Conditional Expectation :

ⓐ Let $\underline{X}$ be a p-dimensional r.v. and let $\underline{Y}$ be a q-dimensional r.v.

(i) The conditional expectation of $\Psi(\underline{X})$ given $\underline{Y}=\underline{y}$ is denoted by $E(\Psi(\underline{x})\,|\,\underline{Y}=\underline{y})$ is the expectation under the conditional distn of $\underline{X}$ given $\underline{Y}=\underline{y}$

(ii) The conditional variance of $\Psi(\underline{X})$ given $\underline{Y}=\underline{y}$ is the variance of $\Psi(\underline{x})$ under the conditional distn of $\underline{X}$ given $\underline{Y}=\underline{y}$

$$E(\Psi(x)\,|\,\underline{Y}=\underline{y}) = \begin{cases} \sum \Psi(x)\, f_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y}) \\ \int \Psi(\underline{x})\, f_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y})\, d\underline{x} \end{cases}$$

$$\mathrm{Var}(\Psi(x)\,|\,\underline{Y}=\underline{y}) = E\{(\Psi(\underline{x}) - E(\Psi(\underline{x})\,|\,\underline{Y}=\underline{y})\,|\,\underline{Y}=\underline{y})\}$$

(b) Let X_1 & X_2 be two random variables and $\underline{Y}$ be a q-dimensional r.v. Then the conditional covariance between X_1 & X_2 given $\underline{Y}=\underline{y}$ $(\mathrm{Cov}(X_1, X_2\,|\,\underline{Y}=\underline{y}))$ is the covariance between X_1 & X_2 under the conditional distn of X_1 & X_2 given $\underline{Y}=\underline{y}$

Function of Random vector :

Let $\underline{X} = (X_1, X_2, \ldots, X_p)$ be a discrete type random vector with support $S_{\underline{X}}$ and p.m.f. $f_{\underline{X}}(\cdot)$. Let $g_i : \mathbb{R}^p \to \mathbb{R}$ and let $Y_i = g_i(\underline{X})$, $i = 1, 2, \ldots K$. Define for $\underline{y} = (y_1, \ldots, y_K)$

$B_{\underline{y}} = \{ \underline{x} \in S_{\underline{x}} : g_1(\underline{x}) = y_1, \ldots, g_K(\underline{x}) = y_K \}$. Then the random vector $\underline{Y} = (Y_1, \ldots, Y_K)$ is a discrete type

and p.m.f. of $\underline{Y}$ is

$$f_{\underline{Y}}(\underline{y}) = \sum_{x \in B_{\underline{y}}} f_{\underline{X}}(\underline{x}), \quad \underline{y} \in \mathbb{R}^K$$

Example: Let $\underline{X} = (X_1, X_2, X_3)$ be a discrete type random vector with p.m.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \begin{cases} \frac{2}{9} & \text{if } \underline{x} \in \{(1,1,0), (1,0,1), (0,1,1)\} \\ \frac{1}{3} & \text{if } \underline{x} \in \{(1,1,1)\} \\ 0, & \text{o/w.} \end{cases}$$

Define $Y_1 = X_1 + X_2$, $Y_2 = X_2 + X_3$

Find the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$

So

$(y_1, y_2) \in \{(1,1), (1,2), (2,1), (2,2)\} = S_{\underline{Y}}$

So the joint p.m.f is

$$f_{\underline{Y}}(y_1, y_2) = P(X_1 + X_2 = y_1, X_2 + X_3 = y_2$$
$$= 0 \quad \text{if } (y_1, y_2) \notin S_{\underline{Y}}$$

Now $f_{\underline{Y}}(1,1) = P(X_1 + X_2 = 1, X_2 + X_3 = 1)$

$$= P(\underline{X} = (1,0,1)) = \frac{2}{9}$$

$$f_{\underline{Y}}(1,2) = P(X_1 + X_2 = 1, X_2 + X_3 = 2)$$
$$= P(\underline{X} = (0,1,1)) = 2/9$$

$$f_{\underline{Y}}(2,1) = P(\underline{X} = (1,1,0)) = 2/9$$

$$f_{\underline{Y}}(2,2) = P(\underline{X} = (1,1,1)) = 1/3$$

$$f_{\underline{Y}}(\underline{y}) = \begin{cases} \frac{2}{9}, & \text{if } \underline{y} \in \{(1,1), (1,2), (2,1)\} \\ \frac{1}{3} & \text{if } \underline{y} = (2,2) \\ 0, & \text{o/w.} \end{cases}$$

Let $\underline{X} = (X_1, X_2, \cdots, X_p)$ be a random vector of continuous type with joint pdf $f_X(\cdot)$ and support $S_{\underline{X}} = \{\underline{x} \in \mathbb{R}^p : f_{\underline{X}}(\underline{x}) > 0\}$. Suppose $t_j = h_j : \mathbb{R}^p \to \mathbb{R}$, $j = 1, 2, \cdots p$ are functions such that $\underline{h} = (h_1, \cdots, h_p) : S_{\underline{X}} \to \mathbb{R}$ one-one with inverses $h_1^{-1}(\underline{t}) = (h_1^{-1}(\underline{t}), \cdots, h_p^{-1}(t))$.

