Random vectors

Random vectors and their distribution

Let $(\Omega, \mathcal{F}, P)$ be a given probability space. In many situations we may be interested in simultaneous studying two or more numerical characteristics of outcomes of a random experiment. Define a fu^n

$$\underline{X} : (X_1, \ldots, X_p) : \Omega \longrightarrow \mathbb{R}^p$$

Ex: A fair coin tossed three times independently. The

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

and $P(\{\omega\}) = \frac{1}{8}, \ \forall \ \omega \in \Omega.$

Suppose that we are simultaneously interested in

- number of heads in three tosses.
- number of heads in first two tosses.

Here we are interested in the fu^n $(X, Y) : \Omega \to \mathbb{R}^2$ defined by

$$(X(\omega), Y(\omega)) = \begin{cases} (0,0) & \text{if } \omega = TTT \\ (1,0) & \text{if } \omega = TTH \\ (1,1) & \text{if } \omega = HTT, THT \\ (2,1) & \text{if } \omega = HTH, THH \\ (2,2) & \text{if } \omega = HHT \\ (3,2) & \text{if } \omega = HHH. \end{cases}$$

The values assumed by (X, Y) are random with

$$P_r\big((X,Y)=(x,y)\big) = \begin{cases} \frac{1}{8}, & \text{if } (x,y) \in \{(0,0),(1,0),(2,2),(3,2)\} \\ \frac{1}{4}, & \text{if } (x,y) \in \{(1,1),(2,1)\} \\ 0, & \text{o/w.} \end{cases}$$

Def^n: Let $(\Omega, \mathcal{S}, P)$ be a given prob. space. A fu^n $\underline{X} = (X_1, \ldots, X_p) : \Omega \to \mathbb{R}^p$ defined on the sample space is called a random vector (p-dimensional random vector).

A one dimensional random vector (r.v.) is simply called a random variable.

Def^n: (a) The joint dist^n fu^n of a p-dimensional random vector $\underline{X} = (X_1, X_2, \ldots X_p)$ is defined as

$$F_{\underline{X}}(x_1, x_2, \ldots, x_p) = P_r(X_1 \le x_1,\ X_2 \le x_2, \ldots,\ X_p \le x_p),$$

$$\underline{x} = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p.$$

(b) The joint d.f. of any subset of r.v. $X_1 \ldots X_p$ is called marginal d.f. of $F_{\underline{X}}(\cdot)$ (or $\underline{X} = (X_1, \ldots X_p)$).

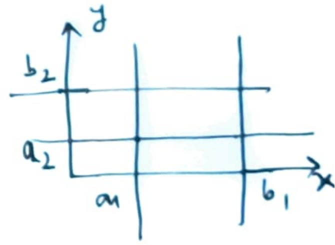
Ex: $F_{X_1,X_2}(x_1,x_2)$, $F_{X_2}(x_2)$, $F_{X_1,X_2,X_3}(x,y,z)$ are marginal d.f. of $F_{\underline{X}}(x_1,x_2,x_3,x_4)$, $\underline{X} = (X_1, \ldots X_4)$

Now we will describe a notation for writing down all the vertices of a p-dimensional rectangle in a compact form

For $-\infty < a_i < b_i < \infty$, $i = 1, 2$, $\underline{a} = (a_1, a_2)$ & $\underline{b} = (b_1, b_2)$

the vertices of two dimensional rectangle

$$[\underline{a}, \underline{b}] = (a_1, b_1] \times (a_2, b_2] = \{(x, y) \in \mathbb{R}^2 : a_1 < x \leq b_1,\ a_2 < y \leq b_2\}$$



are

$$\{(b_1, b_2), (a_1, b_2), (b_1, a_2), (a_1, a_2)\}$$

$$= \{(b_1, b_2)\} \cup \{(a_1, b_2), (b_1, a_2)\} \cup \{(a_1, a_2)\}$$

$$= \Delta_{0,2} \cup \Delta_{1,2} \cup \Delta_{2,2}$$

In general for $-\infty < a_i < b_i < \infty$, $i = 1, 2, \ldots p$, $\underline{a} = (a_1, \ldots, a_p)$ and $\underline{b} = (b_1, \ldots, b_p)$ define

$$\Delta_{k,p} \equiv \Delta_{k,p}\big((\underline{a}, \underline{b}]\big) = \{\underline{z} \in \mathbb{R}^p : z_i \in \{a_i, b_i\},\ i = 1, 2, \ldots, p, \text{ and exactly } k \text{ of } z_j\text{'s are } a_j\text{'s}\}.$$

where

$$(\underline{a}, \underline{b}] = (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_p, b_p].$$

