Q 10) i) for W.

c) $\Rightarrow$ $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$.

$\Rightarrow$ $u_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}^T - \frac{1}{2}\left(\langle(1,2,1).(1,0,i)\rangle\right)(1,0,i)$

$\Rightarrow$ $u_2 = (1,2,1) - \frac{1}{2}(1-i)(1,0,i)$

$\Rightarrow$ $u_2 = \left(\frac{1}{2}+\frac{i}{2}, 2, 1 - \frac{i}{2} + \frac{i^2}{2}\right)$

$\Rightarrow$ $u_2 = \left(\frac{1+i}{2}, 2, \frac{1-i}{2}\right)$

$\Rightarrow$ $u_2 \equiv (1+i, 2, 1-i)$.

$\Rightarrow$ $V_2 = \frac{1}{2\sqrt{2}}(1+i, 2, 1-i)$.

for $W^\perp$

$\langle(x, y, z)(1,0,i)\rangle = 0$

$x - zi = 0 \quad \therefore x = zi$

$\langle(x,y,z)(1,2,1)\rangle = x + 2y + z = 0$.

$\therefore y = -\frac{(i+1)}{2} z$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} i \\ -\frac{(1+i)}{2} \\ 1 \end{pmatrix}$; Orthonormal basis $= \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ -\left(\frac{i+1}{2}\right) \\ 1 \end{pmatrix}$

Q8) $S^{\perp} = \{ v : \langle v, u \rangle = 0 ; u \in S \}$.

let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^{T}$.

$(x, y, z) \cdot (1, 0, i) = \quad x - iz = 0$

$x = iz$

$(x, y, z) \cdot (1, 2, 1) = x + 2y + z = 0$.

$= -\frac{(i+1)z}{2} = y$.

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} i \\ -\left(\frac{i+1}{2}\right) \\ 1 \end{bmatrix}$

$S^{\perp} = L\left(\begin{bmatrix} i \\ -\frac{i-1}{2} \\ 1 \end{bmatrix}\right)$.

Q9)

if $S_0 = \{x_0\} \rightarrow$ line.

$\therefore S^{\perp}$ = plane which has normal vector as x_0;

if $S_0 = \{x_0, x_1\} \Rightarrow$ plane containg vector x_0 & x_1 $\therefore S^{\perp} = \perp$ vector of plane S_0

d)

$$u_1 = \sin t .$$

$$v_1 = \frac{\sin t}{|u_1|} \quad ; \quad |u_1| = \sqrt{\int_0^{\pi} \sin^2 t \, dt} .$$

$$\Rightarrow |u_1| = \sqrt{\int_0^{\pi} \frac{1-\cos 2t}{2} \, dt} = \sqrt{\tfrac{1}{2}\pi - \tfrac{1}{2}(0)} = \sqrt{\frac{\pi}{2}} .$$

$$v_1 = \frac{\sin t \sqrt{2}}{\sqrt{\pi}} .$$

$$u_2 = \cos t - \frac{2}{\pi} \int_0^{\pi} \sin 2t \, dt \, \sin t = \cos t .$$

$$|u_2| = \sqrt{\int_0^{\pi} \frac{\cos 2t - 1}{2} \, dt} = \sqrt{\frac{\pi}{2}} .$$

$$v_2 = \frac{\sqrt{2}}{\sqrt{\pi}} \cos t .$$

$$u_3 = t - \frac{2}{\pi} \int_0^{\pi} t \sin t \, dt \, \sin t - \frac{2}{\pi} \int_0^{\pi} t \cos t \, dt \, \cos t .$$

$$= t - \frac{2}{\pi} (\pi - 2) \sin t + \frac{4}{\pi} \cos t .$$

$$v_3 = t - \frac{2}{\pi} (\pi - 2) \sin t + \frac{4}{\pi} \cos t .$$

$$c_1 = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\pi} (t+1) \sin t = \frac{\sqrt{2}}{\sqrt{\pi}} \left(\int_0^{\pi} t \sin t + \int_0^{\pi} \sin t \right)$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} (\pi + 2)$$

$$c_2 = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\pi} (t+1) \cos t \, dt = \frac{\sqrt{2}}{\sqrt{\pi}} (-2) = \frac{-2\sqrt{2}}{\sqrt{\pi}}$$

