Vector Space

Definition: A vector space or linear space consists of the following.

1. a field $\mathbb{F}$ of scalars
2. a non-empty set V, (elements are called vectors)
3. a rule called vector addition denoted as '+', i.e.
   $+ : V \times V \to V$ satisfying
   a) $\alpha + \beta = \beta + \alpha$ (commutativity) $\forall \alpha, \beta \in V$
   b) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ (Associativity) $\forall \alpha, \beta, \gamma \in V$
   c) There exists a unique vector called zero vector '0' s.t.
   $$\alpha + 0 = 0 + \alpha = \alpha \;\; \forall \alpha \in V$$
   d) for each vector $\alpha \in V$, there exists a vector '$-\alpha$' such that $\alpha + (-\alpha) = 0$ (zero vector)

4. a rule (operation), called scalar multiplication, denoted as '$\cdot$'
   i.e. '$\cdot$' : $\mathbb{F} \times V \to V$ satisfying
   a) $1 . \alpha = \alpha \;\; \forall \alpha \in V$
   b) $(c_1 c_2) . \alpha = c_1 \cdot (c_2 \cdot \alpha)$
   c) $c \cdot (\alpha + \beta) = c\alpha + c \cdot \beta$
   d) $(c_1 + c_2) \cdot \alpha = c_1 \cdot \alpha + c_2 \cdot \alpha$

Thus in general, a vector space can be completely identified by $(V, \mathbb{F}, +, \cdot)$ or $V(\mathbb{F})$, read as V over $\mathbb{F}$.

Remark: The '$\cdot$' between ~~c & α~~ scalars from the field and vectors from V can be removed if we know the underlying operation.

Let us define field $\mathbb{F}$ of either of real numbers, complex numbers or rationals ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{Q}$.)

We call $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{Q}$) a field if

1. $x + y = y + x \quad \forall x, y \in \mathbb{F}$ (commutativity)

2. $x + (y + z) = (x + y) + z \quad \forall x, y, z \in \mathbb{F}$ (Associativity)

3. there is a unique element '0' in $\mathbb{F}$ such that $x + 0 = x \quad \forall x \in \mathbb{F}$

4. for each $x \in \mathbb{F}$, $\exists !$ (there exists a unique) '$-x$' in $\mathbb{F}$ s.t.
$$x + (-x) = 0$$

5. $xy = yx \quad \forall x, y \in \mathbb{F}$ (commutativity)

6. $x(yz) = (xy)z \quad \forall x, y, z \in \mathbb{F}$ (Associativity)

7. there exists a unique element 1 in $\mathbb{F}$ such that
$$1.x = x \quad \forall x \in \mathbb{F}$$

8. for each nonzero $x \in \mathbb{F}$ $\exists$ a unique element x^{-1} or $1/x$ such that
$$x . x^{-1} = 1$$

9. $x(y+z) = xy + xz$ distributive property of multiplication over addition.

Remark: It is easy to check that $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{Q}$ satisfy these properties, with usual addition and multiplication operation.

Let us see following examples of vector spaces.

Example: The space of n-tuple, $\mathbb{R}^n$ (or $\mathbb{C}^n$) is a vector space over $\mathbb{R}$ under the following operations

Vector addition is defined as,

for $x = (x_1, x_2, x_3 \dots x_n)$ &

$y = (y_1, y_2, y_3 \dots y_n)$,

$$x + y = (x_1 + y_1, x_2 + y_2, \dots x_n + y_n)$$

and $c \cdot x = (cx_1, cx_2, \dots cx_n)$

for any $c \in \mathbb{R}$, defines scalar multiplication.

Example: The space $M_{m \times n}(\mathbb{R})$ defines a vector space over $\mathbb{R}$. The operations of vector addition and scalar multiplication are as follows.

Let $A = (a_{ij})$, $1 \le i \le m$, $1 \le j \le n$

and $B = (b_{ij})$, $1 \le i \le m$, $1 \le j \le n$

then

$$(A + B) = (c_{ij}), 1 \le i \le m, \quad 1 \le j \le n$$

where $c_{ij} = a_{ij} + b_{ij}$

and

$$cA = (c\, a_{ij}), \quad 1 \le i \le m, 1 \le j \le n$$

Example. Let $F(S, \mathbb{F})$ denote the collection of all functions from S to a field $\mathbb{F}$. Then $F(S, \mathbb{F})$ forms a vector space over $\mathbb{F}$ under operations

$$(f + g)(s) = f(s) + g(s)$$

$$(cf)(s) = cf(s)$$

$\forall f, g \in F(S, \mathbb{F})$ and $s \in S$.

