Probability distribution of a function of discrete r.v.

Let $(\Omega, \mathcal{S}, P)$ be a prob space and let $X : \Omega \to \mathbb{R}$ be a r.v. with F and p.m.f $f(x)$ and support S. Let $h : \mathbb{R} \to \mathbb{R}$ be a given function. Define $Z : \Omega \to \mathbb{R}$ as

$$Z(\omega) = h(X(\omega)), \quad \omega \in \Omega$$

Then Z is a r.v. and it is a fun of r.v. X.

We have
$$F(x) = P_r(X \leq x), \quad x \in \mathbb{R}$$
$$f(x) = P_r(X = x), \quad x \in \mathbb{R}, \quad P_r(X \in S) = 1$$

Define $T = h(S) = \{h(x) : x \in S\}$. For any set $A \subseteq \mathbb{R}$, Define

$$h^{-1}(A) = \{x \in S : h(x) \in A\}.$$

Then T is a countable set also

$$P_r(Z = z) > 0 \quad \forall z \in T$$

and
$$P_r(Z \in T) = 1 \quad (\because P_r(X \in S) = 1)$$

It follows that Z is a discrete r.v. Moreover for $z \in T$

$$P_r(Z = z) = P_r(h(X) = z) = \sum_{\{x \in S : h(x) = z\}} P_r(X = x)$$
$$= \sum_{x \in h^{-1}(\{z\})} P_r(X = x)$$

and for any $z \in T$, $P_r(Z = z) = 0$, $z \in T$.

Thus we have the following Theorem

Theorem: Let X be a discrete r.v. with support S and d.f. F and p.m.f. $f(x)$. Let $h: \mathbb{R} \to \mathbb{R}$ be a given function. Then $Z = h(x)$ is a discrete r.v. with support $T = \{h(x) : x \in S\}$ and p.m.f

$$g(z) = \begin{cases} \sum\limits_{x \in h^{-1}(\{z\})} f(x) & \text{if } z \in T \\ 0 & \text{otherwise} \end{cases}$$

and d.f.

$$G(z) = P_r(Z \le z) = \sum_{\{t \in T : t \le z\}} g(t) = \sum_{\{x \in S : h(x) \le z\}} f(x)$$

In particulare where $h: S \to \mathbb{R}$ one-one then

$$g(z) = \begin{cases} f(h^{-1}(\{z\})), & \text{if } z \in T \\ 0, & \text{otherwise} \end{cases}$$

Example: Let X be a discrete r.v. with p.m.f

$$f(x) = \begin{cases} \frac{1}{7} & \text{if } x \in \{-2,-1,0,1\} \\ \frac{3}{14} & \text{if } x \in \{2,3\} \\ 0 & \text{o/w} \end{cases}$$

Find the p.m.f of $Y = X^2$

Solu^n: Here the support of X is $S = \{-2,-1,0,1,2,3\}$.

Then $Y = X^2$ is d.r. with support

$$T = \{0, 1, 4, 9\}$$

and p.m.f

$$g(z) = P_r(X^2 = z) = \begin{cases} P_r(X=0), & \text{if } z=0 \\ P_r(X=-1) + P(X=1), & \text{if } z=1 \\ P_r(X=-2) + P(X=2) & \text{if } z=4 \\ P_r(X=-3) + P(X=3) & \text{if } z=9 \end{cases}$$

$$= \begin{cases} \frac{1}{7} & \text{if } z=0 \\ 2/7 & \text{if } z=1 \\ 5/14 & \text{if } z=4 \\ \frac{3}{14} & \text{if } z=9 \\ 0, & \text{o/w}. \end{cases}$$

The d.f of Y is

$$G(z) = P_r(Y \le z) = \begin{cases} 0, & z < 0 \\ 1/7, & 0 \le z < 1 \\ 3/7, & 1 \le z < 4 \\ \frac{11}{14}, & 4 \le z < 9 \\ 1, & z \geq 9 \end{cases}$$

Probability distribution of a function of a continuous random variable

Theorem: Let X be a continuous random variable with pdf $f(x)$. Let $Y = g(x)$ be a fun of r.v. X. Suppose g is differentiable and strictly monotone for all x. Then i.e. $g'(x) > 0$ or $g'(x) < 0$

$Y = g(x)$ is an r.v. of the continuous type with pdf given by

$$f_Y(y) = \begin{cases} f^{-1}(g^{-1}(y)) \left| \frac{d\, g^{-1}(y)}{dy} \right|, & \alpha < y < \beta \\ 0 & o/w. \end{cases}$$

where α, β are respectively, the lower and upper bounds of the range of Y

Example: Let X be a r.v. with p.d.f

$$f(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0 & o/w. \end{cases}$$

Find the p.d.f and d.f. of $Y = \frac{1}{X^2}$. What is the support of d.f. of Y.

soln: $S = \{x : f(x) > 0\} = (0,1)$

$g(x) = \frac{1}{x^2}$ is differentiable and strictly monotone on $(0,1)$, $h((0,1)) = (1, \infty)$. Now

$y = \frac{1}{x^2} \Rightarrow x = \frac{1}{\sqrt{y}}$ i.e. $g^{-1}(y) = \frac{1}{\sqrt{y}}$

$$\frac{d}{dy} g^{-1}(y) = -\frac{1}{2y\sqrt{y}}$$

So The pdf Y is

$$f_Y(y) = \begin{cases} 3\frac{1}{y}\frac{1}{2y\sqrt{y}} & \text{if } y > 1 \\ 0, & o/w. \end{cases}$$

$$= \begin{cases} \frac{3}{2y^2\sqrt{y}}, & \text{if } y > 1 \\ 0, & o/w. \end{cases}$$