Q8) $S^{\perp} = \{ v : \langle v, u \rangle = 0 \, ; \, u \in S \}$.

$$|u_2| = \sqrt{25 + 8 + 8 + 4} = \sqrt{45}.$$

$$v_2 = \frac{1}{\sqrt{45}} \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix}.$$

$$u_3 = \begin{pmatrix} 4 & -11 \\ 3 & -11 \end{pmatrix} - \frac{1}{45} \operatorname{tr}\left(\begin{pmatrix} 4 & -11 \\ 3 & -11 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix} \right) \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix}$$

$$- \frac{1}{13} \operatorname{tr}\left(\begin{pmatrix} 4 & -11 \\ 3 & -11 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 4 & -11 \\ 3 & -11 \end{pmatrix} - \frac{1}{45} \left(20 + 44 - 6 - 22 \right) \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix}$$

$$- \frac{1}{13} \left(8 - 22 + 6 - 11 \right) \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -11 \\ 3 & -11 \end{pmatrix} - \frac{36}{45} \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix} + \frac{19}{13} \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

$$v_3 = \frac{u_3}{|u_3|} \quad ; \quad c_3 = \langle h(x), v_3 \rangle.$$

$$c_2 = \operatorname{tr}\left(\begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix} \right) \times \frac{1}{\sqrt{45}}$$

$$= (40 - 24 - 50 - 26) \times \frac{1}{\sqrt{45}} = \frac{-60}{\sqrt{45}}$$

$$c_1 = \operatorname{tr}\left(\begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \right) \times \frac{1}{\sqrt{13}} = (16 + 12 + 50 - 13) \times \frac{1}{\sqrt{13}}$$

$$= \frac{65}{\sqrt{13}}.$$

$$h(x) = \langle h(x), v_1 \rangle v_1 + \langle h(x), v_2 \rangle (v_2) + \langle h(x), v_3 \rangle v_3$$

$$c_1 = \langle h(x), v_1 \rangle = \langle 1+x, 1 \rangle$$

$$= \int_0^1 1+x = 1 + \frac{1}{2} = \frac{3}{2}.$$

$$c_2 = \langle h(x), v_2 \rangle = \int_0^1 (1+x) v_2 \, dx$$

$$c_3 = \langle h(x), v_3 \rangle = \int_0^1 (1+x) v_3 \, dx.$$

c) $\quad u_1 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$

$$\cancel{|u_1| = \sqrt{\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}}} \qquad |u_1| = \sqrt{tr\left(\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}\right)}$$

$$|u_1| = \sqrt{8+5} = \sqrt{13}$$

$$v_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix} - \frac{1}{13} tr\left(\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix}\right)\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 4 \\ 2 & 5 \end{pmatrix} - \frac{1}{13}(26 + 13)\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 11-6 & 4-6 \\ 2-6 & 5-3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -2 \\ -4 & 2 \end{pmatrix}$$

$v_3 = \frac{1}{\sqrt{2}}(0,-1,1)$. ; $\beta = \{v_1, v_2, v_3\}$.

$c_1 = \langle (1,0,1), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \rangle = \frac{2}{\sqrt{3}}$

$c_2 = \langle (1,0,1), \frac{1}{\sqrt{6}}(-2,1,1) \rangle = -\frac{1}{\sqrt{6}}$

$c_3 = \langle (1,0,1), (0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \rangle = \frac{1}{\sqrt{2}}$.

b)

i) $h(x) = \cancel{c_1} + c\cancel{2}x + c^3/x^2$

ii) $v_1 = 1$

$w_2 = x - \langle 1, x \rangle = x - \frac{1}{2} = x - \frac{1}{2}$

$|x - \cancel{\tfrac{1}{2}}| = \cancel{\sqrt{\int_0^1 (x-\frac{1}{2})^2 dx}} = \cancel{\sqrt{\int \int (x-\frac{1}{2})^2 \Big|_0^1}} = \cancel{\sqrt{\frac{1}{2} - \frac{1}{2}}}$

