Continuous Random Vectors

Let $\underline{X} = (X_1, X_2, \cdots, X_p)$ be a p-dimensional random vector with d.f. F.

Def^n: The r.v. $\underline{X}$ is called a continuous r.v. if there exists a no-negative fu^n $f : \mathbb{R}^p \to \mathbb{R}$ s.t. for any rectengle set A in $\mathbb{R}^p$

$$P_r(\underline{X} \in A) = \iiint_A \cdots \int f(\underline{t})\, dt_1 dt_2 \cdots dt_p.$$

Where the fu^n $f(\cdot)$ is called prob density fu^n.

In particulare if for fixed $\underline{x} = (x_1, \cdots, x_p) \in \mathbb{R}^p$ then

if $A = (-\infty, x_1] \times (-\infty, x_2] \times \cdots \times (-\infty, x_p]$

$$F(x_1, \cdots, x_p) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_p} f(t_1, t_2, \cdots, t_p)\, dt_1 dt_2 \cdots dt_p.$$

(1) If $\underline{X}$ is a continuous r.v. then its d.f. F is a continuous fu^n.

(2) For a continuous r.v. if its p.d.f $f(\underline{x})$ is a pice-wi continuous function then from the fundamental theorem of

multivariable calculus

$$f(\underline{x}) = \frac{\partial^p}{\partial x_1 \partial x_2 \cdots \partial x_p} F(\underline{x}), \quad \underline{x} \in \mathbb{R}^p.$$

Whenever the derivative is defined.

(3) If $\underline{X}$ is a continuous r.v. with p.d.f. $f(\cdot)$

then $P(\underline{X} = \underline{a}) = 0$.

(4) It can be shown that if $\underline{X}$ is a p-dimensional r.v. with continuous d.f. $F(\cdot)$ s.t.

$$\frac{\partial^p}{\partial x_1 \cdots \partial x_p} F(x_1, \ldots, x_p)$$

exists everywhere except (possibly) on a set c comprising of countable number of curves (having 0 volume in $\mathbb{R}^p$) and

$$\int_{\mathbb{R}^p - c} \frac{\partial^p}{\partial x_1 \cdots \partial x_p} F(x_1, \ldots, x_p)\, dx_1 \cdots dx_p = 1$$

Then $\underline{X}$ is a continuous r.v. with p.d.f

$$f_{\underline{X}}(\underline{x}) = \begin{cases} \dfrac{\partial^p}{\partial x_1 \cdots \partial x_p} F(x_1, \ldots, x_p) & \text{if } \underline{x} \in \mathbb{R}^p - c \\ 0 & \text{if } \underline{x} \in c. \end{cases}$$

(5) Let $\underline{X} = (X_1, \dots, X_p)$ be a continuous r.v. with joint p.d.f $f_{\underline{X}}(\underline{x})$. Then for $q \in \{1, \dots, p-1\}$ and $\underline{x} = (x_1, \dots, x_q) \in \mathbb{R}^q$, the marginal of $(X_1, \dots, X_q)$ is a continuous r.v. with pdf

$$f_{X_1, \dots, X_q}(x_1, \dots, x_q) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\underline{X}}(x_1, \dots, x_q, t_{q+1}, t_{q+2}, \dots, t_p)\, dt_p \dots dt_{q+1}$$

Thus marginal distⁿ of a continuous r.v. $\underline{X}$ are continuous with pdf of marginal distribution obtained by integrating out unwanted variables in the pdf of $\underline{X}$.

<u>Conditional Distribution</u>: Let $\underline{X} = (X_1, X_2, \dots, X_p)$ be a p-dimensional r.v. with joint pdf $f_{\underline{X}}(\cdot)$. Let $q \in \{1, 2, \dots, p-1\}$, $\underline{X}_1 = (X_1, \dots, X_q)$ & $\underline{X}_2 = (X_{q+1}, \dots, X_p)$. Then the conditional pdf of $\underline{X}_2$ given $\underline{X}_1 = \underline{x}_1$ is defined by

$$f_{\underline{X}_2 | \underline{X}_1}(\underline{x}_2 | \underline{x}_1) = \frac{f_{\underline{X}_1, \underline{X}_2}(\underline{x}_1, \underline{x}_2)}{f_{\underline{X}_1}(\underline{x}_1)} = \frac{f_{\underline{X}}(\underline{x}_1, \underline{x}_2)}{f_{\underline{X}_1}(\underline{x}_1)}$$

