Sum and Direct Sum of subspaces

Theorem: If W_1 and W_2 are finite dimensional subspaces of vector space V, then W_1+W_2 is finite dimensional and

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1+W_2)$$

Proof: As $W_1 \cap W_2$ is a subspace of W_1 and thus has a basis $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$ of less than $\dim W_1$ elements which is a part of basis of W_1 (say)

$\{\alpha_1, \alpha_2, \ldots \alpha_k, \beta_1, \ldots \beta_m\}$

and also $W_1 \cap W_2$ is a subspace of W_2 and thus a basis of $W_1 \cap W_2$ will be a part of W_2 (say)

$\{\alpha_1, \alpha_2, \ldots \alpha_k, \gamma_1, \gamma_2, \ldots \gamma_n\}$

Note that the subspace W_1+W_2 is spanned by

$\{\alpha_1, \alpha_2, \ldots \alpha_k, \beta_1, \beta_2, \ldots \beta_m, \gamma_1, \gamma_2, \ldots \gamma_n\}$

Our claim is these vectors are linearly independent and hence form a basis for W_1+W_2 (Try to prove this claim)

Now compare the dimensions of these subspaces as

$$\dim W_1 + \dim W_2 = (k+m)+(k+n)$$
$$= k+(m+k+n)$$
$$= \dim(W_1 \cap W_2) + \dim(W_1+W_2)$$

Remark: If $W_1+W_2 = V$ and $W_1 \cap W_2 = \{0\}$, then the sum of W_1 & W_2 is called direct sum and is denoted as $W_1 \oplus W_2 = V$. Moreover if V is a direct sum of W_1 & W_2 then any $v \in V$ can be uniquely written as $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$.

Let us see the following examples.

Example: Let $V=\mathbb{R}^2$ and

$$W_1=\{(x,x) : x\in\mathbb{R}\}$$
$$W_2=\{(y,-y) : y\in\mathbb{R}\}$$

Then $W_1+W_2=\mathbb{R}^2$, as any $(a,b)\in\mathbb{R}^2$ can be written as

$$(a,b)=\left(\frac{a+b}{2},\frac{a+b}{2}\right)+\left(\frac{a-b}{2},-\frac{a-b}{2}\right)$$

Example: Let $V=\mathbb{R}^2$, and

$$W_1=\{(x,2x) : x\in\mathbb{R}\}$$
$$W_2=\{(y,3y) : y\in\mathbb{R}\}$$

Then $V=W_1\oplus W_2$

Solution: We have to verify two things. First any $(a,b)\in\mathbb{R}^2$ can be written in the sum of two vectors coming of W_1 & W_2 respectively. Secondly $W_1\cap W_2=\{0\}$. Let us see one by one.

Let $(a,b)\in\mathbb{R}^2$, then

$$(a,b)=(x,2x)+(y,3y)$$
$$\Rightarrow x+y=a,\quad 2x+3y=b$$
$$\Rightarrow y=b-2a \;\&\; x=a-b+2a$$
$$x=3a-b$$

Thus

$$(a,b)=(3a-b,6a-2b)+(b-2a,3b-6a)$$

Also let $(u,v)\in W_1\cap W_2$ then $(u,v)\in W_1$ & $(u,v)\in W_2$

$$\Rightarrow (u,v)=(c,2c) \;\&\; (u,v)=(d,3d)$$
$$\Rightarrow c=d \;\&\; 2c=3d \Rightarrow c=d=0.$$

Ordered basis

Definition (Ordered basis) Let V be a vector space & B be it's basis. Then B is called ordered basis if we impose some order on it.

For example if we fix the position of every element in the basis.

Let $\mathcal{B}$ be an ordered basis written as

$$\mathcal{B} = \{\alpha_1, \alpha_2, \dots \alpha_n\}$$

Take $\alpha \in V$, then

$$\alpha = \sum_{i=1}^{n} x_i \alpha_i, \text{ for } x_i \in \mathbb{F}, \ i=1,2,\dots n$$

The above linear combination of vectors from $\mathcal{B}$ is unique as

if $\alpha = \sum_{i=1}^{n} y_i \alpha_i$, for some $y_i \in \mathbb{F}$, $i=1,2,\dots n$

then $\sum_{i=1}^{n} (x_i - y_i)\alpha_i = 0$

As $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ are L.I., we get $x_i = y_i \ \forall i=1,2,\dots n$.

