Let us see some results related with subspaces.

Theorem: Let V be a vector space over a field $\mathbb{F}$. The intersection of any collection of subspaces of V is a subspace of V

Proof: Let $\{W : \in I\}$ be a collection of subspaces of V. We denote,
$W = \bigcap_{i \in I} W_i$. As $0 \in W_i \ \forall i \in I$ implies W is non-empty. Let $c \in F$,
& $\alpha, \beta \in W$ implies $\alpha, \beta \in W_i \ \forall i \in I$
As $W_i, i \in I$ is a subspace of V, the vector $c\alpha + \beta \in W_i \ \forall i \in I$
and hence $c\alpha + \beta \in \bigcap_{i \in I} W_i = W$.
Thus W is a vector subspace of V.

The next definition is about sum of subsets of a vector space.

Definition: Let $S_1, S_2, \ldots S_k$ are subsets of vector space V. Then sum of $S_1, S_2, \ldots S_k$ denoted as $S_1 + S_2 + S_3 + \cdots S_k$ is defined as

$$S_1 + S_2 + \cdots S_k = \{\alpha_1 + \alpha_2 + \cdots \alpha_k : \alpha_i \in S_i, \forall 1 \le i \le k\}$$

or

$$\sum_{i=1}^{k} S_i = \{\alpha = \alpha_1 + \alpha_2 + \cdots \alpha_k : \alpha_i \in S_i, 1 \le i \le k\}$$

Remark: If $W_1, W_2, \ldots W_k$ are subspaces of vector space V, then $\sum_{i=1}^{k} W_i$ is a subspace of V containing each of W_i's.

Definition: (Subspace spanned by a set)

Let S be a nonempty subset of a vector space, then the subspace spanned by S is the set of all linear combination of vectors in S and is denoted as $\langle S \rangle$.

Let $S = \{\alpha_1, \alpha_2, \dots \alpha_k\}$ then the subspace (spanned by S)

$$\langle S \rangle = \{c_1\alpha_1 + c_2\alpha_2 + \dots c_k\alpha_k \ \forall c_i \in F, 1 \le i \le k\}$$

Example: Construct a subspace spanned by $\alpha_1 = (1,0,1)$, $\alpha_2 = (0,2,1)$, $\alpha_3 = (0,1,0)$

Solution: Any vector in $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ will be of the following type

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$$
$$= (c_1, 2c_2 + c_3, c_1 + c_2)$$

where c_1, c_2, c_3 are arbitrary.

In particular (take $c_1 = c_2 = c_3 = 2$)

$(2,6,4)$ belongs to $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$.

Is there any vector in $\mathbb{R}^3$ which does not belong to $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$

Example. Take $\alpha_1 = (1,1,2)$, $\alpha_2 = (0,1,1)$, $\alpha_3 = (1,0,1)$. Find $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$

Solution: A general vector in $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is

$$\alpha = c_1(1,1,2) + c_2(0,1,1) + c_3(1,0,1)$$
$$= (c_1 + c_3, c_1 + c_2, 2c_1 + c_2 + c_3)$$

In particular ($c_1 = 0, c_2 = c_3 = 4$)

$(4,4,8)$ belongs to $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$

Is $(2,2,5)$ belongs to $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$?

Question: Is the vector $(3,-1,0,-1)$ in the subspace of $\mathbb{R}^4$ spanned by $(2,-1,3,2)$, $(-1,1,1,-3)$ and $(1,1,9,-5)$

* Solution:- Any vector α belongs to the ~~span~~ subspace spanned by $\{\alpha_1, \alpha_2, \cdots \alpha_k\}$ if $\exists\ c_1, c_2, \cdots c_k \in \mathbb{F}$ such that

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots c_k\alpha_k$$

So the question is can you prove the existence of $c_1, c_2, \cdots c_k$ such that

$$(3,-1,0,-1) = c_1(2,-1,3,2) + c_2(-1,1,1,-3) + c_3(1,1,9,-5)$$

or

$$(3,-1,0,-1) = (2c_1 - c_2 + c_3, -c_1 + c_2 + c_3, 3c_1 + c_2 + 9c_3, 2c_1 - 3c_2 - 5c_3)$$

or

$$2c_1 - c_2 + c_3 = 3$$
$$-c_1 + c_2 + c_3 = -1$$
$$3c_1 + c_2 + 9c_3 = 0$$
$$2c_1 - 3c_2 - 5c_3 = -1$$

Thus the problem is reduced into solving a system of 4 equations in 3 unknowns. (c_1, c_2, c_3) and showing it's consistency.

Thus you need to solve using GJE the following $AC = B$, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 3 & 1 & 9 \\ 2 & -3 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

If solution exists for this system then the answer will be YES.

Basis and dimension of vector space

Our next objective is to study the dimension of a vector space. We start with the following definition

Definition (linearly independent set)

Let V be a vector space over F. Then a set of vectors $\{\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_k\}$ is called linearly independent if the following equation

$$c_1\alpha_1 + c_2\alpha_2 + \cdots c_k\alpha_k = 0$$

has the only solution $c_1 = c_2 = \cdots c_k = 0$

If the set has infinitly many vectors, then it will be called linearly independent set if every finite subset of this set is linearly independent.

A set is linearly dependent if it's not linearly independent.

Example

Show that the vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ are linearly independent in $\mathbb{R}^3$ over $\mathbb{R}$.

Solution: Let us consider

$$c_1e_1 + c_2e_2 + c_3e_3 = (0,0,0)$$

$$\Rightarrow (c_1, c_2, c_3) = (0,0,0)$$

$$\Rightarrow c_1 = c_2 = c_3 = 0$$

Hence $\{e_1, e_2, e_3\}$ are linearly independent.

Example: Show that $\{(1,1,2), (2,1,0), (4,3,4)\}$ are linearly dependent in $\mathbb{R}^3$ over $\mathbb{R}$

Solution: Consider

$$c_1(1,1,2)+c_2(2,1,0)+c_3(4,3,4)=(0,0,0)$$

$$\Rightarrow (c_1+2c_2+4c_3, c_1+c_2+3c_3, 2c_1+4c_3)=(0,0,0)$$

which implies

$$c_1+2c_2+4c_3=0$$
$$c_1+c_2+3c_3=0$$
$$2c_1+4c_3=0$$

Therefore we get a homogeneous system of linear equations in three unknowns. If this system has only the trivial solution then $c_1=c_2=c_3=0$ and vectors will be linearly independent and if system has non-trivial solution, then vectors will be linearly dependent.

We know that if the coefficient matrix has determinant zero then system will have non-trivial solution which is the case here as

$$\det\begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 3 \\ 2 & 0 & 4 \end{pmatrix}=0$$

Note the following observations

1. Any set which contains a linearly dependent set is linearly dependent
2. Any subset of linearly independent set is linearly independent.
3. Any subset which contains zero vector will be linearly dependent.

Now we move to the definition of basis for a vector space.

Definition: Let V be a vector space of $\mathbb{F}$. Then a subset B of V is called basis for V if the following two conditions are satisfied

1) B spans V
2) B is linearly independent.

The no of elements in the basis set represents the dimension of vector space.

Example: Prove that

$$e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)$$

is a basis for $V = \mathbb{R}^3$ over $\mathbb{R}$.