Let $(X_1, X_2, \ldots X_p) = \underline{X}$ be a p-dimensional r.v. with d.f. $F_{\underline{X}}(\cdot)$. Then

(a) $\lim\limits_{\substack{x_i \to \infty \\ i = 1, 2, \ldots p}} F_{\underline{X}}(x_1, x_2, \ldots, x_p) = 1$

(b) for each $i = 1, 2, \ldots p$, $\lim\limits_{x_i \to -\infty} F_{\underline{X}}(x_1, x_2, \ldots, x_p) = 0$

(c) $F_{\underline{x}}(\underline{x})$ is right continuous in each argument (keeping other arguments fixed)

(d) For each rectangle $(\underline{a}, \underline{b}] \subseteq \mathbb{R}^p$

$$\sum_{k=0}^{p} (-1)^k \sum_{\underline{z} \in \Delta_{k,p}} F(\underline{z}) \geq 0.$$

[$p=2$, $\sum_{k=0}^{2} (-1)^2 \sum_{\underline{z} \in \Delta_{k,p}} F(\underline{z}) = F(b_1, b_2) - F(b_1, a_2) - F(a_1, b_2) + F(a_1, a_2)$

$= P(a_1 < X_1 \leq b_1, a_2 < X_2 \leq b_2) \geq 0.$

For $p=1$, (d) reduces to $F(b) - F(a) \geq 0$, i.e. $F(\cdot)$ is non decreasing

Result: Let $F_{\underline{X}}(\underline{x})$, $\underline{x} \in \mathbb{R}^p$ be a distn fun of $\underline{X} = (X_1, X_2, \ldots, X_p)$. Then the marginal d.f. of $(X_1, \ldots, X_{p-1})$ is

$$G(x_1, x_2, \ldots, x_{p-1}) = \lim_{t \to \infty} F_{\underline{X}}(x_1, x_2, \ldots, x_{p-1}, t)$$

Independent of Random Variables:

For an arbitrary (countable or uncountable) set Λ. let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of random variables.

Defn: The random variables X_λ, $\lambda \in \Lambda$ are said to be mutually independent if for any finite sub collection $\{X_{\lambda_1}, \ldots, X_{\lambda_p}\}$ in $\{X_\lambda : \lambda \in \Lambda\}$

$$F_{X_{\lambda_1}, \ldots X_{\lambda_p}}(x_1, \ldots, x_p) = \prod_{i=1}^{p} F_{X_{\lambda_i}}(x_i) \quad \forall \; \underline{x} = (x_1, \ldots x_p) \in \mathbb{R}^p$$

where $F_{X_{\lambda_1},\dots,X_{\lambda_p}}(\cdot)$ denoteds the c.d.f of $(X_{\lambda_1},\dots,X_{\lambda_p})$ and $F_{X_{\lambda_i}}(\cdot)$, $i=1,\dots p$ denotes the marginal d.f. of X_{λ_i}.

Result: For any positive integer $p(\geq 2)$ the r.v's $X_1, X_2, \dots, X_p$ are independent iff

$$F_{\underline{X}}(x_1,\dots,x_p) = \prod_{i=1}^{p} F_{X_i}(x_i)$$

Discrete random vectors: Let $\underline{X} = (X_1,\dots,X_p)$ be p-dim r.v. with d.f. $F(\cdot)$.