Further suppose that h_i^{-1}, $i = 1, 2, \cdots, p$ have continuous partial derivatives and the jacobian

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}(\underline{t})}{\partial t_1}, & \cdots & \frac{\partial h_1^{-1}(\underline{t})}{\partial t_p} \\ \vdots & & \\ \frac{\partial h_p^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_p^{-1}(\underline{t})}{\partial t_p} \end{vmatrix} \neq 0$$

Define $\underline{h}(S_{\underline{X}}) = \{\underline{h}(\underline{x}) = (h_1(\underline{x}), \cdots, h_p(\underline{x})) : \underline{x} \in S_{\underline{X}}\}$ and $T_j = h_j(\underline{X})$, $j = 1, 2, \cdots p$. Then the random vector $\underline{T} = (T_1, T_2, \cdots, T_p)$ has pdf

$$f_T(\underline{t}) = f_{\underline{X}}\left(h_1^{-1}(\underline{t}), \cdots, h_p^{-1}(\underline{t})\right) |J| \, I_{h(S_{\underline{X}})}(\underline{t})$$

Example (1) Let $X_1, X_2, X_3 \overset{iid}{\sim} \text{Exp}(1)$

$$Y_1 = X_1 + X_2 + X_3, \quad Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}, \quad Y_3 = \frac{X_1}{X_1 + X_2}$$

The inverses are

$$x_1 = y_1 y_2 y_3$$
$$x_2 = y_1 y_2 (1 - y_3)$$
$$x_3 = y_1 (1 - y_2)$$

$$J = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ y_2 (1-y_3) & y_1(1-y_3) & -y_1 y_2 \\ 1-y_2 & -y_1 & 0 \end{vmatrix}$$

$$= -y_1^2 y_2$$

So the joint pdf of $(X_1, X_2\ X_3)$ is

$$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{3} f_{X_i}(x_i) = \begin{cases} e^{-\Sigma x_i}, & x_i > 0, \quad i = 1, 2, 3 \\ 0, & o/w. \end{cases}$$

So the joint pdf of $\underline{Y} = (Y_1, Y_2, Y_3)$

$$f_{\underline{Y}}(\underline{y}) = \begin{cases} e^{-y_1} y_1^2 y_2, & y_1 > 0, \quad y_2, y_3 \in (0,1) \\ 0, & o/w \end{cases}$$

The marginal densities of Y_1, Y_2, Y_3 are

$$f_{Y_1}(y_1) = \frac{1}{2} y_1^2 e^{-y_1}, \quad y_1 > 0$$

$$f_{Y_2}(y_2) = \begin{cases} 2y_2, & 0 < y_2 < 1 \\ 0, & o/w \end{cases}, \quad f_{Y_3}(y_3) = \begin{cases} 1, & 0 < y_3 < 1 \\ 0, & o/w \end{cases}$$

Note that $f_{\underline{Y}}(\underline{y}) = \prod_{i=1}^{3} f_{Y_i}(y_i)$.

So Y_1, Y_2, Y_3 are independent.

Ex ② Let $X, Y \overset{i.i.d}{\sim} U(0,1)$

$U = X + Y, \quad V = X - Y$

Then $x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}$ $\quad J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$

$= \frac{1}{2}$

The joint pdf of X & Y

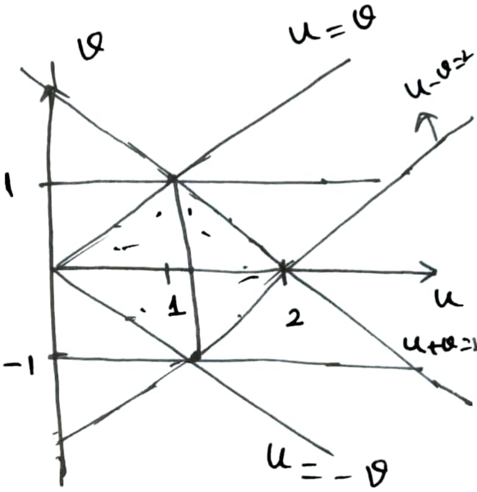
$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} 1, & 0 < x, y < 1 \\ 0, & o/w \end{cases}$$