Similarly if $\beta \in V$, then

$$\beta = \sum_{i=1}^{n} y_i \alpha_i$$

Note that

$$\alpha + \beta = \sum_{i=1}^{k} (x_i + y_i)\alpha_i$$

and $c\alpha = \sum_{i=1}^{k} (c x_i)\alpha_i$ for $c \in \mathbb{F}$

Thus each ordered basis of V establishes a one to one correspondance between V and $\mathbb{F}^n$ as $\alpha \mapsto (x_1, x_2, \dots x_n)$

We call x_i, the ith coordinate of α with respect to ordered basis $\mathcal{B}$.
It means that, if you have coordinate of any vector in V, you can identify the vector simply using the ordered basis.
We use the notation $[\alpha]_{\mathcal{B}}$ (a column vector) to denote the coordinate matrix of α with respect to ordered basis $\mathcal{B}$.

Let V be a finite dimensional vector space and $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and $\mathcal{B}' = \{\alpha_1', \alpha_2', \ldots \alpha_n'\}$ be two ordered basis of V.
Then
$$\alpha_j' = \sum_{i=1}^{n} P_{ij} \alpha_i \quad \text{for } 1 \leq j \leq n$$
Thus
$$[\alpha_j']_{\mathcal{B}} = (P_{1j}, P_{2j}, \ldots P_{nj})^T$$
Now let $(x_1', x_2', \ldots x_n')^T$ be the coordinate of a vector $\alpha \in V$ with respect to $\mathcal{B}'$, i.e.
$$\alpha = \sum_{j=1}^{n} x_j' \alpha_j'$$
$$\alpha = \sum_{j=1}^{n} x_j' \sum_{i=1}^{n} P_{ij} \alpha_i$$
$$= \sum_{j=1}^{n} \sum_{i=1}^{n} x_j' P_{ij} \alpha_i$$
$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} P_{ij} x_j' \right) \alpha_i$$

As we discussed earlier, the coordinates of a vector $\alpha \in V$ is unique w.r. to the basis $\mathcal{B}$ say $(x_1, x_2, \ldots x_n)^T$, we get

$$x_i = \sum_{j=1}^{n} P_{ij} x_j'$$

Or

$$x_1 = P_{11} x_1' + P_{12} x_2' + \ldots P_{1n} x_n'$$
$$x_2 = P_{21} x_1' + P_{22} x_2' + \ldots P_{2n} x_n'$$
$$\vdots$$
$$x_n = P_{n1} x_1' + P_{n2} x_2' + \ldots P_{nn} x_n'$$

or

$$[\alpha]_{\mathcal{B}} = P [\alpha]_{\mathcal{B}'}$$

Moreover, the matrix P is invertible and therefore we can switch between one coordinate to another coordinate (for different basis) i.e.

$$[\alpha]_{\mathcal{B}'} = P^{-1} [\alpha]_{\mathcal{B}}$$

From above discussion, we get the following result

Theorem: Let V be an n-dimensional vector space over the field F and let $\mathcal{B}$ and $\mathcal{B}'$ be two ordered bases of V. Then there exists a unique, invertible $n \times n$ matrix P $(P \in M_{n \times n}(\mathbb{F}))$ such that

$$[\alpha]_{\mathcal{B}} = P [\alpha]_{\mathcal{B}'} \text{ or}$$

$$[\alpha]_{\mathcal{B}'} = P^{-1} [\alpha]_{\mathcal{B}}$$

for every $\alpha \in V$. Moreover the columns of P are $P_j = [\alpha_j']_{\mathcal{B}}$ $j = 1, 2, \ldots n$ where $\mathcal{B} = \{\alpha_1, \ldots \alpha_n\}$ and $\mathcal{B}' = \{\alpha_1', \alpha_2', \ldots \alpha_n'\}$

Let us see some examples.