Example: The space of polynomials ~~over~~ of degree 'k' over a field F defines a vector space under the vector addition and scalar multiplication defined as

$$(p+q)[x] = (p_0+q_0) + (p_1+q_1)x + (p_2+q_2)x^2 + \cdots (p_k+q_k)x^k$$

and

$$(cp)[x] = (cp_0) + (cp_1)x + (cp_2)x^2 + \cdots (cp_k)x^k$$

where

$$p[x] = p_0 + p_1 x + p_2 x^2 + \cdots p_k x^k$$

and $q[x] = q_0 + q_1 x + q_2 x^2 + \cdots q_k x^k$

$\forall\, p, q \in V$ & $c \in \mathbb{F}$.

Let us see the following observations.

i) $0.\alpha = 0 \quad \forall\, \alpha \in V(\mathbb{F})$

ii) $c.0 = 0 \quad \forall\, c \in \mathbb{F}$

iii) $c.\alpha = 0$ implies either $\alpha = 0$ or $c = 0$

iv) $(-1).\alpha = -\alpha \quad \forall\, \alpha \in V$

We know that sum of any vectors of vector space is again in the vector space (closed under vector addition).

Let us now see the following definition.

Definition: Let $\beta \in V$, then β is called to be a linear combination of vectors $\alpha_1, \alpha_2, \ldots \alpha_n$ if there exists $c_1, c_2, \ldots c_n$ from the field such that $\beta = c_1\alpha_1 + c_2\alpha_2 + \ldots c_n\alpha_n$

or $\beta = \sum_{i=1}^{n} c_i \alpha_i$.

Next, we define vector subspaces.

Definition: Let V be a vector space over a field $\mathbb{F}$. Let $W \subseteq V$, then W is called vector subspace of vector space V if W is a vector space in itself under the vector addition and scalar multiplication inherited from V.

The following Theorem gives a characterization of vector subspace of a vector space

Theorem: A non-empty subset W of a vector space V over $\mathbb{F}$ is a subspace of V if and only if $c\alpha + \beta \in W$ $\forall\, \alpha, \beta \in W$ & $c \in \mathbb{F}$

Proof:- To prove this theorem we need to verify all the conditions of a vector space for W.

First assume that $c\alpha + \beta \in W$ whenever $\alpha, \beta \in W$ and $c \in \mathbb{F}$.

In particular for $c=1$, $\alpha+\beta \in W$ & $\forall\, \alpha, \beta \in W$ implies W is closed under vector addition. Now take $c=-1$ and $\beta=\alpha$, we have $-\alpha+\alpha=0 \in W$. hence zero vector belongs to W. Now take $\beta=0$, $c\alpha \in W$, $\forall\, \alpha \in W$ & $c \in \mathbb{F}$.
For $\beta=0$, $-\alpha+0=-\alpha \in W$ for each $\alpha \in W$ implies additive inverse belongs to W. Rest of the properties are independent of the choice of W as they hold true for every element (vector) of vector space V. Thus W is a vector space in itself and hence is a subspace of V.

Conversely assume that W is a subspace of V. Then for every $\alpha, \beta \in W$ and $c \in \mathbb{F}$,
$c\alpha \in W$ (closed under scalar multiplication) and
$c\alpha+\beta \in W$ (closed under vector addition).

<u>Examples</u> (i) Let V be a vector space over $\mathbb{F}$, then
$W=\{0\}$ and $W=V$ are trivial subspaces of V.

(ii) Let $V=\mathbb{R}^n$ be a vector space over $\mathbb{R}$ and
$$W=\{x \in \mathbb{R}^n : x_1=0\}$$
then W is a subspace of V as for $x, y \in W$
$$x=(0, x_2, \dots x_n) \text{ \& } y=(0, y_2, \dots y_n)$$
then for any $c \in \mathbb{R}$
$$cx+y=(0, cx_2+y_2, cx_3+y_3 \dots cx_n+y_n)$$
which belongs to W again.

Examples

Let A be a $m \times n$ matrix over $\mathbb{F}$
Let S denotes the solution set of $Ax=0$. Then it is easy to verify that S is a vector subspace of $\mathbb{F}^n$, $n \times 1$ matrices over $\mathbb{F}$.

Infact, if $x, y \in S$ then for $c \in \mathbb{F}$

$$A(cx+y) = cAx + Ay$$
$$= c.0 + 0 = 0$$

Hence $cx+y \in S$. Note that we have used the distributive property of matrices.