$\int_0^1 (x - \frac{1}{2})^2 dx = \int_{-\frac{1}{2}}^{1/2} t^2 dt = \frac{t^3}{3} = \frac{1}{3}\left[\frac{1}{2^3} + \frac{1}{2^3}\right] = \frac{1}{3 \times 2^2}$

$|w_2| = \frac{1}{2\sqrt{3}}$.

$v_2 = (x - \frac{1}{2})\frac{1}{2\sqrt{3}}$

$w_3 = x^2 - \langle 1, x^2 \rangle - \frac{1}{12}\langle x - \frac{1}{2}, x^2 \rangle (x - \frac{1}{2})$

$w_3 = x^3 - \frac{1}{3} - \frac{x}{144} + \frac{1}{288}$.

$|w_3| = \sqrt{\langle w_3, w_3 \rangle}$

$\therefore$ both Eigen Space are Same.

c) if T is diagonalisable.

$TP = PD \Rightarrow P^{-1}T^{-1} = D^{-1}P^{-1}$

$\Rightarrow T^{-1} = PD^{-1}P^{-1}$

$\Rightarrow T^{-1}P = PD^{-1}$

$\therefore T^{-1} = PD^{-1}P^{-1} \rightarrow T^{-1}$ is diagonalisable.

Q7)

a) $v_1 = \frac{1}{\sqrt{3}}(1,1,1)$

$w_2 = (0,1,1) - \frac{1}{3}(2)(1,1,1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \equiv (-2,1,1)$

$v_2 = \frac{1}{\sqrt{6}}(-2,1,1)$

$w_3 = (0,0,1) - \frac{1}{3}(1)(1,1,1) - \frac{1}{6}(1)(-2,1,1)$

$w_3 = (0,0,1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(\frac{-2}{6}, \frac{1}{6}, \frac{1}{6}\right)$

$= (0,0,1) - \left(0, \frac{1}{2}, \frac{1}{2}\right)$

$= \left(0, -\frac{1}{2}, \frac{1}{2}\right) \equiv (0,-1,1)$

c) $\beta = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$.

b) for $\lambda = 1$ $\quad A^t = A$ } Symetric Matrix

for $\lambda = -1$. $\quad A^t = -A$ } Skew - Symetric Matrix.

b) c) Basis β consist of all Symetric and Skew Symetric Matrix.

$\Rightarrow$ $\beta = \{$ set of all Symetric + Skew Symetric Matrix $\}$ -

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Q6)

a) $TA = \lambda A \Rightarrow T^{-1}TA = \lambda T^{-1}A$.

$\Rightarrow A = \lambda T^{-1}A \Rightarrow T^{-1}A = \frac{1}{\lambda}A = \lambda^{-1}A$.

$\therefore$ Eigen Value $= -\lambda$;

b) let $v \in \{v_i : i = 1(1)n\} \rightarrow$ set of all eigen vecto.

$Tv = \lambda v$; $\quad T^{-1}Tv = \lambda T^{-1}v$

$\Rightarrow T^{-1}v = \lambda^{-1}v$.

$\Rightarrow v$ corresponds to Eigen Space of T^{-1}.

Q5) $T(A) = A^t = \lambda A$ — (i)

a)

Taking Transpose in eq. (i)

$A = \lambda A^t \Rightarrow A = \lambda(\lambda A)$ from (i)

$A = \lambda^2 A \quad \therefore \quad \lambda = \pm 1$

for $\lambda = 1$

c)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$b = c$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

for $\lambda = -1$

$$\begin{bmatrix} 1+1 & 0 & 0 & 0 \\ 0 & 0+1 & 1 & 0 \\ 0 & 1 & 0+1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$a = 0$

