Tutorial - 4

(1) Let $\epsilon > 0$. Since g is continuous at 0, there exists $\delta > 0$ s.t. $|g(x)| = |g(x) - g(0)| < \epsilon$ for all $x \in \mathbb{R}$ with $|x-0| < \delta$

So $|f(x) - f(0)| \leq |f(x)| + |f(0)| \leq |g(x)| + |g(0)| = |g(x)| < \epsilon$

$\forall\, x \in \mathbb{R}$ with $|x-0| < \delta$.

② $\lim\limits_{x \to 1-} f(x) = \lim\limits_{x \to 1-} (4x+2) = 6 \neq 5 = f(1)$.

So $f(x)$ is not continuous.

③ Let $g(x) = f(x) - x \quad \forall\, x \in [0,1]$. So $g(x)$ is continuous on $[0,1]$, since $f(x)$ is continuous on $[0,1]$.

$g(0) = f(0)$, $\quad g(1) = f(1) - 1$

If $f(0) = 0$ or $f(1) = 1$, then we get the result by taking $c = 0$ or $c = 1$. respectevely. Otherwise $\because\ 0 \leq f(x) \leq 1$

$g(0) > 0$ & $g(1) = f(1) - 1 < 0$

Hence by intermediate value theorem there exists $c \in (0,1)$ such that $g(c) = 0$, i.e. $f(c) = c$.

(4) Let $g(x) = f(x+1) - f(x)$ on $[0,1]$. Since f is continuous so $g : [0,1] \to \mathbb{R}$ is continuous.

$g(0) = f(1) - f(0)$

$g(1) = f(2) - f(1) = f(0) - f(1)$

So $g(0)$ & $g(1)$ have opposite signs and hence by intermediate value theorem $\exists\, c \in (0,1) \ni$

$g(c) = 0 \Rightarrow f(c+1) = f(c)$, take $x_1 = c+1$ & $x_2 = c$.

(5) ~~If $f : [0,1] \to (0,\infty)$ is continuous, then f must be bdd set in $\mathbb{R}$, bdd. Since $(0,\infty)$ is not bdd~~

(5) $f(x) = 1 - x^{2/3}$, $f(x)$ is diff for all $x \neq 0$

$f'(x) = -\frac{2}{3} x^{-1/3} \neq 0 \quad \forall x \neq 0.$

Hence f does not have local maximum or local minimum at any $x\,(\neq 0) \in \mathbb{R}$. Again $f(x) \leq 1 = f(0)$ $\forall x \in \mathbb{R}$. f has local maximum at 0.

(6) There exists a sequ^n $\{x_n\} \in \mathbb{Q}$ s.t. $x_n \to \sqrt{2}$

Since f is continuous at $\sqrt{2}$ we have

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = (\sqrt{2})^2 + 5 = 7.$$

(7) Let $g(x) = e^x \cos x + 1$. Take $f(x) = e^{-x} g(x)$

i.e. $f(x) = \cos x + e^{-x}$

Let a, b be two real root of $g(x)$ then

$$g(a) = g(b) = 0 \Rightarrow f(a) = f(b) = 0.$$

So apply Rolle's Theorem on $f(x)$.

(8) Let $g(x) = x^3$, which is differentiable on $[0,1]$.
So by Cauchy's mean value theorem $\exists\ \xi \in (0,1)$

$$g'(\xi)[f(1) - f(0)] = f'(\xi)[g(1) - g(0)]$$

or $$3\xi^2 [f(1) - f(0)] = f'(\xi)$$

(9) Consider the fun f defined as

$$f(x) = \frac{a_0}{n+1} x^{n+1} + \frac{a_1}{n} x^n + \cdots + \frac{a_{n-1}}{2} x^2 + a_n x, \quad x \in [0,1]$$

(i) f is continuous on $[0,1]$ (ii) f is differentiable in $(0,1)$

(iii) $f(0) = 0$ & $f(1) = 0$ by the given condition

so $f(0) = f(1)$.

Hence $\exists$ ~~some~~ $x \in (0,1)$ s.t. $f'(x) = 0$.

i.e. $a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$.

(10) Let $f(x) = e^{\alpha x} p(x)$. Let α, β be two roots of $p(x) = 0$ i.e. $p(\alpha) = p(\beta) = 0$.

Then $f(x)$ is continuous and derivable in any interval say $[\alpha, \beta]$.

Further $F(\alpha) = 0$ & $F(\beta) = 0$.

By Rolle's theorem $\exists$ a c between α, β such that

$$f'(c) = 0 \Rightarrow \alpha\, p(c) + p'(c) = 0.$$

$\Rightarrow$ There exist a root of $p'(x) + \alpha p(x) = 0$ between a pair of roots of $p(x) = 0$.

(ii) Since $f''(x) \geq 0 \Rightarrow f'(x)$ is increasing $[a, b]$.

Let $a \leq x_1 < x_2 \leq b$. In $\left[x_1, \frac{x_1+x_2}{2}\right]$ apply MVT we get

$$f\left(\frac{x_1+x_2}{2}\right) - f(x_1) = \frac{x_2-x_1}{2} f'(c)$$

$$x_1 < c < \frac{x_1+x_2}{2}$$

Again in $\left[\frac{x_1+x_2}{2}, x_2\right]$ apply MVT we get

$$f(x_2) - f\left(\frac{x_1+x_2}{2}\right) = \frac{x_2-x_1}{2} f'(d), \quad \frac{x_1+x_2}{2} < d < x_2$$

$\because$ $f'(x)$ is increasing so $f'(d) \geq f'(c)$

$$\Rightarrow \quad f(x_2) - f\left(\frac{x_1+x_2}{2}\right) \geq f\left(\frac{x_1+x_2}{2}\right) - f(x_1)$$

$$\Rightarrow \frac{1}{2}\left[f(x_1) + f(x_2)\right] \geq f\left(\frac{x_1+x_2}{2}\right)$$