Department of Mathematics
Indian Institute of Technology Bhilai
IC104: Linear Algebra-I
Hints of Tutorial Sheet 2: Vector Space

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1. Let (x, y, z) be a linear combination of the vectors $(1, 0, -1), (0, 1, 1)$ and $(1, 1, 1)$ then there exist $a, b, c \in \mathbb{C}$ such that

$$(x, y, z) = a(1, 0, -1) + b(0, 1, 1) + c(1, 1, 1) = (a + c, b + c, -a + c).$$

Hence any vector which is a linear combination of $(1, 0, -1), (0, 1, 1)$ and $(1, 1, 1)$ must be of the type $(a + c, b + c, -a + c)$, where $a, b, c \in \mathbb{C}$ are arbitrary.

2. No, V with these operations is not a vector space because the vector addition does not satisfy the property $3c)$ as for any $(a, b) \in V$ and zero vector $(0, 0) \in V$, $(a, b)+(0, 0) = (a, 0) \neq (a, b)$ and the scalar multiplication fails to satisfy $4a)$ as for any $(a, b) \in V$ and $1 \in \mathbb{F}$, $1.(a, b) = (1a, 0) = (a, 0) \neq (a, b)$.

3. First, we will check that whether the addition and scalar multiplication is closed i.e, if $f \in V$ and $g \in V$ then $f + g \in V$. Also, if $a \in \mathbb{R}$ and $f \in V$ then $af \in V$. This is obvious because $(f + g)(-t) = f(-t) + g(-t) = \overline{f(t)} + \overline{g(t)} = \overline{f(t) + g(t)} = \overline{(f + g)(t)}$ and $cf(-t) = c(f(-t)) = c\overline{f(t)} = \overline{cf(t)}$ if field is real. **(What if we have complex field!!)**

   (i) Commutativity is obvious, since $\mathbb{C}$ is commutative.

   (ii) Associativity is obvious, since $\mathbb{C}$ is associative.

   (iii) Additive identity $g(t) = 0 \in V$, since $-0 = \overline{0}$.

   (iv) $g(t) = -f(t)$ is the additive inverse of $f(t)$, since $g(-t) = -f(-t) = -\overline{f(t)} = \overline{-f(t)} = \overline{g(t)}$.

   (v) Similarly, scalar multiplication properties are easy to prove.

   Example: $f(t) = \cos(t) + i \sin(t)$ is not a real valued function and $f(t) \in V$.

4. Let $< S >$ denotes the subspace of V which is spanned by a set S. Assume, if possible, that S' is a subspace of V which which contains S and is a proper subset of $< S >$. Since S is contained in S' and S' is a vector space in itself, all the linear combinations of elements of S must belong to S'. It means $< S > \subset S'$, which is a contradiction. Hence $< S > \subseteq S'$, that is any subspace which contains S must be bigger or equal to $< S >$.

5. (a) $W_1 = \{f \in V : f(t^2) = f(t)^2\}$. Consider $f(x) = 1$ and $g(x) = x$, for all $x \in \mathbb{R}$. It is clear that $f, g \in W_1$. But one can check that $f + g \notin W_1$. Thus W_1 is not a subspace of V.

(b) $W_2 = \{f \in V : f(0) = f(1)\}$. It is easy to check that W_2 is non-empty as zero function is in W_2. Consider $f, g \in W_2$ and $c \in \mathbb{R}$. Now $(cf+g)(0) = (cf)(0) + g(0) = c(f(0)) + g(0) = c(f(1)) + g(1) = (cf)(1) + g(1) = (cf+g)(1)$. Thus W_2 is a subspace of V.

(c) $W_3 = \{f \in V : f(3) = 1 + f(-5)\}$. Is zero function belongs to W_3? No as $0(3) = 0$ and $1 + 0(-5) = 1$.

(d) $W_4 = \{f \in V : f(-1) = 0\}$. It is non-empty. Let $f, g \in W_4$ and $\alpha \in \mathbb{R}$. Now $(\alpha f + g)(-1) = (\alpha f)(-1) + g(-1) = \alpha(f(-1)) + g(-1) = \alpha.0 + 0 = 0$. That is, $\alpha f + g \in W_4$. Hence W_4 is a subspace of V.

6. (a) Here $W_1 = \{(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^n : \alpha_1 \geq 0\}$. Let $\alpha = \{(1, 1, \cdots, 1)\} \in \mathbb{R}^n$ and $c = -1 \in \mathbb{R}$. Now $c\alpha = \{(-1, -1, \cdots, -1)\} \notin W_1$. Therefore W_1 is not a vector subspace of $\mathbb{R}^n$.

