Department of Mathematics

Indian Institute of Technology Bhilai

IC104: Linear Algebra-I

Tutorial Sheet 2: Vector Space

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1. Let $V = \mathbb{C}^3$ be a vector space over $\mathbb{C}$. Find the vectors which are the linear combinations of the vectors $\{(1, 0, -1),\ (0, 1, 1),\ (1, 1, 1)\}$

2. Let V be the set of pairs (x, y) of real numbers and $\mathbb{F}$ be the field of real numbers. Define

$$\begin{aligned}(x, y) + (x_1, y_1) &= (x + x_1, 0)\\ c(x, y) &= (cx, 0)\end{aligned}$$

Is V, with these operations, a vector space?

3. Let V be the set of all complex valued functions f on $\mathbb{R}$ such that $f(-t) = \overline{f(t)}$, for all $t \in \mathbb{R}$, with bar denoting complex conjugation. Define

$$\begin{aligned}(f + g)(t) &= f(t) + g(t)\\ (cf)(t) &= cf(t)\end{aligned}$$

Determine if V is a vector space under the above operations of vector addition and scalar multiplication. Give an example of a function in V which is not real valued.

4. Prove that th subspace spanned by a set $S \subset V$ is the smallest subspace of vector space V which contains S.

5. Let V be the vector space of real valued functions on $\mathbb{R}$. Which of the following subsets of V are subspaces of V

$$\begin{aligned}&W_1 = \{f \in V : f(t^2) = f(t)^2\}, &&W_2 = \{f \in V : f(0) = f(1)\}\\ &W_3 = \{f \in V : f(3) = 1 + f(-5)\}, &&W_4 = \{f \in V : f(-1) = 0\}\end{aligned}$$

6. Which of the following subsets of $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$

$W_1 = \{\alpha = (\alpha_1, \alpha_2, \cdots \alpha_n) \in \mathbb{R}^n : \alpha_1 \geq 0\}$, $W_2 = \{\alpha = (\alpha_1, \alpha_2, \cdots \alpha_n) \in \mathbb{R}^n : \alpha_1 + 3\alpha_2 = \alpha_3\}$,
$W_3 = \{\alpha = (\alpha_1, \alpha_2, \cdots \alpha_n) \in \mathbb{R}^n : \alpha_2 = \alpha_1^2\}$, $W_4 = \{\alpha = (\alpha_1, \alpha_2, \cdots \alpha_n) \in \mathbb{R}^n : \alpha_1.\alpha_2 = 0\}$

7. Let

$$W = \{x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : Ax = 0\},$$

where

$$A = \begin{bmatrix} 2 & -1 & 4/3 & -1 & 0\\ 1 & 0 & 2/3 & 0 & -1\\ 9 & -3 & 6 & -3 & -3 \end{bmatrix}.$$

Find a finite set of vectors which spans W.

8. Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let

$$V_e = \{f \in V : f(-x) = f(x)\} \text{ and } V_o = \{f \in V : f(-x) = -f(x)\}$$

Prove that

(a) V_e and V_o are subspaces of V.

(b) $V_e + V_o = V$

(c) $V_e \cap V_o = \{0\}$

9. Show that the vectors $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$ and $\alpha_3 = (0, -3, 2)$ form a basis for $\mathbb{R}^3$. Express each of the standard basis vectors as linear combinations of α_1, α_2 and α_3. (Recall that standard basis vectors of $\mathbb{R}^3$ are $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$).

10. Find three vectors in $\mathbb{R}^3$ which are linearly dependent such that any two of them are linearly independent.

11. Let V be the vector space of all 2×2 matrices over $\mathbb{R}$. Let

$$W_1 = \left\{ A \in V : A = \begin{bmatrix} x & -x \\ y & z \end{bmatrix} \right\}$$
$$W_2 = \left\{ A \in V : A = \begin{bmatrix} a & b \\ -a & c \end{bmatrix} \right\}$$

be subsets of V. Then prove that W_1 and W_2 are subspaces of V. Also find the dimensions of $W_1, W_2, W_1 + W_2$ and $W_! \cap W_2$.

12. Let V be a vector space over a field $\mathbb{F}$. If there are finite number of vectors which spans V, then prove that V is finite dimensional.

13. Find the coordinate matrix of the vector $(1, 0, 1)$ in the basis of $\mathbb{C}^3$ consisting of the vectors $(2i, 1, 0)$, $(2, -1, 1)$, $(0, 1 + i, 1 - i)$ in that order.

14. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for $\mathbb{R}^3$ consisting of $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1)$, $\alpha_3 = (1, 0, 0)$. What are the coordinates of the vector (a, b, c) in the ordered basis $\mathcal{B}$.

15. Using the idea of row spaces, prove that the following matrices are row equivalent

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 5 & -1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 10 \\ 2 & 0 & 4 \end{bmatrix}$$