$b = -c$

$d = 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\text{Eigen vectors} \equiv \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\lambda = 1}, \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\lambda = -1}$$

for $\lambda = 5$

$$\begin{bmatrix} -2 & 0 & 0 & 2 \\ 0 & -5 & 1 & 0 \\ 0 & 1 & -5 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

$x = w$

$\left.\begin{aligned} y &= 5\cancel{z} \\ z &= 5y \end{aligned}\right\} y = z = 0$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Q4)

a) As A Similar to λI $\therefore$ there exists a P such that

$A = P(\lambda I)P^{-1}$

$A = \lambda PP^{-1} = \lambda I$

b) let γ be Eigen value.

$$\begin{vmatrix} \lambda-\gamma & 0 \cdots \\ 0 & \lambda-\gamma \\ \vdots & & \ddots \\ & & & \lambda-\gamma \end{vmatrix}_{n \times n} \qquad (\lambda-\gamma)^n = 0 \qquad \lambda = \gamma \text{ Only Solution.}$$

c) $\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0 \quad (1-\lambda)^2 = 0 \quad \therefore \lambda = 1$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \therefore x = 0,\ y = 0$ no. Eigen vector $\therefore$ not dia

$|w_3| = \sqrt{\langle w_3, w_3 \rangle}$

$$\begin{vmatrix} 3-\lambda & 0 & 0 & 2 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 2 & 0 & 0 & 3-\lambda \end{vmatrix} = 0 .$$

$$(3-\lambda)\left[(3-\lambda)\left[\lambda^2-1\right]\right] - 2\left[2\left[\lambda^2-1\right]\right].$$

$$\Rightarrow (3-\lambda)^2(\lambda^2-1) - 4(\lambda^2-1) = 0$$

$$\Rightarrow (\lambda^2-1)\left((3-\lambda)^2 - 2^2\right) = 0$$

$$\Rightarrow (\lambda+1)(\lambda-1)\,(5-\lambda)(1-\lambda) = 0 .$$

$$(1-\lambda)^2(\lambda+1)(5-\lambda) = 0$$

$$\therefore \lambda = 5, 1, -1 .$$

for $\lambda = 1$

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0 .$$

$x + w = 0 \quad x = -w .$

$y = z$

ef. y=t) a

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} .$$

for $\lambda = -1$

$$\begin{bmatrix} 4 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

$$\left.\begin{aligned} 2x + w &= 0 \\ y &= -z \\ x + 2w &= 0 \\ \Rightarrow x = w &= 0 \end{aligned}\right\} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

for $\lambda = -1$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 .$$

$+1\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$a = -d$; $b = 0$, $c = 0$,

$$e.v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} .$$

d) $$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} + 2(a+d)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a + 2d. & c \\ b & 2a+3d \end{bmatrix}$$

b) $$T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} .$$

$$T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 3 \end{bmatrix}$$

ord. $\beta = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -3/2 \\ 0 \end{pmatrix}, \begin{pmatrix} -7/2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

b)

c) $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = +\lambda\left((1-\lambda)^2\lambda\right) - 1\left((1-\lambda)^2\right)$$

$$= \lambda^2(1-\lambda)^2 - (1-\lambda)^2$$

$$(1-\lambda)^2(\lambda^2-1) = (\lambda-1)^2(\lambda-1)(\lambda+1)$$

$\lambda = \pm 1$

for $\lambda = 1$

$$\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \quad \Big\} \quad a = d$$

$$e.v = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

for $\lambda = -1$

$$\begin{bmatrix} 0 & -2 & -2 & -8 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = 0 .$$

$y + z + 4w = 0$

$y + 3w = 0 \Rightarrow y = 0$

$z = 0$

$w = 0$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

for $\lambda = 2$

$$\begin{bmatrix} -3 & -2 & -2 & -8 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0 .$$

a

$\Rightarrow$ $-3x + 2y + 2z + 8w = 0$. $\quad z = -\frac{3}{2}x$.

