Elementarey Probability Theory

Example: The production manger of a bulb manufacturing company wishes to study the effect of new manufacturing process on the lifetimes of bulbs produceds through it.

Here the population under study is

P: collection of lifetimes of all bulbs produced by using new process.

In most practical situation P is genereally large.

Probability theory: A mathematical tool for modeling uncertainity. (e.g. to describe the law according to which values of the life time a bulb vary across P).

The only way to collect information about any random phenomenon is to perform experiment. As an example selecting a set of bulbs manufactured by the new process and putting them on test for measuring their lifetimes. Each experiments terminates in an outcome which can not be predicted in advance prior to the performence of experiment.

Def^n (Random experiment): A random (or statistical) experiment is an experiment in which

(a) All outcomes of the experiment are known in advance

(b) Outcome of a particular performance of the experiment can not be predicted in advance.

(c) the experiment can be repeated under identical conditions.

Sample space: The collection of all possible outcomes of a random experiment is called its sample space. Sample space is denoted by Ω.

Ex (i) Tossing a coin: $\Omega = \{H, T\}$

(ii) Throwing a die $\Omega = \{1, 2, 3, 4, 5, 6\}$

(iii) Birth of child, $\Omega = \{M, F\}$

If we consider weight $\Omega = (0, 7)$

(iv) Age at death of a person: $\Omega = (0, 120)$

(v) Life of a bulb: $\Omega = (0, \infty)$

(vi) Throughing two dies

$$\Omega = \{ (1,1), (1,2), \ldots (1,6), (2,1), (2,2) \ldots, (2,6) \ldots (6,1), (6,2), \ldots, (6,6)\}$$

$$= \{(i,j),\ i,j \in \{1,2,3,\ldots 6\}\}$$

Event: An event is any subset of the sample space. If the outcome of a random experiment is a member of the set $E \subseteq \Omega$ we say that event E has occured.

Impossible event: ϕ, Sure event: Ω.

Let A and B are two event

1) $A \cup B \to$ occurence of at least one of A & B

II) $\bigcup_{i=1}^{n} A_i \to$ occurence of at least one A_i, $i=1,2,\ldots n$

III) $A \cap B \to$ simultaneous occurence of A & B.

(IV) $\bigcap_{i=1}^{n} A_i \to$ simultaneous occurence of A_i, $i=1,2,\ldots n$

Exhaustive events: If $\bigcup_{i=1}^{n} A_i = \Omega$ we call $A_1, A_2, \ldots A_n$ to be exhaustive events

If $A \cap B = \phi$ then A & B are called mutually exclusive events i.e. happenning or occurence of one of them excludes the possibility of occurence of other

Let $A_1, A_2, \ldots$ are event then

$A_i \cap A_j = \phi$ $i \neq j$, then we say $A_1, A_2, \ldots$ are pair wise disjoint or mutually exclusive

$A^C \to$ not happening of A

$A - B \to$ happening of A not B $= A \cap B^C$

~~In general~~ Generally we are interested in specific subsets of Ω which we will treated as event. So the event space (events under consideration) $\mathfrak{S}$ is a subsets of power set of Ω.

So the event space is $\mathfrak{S} \subseteq \mathcal{P}(\Omega)$, Here $\mathcal{P}(\Omega)$ is the power set of Ω.

The choice of $\mathfrak{S}$ is an importent one

(I) If Ω contains at most a countable number of points we can alwys take $\mathfrak{S}$ to be the $\mathcal{P}(\Omega)$. (This is certainly a σ-field). In this case each point set is a member of $\mathfrak{S}$ and is the fundamental object of interest. Every subset of Ω is an event.

(II) If Ω has uncountably many points the class of all subsets of Ω is still a σ-field but it is much too ~~to~~ large a class of sets to be of interest. If $\Omega = \mathbb{R}$ or any interval then Ω is uncountable. In this case we would like to consider all one point subsets of Ω all all intervals (closed, open, or semiclosed) to be events. We consider the Borel σ-field $\mathcal{B}_1$ generated by the class of all semi closed intervals $(a, b]$, which is a σ-field in $\mathbb{R}$.

We say that the event space $\mathcal{S} \subseteq \mathcal{P}(\Omega)$ contains all subsets of Ω actually encountered in ordinary analysis and probability. It is large enough for all practicle purposes.