Result: Let $\underline{X} = (X_1, \dots, X_p)$ be a continuous r.v. with joint p.d.f $f_{\underline{X}}(\cdot)$ and marginal pdfs $f_{X_i}(\cdot)$ $i = 1, 2, \dots p$. Then $X_1, \dots, X_p$ are independent iff

$$f_{X_1, \dots, X_p}(x_1, \dots x_p) = \prod_{i=1}^{p} f_{X_i}(x_i)$$

Example: Let $\underline{X} = (X_1, X_2, X_3)$ have the joint p.d.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \begin{cases} \dfrac{1}{x_1 x_2} & , 0 < x_3 < x_2 < x_1 < 1 \\ 0 & , \text{o/w.} \end{cases}$$

(a) Show that $f_{\underline{X}}(\cdot)$ is a proper pdf.

(b) Find the marginal p.d.f. of (X_2, X_3)

(c) Find the marginal p.d.f of X_1

(d) Find the conditional pdf of X_1 given $(X_2, X_3) = (x_2, x_3)$ where $0 < x_3 < x_2 < 1$.

(e) Are X_1, X_2 & X_3 independent.

solu^n (a) clearly $f_{\underline{X}}(\underline{x}) \geq 0 \quad \forall\, \underline{x} \in \mathbb{R}^3$, Also

$$\int_{\mathbb{R}^3} f_{\underline{X}}(\underline{x})\, d\underline{x} = \int_0^1 \int_0^{x_1} \int_0^{x_2} \frac{1}{x_1 x_2}\, dx_3\, dx_2\, dx_1 = 1$$

So $f_{\underline{X}}(\underline{x})$ is a pdf.

(b) The marginal p.d.f of (X_2, X_3) is obtained as

$$f_{X_2,X_3}(x_2, x_3) = \int_{-\infty}^{\infty} f_{\underline{X}}(x)\, dx = \int_{x_2}^{1} \frac{1}{x_1 x_2}\, dx_1,$$

$0 < x_3 < x_2 < 1.$

$$= -\frac{\ln x_2}{x_2}, \quad 0 < x_3 < < x_2 < 1.$$

So
$$f_{X_2,X_3}(x_2,x_3) = \begin{cases} -\dfrac{\ln x_2}{x_2}, & 0 < x_3 < x_2 < 1 \\ 0, & o/\omega \end{cases}$$

(c) The marginal of X_1

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\underline{X}}(\underline{x})\, dx_2\, dx_3,$$

Now $0 < x_1 < 1,$

$$f_{X_1}(x_1) = \int_0^{x_1}\int_0^{x_2} \frac{1}{x_1 x_2}\, dx_3\, dx_2 = 1$$

Thus
$$f_{X_1}(x_1) = \begin{cases} 1, & 0 < x_1 < 1 \\ 0, & o/w. \end{cases}$$

(d) The condition of X_1 given $(X_2, X_3) = (x_2, x_3)$ is

$$f_{X_1|(X_2,X_3)}(x_1|x_2,x_3) = \frac{f_{\underline{X}}(x_1, x_2, x_3)}{f_{X_2,X_3}(x_2,x_3)}, \quad x_2 < x_1 < 1$$

$$= \frac{\frac{1}{x_1 x_2}}{-\frac{\ln x_2}{x_2}}, \quad x_2 < x_1 < 1.$$

$$= -\frac{1}{x_1 \ln x_2}, \quad x_2 < x_1 < 1.$$

So the conditional dist^n of X_1 given $(X_2, X_3) = (x_2, x_3)$ is

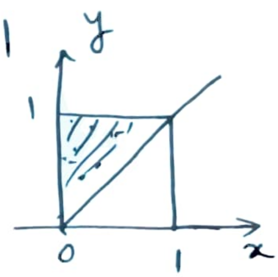
$$f_{X_1|(X_2,X_3)}(x_1|x_2,x_3) = \begin{cases} -\frac{1}{x_1 \ln x_2}, & x_2 < x_1 < 1 \\ 0 & o/w. \end{cases}$$