So the joint pdf of U & V is

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{2}, & 0 < u+v < 2, \quad 0 < u-v < 2 \\ & 0 < u < 2, \quad -1 < v < 1 \\ 0, & o/w. \end{cases}$$

The marginal density of U is obtained as

$$f_U(u) = \begin{cases} \frac{1}{2}\int_{-u}^{u} dv, & 0 < u < 1 \\ \frac{1}{2}\int_{u-2}^{2-u} dv, & 1 < u < 2 \\ 0, & o/w. \end{cases}$$



$$= \begin{cases} u, & 0 < u < 1 \\ 2-u, & 1 < u < 2 \\ 0, & o/w. \end{cases}$$

The marginal of V is

$$f_V(v) = \begin{cases} \frac{1}{2}\int_{-v}^{v+2} du, & -1 < v < 0 \\ \frac{1}{2}\int_{v}^{2-v} du, & 0 < v < 1 \\ 0, & o/w \end{cases}$$

$$= \begin{cases} 1+v, & -1 < v < 0 \\ 1-v, & 0 < v < 1 \\ 0, & o/w. \end{cases}$$

Order statistics :

Let $X_1, X_2, X_3, \dots, X_n$ be i.i.d. with c.d.f $F(x)$ and pdf $f_X(x)$ (we consider the continuous case)

$$X_{(1)} = \min\{X_1, X_2, \dots, X_n\}$$

$$X_{(2)} = \text{second smallest } \{X_1, X_2, \dots, X_n\}$$

$$\vdots$$

$$X_{(n)} = \max\{X_1, X_2, \dots, X_n\}$$

Let $Y_n = X_{(n)} = \max\{X_1, \dots, X_n\}$

$$F_{X_{(n)}}(y_n) = P(X_{(n)} \le y_n)$$

$$= P(X_1 \le y_n, X_2 \le y_n, \dots, X_n \le y_n)$$

$$= \prod_{i=1}^{n} P(X_i \le y_n) = [F_X(y_n)]^n$$

So the pdf of $X_{(n)}$ is

$$f_{X_{(n)}}(y_n) = n[F_X(y_n)]^{n-1} f(y_n).$$

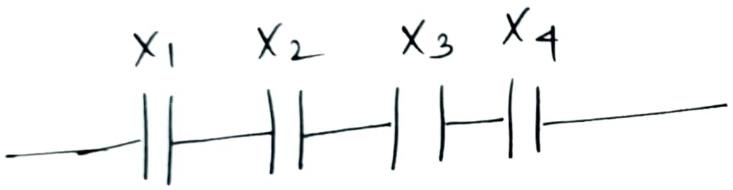
Next consider $Y_1 = X_{(1)}$

$$P(X_{(1)} > y_1) = P(X_1 > y_1, \dots, X_n > y_1)$$

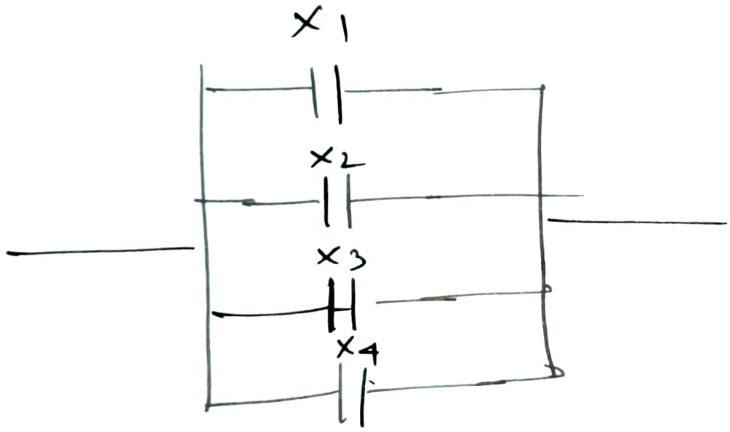
$$= \prod_{i=1}^{n} P(X_i > y_1) = \prod_{i=1}^{n} [1 - P(X_i \le y_1)]$$

$$F_{X_{(1)}}(y_1) = 1 - \left[1 - F_X(y_1)\right]^n$$

$$f_{X_{(1)}}(y_1) = n\left[1 - F_X(y_1)\right]^{n-1} f_X(y_1).$$



← life $X_{(1)}$



life $X_{(4)}$

In general the dist^n of r^{th} order statistic $X_{(r)}$ beyond the scope.

<u>Ex</u> Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} Exp(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0,\ \lambda > 0 \\ 0, & o/w \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0 \\ 0, & o/w. \end{cases}$$

consider $X_{(1)}$ then the dist^n of $X_{(1)}$ is

$$f_{X_{(1)}}(y_1) = n\left(e^{-\lambda y_1}\right)^{n-1} \lambda e^{-\lambda y_1}$$

$$= n\lambda e^{n\lambda y_1}, \quad y_1 > 0, \lambda > 0.$$

So $X_{(1)} \sim \text{Exp}(n\lambda)$.

