Tutorial 2: Calculus I (IC153)

Indian Institute of Technology Bhilai

---

1. If $x_1 = \sqrt{6}$ and $x_{n+1} = \sqrt{x_n + 6}$, $n = 1, 2, 3 \ldots$, show that $\{x_n\}$ is monotonically increasing.

2. Let $\{x_n\}$ be a sequence of real numbers be defined as $x_{n+1} = x_n(2 - x_n)$, $0 < x_1 < 1$. Show that the sequence is convergent. Find the limit of the sequence.

3. Show that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{7}$ and $x_{n+1} = \sqrt{7 + x_n}$ converges to the positive root of the equation $x^2 - x - 7 = 0$.

4. If the subsequences of a sequence x_{2n} and $\{x_{2n-1}\}$ of a sequence $\{x_n\}$ converges to the same limit l then prove that the sequence $\{x_n\}$ converges to the limit l.

5. Every subsequence of a monotone increasing (decreasing) sequence of real numbers is monotone increasing (decreasing).

6. A monotone sequence of real numbers having convergent subsequence with limit l, is convergent with limit l.

7. Examine whether the sequence $\{x_n\}$ where $x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ is a Cauchy sequence.

8. Using Cauchy's general principle of convergence show that the sequence $\{x_n\}$ where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ not convergent.

9. Prove tht $\{x_n\}$ is a Cauchy sequence if $\{x_n\}$ satisfies either of the following conditions.

   (a) $|x_{n+1} - x_n| \leq \alpha^n$, (b) $|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|$, (In this case $\{x_n\}$ is called Contractive sequence), where $0 < \alpha < 1$.

10. Let the sequence $\{x_n\}$ is defined as $x_1 = 1$ and $x_{n+1} = \frac{1}{x_n + 2}$ for all $n \in \mathbb{N}$. Then show that the $\{x_n\}$ sequence is convergent and find its limit.

11. Show that the sequence $\{1, \frac{1}{2}, 1, \frac{2}{3}, \frac{3}{4}, \ldots\}$ converges to 1.

12. Discuss the convergence of the following sequences
    (i) $\left\{\frac{2n^2-3n}{3n^2+5n+3}\right\}$ (ii) $\left\{\sqrt{n+1} - \sqrt{n}\right\}$ (iii) $\{(n^3+1)\}$ (iv) $\left\{(2^n + 3^n)^{1/n}\right\}$ (v) $\{a^n + b^n\}^{1/n}$, where $0 < a < b$ (vi) $\left\{\frac{1}{(1+n)^2} + \frac{1}{(2+n)^2} + \cdots + \frac{1}{(n+n)^2}\right\}$ (vii) $\left\{\frac{3n^2 + \sin n - 4}{2n^2+3}\right\}$

13. Let $l \in \mathbb{R}$. Prove that there exists a sequence $\{x_n\}$ of rational numbers converges to l. Also prove that that there exists a sequence $\{y_n\}$ of irrational numbers converges to l.

14. Determine whether the following sequences are is increasing, decreasing, or neither.
    (i) $\left\{\frac{3^n}{3^n+1}\right\}$ (ii) $\left\{\frac{5n-2}{4n+1}\right\}$ (iii) $\{(n^3+1)\}$ (iv) $\left\{\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right\}$ (v) $\{\cos\frac{n\pi}{3}\}$

15. Find the limsup and liminf of the following sequences: (i) $\left\{(-1)^n \frac{n+1}{n}\right\}$ (ii) $\{0, 1, 0, 1, 0, 1 \cdots\}$ (iii) $\left\{\sin\frac{n\pi}{3}\right\}$ (iv) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}, \cdots\right\}$ (v) $\{x_n\}$, where $x_n = \begin{cases} (-1)^{n/2}\frac{n}{n+1} & \text{if } n \text{ is even} \\ \frac{n^2-1}{2n^2+1} & \text{if } n \text{ is odd} \end{cases}$