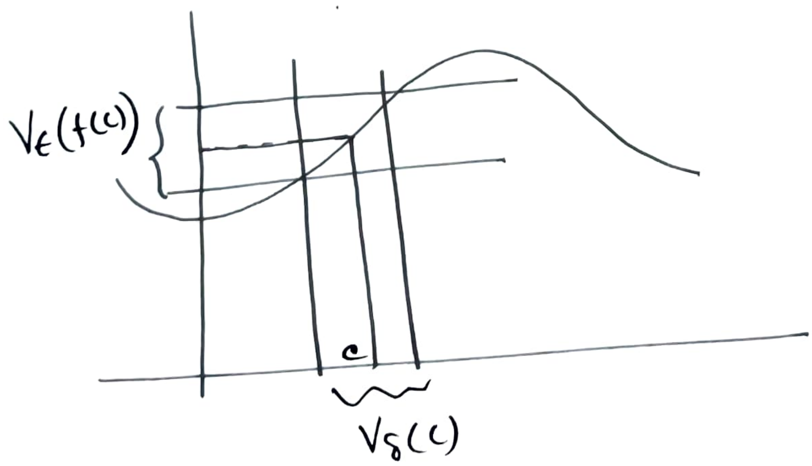
Continuous Functions

Defn: Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and $c \in A$. We say that f is continuous at c if, given any $\epsilon > 0$ there exists $\delta > 0$ such that if x is any point of A satisfying $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$

Equivalently: A function $f: A \to \mathbb{R}$ is continuous at a point $c \in A$ if and only if given any ϵ nbd $V_\epsilon(f(c))$ of $f(c)$ there exists a δ- nbd $V_\delta(c)$ of c such that x is any point of $A \cap V_\delta(c)$ then $f(x)$ belongs to $V_\epsilon(f(c))$



Remark: 1 If $c \in A$ is a cluster point or limit point of A, then by comparing the defn of limit we get f is continuous at c iff

$$f(c) = \lim_{x \to c} f(x).$$

Thus c is a limit point of A the following conditions must hold for f to be continuous at c:

(i) f must be defined at c so that $f(c)$ makes sense.

(ii) $\lim_{x \to c} f(x)$ exists

(iii) $\lim_{x \to c} f(x) = f(c)$.

Sequential Criterion for continuity:

A function $f : A \to \mathbb{R}$ is continuous at the point $c \in A$ iff for every sequence $\{x_n\}$ in A that converges to c the sequence $\{f(x_n)\}$ converges to $f(c)$.

A function $f(x)$ is not continuous at a point c then we say f is discontinuous at c.

Sequential Criterion for discontinuity: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$. Then $f(x)$ is discontinuous at c iff there exists a sequ^n $\{x_n\}$ in A such that $\{x_n\}$ converges to c but the sequ^n $\{f(x_n)\}$ does not converges to $f(c)$.

Def^n: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$. If $B \subset A$, we say that f is continuous on the set B if f is continuous at every point of B.

Example: (a) $f(x) = b$, $x \in \mathbb{R}$, Let $c \in \mathbb{R}$

$$\lim_{x \to c} f(x) = b = f(c) \Rightarrow f(x) \text{ is continuous}$$

(b) $f(x) = x^n$, $x \in \mathbb{R}$, Let $c \in \mathbb{R}$

$$\lim_{x \to c} f(x) = c^n = f(c) \Rightarrow f(x) \text{ is continuous}$$

at c.

(c) $f(x) = \frac{1}{x}$ is not continuous at $x = 0$.

(D) Let $A = \mathbb{R}$. $f : A \to \mathbb{R}$ is defined as

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

This is called Dirichlet's funⁿ.

Let c is a rational. Let $\{x_n\}$ be a sequⁿ of irrational then by problem-13 of tutorial-2, $x_n \to c$.

Now we have $f(x_n) = 0 \ \forall n$, $f(c) = 1$

So $f(x_n) \not\to f(c) \Rightarrow f(x)$ not continuous at the reational number c.

$||^{ly}$ let c is irrational. Let $\{y_n\}$ be a sequⁿ of rational then by problem-13 of tutorial-2, $y_n \to c$.

