(1)

$$f(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

Let $P$ be any partition on $[0,2]$. Then

$$U(P,f) = \sum_{k=1}^{n} M_k (x_k - x_{k-1}) = 1 \quad [\because f(x) = 1,\ x \neq]$$

$L(P,f)$ will be less than 2 because any subinterval of $P$ that contains $x=1$ will contribute zero to the value of the lower sum. They way to to show $f$ is integrable is to construct a partition that minimizes the effect of discontinuity by embedding $x=1$ into a very small subinterval. Let $\epsilon > 0$ and consider the partition $P_\epsilon = \{0, 1-\epsilon/3, 1+\epsilon/3, 2\}$

Then $U(P_\epsilon, f) = 2$

$$L(P_\epsilon, f) = 1(1-\epsilon/3) + 0 \cdot \frac{2\epsilon}{3} + 1(2 - (1+\epsilon/3))$$

$$= 2 - 2\epsilon/3$$

$$U(P_\epsilon, f) - L(P_\epsilon, f) = \cancel{2} + 2\epsilon/3 - \cancel{2} = \frac{2\epsilon}{3} < \epsilon$$

$\Rightarrow f$ is integrable.

② Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Since $U(P,f) = L(P,f)$

$$\Rightarrow \sum_{K=1}^{n} (M_K - m_K)(x_K - x_{K-1}) = 0$$

Since $M_K \geq m_K$ so $(M_K - m_K) \geq 0$ & also $x_K - x_{K-1} > 0$

$\Rightarrow$ $M_K - m_K = 0$ i.e. $M_K = m_K$ for

$K = 1, 2, \ldots n$.

$\Rightarrow$ $f$ is constant on $[x_{K-1}, x_K]$ for each $K = 1, 2, \ldots, n$

$\Rightarrow f(x_{K-1}) = f(x_K) = f(x) \quad \forall\, x \in [x_{K-1}, x_K]$, $K = 1(1)n$

consequently $f(x) = f(a) \quad \forall\, x \in [a, b]$.

Therefore $f$ is a constant fun$^n$

③
$$f(x) = \begin{cases} 0 & \text{if } x \in [a,b] \cap \mathbb{Q} \\ x & \text{if } x \in [a,b] \cap \mathbb{Q}^c \end{cases}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

We have $m_K = 0$, $M_K = x_K$, $K = 1, 2, \ldots n$

$\Rightarrow L(P,f) = 0 \Rightarrow \sup_{P \in \mathcal{P}} L(P,f) = 0$

$$U(P,f) = x_2(x_2 - x_1) + \cdots\cdots + x_k(x_k - x_{k-1}) + \cdots + x_n(x_n - x_{n-1})$$

$$= U(P, g), \quad \text{where } g(x) = x.$$

$g : [a,b] \to \mathbb{R}$ is a integrable fu$^n$. since $g(x) = x$ is continuous

$$\inf_{P \in \mathcal{P}} U(P,g) = \frac{b^2 - a^2}{2} = \int_a^b g(x)\,dx$$

$$\Rightarrow \quad \inf_{P \in \mathcal{P}} U(P,f) = \frac{b^2 - a^2}{2}$$

Since $0 < \frac{b^2 - a^2}{2}$. So $f(x)$ is not integrable

(4) $$\frac{1}{n^2} \sum_{k=1}^{n} \sqrt{n^2 - k^2} = \frac{n}{n^2} \sum_{k=1}^{n} \sqrt{1 - \left(\frac{k}{n}\right)^2}$$

consider $f(x) = \sqrt{1 - x^2}$, $x \in [0,1]$

consider the standard partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots \frac{n}{n} = 1\}$

on $[0,1]$. Then

$$S_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^2} \sum_{k=1}^{n} \sqrt{n^2 - k^2}$$

So $$\lim_{n \to \infty} S_n = \int_0^1 f(x)\,dx = \frac{1}{2}\left(x\sqrt{1-x^2} + \sin^{-1} x\right)\Big|_0^1 = \pi/4$$

(5) Since $f$ is continuous on $[a,b]$, $f$ is integrable on $[a,b]$ and so have

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Where $m = \inf\limits_{x\in[a,b]} f(x)$, $M = \sup\limits_{x\in[a,b]} f(x)$

Since $f$ is continuous on $[a,b]$ $\exists\ \alpha, \beta \in [a,b]$ s.t $f(\alpha) = m$, $f(\beta) = M$.

Hence

$$f(\alpha) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq f(\beta)$$

Now by the intermediate value property of continuous fun$^n$ $\exists$ $c \in [\alpha,\beta]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \Rightarrow \int_a^b f(x)\,dx = (b-a)f(c)$$

(6) Since $f'$ is continuous on $[0,\frac{1}{3}]$ then $f'$ is integrable on $[0,1/3]$ and $\int_0^{1/3} f'(t)\,dt = \lim\limits_{\|P_n\|\to 0} \sigma(P_n, f', x^*)$

Where $P_n = \left\{0, \frac{1}{3n}, \frac{2}{3n}, \dots \frac{n}{3n} = \frac{1}{3}\right\}$ is a partition of $[0,1/3]$ and $\sigma(P_n, f', x^*) = \sum\limits_{k=1}^{n}\left(\frac{k}{3n} - \frac{k-1}{3n}\right) f'\left(\frac{k}{3n}\right)$

$$= \frac{1}{3n}\sum_{k=1}^{n} f'\left(\frac{k}{3n}\right)$$

So $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f'\left(\frac{k}{3n}\right) = 3\int_0^{1/3} f'(t)\,dt = 3\left[f(1/3) - f(0)\right].$