$-y + 6w = 0 \Rightarrow y = 0$

$w = 0$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -3/2 \\ 0 \end{bmatrix}$$

b)

Tv:

for $\lambda = 3$

$$\Rightarrow \begin{bmatrix} -4 & -2 & -2 & -8 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0 \quad \left\} \begin{aligned} 2x + y + z + 4w &= 0 \\ -y + 3w &= 0 \\ z &= 0 \\ y &= 3w \\ x &= -\tfrac{7}{2}w \end{aligned} \right\} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = w\begin{bmatrix} -7/2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

ord. basis = $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -3/4 \\ -13/4 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

b) $T(1) = -1 = -1(1) + 0(x) + 0(x^2) + 0(x^3)$

$T(x) = x - 2 = -2(1) + 1(x) + 0(x^2) + 0(x^3)$

$T(x^2) = 2x^2 + 2 - 4 = 2x^2 - 2 = -2(1) + 0(x) + 2(x^2)$

$T(x^3) = 3x^3 + 6x - 8 = -8(1) + 6(x) + 0(x^2) + 3(x^3)$

$$A = \begin{bmatrix} -1 & -2 & -2 & -8 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = 0.$$

$$\begin{vmatrix} -1-\lambda & -2 & -2 & -8 \\ 0 & 1-\lambda & 0 & 6 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{vmatrix} = (-1+\lambda)(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$\therefore \lambda = 1, -1, 2, 3.$

for $\lambda = 1$

$$\begin{bmatrix} -2 & -2 & -2 & -8 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

$x + y + z + 4w = 0$
$6w = 0$
$z = 0$

$w = 0$
$z = 0$
$x + y = 0 \therefore x = -y.$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{vmatrix} 1-\lambda & 3 & 9 \\ 1 & 3-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)\left[(3-\lambda)(2-\lambda)\right] - 3\left[2-\lambda\right] + 9(0)$$

$$= (2-\lambda)\left(((1-\lambda)(3-\lambda) - 3\right)$$

$$= (2-\lambda)(\cancel{3} - 4\lambda + \lambda^2 - \cancel{3}) = (2-\lambda)\lambda(\lambda-4)$$

$\lambda = 0, 2, 4.$

for

$\lambda = 0$

$$\begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad ; \quad \begin{aligned} x + 3y + 9z &= 0 \\ x + 3y + 4z &= 0 \\ 2z &= 0 \quad \therefore z = 0. \end{aligned}$$

$x = -3y$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

for; $\lambda = 2$

a

$$\begin{bmatrix} -1 & 3 & 9 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \Big\} \quad \begin{aligned} -x + 3y + 9z &= 0 \\ x + y + 4z &= 0. \end{aligned}$$

$4y + 13z = 0$

$y = -\frac{13}{4}z.$

$\Rightarrow \quad x - \frac{13}{4}z + 4z = 0 \Rightarrow x = -\frac{3}{4}z.$

b) $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} -3/4 \\ -13/4 \\ 1 \end{bmatrix}.$$

Tv

$\lambda = 4.$

$$\Rightarrow \begin{bmatrix} -3 & 3 & 9 \\ 1 & -1 & 4 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \left\{ \begin{aligned} -3x + 3y + 9z &= 0 \\ x - y + 4z &= 0 \\ 2z &= 0 \\ \therefore z &= 0 \\ x &= y. \end{aligned} \right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = -1$

$$\begin{bmatrix} i+1 & 1 \\ 2 & -i+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$(i+1)x + y = 0$ $\cdot$ $\left\{ -(i+1)(i-1)\frac{y}{2} + y = 0 \right.$

$2x = \frac{y}{2}(1-i)$ $\quad$ $\left. -(\cdot\cdot) + y = 0 \right\}$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} \frac{1-i}{2} \\ 1 \end{bmatrix}$$

eigenvectors $\rightsquigarrow$ $\begin{pmatrix} \frac{i+1}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1-i}{2} \\ 1 \end{pmatrix}$

Basis $= \left\{ \begin{pmatrix} \frac{i+1}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1-i}{2} \\ 1 \end{pmatrix} \right\}$.

$$Q = \begin{bmatrix} \frac{i+1}{2} & \frac{1-i}{2} \\ 1 & 1 \end{bmatrix}$$

Q3) with respecto standard basis evaluating A.