Def^n (a) The random vector $\underline{X}$ is said to be a discrete random vector if there exists a countable set S such that

$$P_r(\underline{X} = \underline{x}) > 0 \quad \forall\ \underline{x} \in S$$

$$P_r(\underline{X} \in S) = 1$$

The set S is called support of r.v. $\underline{X}$

(b) The joint p.m.f. of $\underline{X}$ is defined as

$$f_{\underline{X}}(\underline{x}) = \begin{cases} P_r(\underline{X} = \underline{x}) & \text{if } \underline{x} \in S \\ 0 & \text{o/w} \end{cases}$$

Let $\underline{X} = (X_1, \ldots X_p)$ be a p-dimensional discrete r.v with p.m.f $f_{\underline{X}}(\cdot)$ and d.f $F(\cdot)$ and support S. Then for any $A \subseteq \mathbb{R}$

ⓐ $P_r(\underline{X} \in A) = P_r(\underline{X} \in A \cap S)$

$$= \sum_{\underline{x} \in A \cap S} f_X(\underline{x})$$

ⓑ $F_X(\underline{x}) = \sum_{\underline{y} \in S \cap (-\infty, \underline{x}]} f_{\underline{X}}(\underline{y})$

ⓒ The p.m.f $f_{\underline{X}}(\underline{x})$ satisfies (i) $f_{\underline{X}}(\underline{x}) > 0$ $\forall x \in S$ and $f_{\underline{X}}(\underline{x}) = 0$, $x \in S^c$. (ii) $\sum_{\underline{x} \in S} f_{\underline{X}}(\underline{x}) = 1$

ⓓ Marginal $dist^n$ of discrete r.v. are discrete.

(e) The marginal distribution of any subset of $\{X_1, \ldots X_p\}$ say $\underline{Y} = (X_1, \ldots X_q)$ $1 \le q \le p$ is again discrete with p.m.f.

$$g(x_1, \ldots x_q) = \sum_{x_{q+1}} \sum_{x_{q+2}} \cdots \sum_{x_q} f_{\underline{X}}(\underline{x})$$

Conditional distⁿ of discrete r.v.

Let $\underline{Y} = (Y_1, \ldots, Y_p)$, $\underline{Z} = (Z_1, \ldots, Z_q)$ and $\underline{X} = (\underline{Y}, \underline{Z}) = (Y_1, Y_2, \ldots, Y_p, Z_1, \ldots, Z_q)$ are r.v. with p.m.f f_1, f_2 and f respectively. Suppose $\underline{X}$, $\underline{Y}$, & $\underline{Z}$ have support S, S_1, S_2 respectively. For fixed $\underline{z} \in S_2$ define

$$T_{\underline{z}} = \{ \underline{y} = (y_1, \ldots, y_p) \in \mathbb{R}^p : (\underline{y}, \underline{z}) \in S \}$$

For fixed $z \in S_2$ the conditional p.m.f. of $\underline{Y}$ given $\underline{Z} = \underline{z}$ is defined by

$$f(\underline{y} \mid \underline{z}) = P_r(\underline{Y} = \underline{y} \mid \underline{Z} = \underline{z}) = \frac{P_r(\underline{X} = (\underline{y}, \underline{z}))}{P_r(\underline{Z} = \underline{z})}$$

$$= \begin{cases} \dfrac{f(\underline{y}, \underline{z})}{f_2(\underline{z})}, & \underline{y} \in T_{\underline{z}} \\ 0 & o/w. \end{cases}$$

<u>Theorem</u>: Let $\underline{x} = (x_1, \ldots, x_p)$ be a p-dimensional r.v. with support S and p.m.f $f(\cdot)$. Let $f_i(\cdot)$ denote the marginal p.m.f. of X_i, $i = 1, 2, \ldots, p$. Then $X_1, \ldots, X_p$ are independent iff

$$f(x_1, \ldots, x_p) = \prod_{i=1}^{p} f_i(x_i) \quad \forall\, \underline{x} \in S.$$

Example Let $\underline{X} = (X_1, X_2, X_3)$ have the joint p.m.f.

$$f(x_1, x_2, x_3) = \begin{cases} c\, x_1 x_2 x_3, & x_1 = 1,2,\ x_2 = 1,2,3,\ x_3 = 1,3 \\ 0, & o/w \end{cases}$$

where c is a real constant

(a) Find the value of c.