(7) Since $f$ is continuous on $[a,b]$, $f$ is bdd on $[a,b]$ and $\exists\ \alpha, \beta \in [a,b]$ s.t. $f(\alpha) = m$, $f(\beta) = M$

$M = \sup_{x\in[a,b]} f(x)$, $m = \inf_{x\in[a,b]} f(x)$.

we have

$$f(\alpha) \le f(x) \le f(\beta) \quad \forall\ x \in [a,b]$$

$$\Rightarrow f(\alpha)\,g(x) \le f(x)\,g(x) \le f(\beta)\,g(x) \quad \forall\ x \in [a,b]$$

$$(\because g(x) > 0)$$

Since $f, g$ are continuous on $[a,b]$, so $g$, $fg$ are integrable on $[a,b]$. Hence we have

$$f(\alpha)\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le f(\beta)\int_a^b g(x)\,dx$$

If $\int_a^b g(x) = 0$ then $\int_a^b f(x)\,g(x) = 0$ and so we can choose any $c \in [a,b]$. If $\int_a^b g(x)dx \neq 0$ then $\int_a^b g(x)\,dx > 0 \Rightarrow f(\alpha) \le \dfrac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \le f(\beta)$

By intermediate value prop of continuous fun $f$

$\exists\ c \in [\alpha, \beta]$ s.t.

$$f(c) = \frac{\int_a^b f(x)\, g(x)\, dx}{\int_a^b g(x)\, dx}.$$

$$\Rightarrow f(c) \int_a^b g(x)\, dx = \int_a^b f(x)\, g(x)\, dx.$$

(8) (1) Let $f(x) = \frac{x}{\sin x}$ $\forall x \in (0, \pi/2]$

$$f'(x) = \frac{\sin x - x\cos x}{\sin^2 x} \quad \forall x \in (0, \pi/2].$$

Let $g(x) = \sin x - x\cos x$ $\forall x \in [0, \pi/2]$

$g'(x) = x\sin x \geq 0$ $\forall x \in [0, \pi/2]$

So $g(x)$ is increasing on $[0, \pi/2]$. Hence for all $x \in [0, \pi/2]$ $g(x) \geq g(0) = 0$ consequently $f'(x) \geq 0$ $\forall x \in (0, \pi/2] \Rightarrow f(x)$ is increasing on $(0, \pi/2]$ and so $f(\pi/6) \leq f(x) \leq f(\pi/2)$

Since $f(x)$ is continuous on $[\pi/6, \pi/2]$

$$\Rightarrow \int_{\pi/6}^{\pi/2} f(\pi/6)\, dx \leq \int_{\pi/6}^{\pi/2} f(x)\, dx \leq \int_{\pi/6}^{\pi/2} f(\pi/2)\, dx$$

$$\Rightarrow \quad \frac{\pi^2}{9} \le \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\, dx \le \frac{2\pi^2}{9}.$$

(ii) $f(x) = \frac{\sin x}{x} \quad \forall\, x \in (0, \pi/2]$. Do same way.

(9) we have

$$g(x) = \int_0^x (x-t) f(t)\, dt \quad \text{for all } x \in \mathbb{R}$$

$$= x\int_0^x f(t)\, dt - \int_0^x t f(t)\, dt$$

$$= x\, F(x) - G(x)$$

where $F(x) = \int_0^x f(t)\, dt$, $G(x) = \int_0^x t f(t)\, dt$

Since $f(x)$ is continuous so $F(x)$ and $G(x)$ are diffⁿ $\Rightarrow$ $F'(x) = f(x)$.

$G'(x) = x f(x)$.

$$\Rightarrow g'(x) = \int_0^x f(t)\, dt + \cancel{x f(x)} - \cancel{x f(x)}$$

$$= \int_0^x f(t)\, dt = F(x).$$

$$\Rightarrow g''(x) = f(x).$$

(10) $\phi(x) = \int_{x^2}^{x^3} \frac{1}{(1+t^2)^3} dt$ , $x \in [1, \infty)$. Find $\phi'(x)$

Let $u = x^2$, $x \in [1, \infty)$, $v = x^3 \in [1, \infty)$, $f(t) = \frac{1}{(1+t^2)^3}$

with $t \in \mathbb{R}$.

So $\phi(x) = \int_u^v f(t) dt = \int_0^v f(t) dt - \int_0^u f(t) dt \quad \because v > u$

as $x \geq 1 \Rightarrow v \geq 1$ & $u \geq 1$.

$F(u) = \int_0^u f(t) dt$, $G(v) = \int_0^v f(t) dt$.

Since $f$ is continuous on $\mathbb{R}$ so on $[0, u]$, $u \geq 1$

and also continuous on $[0, v]$, $v \geq 1$.

$\Rightarrow F'(u) = f(u)$, $G'(v) = f(v)$.

Now for all $x \in [1, \infty)$

$\phi'(x) = G'(v) \frac{dv}{dx} - F'(u) \frac{du}{dx}$

$= \frac{3x^2}{(1+x^6)^3} - \frac{2x}{(1+x^4)^3}$.