$T(1) = 1 + x = 1(1) + 1(x) + 0(x^2)$

$T(x) = \cancel{3}x + 2x + 3 = 3(1) + 3(x) + 0(x^2)$

$T(x^2) = 2x^2 + 4x + 9 = 9(1) + 4(x) + 2(x^2)$

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

using $Ax - \lambda x = 0$

for $\lambda = \cancel{8}\ 4$

$$\begin{bmatrix} -\cancel{4} & -2 & -3 \\ -1 & -\cancel{3} & -1 \\ 2 & 2 & \cancel{0}\,1 \end{bmatrix}\begin{bmatrix} x \\ \cancel{y} \\ z \end{bmatrix} = 0$$

$$\left.\begin{aligned} 4x + 2y + 3z &= 0 \\ x + 3y + z &= 0 \\ 2x + 2y + z &= 0 . \end{aligned}\right\} \quad \begin{aligned} x + 3y - 2x - 2y &= 0 \\ y &= x . \\ z &= -4x \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$$

Eigen Vectors are $\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$$\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$Q = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & -4 \end{bmatrix} .$$

(b) $A = \begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix}$; $\begin{vmatrix} i-\lambda & 1 \\ 2 & -i-\lambda \end{vmatrix} = 0$

$$(i-\lambda)(-i-\lambda) - 2 = -i^2 - \cancel{\lambda i} + \cancel{\lambda i} + \lambda^2 - 2 = 0 .$$

$1 + \lambda^2 - 2 = 0 \qquad \lambda^2 - 1 = 0 \therefore \quad \lambda = \pm 1$

for $\lambda = 1$

$$\begin{bmatrix} i-1 & 1 \\ 2 & -i-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \left\{ \begin{aligned} (i-1)x + y &= 0 \\ \cancel{2}x &= (i+1)\frac{y}{2} \end{aligned} \right\} \quad \begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} \frac{i+1}{2} \\ 1 \end{bmatrix}$$

$\Rightarrow (\lambda-1)(\lambda-3)(4-\lambda)=0.$

$\therefore \lambda_1 = 1;\ \lambda_2 = 3,\ \lambda_3 = 4.$

Finding eigen vectors:

① for $\lambda=1$

$[A-\lambda I]X=0$;
$$\begin{bmatrix} -1 & -2 & -3 \\ -1 & 1-1 & -1 \\ 2 & 2 & 5-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-x-2y-3z=0 \quad \therefore x+2y+3z=0.$

$-x-z=0 \quad \therefore x=-z.$

$x+y+2z=0; \quad y+z=0 \quad \therefore y=-z.$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix} = -z\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

② for $\lambda=3$

$$\begin{bmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\left.\begin{aligned} 3x+2y+3z&=0 \\ x+2y+z&=0 \\ x+y+z&=0 \end{aligned}\right\} \quad \begin{aligned} y&=0. \\ x&=-z \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$T\begin{pmatrix}-1 & 2\\ 0 & 0\end{pmatrix} = \begin{pmatrix}7-8 & 2\\ 0 & 0\end{pmatrix} = \begin{pmatrix}-1 & 2\\ 0 & 0\end{pmatrix}$$

$$= 0\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} + 1\begin{pmatrix}-1 & 2\\ 0 & 0\end{pmatrix} + 0\begin{pmatrix}1 & 0\\ 2 & 0\end{pmatrix} + 0\begin{pmatrix}-1 & 0\\ 0 & 2\end{pmatrix}$$

$$T\begin{pmatrix}1 & 0\\ 2 & 0\end{pmatrix} = \begin{pmatrix}-7+8 & 0\\ -8+10 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 2 & 0\end{pmatrix} = 0\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} + 0\begin{pmatrix}-1 & 2\\ 0 & 0\end{pmatrix} + 1\begin{pmatrix}1 & 0\\ 2 & 0\end{pmatrix}$$
$$+ 0\begin{pmatrix}-1 & 0\\ 0 & 2\end{pmatrix}$$

$$T\begin{pmatrix}-1 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}7-8 & 0\\ 8-8 & 2\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & 2\end{pmatrix} = 0\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} + 0\begin{pmatrix}-1 & 2\\ 0 & 0\end{pmatrix}$$
$$+ 0\begin{pmatrix}1 & 0\\ 2 & 0\end{pmatrix} + 1\begin{pmatrix}-1 & 0\\ 0 & 2\end{pmatrix}$$

