Assignment-1

Linear Algebra-(IC152)
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Question 1. For each of the following linear operator T on a vector space V and an order basis β, compute $[T]_\beta$ and determine whether β is a basis consisting of eigenvectors of T.

(a) $V = P_1(\mathbb{R}),\ T(a+bx) = 6(a-b) + (12a - 11b)x$ and $\beta = \{3+4x, 2+3x\}$.

(b) $V = p_3(\mathbb{R})$,

$$T(a+bx+cx^2) = (-4a+2b-2c) - (7a+3b+7c)x + (7a+b+5c)x^2$$

and $\beta = \{x - x^2, -1 + x^2, -1 - x + x^2\}$.

(c) $V = M_{2\times 2}(\mathbb{R})$,

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a-4b+4c-4d & b \\ -8a-4b+5c-4d & d \end{pmatrix}$$

and

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\}.$$

Question 2. For each of the following matrices $A \in M_{n\times n}(\mathbb{F})$:

1. Determine all the eigenvalues of A.
2. For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.
3. If possible find a basis of $\mathbb{F}^n$ consisting of eigenvectors of A.
4. If successful in finding such basis determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $\mathbb{F} = \mathbb{R}$.

(b) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for $\mathbb{F} = \mathbb{C}$.

Question 3. For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

(a) $V = \mathrm{P}_2(\mathbb{R})$ and $T(f(x)) = xf'(x) + f(2)x + f(3)$.

(b) $\mathrm{P}_3(\mathbb{R})$ and $T(f(x)) = xf'(x) + f"(x) - f(2)$.

(c) $V = M_{2\times 2}(\mathbb{R})$ and
$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & b \\ e & a \end{pmatrix}.$$

(d) $V = M_{2\times 2}(\mathbb{R})$ and $T(A) = A^t + 2.tr(A).I$.

Question 4. A scalar matrix is a square matrix of the form λI for some scalar λ , i.e., scalar matrix in which all the diagonal entries are equal.

(a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

(b) Show that a diagonal matrix have only one eigen value is a scalar matrix.

(c) Prove that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Question 5. Let T be a linear operator on $M_{n\times n}(\mathbb{R})$ defined by $T(A) = A^t$.

(a) Show that are the only eigenvalues of T.

(b) Describe the eigenvectors corresponding to each eigenvalue of T.

(c) Find the ordered basis β for $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.

(d) Find an ordered basis β for $M_{n\times n}(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix for $n > 2$.

Question 6. Let T be an invertible linear operator on a finite dimensional vector space V.

(a) Prove that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1}.

(b) Prove that the eigen space corresponding to λ is same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(c) Prove that if T is diagonalizable then T^{-1} is digonalizable.

Question 7. In each part, apply the Gram Schmidt process to the given subset S of the inner product space V to obtain the orthogonal basis for span(S).

Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonarmal basis for S. Then any vector $x \in S$, we write $u = \sum_{i=1}^{n} < x, u_i > u_i$, where $c =< x, u_i >$ for $i = 1, 2, \ldots, n$. Compute the coefficient c_is of the given vectors relative to β.

(a) $V = \mathbb{R}^3$, $S = \{(1,1,1), (0,1,1), (0,0,1)\}$ and $x = (1,0,1)$. Then normalize the vectors to obtain the orthonormal basis β for span(S).

(b) $V = P_2(\mathbb{R})$ with the inner product

$$< f(x), g(x) >= \int_0^\infty f(t)g(t)\mathrm{d}t,$$

$S = \{1, x, x^2\}$ and $h(x) = 1 + x$.

(c) $V = M_{2\times 2}(\mathbb{R})$, $S = \left\{ \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 4 & -11 \\ 3 & -11 \end{pmatrix} \right\}$ and $A = \begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}$.

(d) Span S with the inner product

$$< f, g >= \int_0^\infty f(t)g(t)\mathrm{d}t$$

$S = \{\sin t, \cos t, t\}$ and $h(t) = t + 1$.

Question 8. Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{C}^3$, then compute $S^\perp$.

Question 9. Let $S_o = \{x_0\}$, where x_0 is a non-zero lvector in $\mathbb{R}^3$. Describe $S_0^\perp$ geometrically. Now suppose that $S = \{x_0, x_1\}$ is linearly independent subset of $\mathbb{R}^3$. Describe $S_0^\perp$ geometrically.

Question 10. Let $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{C}^3$. Find orthonormal base for W and $W^\perp$.