The algebra of set theory is applicable in prob. Theory

Probability is a measure of uncertainty. We are interested in quantifying uncertainity associated with various outcomes of a random experiment by assiging probility to these outcomes.

Here we will not discuss how probabilities are assigned (which is a part of prob. modeling) rather we will discuss properties of a probability measure.

Defn (Probability function or Probability measure)

A probability fun (or prob measure) is a real valued set function, defined on the event space $\mathcal{S}$ satisfying the following axioms

(a) $P(A) \geq 0 \quad \forall A \in \mathcal{S}$

(b) $P(\Omega) = 1$

(c) If $\{A_n\}$ is a sequence of mutually exclusive (disjoint) sets in $\mathcal{S}$ i.e. $A_i \cap A_j = \phi$, $i \neq j$

then

$$P\left(\bigcup_{i\geq 1}^{\infty} A_i\right) = \sum_{i\geq 1}^{\infty} P(A_i) \quad (\text{countable additivity}).$$

We call $P(A)$ the probability of event A.

The triplet $(\Omega, \mathcal{S}, P)$ is called probability space.

Properties of Probability Measure :

(P_1) $\quad P(\phi) = 0$

Proof: Let $A_1 = \Omega$ and $A_i = \phi$, $i = 2, 3, \ldots$

Then $P(A_1) = 1$. Also we have $A_1 = \bigcup_{i\geq 1}^{\infty} A_i$, $A_i \cap A_j = \phi$

$i \neq j$. Therefore

$$1 = P(A_1) = P\left(\bigcup_{i\geq 1}^{\infty} A_i\right) = \sum_{i\geq 1}^{\infty} P(A_i) = 1 + \sum_{i\geq 2}^{\infty} P(A_i)$$

$$\Rightarrow \sum_{i=2}^{\infty} P(\phi) = 0 \Rightarrow P(\phi) = 0$$

(P2) Let $A_1, A_2, \ldots, A_n \in \mathcal{S}$, $\ni$ $A_i \cap A_j = \phi$, $i \neq j \Rightarrow$

$$P\left(\bigcup_{i\geq 1}^{n} A_i\right) = \sum_{i\geq 1}^{\infty} P(A_i)$$

Proof: Take $A_i = \phi$, $i = n+1, n+2, \ldots$

Then $$P\left(\bigcup_{i\geq 1}^{n} A_i\right) = P\left(\bigcup_{i\geq 1}^{\infty} A_i\right) = \sum_{i\geq 1}^{\infty} P(A_i) = \sum_{i\geq 1}^{n} P(A_i)$$

(P3) $\forall A \in \mathcal{S}$, $0 \le P(A) \le 1$ and $P(A^c) = 1 - P(A)$

Proof: $1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$

$\Rightarrow$ $P(A^c) = 1 - P(A)$

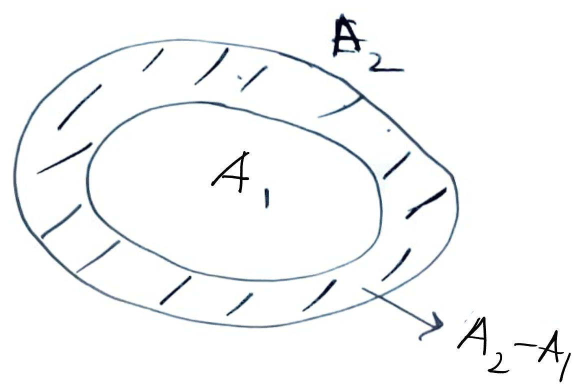
Also $P(A) \le 1$

So $0 \le P(A) \le 1$ & $P(A^c) = 1 - P(A)$

(P4) $A_1, A_2 \in \mathcal{S}$ and $A_1 \subseteq A_2 \Rightarrow P(A_2 - A_1) = P(A_2) - P(A_1)$

and $P(A_1) \le P(A_2)$.

Proof: $A_2 = A_1 \cup (A_2 - A_1)$



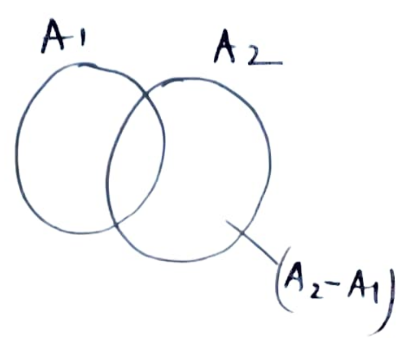
$A_1 \cap (A_2 - A_1) = \phi$.

