Tutorial 1: Calculus I (IC153)

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1. Prove that $(n+1)! > 2^n$ for each $n \geq 2$
2. Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$
3. Prove that $3 + 11 + \cdots + (8n-5) = (4n^2 - n)$ for all positive integer.
4. Let $a, b \in \mathbb{R}$. Then $a < b \implies a < \frac{a+b}{2} < b$. Hence prove that there is no least positive real.
5. Prove that the sum of a rational and an irrational is always irrational. What can you say about the product of a rational and an irrational?
6. Prove that if $a < b$ are real numbers, then there is an irrational $\xi \in \mathbb{R}$ such that $a < \xi < b$.
7. If $S = \left\{\frac{5}{n} : n \in \mathbb{N}\right\}$. Show that $\inf S = 0$.
8. Let S be a non-empty bounded subset of $\mathbb{R}$. Prove that $\inf S \leq \sup S$. What can you say about S if $\inf S = \sup S$.
9. Let S and T be non-empty subsets of $\mathbb{R}$ with the following property: $s \leq t$ for all $s \in S$ and $t \in T$.
   (a) Prove that S is bounded above and T is bounded below.
   (b) Prove that $\sup S \leq \inf T$.
   (c) Give an example of such sets S and T where $S \cap T$ is nonempty.
   (d) Give an example of such sets S and T where $\sup S = \inf T$ and $S \cap T$ is empty.
10. Show $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$
11. Prove the following using definition
    (a) $\lim \frac{(-1)^n}{n} = 0$ (b) $\lim \frac{1}{n^{1/3}} = 0$ (c) $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$ (d) $\lim \frac{n+6}{n^2-6} = 0$
12. Let $\{x_n\}$ be a bounded sequence, i.e., there exists M such that $|x_n| \leq M$ for all n, and let $\{y_n\}$ be a sequence such that $\lim y_n = 0$. Prove $\lim(x_n y_n) = 0$.
13. If $\lim a_n = a$ then prove that $\lim |a_n| = |a|$. Show by an example that the converse may not be true. When will be the converse is true. (A sequence $\{a_n\}$ is said to be null sequence if $\lim a_n = 0$)
14. Every bounded sequence is not convergent. Justify your answer.
15. Using the limit Theorems, prove the following. Justify all steps.
    (a) $\lim \frac{n+1}{n} = 1$ (b) $\lim \frac{3n+7}{6n-5} = \frac{1}{2}$ (c) $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \frac{17}{23}$.
16. Suppose $\lim x_n = 3$, $\lim y_n = 7$ and all y_n are nonzero. Determine the following limits
    (a) $\lim(x_n + y_n)$ (b) $\lim \frac{3y_n - x_n}{y_n^2}$