(e) Find the marginal of X_2, & X_3 and check

$$f_{\underline{X}}(\underline{x}) \neq f_{X_1}(x_1)\, f_{X_2}(x_2)\, f_{X_3}(x_3).$$

So X_1, X_2, X_3 are not independent

Example: $f_{X,Y}(x,y) = \begin{cases} 10xy^2, & 0<x<y<1 \\ 0, & o/w \end{cases}$



(a) check it is a pdf.

$$\int_0^1 \left[\int_0^y 10xy^2\, dx\right] dy = 1. \quad \text{also } f_{X,Y}(x,y) \geq 0.$$

So it is a pdf.

(b) Find marginal distⁿ of X and Y.

Marginal distⁿ of X is

$$f_X(x) = \int_x^1 10xy^2\, dy, \qquad 0<x<1$$

$$= \frac{10}{3}x(1-x^3), \quad 0<x<1$$

So $f_X(x) = \begin{cases} \frac{10}{3}x(1-x^3), & 0<x<1 \\ 0 & o/w. \end{cases}$

Marginal pdf of y is

$$f_y(y) = \int_0^y 10xy^2\,dx \quad , \quad 0 < y < 1$$

$$= 5y^4, \quad 0 < y < 1$$

So $f_y(y) = \begin{cases} 5y^4, & 0 < y < 1 \\ 0, & \text{o/w.} \end{cases}$

(c) Find the condition distribution X given $= y$ and Y given $X = x$.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad f_Y(y) \neq 0.$$

So for $0 < y < 1$

$$f_{X|Y}(x|y) = \frac{10xy^2}{5y^4}, \quad 0 < x < y.$$

$$= \begin{cases} \dfrac{2x}{y^2}, & 0 < x < y \\ 0, & \text{o/w.} \end{cases}$$

Similarly the marginal of Y given $X = x$.

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, \quad f_X(x) \neq 0.$$

For $0 < x < 1$.

$$f_{Y|X}(y|x) = \frac{10xy^2}{\frac{10}{3}x(1-x^3)}, \quad x < y < 1$$

So
$$f_{Y|X}(y|x) = \begin{cases} \dfrac{3y^2}{x(1-x^3)}, & x < y < 1 \\ 0, & \text{o/w} \end{cases},$$

for a given $0 < x < 1$.

Find the following probabilities.

(i) $P(X < 1/4)$ (ii) $P(Y > 3/4)$ (iii) $P(0 < X + Y < 1/2)$

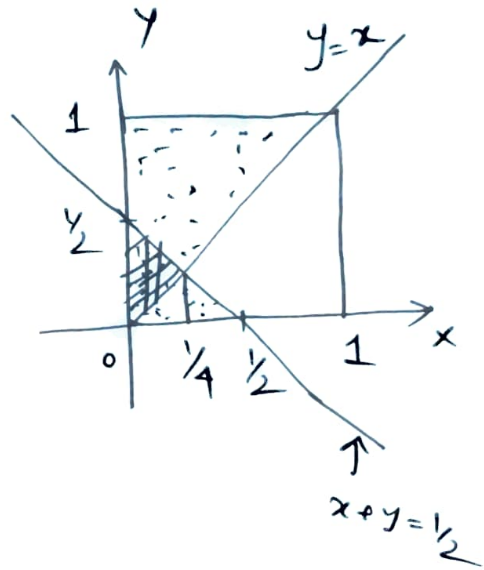
(iv) $P(X < 1/2 \mid Y = 3/4)$ (v) $P(Y < 1/2 \mid X = 1/4)$

(vi) $P(0 < X < 1/2, \frac{1}{4} < Y < 3/4)$

(i) $P(X < 1/4) = \int_0^{1/4} f_X(x)\,dx = \int_0^{1/4} \frac{10}{3} x(1-x^3)\,dx =$

(ii) $P(Y > 3/4) = \int_{3/4}^{1} 5y^4\,dy$

(iii) $P(0 < X+Y < 1/2)$



$$= \int_{x=0}^{1/4} \int_{y=x}^{1/2-x} 10xy^2\,dy\,dx$$

$$= \frac{10}{3} \int_0^{1/4} x\left[\left(\frac{1}{2}-x\right)^3 - x^3\right] dx$$

(iv) $P(X < 1/2 \mid Y = 3/4)$.

