$$f_X(x) = \begin{cases} \dfrac{x^{a-1}(1-x)^{b-1}}{B(a,b)} , & 0 < x < 1 \\ 0 & o/w . \end{cases}$$

$$E(X) = \frac{a}{a+b} , \qquad Var(X) = \frac{a(a+1)}{(a+b)(a+b+1)} .$$

Normal Distribution :

A continuous r.v. X is said to have a normal distⁿ with mean μ and variance σ^2 and written as $X \sim N(\mu, \sigma^2)$ if its pdf given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} , \quad -\infty < x < \infty$$
$$-\infty < \mu < \infty, \ \sigma > 0 .$$

Here

$f_X(x) > 0 \ \forall \ x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} , \quad \text{Let } z = \frac{x-\mu}{\sigma}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-z^2/2} dz$$

take $\frac{z^2}{2} = t \Rightarrow dz = \frac{1}{\sqrt{2t}} dt$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t} \frac{1}{\sqrt{2t}}\,dt = \frac{1}{\sqrt{\pi}}\int_0^{\infty} t^{\frac{1}{2}-1} e^{-t}\,dt$$

$$= \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} = 1.$$

The family $\{N(\mu,\sigma^2) : \mu \in \mathbb{R},\ \sigma > 0\}$ is a two parameter family of dist^n

$$E\left(\frac{X-\mu}{\sigma}\right)^k = \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^k \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$z = \frac{x-\mu}{\sigma}$, then we have

$$E\left(\frac{X-\mu}{\sigma}\right)^k = \int_{-\infty}^{\infty} z^k e^{-z^2/2}\,dz = 0 \quad \text{if } k \text{ is odd}$$

If k is even then $k = 2m$. then we have

$$E\left(\frac{X-\mu}{\sigma}\right)^k = 2\int_0^{\infty} z^{2m} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz, \quad \text{take } \frac{z^2}{2} = t.$$

$$= 2\int_0^{\infty} (2t)^m \frac{1}{\sqrt{2\pi}} e^{-t} \frac{1}{\sqrt{2t}}\,dt$$

$$= \frac{2^m}{\sqrt{\pi}}\int_0^{\infty} t^{m-\frac{1}{2}} e^{-t}\,dt = \frac{2^m}{\sqrt{\pi}}\Gamma\left(m+\tfrac{1}{2}\right)$$

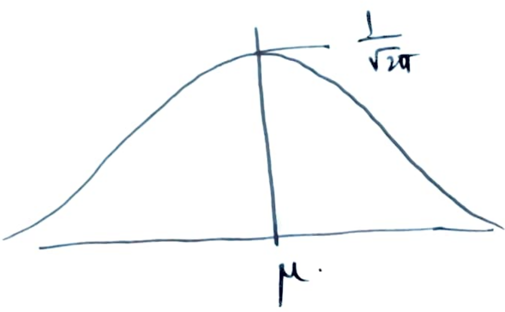
$$= \frac{2^m}{\cancel{\sqrt{\pi}}}\left(m-\tfrac{1}{2}\right)\left(m-\tfrac{3}{2}\right)\cdots\tfrac{3}{2}\cdot\tfrac{1}{2}\,\cancel{\Gamma\tfrac{1}{2}}$$

$$E\left(\frac{X-\mu}{\sigma}\right)^k = \begin{cases} 0, & k \text{ is odd} \\ (2m-1)(2m-3)\cdots 5\cdot 3\cdot 1, & k=2m, \; m=1,2,\ldots \end{cases}$$

$k=1$, $\quad E\left(\frac{X-\mu}{\sigma}\right)=0 \Rightarrow E(X)=\mu.$

So $E(X-\mu)^k = 0$ if k is odd. So all order central moment of a normal distⁿ is zero.

$$E(X-\mu)^{2m} = \sigma^{2m}(2m-1)(2m-3)\cdots 5\cdot 3\cdot 1$$



$\underline{m=1}$

$E(X-\mu)^2 = \sigma^2$, $\quad \mu_3 = 0.$

$\mu_4 = E(X-\mu)^4 = 3\sigma^4$

$\beta_1 = 0$, $\quad \beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = 3-3 = 0.$

m.g.f. of X.