If X_i's denote the lifetime of each observation then average life is $\frac{1}{\lambda}$.

But the average life of a series system is $\frac{1}{n\lambda}$ which is very small.

Def^n: If $X_1, X_2, \ldots X_n$ are independently identically distributed r.v.'s then we call $X_1, \ldots, X_n$ a random sample (r.s.) of size n from a dist^n having cdf $F(\cdot)$ (pdf/pmf $f(\cdot)$). In other words a random sample is a collection of i.i.d. r.v.'s.

Sample Median: If the sample size is odd i.e. $n = 2k+1$ then sample median is

$$M = X_{(k+1)}$$

if n is even then

$$M = \frac{X_{(k+1)} + X_{(k)}}{2}.$$

Additive Property

Ex: ① Let $X_1, X_2, \dots X_k$ be independent and let $X_i \sim Bin(n_i, p)$, $i = 1(1)k$. Let $S_k = \sum_{i=1}^{k} X_i$

$$M_{S_k}(t) = \prod_{i=1}^{k} M_{X_i}(t) = \prod_{i=1}^{k} (q+pe^t)^{n_i} = (q+pe^t)^{\sum n_i}$$

Which is mgf of $Bin(\sum n_i, p)$. So $S_k \sim Bin(\sum n_i, p)$

Ex ② Let $X_1, X_2, \dots, X_k$ be independent Poisson r.v.s with $X_i \sim P(\lambda_i)$, $i = 1(1)k$.

$$S_k = \sum_{i=1}^{k} X_i.$$

$$M_{S_k}(t) = \prod_{i=1}^{k} M_{X_i}(t) = \prod_{i=1}^{k} e^{\lambda_i(e^t-1)}$$

$$= e^{\sum \lambda_i (e^t - 1)}$$

So $S_k \sim P(\sum \lambda_i)$

Ex ③ Let $X_1, X_2, \dots, X_k \underset{\sim}{iid} Geo(p)$

$$S_k = \sum_{i=1}^{k} X_i.$$

$$M_{S_K}(t) = \prod_{i=1}^{K} M_{X_i}(t) = \left(\frac{pe^t}{1-qe^t}\right)^K, \quad qe^t < 1.$$

Which is mgf of $NB(K, p)$. So $S_K \sim NB(K, p)$

Ex (4) $X_1, X_2, \ldots X_K$ are independent and $X_i \sim NB(r_i, p)$. Then $\sum_{i=1}^{K} X_i \sim NB(\sum r_i, p)$

Ex (5) $X_1, \ldots X_K \overset{i.i.d}{\sim} Exp(\lambda)$

$S_K = \sum X_i \sim Gamma(K, \lambda)$

Ex (6) If $X_1, X_2, \ldots X_K$ are independent Gamma r.v's with $X_i \sim Gamma(r_i, \lambda)$

Then $\sum_{i=1}^{K} X_i \sim Gamma(\sum r_i, \lambda)$.

Ex (7) <u>Linearity property of Normal distn.</u>

Let $X_1, X_2, \ldots X_K$ be independent normal r.v's and $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1(1)K$.

Let $Y = \sum_{i=1}^{k} (a_i X_i + b_i)$

$$M_Y(t) = E(e^{tY}) = E\left(e^{t(\Sigma(a_i x_i + b_i))}\right)$$

$$= e^{t\Sigma b_i} E\left(e^{t\Sigma a_i x_i}\right)$$

$$= e^{t\Sigma b_i} E\left(\prod_{i=1}^{k} e^{a_i x_i t}\right) = e^{t\Sigma b_i} \prod_{i=1}^{k} E\left(e^{a_i t X_i}\right)$$

$$= e^{t\sum_{i=1}^{k} b_i} \prod_{i=1}^{k} M_{X_i}(a_i t)$$

$$= e^{t\sum_{i=1}^{k} b_i} e^{\sum_{i=1}^{k}(\mu_i a_i t + \frac{1}{2} a_i^2 \sigma_i^2)}$$

$$= e^{t\sum_{i=1}^{k}(\mu_i a_i + b_i) + \frac{1}{2}\sum_{i=1}^{k} a_i^2 \sigma_i^2}$$

Which is m.g.f of $N\left(\Sigma(a_i \mu_i + b_i),\ \Sigma a_i^2 \sigma_i^2\right)$

$$\Rightarrow Y = \sum_{i=1}^{k}(a_i X_i + b_i) \sim N\left(\sum_{i=1}^{k}(a_i \mu_i + b_i),\ \sum_{i=1}^{k} a_i^2 \sigma_i^2\right)$$