The d.f. of Y is

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)\,dt$$

$$= \begin{cases} 0, & y \le 1 \\ \int_1^y \frac{3}{2t^2\sqrt{t}}\,dt & \text{if } y > 1 \end{cases} = \begin{cases} 0, & y \le 0 \\ 1 - \frac{1}{y^{3/2}}, & y > 1 \end{cases}$$

In the following theorem we give a general result

Theorem: Let X be a continuous r.v. with d.f. F and p.d.f $f(x)$. Suppose that $\{x \in \mathbb{R} : f(x) > 0\}$ $= \bigcup_{i\in 1}^{k} (a_i, b_i)$, where (a_i, b_i) are disjoint. Let $h : \mathbb{R} \to \mathbb{R}$ be a fun^n s.t. on each (a_i, b_i) $h : (a_i, b_i) \to \mathbb{R}$ is strictly monotone, differentiable and $h'(x)$ continuous with inverse $h_i^{-1}(\cdot)$. Let $h((a_i, b_i)) = \{h(x) : x \in (a_i, b_i)\}$. Then the random variable $Z = h(X)$ is continuous type with p.d.f

$$f_Z(z) = \sum_{i=1}^{k} f_X\left(h_i^{-1}(z)\right) \left| \frac{d}{dz} h_i^{-1}(z) \right| I_{h_i(a_i,b_i)}(z),$$

where $I_{h_i(a_i,b_i)} = \begin{cases} 1, & z \in h_i(a_i, b_i) \\ 0, & o/w \end{cases}$

Ex Let X be r.v. with p.d.f

$$f(x) = \begin{cases} \frac{|x|}{2} & \text{if } -1 < x < 1 \\ \frac{x}{3} & \text{if } 1 \le x \le 2 \\ 0 & \text{o/w} \end{cases}$$

and let $Z = X^2$

(a) Find the pdf of Z

Solⁿ we have $S = (-1,0) \cup (0,2)$. Let $S_1 = (-1,0)$, $S_2 = (0,2)$. $h(x) = x^2$, $x \in S$ is strictly decreasing in S_1 and strictly increasing in S_2. Let $Z = h(X) \Rightarrow X = h^{-1}(z)$

Now $h^{-1}(x) = -\sqrt{z}$, $x \in (-1,0)$ &

$h^{-1}(x) = \sqrt{z}$, $x \in (0,2)$

$h(S_1) = (0,1)$, $h(S_2) = (0,4)$. Then the p.d.f $Z = X^2$ is

$$f_Z(z) = f_X(-\sqrt{z}) \left|\frac{d}{dz}(-\sqrt{z})\right| I_{(0,1)}(z) + f_X(\sqrt{z}) \left|\frac{d}{dz}(\sqrt{z})\right| I_{(0,4)}(z)$$

$$= \begin{cases} \frac{1}{2}, & \text{if } 0 < z < 1 \\ 1/6, & \text{if } 1 < z < 4 \\ 0, & \text{o/w.} \end{cases}$$

(b) Find the d.f. of Z and hence find the pdf.

Sol^n

We have $F_Z(z) = P(Z \le z) = P(X^2 \le z) \quad \forall z \in \mathbb{R}$

$$= P(\{-\sqrt{z} \le X \le \sqrt{z}\})$$

$$= \int_{-\sqrt{z}}^{\sqrt{z}} f_X(x)\, dx.$$

we have $P\{X \in (-1,2)\} = 1$ and $P(\{Z \in (0,4)\}) =$

If $z < 0$, $F_Z(z) = P(Z \le z) = 0$ &

If $z \geq 4$ $F_Z(z) = P(Z \le z) = 1$

Now consider $z \in [0,4)$.

$z \in [0,1)$, then $F_Z(z) = \displaystyle\int_{-\sqrt{z}}^{\sqrt{z}} f_X(x)\,dx = \int_{-\sqrt{z}}^{\sqrt{z}} \frac{|x|}{2}$

If $1 \le z < 4$, Then $F_Z(z) = \displaystyle\int_{-\sqrt{z}}^{\sqrt{z}} f_X(x) = \int_{-1}^{1} \frac{|x|}{2} dx + \int_{1}^{\sqrt{z}} \frac{x}{3} dx$

$[\ 1 \le z < 4 \Rightarrow -2 < \sqrt{z} \le -1$ & $1 \le \sqrt{z} < 2)$

So

$$F_Z(z) = \begin{cases} 0, & z < 0 \\ z/2, & 0 \le z < 1 \\ \frac{z+2}{6}, & 1 \le z < 4 \\ 1, & z \geq 4. \end{cases}$$

$F_Z(z)$ is differentiable except at finite number of points then

$$F'(z) = \begin{cases} \frac{1}{2}, & 0 < z < 1 \\ 1/6, & 1 < z < 4 \\ 0, & \text{otherwise} \end{cases}$$

Moreover

$$\int_{-\infty}^{\infty} F'(z)dz = \int_0^1 1/2\, dz + \int_1^4 \frac{1}{6} dz = 1.$$

Thus Z is a continuous random variable with p.d.f.

$$f_Z(z) = \begin{cases} \frac{1}{2} & \text{if } 0 < z < 1 \\ 1/6 & \text{if } 1 < z < 4 \\ 0 & o/w. \end{cases}$$