So $P(A_2) = P(A_1) + P(A_2 - A_1)$

$\Rightarrow$ $P(A_2 - A_1) = P(A_2) - P(A_1)$

$\because$ $P(A_2 - A_1) \geq 0 \Rightarrow P(A_2) \geq P(A_1)$ (Monotonicity)

(P5) $A_1, A_2 \in \mathcal{S}$, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$



$P(A_1 \cup A_2) = P(A_1 \cup (A_2 - A_1))$

$= P(A_1) + P(A_2 - A_1)$ ——— (i)

Now we have

$$(A_1 \cap A_2) \cap (A_2 - A_1) = \phi \quad \text{and} \quad A_2 = (A_1 \cap A_2) \cup (A_2 - A_1)$$

$\Rightarrow$

$$P(A_2) = P(A_1 \cap A_2) + P(A_2 - A_1)$$

$$\Rightarrow \quad P(A_2 - A_1) = P(A_2) - P(A_1 \cap A_2) \quad \text{——(II)}$$

Using (II) from (I) we get

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

Theorem

Let $A_1, A_2, A_3, \ldots, A_n \in \mathcal{S}$, $n \geq 2$. Define

$$p_{1,n} = P(A_1) + P(A_2) + \cdots + P(A_n) = \sum_{i=1}^{n} P(A_i)$$

$$p_{2,n} = \sum\sum_{1 \leq i < j \leq n} P(A_i \cap A_j)$$ (sum of probabilities of all possible intersections involving 2 events out of n events $A_1, \ldots, A_n$)

$\vdots$

$$p_{i,n} = \sum\sum \cdots \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq n} P(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_i})$$

(sum of probabilities of all possible intersections involving i events out of n events $A_1, A_2, \ldots A_n$)

Then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = p_{1,n} - p_{2,n} + p_{3,n} - p_{4,n} + \cdots + (-1)^{n-1} p_{n,n}$$

Proof: Do yourself. Use induction.

Remark: (i) Let $A_1, A_2, A_3 \in \mathcal{S}$, then

$$P(A_1 \cup A_2 \cup A_3) = p_{1,3} - p_{2,3} + p_{3,3}$$
$$= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

(ii) We have $P(A_1 \cup A_2) \leq 1$

$$\Rightarrow P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq 1$$

$$\Rightarrow P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$$

This inequality is known as Bonferroni's inequality.

Theorem: Let $(\Omega, \mathcal{S}, P)$ be a probability space and let $A_1, A_2, \dots A_n \in \mathcal{S}$ $(n \in \mathbb{N}, n \geq 2)$. Then

(i) Boole's inequality

$$p_{1,n} - p_{2,n} \leq P\left(\bigcup_{i=1}^{n} A_i\right) \leq p_{1,n}$$

(ii) $P\left(\bigcap_{i=1}^{n} A_i\right) \geq p_{1,n} - (n-1)$. (Bonferroni's Inequality)

Example: Consider a random experiment throughing two dies one is red and other one white

Sample space : $\Omega = \{ (i,j) : i = 1, 2, \cdots 6, \; j = 1, 2, \cdots 6 \}$

For $(i,j) \in \Omega$

i : number of spots up on the red die

j : number of spots up on the white die.

Event space $\xi =$ power set of $\Omega = 2^{\Omega}$

For $E \in \xi$ define $Q : \xi \to \mathbb{R}$ as

$$Q(E) = \frac{|E|}{36}, \text{ where } |E| = \# \text{ of elements in } E$$

Then (i) $Q(\Omega) = \dfrac{|\Omega|}{36} = 1$

(ii) $Q(E) = \dfrac{|E|}{36} \geq 0 \quad \forall E \in \xi.$

(iii) For mutually disjoint events $E_1, E_2, \cdots$

$$Q\left(\bigcup_{i=1}^{\infty} E_i\right) = \frac{\left|\bigcup_{i=1}^{\infty} E_i\right|}{36} = \frac{\sum_{i=1}^{\infty} |E_i|}{36} = \sum_{i=1}^{\infty} Q(E_i)$$

Thus (Ω, ξ, Q) is a prob. space.