Now the conditional distⁿ of $X \mid Y = 3/4$ is

$$f_{X|Y}(x|y=3/4) = \begin{cases} \frac{2x}{9/16}, & 0 < x < 3/4 \\ 0, & \text{o/w}. \end{cases}$$

$$= \begin{cases} \frac{32x}{9}, & 0 < x < 3/4 \\ 0, & \text{o/w}. \end{cases}$$

So $$P(X < 1/2 \mid Y = 3/4) = \int_0^{1/2} \frac{32}{9} x \, dx = 4/9.$$

(v) $$P(Y < 1/2 \mid X = 1/4)$$

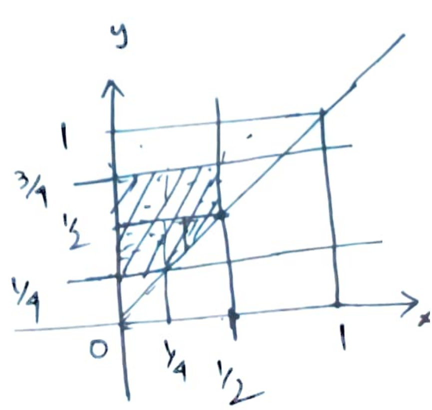
Now $$f_{Y|X}(y \mid x = 1/4) = \begin{cases} \dfrac{3y^2}{1/4\left(1-(1/4)^3\right)}, & 1/4 < y < 1 \\ 0, & o/w \end{cases}$$

$$= \begin{cases} \dfrac{64}{21} y^2, & 1/4 < y < 1 \\ 0 & o/w. \end{cases}$$

$$P(Y < 1/2 \mid X = 1/4) = \int_{1/4}^{1/2} \frac{64 y^2}{21} \, dy$$

(vi) $$P\left(0 < X < 1/2, \ \frac{1}{4} < Y < 3/4\right)$$

$$= \int_{y=1/4}^{1/2} \int_{x=0}^{y} 10xy^2 \, dx \, dy + \int_{x=0}^{1/2} \int_{y=1/2}^{3/4} 10xy^2 \, dy \, dx$$



Expectation and Moments

Let $\underline{X} = (X_1, X_2, \ldots, X_p)$ - p dimensional r.v. with pmf/ pdf $f_{\underline{X}}(\cdot)$ and support S. $g: \mathbb{R}^p \to \mathbb{R}$ is fun

Defn: We say that the expected value of $g(\underline{X})$ (denoted by $E(g(\underline{X}))$) is finite and equals

$$E(g(\underline{X})) = \begin{cases} \sum_{x \in S} g(\underline{x}) f_{\underline{X}}(\underline{x}) & \text{if } \underline{X} \text{ is discrete} \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(\underline{x}) f_{\underline{X}}(\underline{x}) \, d\underline{x} & \text{if } \underline{X} \text{ is continuous.} \end{cases}$$

Some special Expectations:

For non-negative integers $k_1, \ldots, k_p$

ⓐ $\mu'_{k_1, k_2, \ldots, k_p} = E\left(X_1^{k_1} X_2^{k_2} \cdots X_p^{k_p}\right)$ provided it is finite is called a joint moment of order $k_1 + k_2 + \cdots + k_p$ of $\underline{X}$

ⓑ $\mu_{k_1, k_2, \ldots, k_p} = E\left((X_1 - E(X_1))^{k_1} (X_2 - E(X_2))^{k_2} \cdots (X_p - E(X_p))^{k_p}\right)$

provided it is finite is called a joint central moment of order $k_1 + k_2 + \cdots + k_p$ of $\underline{X}$.