(b) $W_2 = \{(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^n : \alpha_1 + 3\alpha_2 = \alpha_3\}$. It is clear that W_2 is non-empty. Let $a = (a_1, a_2, \cdots, a_n)$, $b = (b_1, b_2, \cdots, b_n) \in W_2$. Then $a_1 + 3a_2 = a_3$ and $b_1 + 3b_2 = b_3$. Now consider $c \in \mathbb{R}$. Then one can observe that $ca + b \in W_2$. Therefore W_2 is a subspace of $\mathbb{R}^n$.

(c) $W_3 = \{(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^n : \alpha_1^2 = \alpha_2\}$. Consider $a = (1, 1, 0, \cdots, 0) \in W_3$ and $b = (2, 4, 0, \cdots, 0) \in W_3$. Now $a + b \notin W_3$. Hence W_3 is not a subspace of $\mathbb{R}^3$.

(d) $W_4 = \{(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^n : \alpha_1.\alpha_2 = 0\}$. Consider $a = (1, 0, 0, \cdots, 0) \in W_4$ and $b = (0, 1, 0, \cdots, 0) \in W_4$. But $a + b \notin W_4$. Therefore W_4 is not a subspace of $\mathbb{R}^n$.

7. $W = \{x = (x_1, x_2, x_3, x_4, x_5) : Ax = 0\}$, where $A = \begin{bmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{bmatrix}$. The RRE of $A = \begin{bmatrix} 1 & 0 & \frac{2}{3} & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore $Ax = 0$ can be written as

$$\begin{bmatrix} 1 & 0 & \frac{2}{3} & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$\begin{aligned} x_1 + \frac{2}{3}x_3 - x_5 &= 0 \\ x_2 + x_4 - 2x_5 &= 0. \end{aligned}$$

That is,

$$\begin{aligned}(x_1,x_2,x_3,x_4,x_5) &= (-\frac{2}{3}x_3+x_5,-x_4+2x_5,x_3,x_4,x_5)\\ &= x_3(-\frac{2}{3},0,1,0,0)+x_4(0,-1,0,1,0)+x_5(1,2,0,0,1).\end{aligned}$$

Therefore the set $S=\{(-\frac{2}{3},0,1,0,0),(0,-1,0,1,0),(1,2,0,0,1)\}$ spans W.

8. (a) In this problem, $V_e=\{f\in V: f(-x)=f(x)\}$ and $V_o=\{f\in V: f(-x)=-f(x)\}$. It is easy to check that V_e is non-empty. Now consider $f,g\in V_e$ and $c\in\mathbb{R}$. Then we have $f(-x)=f(x)$ as well as $g(-x)=g(x)$ and

$$(cf+g)(-x)=(cf)(-x)+g(-x)=c(f(-x))+g(-x)=c(f(x))+g(x)=(cf+g)(x),\quad \forall x\in\mathbb{R}.$$

Therefore, $cf+g\in V_e$. Thus V_e is a subspace of V. On the other hand, by the same process one can easily prove that V_o is also a subspace of V.

(b) Again $V_o,V_e\in V$, then $V_e+V_o\subseteq V$. On the other hand, we have to show that $V\subseteq V_e+V_o$. Let $g\in V$, then

$$g(x)=\frac{1}{2}\underbrace{\{g(x)+g(-x)\}}_{\text{even function}}+\frac{1}{2}\underbrace{\{g(x)-g(-x)\}}_{\text{odd function}}.$$

$\therefore g\in V_e+V_o$. Hence $V\subseteq V_e+V_o$. Thus $V=V_e+V_o$.

(c) On the contrary, $g\in V_e\cap V_o$, where $g\neq 0$ (zero map). Then $g\in V_e$ as well as $g\in V_o$. Therefore

$$\begin{aligned}g(x) &= g(-x) && [\because g\in V_e]\\ &= -g(x) && [\because g\in V_0].\end{aligned}$$

Therefore $2g(x)=0$, which implies $g(x)=0$ for all $x\in\mathbb{R}$.

9. Consider the relation

$$\begin{aligned}c_1(1,0,-1)+c_2(1,2,1)+c_3(0,-3,2) &= (0,0,0)\\ \text{or, } (c_1+c_2,2c_2-3c_3,-c_1+c_2+2c_3) &= (0,0,0)\\ \text{This is equivalent to, } \begin{bmatrix}1&1&0\\0&2&-3\\-1&1&2\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} &= \begin{bmatrix}0\\0\\0\end{bmatrix}.\end{aligned}$$

As, $\det\begin{bmatrix}1&1&0\\0&2&-3\\-1&1&2\end{bmatrix}\neq 0$. Then the system has unique solution and the solution is $c_1=c_2=c_3=0$. Therefore the set of vectors $\{(1,0,-1),(1,2,1),(0,-3,2)\}$ are linearly independent in $\mathbb{R}^3$. As 3 vectors are linearly independent in $\mathbb{R}^3$ ($\dim \mathbb{R}^3 = 3$), they form a basis in $\mathbb{R}^3$.