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Take $\frac{x-\mu}{\sigma} = z$, $\quad x = (\mu + \sigma z)$

$$M_X(t) = \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 - 2\sigma t z + \sigma^2 t^2)} \, dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma t)^2} \, dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$\boxed{M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}}$$

Let $X \sim N(\mu, \sigma^2)$, Let $Y = aX + b$, $a \neq 0$, $b \in \mathbb{R}$

$$M_Y(t) = E(e^{tY}) = E(e^{t(aX+b)})$$

$$= e^{bt} E\left(e^{(ta)X}\right) = e^{bt} M_X(at)$$

$$= e^{bt} e^{\mu(at) + \frac{1}{2}\sigma^2 (at)^2}$$

$$= e^{(a\mu+b)t + \frac{1}{2}(a^2\sigma^2)t^2}$$

$$\Rightarrow Y \sim N(a\mu + b, a^2\sigma^2).$$

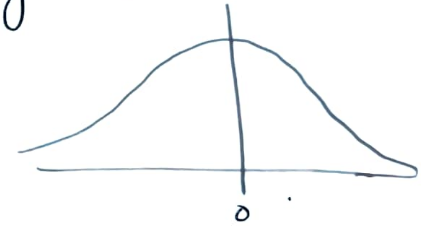
Now take

$$Z = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma} \sim N(0,1)$$

This is called standard normal r.v.

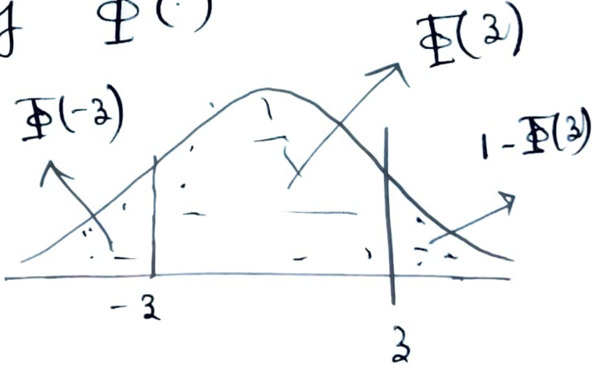
The pdf of Z is denoted by ϕ

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad z \in \mathbb{R}.$$



The cdf of Z is denoted by $\Phi(\cdot)$

$$\Phi(z) = \int_{-\infty}^{z} \phi(t)\,dt.$$



we have from symmetry

$$1-\Phi(z) = \Phi(-z) \Rightarrow \Phi(-z)+\Phi(z) = 1. \quad \text{---} (*)$$

Also we have

$$\phi(z) = \phi(-z).$$

If we take $z=0$ in $(*)$ we have

$$\Phi(0) = \frac{1}{2}.$$

Consider $X \sim N(\mu, \sigma^2)$

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

<u>Example.</u> Assume that the time required for a distance runner to run a mile is the normal r.v. with $\mu = 4$ min 1 sec & $\sigma = 2$ sec. What is the prob that this athelete will run the mile in less than 4 min?

<u>Solu</u>

$$X \sim N(241, 4) \rightarrow X \rightarrow \text{time in sec.}$$

$$P(X < 240) = P\left(\frac{X - 241}{2} < \frac{240 - 241}{2}\right)$$

$$= P(Z < -0.5) = \Phi(-0.5) = 0.3085.$$

Normal Approximation to Binomial

Let $X \sim Bin(n, p)$. as $n \rightarrow \infty$ the dist^n of

$$\frac{X - np}{\sqrt{npq}}$$ is approximately $N(0,1)$

Poisson Approximation to Normal

$X \sim P(\lambda)$. as $\lambda \rightarrow \infty$ $\frac{X - \lambda}{\sqrt{\lambda}} \longrightarrow N(0,1)$.