Note: If we take $k_i \neq 0$, & $k_j = 0$, $i \neq j$

$$\mu'_{0 0 \dots, k_i, 0 \dots 0} = E\left(X_i^{k_i}\right) = \mu'_{k_i}$$

$$\mu_{0,0 \dots k_i, 0 \dots 0} = E\left(\left(X_i - E(X_i)\right)^{k_i}\right) = \mu_{k_i}$$

(c) The quantity

$$\operatorname{Cov}(X_1, X_2) = E\left(\left(X_1 - E(X_1)\right)\left(X_2 - E(X_2)\right)\right) \text{ is}$$

called covariance between X_1 & X_2.

Note: (i) In particular we denote

$$\mu_i = E(X_i), \quad \sigma_i^2 = E(X_i - \mu_i)^2.$$

(ii)
$$\begin{aligned}
\operatorname{Cov}(X_1, X_2) &= E\left(\left(X_1 - E(X_1)\right)\left(X_2 - E(X_2)\right)\right) \\
&= E\left(\left(X_1 - \mu_1\right)\left(X_2 - \mu_2\right)\right) \\
&= E\left(X_1 X_2 - X_1 \mu_2 - \mu_2 X_1 + \mu_1 \mu_2\right) \\
&= E(X_1 X_2) - E(X_1) E(X_2)
\end{aligned}$$

(iii) $\operatorname{Cov}(X_1, X_1) = \operatorname{Var}(X_1)$.

<u>The correlation coefficient</u>: Let X and Y be two random variable then the correlation coefficient between X & Y is defined as

$$\rho_{X,Y} = \frac{\operatorname{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E\big(X - E(X)\big)\big(Y - E(Y)\big)}{\sqrt{E\big(X - E(X)\big)^2 E\big(Y - E(Y)\big)^2}}$$

Consider r.v.s U and V with

$$E(U) = 0, \quad EU^2 = 1 \quad E(V) = 0, \quad E(V^2) = 1.$$

Consider

$$E(U - V)^2 \geq 0 \Rightarrow E(U^2 + V^2 - 2UV) \geq 0$$

$$\Rightarrow E(UV) \leq 1. \quad \text{——— (1)}$$

Similarly, $E(U + V)^2 \geq 0$

$$\Rightarrow E(U^2 + V^2 + 2UV) \geq 0$$

$$\Rightarrow E(UV) \geq -1$$

So $-1 \leq E(UV) \leq 1$.

Hence $E(UV)_1 = 1$ iff $P(U=V)=1$

& $E(UV)=1$ iff $P(U=-V)=1$.

Now for any r.v X & Y.

$E(X)=\mu_x$, $E(Y)=\mu_Y$, $Var(X)=\sigma_x^2$, $Var(Y)=\sigma_y^2$

Define $U = \dfrac{X-\mu_x}{\sigma_x}$, $V = \dfrac{Y-\mu_Y}{\sigma_y}$

$E(U) = E\left(\dfrac{X-\mu_x}{\sigma_x}\right) = 0$, $E(U^2) = E\left(\dfrac{X-\mu_x}{\sigma_x}\right)^2 = 1$

$E(V) = E\left(\dfrac{Y-\mu_Y}{\sigma_y}\right) = 0$ $E(V^2) = E\left(\dfrac{X-\mu_x}{\sigma_x}\right)^2 = 1$

$-1 \le E(UV) \le 1$

$\Rightarrow -1 \le \dfrac{E(X-\mu_x)(Y-\mu_Y)}{\sigma_x \sigma_y} \le 1$

$\Rightarrow -1 \le \rho_{x,y} \le 1$.

$\rho_{x,y}=1 \Leftrightarrow P\left(\dfrac{X-\mu_x}{\sigma_x} = \dfrac{Y-\mu_Y}{\sigma_y}\right) = 1$

or $P(X = aY+b) = 1$, $a => 0$

$$\rho_{X,Y} = -1 \iff P\left(\frac{X-\mu_X}{\sigma_X} = -\frac{Y-\mu_Y}{\sigma_Y}\right) = 1$$

$$\text{or } P(X = aY + b) = 1, \quad a < 0.$$

So correlation coefficient is a measure of linear relationship between two random variable.

