epsilon.$$

$\Rightarrow$ For a pre assigned $\epsilon>0$ $\exists$ $\delta>0$ $\ni$

$|x-c|<\delta \Rightarrow |x^2-c^2|<\epsilon$

$\Rightarrow \lim_{x\to c} x^2 = c^2.$

Ex 5 (i) $\lim_{x\to c} x^n = c^n$. Since $\lim_{x\to c} x = c$ so by limit theorem its true.

(ii) $\lim_{x\to 2} (x^2+1)(x^3-4) = 20$

$\lim_{x\to 2} (x^2+1) = 5$, $\lim_{x\to 2} (x^3-4) = 4$

So the limit is $4\times 5 = 20$

(iii) $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ x\in\mathbb{R}.$

$\lim_{x\to c} p(x) = p(c)$

(iv) If $p(x)$ and $q(x)$ are two polynomial then

$$\lim_{x \to c} p(x)/q(x) = \frac{p(c)}{q(c)} \quad \text{if } q(c) \neq 0.$$

Limit at infinity

Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. We say $l \in \mathbb{R}$ is a limit of f as $x \to \infty$ and write $\lim_{x \to \infty} f(x) = l$ if given $\epsilon > 0$ $\exists$ $M > a$ such that for any $x > M$

$$|f(x) - l| < \epsilon.$$

Similarly we can define for $x \to -\infty$.

Theorem: Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and suppose that $(a, \infty) \subseteq A$ for som $a \in \mathbb{R}$. Then the following statements are equivalent:

(i) $\lim_{x \to \infty} f = l$

(ii) For every sequence $\{x_n\}$ in $A \cap (a, \infty)$ $\ni$ $\lim x_n = \infty$ the sequ^n $\{f(x_n)\}$ converges to l.

Similarly write for the case $x \to -\infty$

Ex. $\lim_{x\to\infty} \frac{1}{x} = 0$. Here $f(x) = \frac{1}{x}$ is defined for $\mathbb{R}\setminus\{0\}$

$(0,\infty) \subset \mathbb{R}\setminus\{0\}$. Let $\epsilon > 0$, when $x \neq 0$

$$|f(x) - 0| = \frac{1}{x} < \epsilon \Rightarrow x > \frac{1}{\epsilon} \quad \text{Take } M = \frac{1}{\epsilon} > 0$$

$\Rightarrow$ For given $\epsilon > 0$, $|f(x) - 0| < \epsilon$ whenever $x > M$

$$\Rightarrow \lim_{x\to\infty} f(x) = 0.$$

Ex: $\lim_{x\to\infty} x \sin x$ does not exist

Take $x_n = \pi/2 + 2n\pi$, $n \in \mathbb{N}$, $y_n = -\pi/2 + 2n\pi$

[Note in this case the limit also not exist for extended real line]

Ex: $\lim_{x\to\infty} \sin x$ does not exist

$x_n = n\pi$ then $x_n \to \infty$ and $\sin x_n = 0 \quad \forall n$

$y_n = \pi/2 + 2n\pi$ then $y_n \to \infty$ and $\sin y_n = 1$

Hence the limit does not exist.

Infinite limit: Let $A \subseteq \mathbb{R}$ let $f: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of A.

(1) We say that f tends to ∞ as $x \to c$ and write

$$\lim_{x\to c} f = \infty$$

if for every $\alpha > 0$ $\exists$ $\delta > 0$ such that $\forall x \in A$ with $0 < |x - c| < \delta$ then $f(x) > \alpha$.

(ii) We say $f \to -\infty$ as $x \to c$ and write

$$\lim_{x \to c} f = -\infty$$

if for every $\beta < 0$ $\exists\ \delta > 0$ s.t. $\forall\ x \in A$ with $0 < |x - c| < \delta$ then $f(x) < \beta$

<u>Sequential Criterion</u>:

(1) Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. Let c be a limit point of A. Then

(i) $\lim_{x \to c} f(x) = \infty$ iff for every $\{x_n\} \in A$, $x_n \neq c$ $\forall n$ converging to c, $\{f(x_n)\}$ diverges to ∞

(ii) $\lim_{x \to c} f(x) = -\infty$ iff for every $\{x_n\} \in A$, $x_n \neq c$ $\forall n$ converging to c, $\{f(x_n)\}$ diverges to $-\infty$.

<u>Ex</u>: The funⁿ $f(x) = \frac{1}{x}$ does not tend to either ∞ or $-\infty$ as $x \to 0$.

<u>Solⁿ</u>: $x_n = \frac{1}{n} \to 0$ as $n \to \infty$, $f(x_n) = n \to \infty$

$y_n = -\frac{1}{n} \to 0$ as $n \to \infty$, $f(y_n) = -n \to -\infty$

So the result proved.