(b) Find marginal p.m.f of $X_1, X_2, \& X_3$

(c) Are $X_1, X_2 \& X_3$ independent.

(d) Find marginal pm.f. of (X_1, X_3)

(e) Find conditional p.m.f. of X_1 given $(X_2, X_3) = (2,1)$

(f) Are $X_1 \& X_3$ independent

(g) compute $P_r(X_1 = X_2 = X_3)$.

solⁿ. Here support of r.v. $\underline{X}$ is

$$S_{\underline{x}} = \{1,2\} \times \{1,2,3\} \times \{1,3\}.$$

$$\sum_{\underline{x} \in S_{\underline{x}}} f_{\underline{x}}(\underline{x}) = 1 \Rightarrow c\left[1+3+2+6+3+9+2+6+4+12+6+18\right] = 1$$

$$\Rightarrow c = 1/72$$

(b) $$f_{X_1}(x_1) = \sum_{(x_2, x_3) \in \{1,2,3\} \times \{1,3\}} \frac{x_1 x_2 x_3}{72} = \frac{x_1}{72}\left(\sum_{x_2=1}^{3} x_2\right)\left(\sum_{x_3 = 1,3} x_3\right)$$

$$= \frac{x_1}{3}.$$

$$f_{X_1}(x_1) = \begin{cases} \frac{x_1}{3}, & x_1 \in \{1,2\} \\ 0 & o/w \end{cases}$$

$||^y$

$$f_{X_2}(x_2) = \begin{cases} \frac{x_2}{6}, & x_2 = 1,2,3 \\ 0, & o/w \end{cases}, \quad f_{X_3}(x_3) = \begin{cases} \frac{x_3}{4}, & x_3 = 1,3 \\ 0, & o/w \end{cases}$$

(c) We have

$$f_{\underline{X}}(x_1, x_2, x_3) = f_{X_1}(x_1)\, f_{X_2}(x_2)\, f_{X_3}(x_3) \;\; \forall (x_1, x_2, x_3) \in \mathbb{R}^3$$

$\Rightarrow$ X_1, X_2, X_3 are independent.

(d) Marginal of (X_1, X_3) is

$$f_{X_1, X_3}(x_1, x_3) = \sum_{x_2} f_{\underline{X}}(x_1, x_2, x_3) = \frac{x_1 x_3}{72} \times 6 = \frac{x_1 x_3}{12}$$

$$f_{X_1, X_3}(x_1, x_3) = \begin{cases} \frac{x_1 x_3}{12}, & (x_1, x_3) \in \{1,2\} \times \{1,3\} \\ 0, & o/w. \end{cases}$$

(e) For $x_1 \in \{1,2\}$

$$\Pr(X_1 = x_1 \mid X_2 = 2, X_3 = 1) = \frac{\Pr(X_1 = x_1, X_2 = 2, X_3 = 1)}{\Pr(X_2 = 2, X_3 = 1)}$$

$$= \frac{x_1 \cdot 2 \cdot 1}{72} \Big/ \frac{1}{12} = \frac{x_1}{3}$$

Thus

$$f_{X_1|(X_2,X_3)}\left(x_1 \mid (2,1)\right) = \begin{cases} \frac{x_1}{3} & , x_1 \in \{1,2\} \\ 0 & o/w. \end{cases}$$

(g)

$$P_r\left(X_1 = X_2 = X_3\right) = \sum_{\substack{\underline{x} \in S_X \\ x_1 = x_2 = x_3}} \frac{x_1 x_2 x_3}{72} = P\left(X_1 = X_2 = X_3 = 1\right)$$

$$= \frac{1}{72}.$$

Example. Suppose a car showroom has 10 cars of a brand out of which 5 are good, 2 have defective transmission (dt) and 3 have defective steerings (ds). If two cars are selected at random

Let $X \rightarrow$ no of cars with dt

$Y \rightarrow$ no of cars with ds

(i) Find the p.m.f of X, Y

(ii) Find the marginals of X, Y.

solun: H.W.