$f(y_n) = 1 \ \forall n \in \mathbb{N}$ & $f(c) = 0$

$\Rightarrow f(y_n) \not\to f(c)$

$\Rightarrow$ $f(x)$ is not continuous at irrational number c.

$\Rightarrow$ $f(x)$ is not continuous at any point of $\mathbb{R}$.

Theorem (1) Let $f: A \to \mathbb{R}$ be a real valued function continuous at c, then $|f(x)|$ and $Kf(x)$, $K \in \mathbb{R}$ are continuous at c.

Theorem: Let $f: A \to \mathbb{R}$ & $g: A \to \mathbb{R}$. Suppose $c \in A$ and f and g are continuous at c then

(1) $f \pm g$, fg are continuous at c

(ii) If $h: A \to \mathbb{R}$ is continuous at $c \in A$ & $h(x) \neq 0$ $\forall x \in A$, then f/h is continuous at c.

Ex (i) $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is continuous on $\mathbb{R}$.

(ii) $\sin x$, $\cos x$ are continuous on $\mathbb{R}$.

(iii) $\tan x$, $\cot x$, $\sec x$ are continuous where they are defined.

Theorem: If f is continuous at c and g is continuous at $f(c)$, then the function $g \circ f$ is continuous at c.

Continuous function on interval

Def^n: A function $f : A \to \mathbb{R}$ is said to be bounded on A if there exists a constant $M > 0$ such that

$$|f(x)| \leq M \quad \forall\, x \in A.$$

Def^n: (i) $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. We say f has an maximum on A if there exists a point $x^* \in A \ni$

$$f(x^*) \geq f(x) \quad \forall\, x \in A.$$

(ii) We say f has an minimum on A if $\exists\, x_* \in A \ni$

$$f(x_*) \leq f(x) \quad \forall\, x \in A.$$

Properties of continuous function

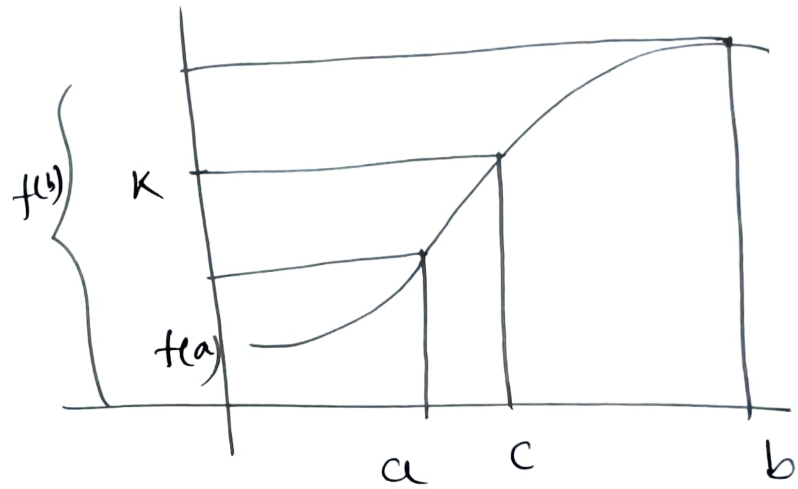
Bounded Ness theorem: Let $I = [a,b]$ be a closed interval and let $f : I \to \mathbb{R}$ be continuous on I, then f is bounded on I.

Maximum - Minimum Theorem: Let f be a continuous on $I = [a,b]$. Then f assumes its maximum an minimum on I i.e. there x^* & x_* in I s.t.

$$f(x_*) \leq f(x) \leq f(x^*) \quad \forall\, x \in I.$$

Bolzano's Intermediate Value theorem: Let J be an interval and let $f: J \to \mathbb{R}$ be continuous on J. If $a, b \in J$ and if $K \in \mathbb{R}$ satisfies $f(a) < K < f(b)$ ~~there~~ then there exist a point $c \in J$ between a, b such that

$f(c) = K$



Ex Show that the equ^n $x^2 = x\sin x + \cos x$ has at least two ~~one~~ real roots

soln: $f(x) = x^2 - x\sin x - \cos x$, $f(x)$ is continuous

and $f(-\pi) = \pi^2 + 1 > 0$, $f(0) = -1 < 0$, $f(\pi) = \pi^2 + 1 > 0$
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Hence by intermediate value theorem the equ^n $f(x) = 0$ has at least one root in $(-\pi, 0)$ and at least one root in $(0, \pi)$. Thus the equ^n $f(x) = 0$ has at least two real roots.