If $\rho_{X,Y} = 0$ we say that X & Y are uncorrelated

Result ①: Let a_i, $i=1,2,\ldots p$ and b_j, $j=1,2,\ldots p$ are real constants and let X_i, $i=1,2,\ldots p$, Y_j, $j=1,\ldots p$ be r.v.s then

ⓐ $$E\left(\sum_{i=1}^{p} a_i X_i\right) = \sum_{i=1}^{p} a_i E(X_i)$$

(b) $$\text{cov}\left(\sum_{i=1}^{p} a_i X_i, \sum_{j=1}^{p} b_j Y_j\right) = \sum_{i=1}^{p}\sum_{j=1}^{p} a_i b_j \,\text{cov}(X_i, Y_j)$$

(c) $$\text{Var}\left(\sum_{i=1}^{p} a_i X_i\right) = \sum_{i=1}^{p} a_i^2 \,\text{var}(X_i) + 2 \sum_{1\le i<j\le p}\sum a_i a_j \,\text{cov}(X_i, X_j)$$

Result: Let $X_1, X_2, \cdots X_p$ be independent random variables let $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \cdots p$ be given fun then

(a) $$E\left(\prod_{i=1}^{p} g_i(x_i)\right) = \prod_{i=1}^{p} E(g_i(x_i)).$$ In particular

$$E(x_1 x_2 \cdots x_p) = E(x_1) E(x_2) \cdots E(x_p).$$

(b) For any $A_1, A_2, \cdots A_p$ of subsets of $\mathbb{R}^p$

$$P_r(X_1 \in A_1, \cdots, X_p \in A_p) = \prod_{i=1}^{p} P_r(X_i \in A_i)$$

Remark: (1) Let X & Y are independent then

$$E(XY) = E(X)\, E(Y).$$

$$\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = 0.$$

$$\Rightarrow corr(X,Y) = 0.$$

If X, Y are independent then $Corr(X,Y) = 0$

But the converse is not true.

(II) If $X_1, X_2, \cdots X_p$ are independent Then

$$Cor(X_i, X_j) = 0, \quad i \neq j$$

Also we have

$$Var\left(\sum_{i=1}^{p} a_i X_i\right) = \sum_{i=1}^{p} a^2 \, Var(X_i)$$

Joint Moment Generating funⁿ:

$\underline{X} = (X_1, X_2, \dots, X_p)$, a p dimension r.v. with pdf | pmf $f_{\underline{Y}}(\cdot)$. The joint m.g.f. of $\underline{X}$ is defined as

$$M_{\underline{X}}(t) = E\left(e^{\sum_{i=1}^{p} t_i X_i}\right)$$ is defined in a nbd of $\underline{0} = (0, 0, \dots, 0)$.

(1) If $X_1, X_2, \dots, X_p$ are independent Then

$$M_{\underline{X}}(t) = E\left(e^{\sum_{i=1}^{p} t_i X_i}\right)$$

$$= E\left(\prod_{i=1}^{p} e^{t_i X_i}\right) = \prod_{i=1}^{p} E\left(e^{t_i X_i}\right)$$

$$= \prod_{i=1}^{p} M_{X_i}(t_i).$$

The converse is also true i.e. if $M_{\underline{X}}(t) = \prod_{i=1}^{p} M_{X_i}(t_i)$

Then $X_1, X_2, \ldots, X_p$ are independent.

(ii) Let $X_1, X_2, \ldots X_p$ be independent r.v.s and let

$Y = \sum_{i=1}^{p} X_i$. Then $M_Y(t) = \prod_{i=1}^{p} M_{X_i}(t)$.

<u>Defn</u>: $X_1, X_2, \ldots, X_p$ are called independent and identically distributed (i.i.d) if they are independent and have same probability distribution.

So if $X_1, X_2, \ldots X_p$ are i.i.d. then

$$M_Y(t) = \left(M_{X_1}(t)\right)^p.$$