Again,

$$\begin{aligned}
(1,0,0) &= \frac{7}{10}(1,0,-1) + \frac{3}{10}(1,2,1) + \frac{1}{5}(0,-3,2) \\
(0,1,0) &= \frac{-1}{5}(1,0,-1) + \frac{1}{5}(1,2,1) + \frac{-1}{5}(0,-3,2) \\
(0,0,1) &= \frac{-3}{10}(1,0,-1) + \frac{3}{10}(1,2,1) + \frac{1}{5}(0,-3,2)
\end{aligned}$$

10. Consider three vectors $\{(1,0,1),(0,1,0),(1,1,1)\}$. Here any two vectors are linearly independent but these three vectors are linearly dependent.

11. Let $A, B \in W_1$, then $A = \begin{bmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{bmatrix}$ and $B = \begin{bmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{bmatrix}$, where $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}$. Suppose $\alpha \in \mathbb{R}$. Then $\alpha A + B = \begin{bmatrix} \alpha x_1 + x_2 & -(\alpha x_1 + x_2) \\ \alpha y_1 + y_2 & \alpha z_1 + z_2 \end{bmatrix}$. Therefore $\alpha A + B \in W_1$.

    Now
$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix} = x \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

    It is easy to see that $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is linearly independent and spans the space W_1. Thus it form a basis for W_1. Hence $\dim W_1 = 3$.

    By similar process $\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for W_2. Hence $\dim W_2 = 3$.

    Now, $W_1 \cap W_2 = \left\{ A \in V : A = \begin{bmatrix} x & -x \\ -x & y \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $W_1 \cap W_2$. Hence $\dim (W_1 \cap W_2) = 2$.

    As we know that $\dim (W_1 + W_2) = \dim W_1 + \dim W_2 - \dim (W_1 \cap W_2)$. Therefore $\dim (W_1 + W_2) = 3 + 3 - 2 = 4$

12. Let the set $S = \{a_1, a_2, \cdots, a_n\}$ spans V. Now if this set S is linearly independent, then it forms a basis for V. Therefore V is finite dimensional.

    On the other hand, if S is linearly dependent, then there is $a_i \in S$ such that
$$a_i = c_1 a_1 + c_2 a_2 + \cdots + c_{i-1} a_{i-1} + c_{i+1} a_{i+1} + \cdots c_n a_n.$$

    From here one can observe that the set $W_1 = \{a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n\}$ also spans the set V. If this set is linearly independent then this forms a basis for V. Thus V is finite dimensional. If W_1 is linearly dependent, then proceed the same technique.

Hence moving inductively we get some $B \subseteq S$ such that, elements of B are linearly independent and spans V. Also no of elements in B is less than no of elements of S, which is n. Thus V is finite dimensional.

13. Let $(x, y, z)^T$ be the co-ordinate matrix of $(1, 0, 1)$ corresponding to the ordered basis $\{(2i, 1, 0), (2, -1, 1), (0, 1 + i, 1 - i)\}$. Then we have,

$$\begin{aligned} x(2i, 1, 0) + y(2, -1, 1) + z(0, 1 + i, 1 - i) &= (1, 0, 1) \\ \text{or, } (2ix + 2y, x - y + (1 + i)z, y + (1 - i)z) &= (1, 0, 1). \end{aligned}$$

Then solving we have $x = \frac{1-i}{2i}$; $y = \frac{i}{2}$; $z = \frac{3+i}{4}$. Therefore $(\frac{1-i}{2i}, \frac{i}{2}, \frac{3+i}{4})^T$ is the co-ordinate matrix.

14. Let x, y, z are the co-ordinates of the vector (a, b, c) corresponding to the ordered basis $\{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$. Then we have,

$$\begin{aligned} x(1, 0, -1) + y(1, 1, 1) + z(1, 0, 0) &= (a, b, c) \\ \text{or, } (x + y + z, y, -x) &= (a, b, c). \end{aligned}$$

Therefore $x = -c$; $y = b$; $z = a - b + c$.

15. The RRE of A is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$. Again the RRE of B is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$. Hence the row spaces of A and B is spanned by $\{(1, 0, 2), (0, 1, 5)\}$. Since both A and B span a same two dimensional subspaces of $\mathbb{R}^3$, hence both are row equivalent.