<u>Example</u>. Let $\mathbb{R}^n$ be a vector space over $\mathbb{R}$ and

$\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n) \in \mathbb{R}^n$, then the co-ordinate matrix of α in the ordered basis of V

$\mathcal{B} = \{e_1, e_2, \ldots e_n\}$ is

$$[\alpha]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

<u>Example</u>: Let $\mathcal{B} = \{\beta_1, \beta_2\}$

where $\beta_1 = (1,1,0)$ & $\beta_2 = (1, i, 1+i)$

be a basis for a subspace W of $\mathbb{C}^3$. Determine the co-ordinate matrix of $\alpha_1 = (1, 0, i)$ & $\alpha_2 = (1+i, 1, -1)$ with respect to ordered basis $\mathcal{B}$.

<u>Solution</u>

$$(1, 0, i) = x_1(1,1,0) + x_2(1, i, 1+i)$$

$$\Rightarrow x_1 + x_2 = 1,$$

$$x_1 + i x_2 = 0$$

$$(1+i)\, x_2 = i$$

$$\Rightarrow x_2 = \frac{i}{1+i} = \frac{i(1-i)}{2}$$

$$\Rightarrow x_1 = -i x_2 = \frac{-i+1}{2} = \frac{1-i}{2}$$

$$(x_1 + x_2 = \frac{1-i}{2}(i+1) = \frac{1+1}{2} = 1$$

Thus $[\alpha_1]_{\mathcal{B}} = \begin{bmatrix} \frac{1-i}{2} \\ \frac{i}{2}(1-i) \end{bmatrix}$

Similarly $(1+i, i, -1) = y_1(1,1,0) + y_2(1, i, 1+i)$

$\Rightarrow 1+i = y_1 + y_2,\ 1 = y_1 + i y_2,\ -1 = (1+i) y_2$

$\Rightarrow y_2 = -1/1+i = \frac{-(1-i)}{2},\ y_1 = 1+i+\frac{1-i}{2} = \frac{2+2i+1-i}{2} = \frac{3+i}{2},\ [\alpha_2]_{\mathcal{B}} = \begin{bmatrix} \frac{3+i}{2} \\ -\frac{(1-i)}{2} \end{bmatrix}$

Example. Observe that
$\mathcal{B} = \{(1,1), (1,-1)\}$ forms a basis for $\mathbb{R}^2$, Let $\alpha \in \mathbb{R}^2$ be $\alpha = (\alpha_1, \alpha_2)$ then
$[\alpha]_{\mathcal{B}} = ?$

Solution. $(\alpha_1, \alpha_2) = x_1(1,1) + x_2(1,-1)$

$$\Rightarrow x_1 + x_2 = \alpha_1$$
$$x_1 - x_2 = \alpha_2$$
$$\Rightarrow x_1 = \frac{\alpha_1+\alpha_2}{2}$$
$$\& \ x_2 = \frac{\alpha_1-\alpha_2}{2}$$

Thus $[\alpha]_{\mathcal{B}} = \begin{bmatrix} \frac{\alpha_1+\alpha_2}{2} \\ \frac{\alpha_1-\alpha_2}{2} \end{bmatrix}$

Let $\mathcal{B}' = \{(1,3), (3,1)\}$ be another basis of $\mathbb{R}^2$, then
$[\alpha]_{\mathcal{B}'} = ?$

For this we can find the invertible matrix P as

$$P_1 = [(1,3)]_{\mathcal{B}} \quad P_2 = [(3,1)]_{\mathcal{B}}$$

Note that
$(1,3) = y_1(1,1) + y_2(1,-1)$
$$\Rightarrow y_1 + y_2 = 1$$
$$y_1 - y_2 = 3$$
$$\Rightarrow y_1 = 2, \ y_2 = -1 \Rightarrow [1,3]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Thus Similarly,
$$[(3,1)]_{\mathcal{B}} = y_1 + y_2 = 3, \ y_1 - y_2 = 1$$
$$\Rightarrow y_1 = 4, \ y_2 = -1$$
$[3,1]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ Hence $P = \begin{bmatrix} 2 & 4 \\ -1 & -1 \end{bmatrix}$ Thus $[\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}}$.