Tutorial 1. (Solutions)

Linear Algebra-(IC152)
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Answer 1.

1. Write $\begin{pmatrix} a \\ b \end{pmatrix} = \ln(a)\begin{pmatrix} e \\ 1 \end{pmatrix} + \ln(b)\begin{pmatrix} 1 \\ e \end{pmatrix}$

where $\ln(a)$ denote the natural logarithm. Thus $\begin{pmatrix} a \\ b \end{pmatrix} \in \text{Span}\left\{\begin{pmatrix} e \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ e \end{pmatrix}\right\}$

Also if for some real number a, we have

$$a\begin{pmatrix} e \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ e \end{pmatrix}$$

i.e.

$$\begin{pmatrix} e^a \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ e \end{pmatrix}$$

$\implies 1 = e$, not possible. Therefore $\beta = \left\{e^1 = \begin{pmatrix} e \\ 1 \end{pmatrix},\ e^2 = \begin{pmatrix} 1 \\ e \end{pmatrix}\right\}$ is linearly independent subset of V. Hence, β is a basis of V.

2. Let $X = \begin{pmatrix} x \\ y \end{pmatrix}, Y = \begin{pmatrix} u \\ v \end{pmatrix} \in V$ and $\alpha \in \mathbb{R}$, then

$$T(\alpha X + Y) = T\left(\begin{pmatrix} x^\alpha \\ y^\alpha \end{pmatrix} + Y\right) = T\begin{pmatrix} x^\alpha u \\ y^\alpha v \end{pmatrix}$$

$$= \begin{pmatrix} \log_2(y^\alpha v) \\ \log_2(x^\alpha u) \end{pmatrix}$$

$$= \alpha\begin{pmatrix} \log_2(y) \\ \log_2(x) \end{pmatrix} + \begin{pmatrix} \log_2(v) \\ \log_2(u) \end{pmatrix}$$

$$= \alpha T(X) + T(Y)$$

Therefore T is linear.
3.

$$\begin{aligned}\mathrm{Ker(T)} &= \left\{X = \begin{pmatrix} a \\ b \end{pmatrix} \in V \mid T(X) = 0\right\} \\ &= \left\{X = \begin{pmatrix} a \\ b \end{pmatrix} \in V \mid \begin{pmatrix} \log_2(b) \\ \log_2(a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} \\ &= \left\{X = \begin{pmatrix} a \\ b \end{pmatrix} \in V \mid \log_2(b) = 0, \log_2(a) = 0\right\} \\ &= \left\{X = \begin{pmatrix} a \\ b \end{pmatrix} \in V \mid b = 1, a = 1\right\} = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.\end{aligned}$$

4. Here we have

$$\begin{aligned}T(e^1) &= 0.e_1 + \log_2 e.e_2 \\ T(e^2) &= \log_2 e.e_1 + 0.e_2.\end{aligned}$$

Therefore the matrix T is

$$[T]^S_\beta = \begin{bmatrix} 0 & \log_2 e \\ \log_2 e & 0 \end{bmatrix}$$

where $S = \left\{e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

Answer 2.

Let $C[0,1]$ be the vector space of all real valued continuous functions $[0,1]$ under usual addition and scalar multiplication. Define

$$V = \{f \in C[0,1] \mid f(0) = 0\}$$

then V is subspace of $C[0,1]$. The integral operator $f \mapsto \int_0^x f(t)\mathrm{d}t$ on V has no eigenvalue.
Explaination: Suppose $\lambda \in \mathbb{R}$ be an eigenvalue of this operator associated to the eigenvector $f \not\equiv 0$. Then, for all x

$$\int_0^x f(t)\mathrm{d}t = \lambda f(x),\ f(0) = 0.$$

If $\lambda = 0$,

$$\int_0^x f(t)\mathrm{d}t = 0.$$

This implies that $f \equiv 0$, an absurd.

If $\lambda \neq 0$, then differentiating above we have

$$f(x) = \lambda f^{'}(x)$$

On solving this differential equation we have $f(x) = Ae^{\frac{x}{\lambda}}$ and by $f(0) = 0$ we get $f \equiv 0$. Therefore no member of V is an eigen vector of operator corresponding to non-zero λ.

Answer 3.

We have

$$D : P_4 \to P_4$$

such that $D(p(x)) = \frac{\mathrm{d}}{\mathrm{d}x}p(x)$. Let $\beta = \{1, x, x^2, x^3\}$ be a basis of P_4. Then, we have

$$\begin{aligned}
D(1) &= 0.1 + 0.x + 0.x^2 + 0.x^3 \\
D(x) &= 1.1 + 0.x + 0.x^2 + 0.x^3 \\
D(x^2) &= 0.1 + 2.x + 0.x^2 + 0.x^3 \\
D(x^3) &= 0.1 + 0.x + 3.x^2 + 0.x^3.
\end{aligned}$$

So, the matrix of D relative to β is given by

$$[D]_\beta = A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that $A^4 = 0$, hence it is a nilpotent matrix. Therefore, all the eigenvalues of A are 0. Also

$$\begin{aligned}
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= 0 \\
\implies x_2 = 0,\ x_3 = 0,\ x_4 = 0.
\end{aligned}$$

So, $\{(a, 0, 0, 0) : a \in \mathbb{R} \setminus \{0\}\}$ is the set of eigenvectors of A corresponding to the eigenvalue 0. Therefore the set of eigen vector of D is $\{p(x) = a \mid a \in \mathbb{R} \setminus \{0\}\}$.

Answer 4.

The operator T is given as $T(A) = A^T$. Then,

$$T(A) = \lambda A \implies A^T = \lambda A$$
$$\implies (A^T)^T = (\lambda A)^T = \lambda A^T = \lambda(\lambda A) = \lambda^2 A.$$

Thus $\lambda^2 = 1$, that is, $\lambda = \pm 1$.

Answer 5

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1. Let x be a eigenvector corresponding to λ. Then $Ax = \lambda x$. So,

$$\begin{aligned} A^m x &= A^{m-1}(Ax) \\ &= \lambda A^{m-2}(Ax) \end{aligned}$$

   continuing this way, we get
$$A^m x = \lambda^m x.$$

2. - We have

$$\begin{aligned} Ax_i &= \lambda_i x_i, \ i = 1, 2, \dots, n \\ \implies A^{-1}Ax_i &= \lambda_i A^{-1} x_i \\ \implies A^{-1} x_i &= \frac{1}{\lambda_i} x_i \end{aligned}$$

     Since A is non-singular hence each $\lambda_i \neq 0$. Therefore, $1/\lambda_1, 1/\lambda_2, \dots, 1/\lambda_n$ are the eigenvalues of A^{-1}.

   - We have

$$Adj(A) = det(A)A^{-1}.$$

     Hence, $\frac{det(A)}{\lambda_i}$ for $i = 1, 2, \dots, n$ are the eigenvalues of $Adj(A)$.

Answer 6.

Let $A = (a_{ij}), 1 \leq i, j \leq n$. Given that $Ax = \lambda x$ for all $x \in \mathbb{R}^n$. Then $Ae_i = \lambda e_i$ implies $a_{ii} = \lambda$ and $a_{ij} = 0$ if $j \neq i$. That is,

$$a_{ij} = \begin{cases} \lambda & \text{if } i = j, \\ 0 & \text{if } j \neq i. \end{cases}$$

Equivalently, $A = \lambda I$.