$\therefore$ $\forall$ vector element of basis $T(K) = \lambda K$

$\therefore$ basis consists of Eigen Vectors.

Q2)

a) $A = \begin{bmatrix}0 & -2 & -3\\ -1 & 1 & -1\\ 2 & 2 & 5\end{bmatrix}$

for eigen value: $Ax = \lambda x \Rightarrow |A - \lambda I| = 0$.

$$\begin{vmatrix}0-\lambda & -2 & -3\\ -1 & 1 & -1\\ 2 & 2 & 5\end{vmatrix} = -\lambda[(1-\lambda)(5-\lambda) - 2\times 2] + 2[-2(5-\lambda) - 2(-1)] - 3[-2(-1) - (1-\lambda)2]$$

$$= -\lambda^3 + 9\lambda^2 - 19\lambda + 12$$

for $x - x^2$; $a = 0, b = 1, c = -1$

$$T(x - x^2) = 2+4 - (3-7)x + (1-5)x^2$$
$$= 4 + 4x - 4x^2 = 0(x - x^2) + 0(-1 + x^2) - 4(-1 - x + x^2)$$

for $(-1 + x^2) \Rightarrow a = -1, b = 0, c = 1$

$$T(-1 + x^2) = (4-2) - (-7+7)x + (-7+5)x^2$$
$$= 2 + (-2)x^2 = 2(1 - x^2)$$
$$= 0(x - x^2) - 2(-1 + x^2) + 0(-1 - x + x^2)$$

for $-1 - x + x^2 \Rightarrow a = -1, b = -1, c = 0$

$$T(-1 - x + x^2) = (4-2-2) - (-7-3+7)x + (7-1+5)x^2$$
$$= 3x - 3x^2 = 3(x - x^2) + 0(-1 + x^2) + 0(-1 - x + x^2)$$

$$[T]_\beta = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -2 & 0 \\ -4 & 0 & 0 \end{bmatrix}$$

for Basis $x - x^2$ & $-1 - x + x^2$

$T(x) \neq \lambda x$ $\therefore$ β does not consist of Eigen Vector.

c) $V = M_{2\times 2}(\mathbb{R})$ $\quad T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a - 4b + 4c - 4d & b \\ -8a - 4b + 5c - 4d & d \end{pmatrix}$

$$T\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -7+4 & 0 \\ -8+5 & 0 \end{pmatrix} = -3\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 0\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} + 0\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$
$$+ 0\begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix}$$

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Assignment - 1

Q1)

a) $V = P_1(R)$, $T(a+bx) = 6(a-b) + (12a - 11b)x$

$\beta = \{3+4x, 2+3x\}$

$T(3+4x) = 6(3-4) + (12\times3 - 4\times11)x$

$= -6 - 8x = -2(3+4x) + 0(2+3x)$

$T(2+3x) = 6(2-3) + (12\times2 - 3\,11)x$

$= -6 - 9x = 0(3+4x) - 3(2+3x)$

$$[T]_\beta = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

$T(3+4x) = -2(3+4x)$ which by defination means $3+4x$ is Eigen vector corresponding to eigen value -2 as $(Tx = \lambda x)$

Similary $T(2+3x) = -3(2+3x)$ which by defination $2+3x$ is corresponds to Eigen value -3.

b) $V = P_2(R)$ $T(a+bx+cx^2) = (-4a+2b-2c) - (7a+3b+7c)x + (7a+b+5c)x^2$

$\beta = \{x - x^2, -1+x^2, -1-x+x^2\}$