Row equivalent matrices

Let A be a $m \times n$ matrix over the field F. Then row vectors will be the elements of F^n and there will be m such vectors.

For example, $A \in M_{2\times 3}(\mathbb{R})$

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$$

then $(1,3,4) \in \mathbb{R}^3$ & $(2,4,6) \in \mathbb{R}^3$ are the row vectors of A.

Row space of a matrix A is defined as the subspace of F^n spanned by row vectors of the matrix A.

The dimension of row space is equal to the row rank of the matrix.

The above fact can be written in the form of the following theorem

Theorem Let R be a non zero row reduced echelon matrix. Then non zero row vectors of R form a basis for the row space of R.

Proof: Let $f_1, f_2, \ldots f_r$ be the non zero rows of R and denote

$$f_i = (R_{i1}, R_{i2}, \ldots R_{in})$$

To check that $\{f_1, f_2 \ldots f_r\}$ form a basis of row space, we need to ensure that they are linearly independent. Consider an arbitrary linear combination

$$c_1 f_1 + c_2 f_2 + \ldots c_r f_r = 0$$

Then we make use of following observation w.r.to RRE form.

Let $k_1, k_2 \dots k_r$ be the column numbers of $1, 2, \dots r^{th}$ row where the leading entry is found (and equal to 1)

Then for $i \leq r$

i) $R_{ij} = 0$ if $j < k_i$

ii) $R_{ik_j} = \delta_{ij}$

iii) $k_1 < k_2 \dots < k_r$

Under these observations let's look back the linear combination

$$c_1 \rho_1 + c_2 \rho_2 + \dots c_r \rho_r = \beta = (b_1, b_2 \dots b_n)$$

$$\Rightarrow \sum_{i=1}^{r} c_i R_{ik_j} = b_{k_j}$$

$$\Rightarrow \sum_{i=1}^{r} c_i \delta_{ij} = b_{k_j}$$

$$\Rightarrow c_j = b_{k_j}$$

In case $\beta = (0, 0, 0 \dots 0)$

then $c_j = 0 \ \forall \ j = 1, 2, \dots r$.

Let us see the following result.

Theorem: Row-equivalent matrices have the same row space.

Proof: Let A & B be $n \times n$ matrices having row equivalence. It means there exist elementary matrices $E_1, E_2, \cdots E_k$ such that

$$A = E_1 E_2 \cdots E_k B.$$

Denote by $E = E_1 E_2 \cdots E_k$, then

$$A = EB \text{ and } E \text{ is invertible}$$

as E_i is invertible for each $i = 1, 2 \cdots k$.

Let us observe that, if $(A_{i1}, \cdots A_{in})$ be the i^{th} row of matrix A, $(E_{i1}, \cdots E_{in})$ be the i^{th} row of E and $(B_{i1} \cdots B_{in})$ be the i^{th} row of B, then

$$\begin{aligned}
&(A_{i1}, A_{i2}, \cdots A_{in}) \\
&= E_{i1}(B_{11}, B_{12}, \cdots B_{1n}) \\
&\quad + E_{i2}(B_{21}, B_{22}, \cdots B_{2n}) \\
&\quad \vdots \\
&\quad E_{in}(B_{n1}, B_{n2}, \cdots B_{nn})
\end{aligned}$$

Hence row space of $A \subseteq$ row space of B

Now using $E^{-1}A = B$, we can get row space of $B \subseteq$ row space of A implies row space of $A =$ row space of B.

Theorem: Each $m \times n$ matrix is row equivalent to a unique row reduced echelon matrix.

Proof: We know that atleast one RRE can be found using elementary row operation for any matrix A. Let there be another RRE associated with A (say R') other than R. Then R is row equivalent to R' and hence will have the same row space and must be identical.

Theorem: Let A and B be $m \times n$ matrices over the field F. Then A and B are row equivalent iff they have the same row space.

Proof:- In one direction, we already proved that is if A and B are row equivalent then they have the same row space. Conversely assume that they have the same row space, which implies their RRE (say R & R') will also have the same row space and hence will be identical $R = R'$. Therefore A and B will be row equivalent by equivalence relation.

Let us see the following example

Example: Let A & B be given as

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{bmatrix}$$

then we can compute that A and B have the same RRE as

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence the row space of A and B is spanned by $\{(1,0,2),(0,1,5)\}$

As the row space of both the matrices is same, both the matrices are row equivalent.