Differentiation

Defn: Let $I \subseteq \mathbb{R}$ be an interval. Let $f: I \to \mathbb{R}$, we say that the function $f(x)$ is differentiable at a point $c \in I$ if the limit $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists and it is denoted by $f'(c)$.

Right hand and left hand derivative

Let I be an interval and $f: I \to \mathbb{R}$. Let $c \in I$

If $\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$ exist is called reight hand derivative and denoted by $Rf'(c)$

$\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c}$ exist is called left hand derivative and denoted by $Lf'(c)$.

We say that the derivative of f at c exists iff $Rf'(c)$ and $Lf'(c)$ exists and both, aree equal.

Theorem: If $f: I \to \mathbb{R}$ has a derivative at a point $c \in I$ then f is continuous at c.

Proof: Let $x \in I$ & $x \neq c$

$$f(x)-f(c) = \frac{f(x)-f(c)}{x-c}(x-c)$$

Since $f'(c)$ exists. so taking limit both side

$$\lim_{x\to c}\left(f(x)-f(c)\right) = f'(c)\cdot 0 = 0$$

$$\Rightarrow \lim_{x\to c} f(x) = f(c) \Rightarrow f(x) \text{ is continuous at } c.$$

Ex: $f(x) = |x|$ is continuous but not differentiable at 0.

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{|x|}{x} = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$$

$\Rightarrow$ $f'(0)$ does not exist. So $f(x)$ not differentiable at 0.

Theorem: Let $f(x)$ and $g(x)$ be functions that are differentiable at the point c. Each of the functions kf (k is a const), $f+g$, fg, f/g ($g(c)\neq 0$) is also differentiable. The formulars are

(I) $kf'(c) = (kf)'(c)$

(II) $\left(f(c)+g(c)\right)' = f'(c)+g'(c)$

(III) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

(III) $\cancel{\frac{f(c)}{g(c)}}\left(\frac{f}{g}\right)'(c) = \dfrac{g(c)f'(c) - g'(c)f(c)}{g^2(c)}$, $g(c)\neq 0$.

Proof: See Ross.

Chain Rule: If f is differentiable at c and g is differentiable at $f(c)$ then the composite function $g \circ f$ is differentiable at c and we have $(g \circ f)'(c)$
$= g'(f(c))\, f'(c)$.

Proof: Ross.

Example: $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is not diff^n at $x = 0$.

Solu^n: $\lim_{x \to 0} \frac{x \sin \frac{1}{x} - 0}{x - 0} = \lim_{x \to 0} \sin \frac{1}{x}$ not exist.

Ex: $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0 & \text{if } x \neq 0 \end{cases}$. Show that $f(x)$ is differentiable but f' is not continuous at 0.

Solu^n. H.W.

Ex: $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. This function is differentiable only at 0.

Solu^n: First we show that $f(x)$ in not continuous when $x \neq 0$.

Let $x_0 (\neq 0) \in \mathbb{Q}$ then there exist $\{x_n\} \in \mathbb{R} \setminus \mathbb{Q}$ s.t.

$x_n \to x_0$. Since $f(x_n) = 0 \quad \forall n \Rightarrow f(x_n) \to 0$

which not equal to $f(x_0) = x_0^2$. So $f(x)$ not

continuous at x_0.

Again $y_0 (\neq 0) \in \mathbb{R} \setminus \mathbb{Q}$ then $\exists \{y_n\} \in \mathbb{Q} \ni$

$y_n \to y_0$. Since $f(y_n) = y_n^2 \longrightarrow y_0^2 \neq 0$.

But $f(y_0) = 0$. So $f(x)$ not continuous at y_0.

$\Rightarrow$ $f(x)$ not continuous at $x \in \mathbb{R} \setminus \{0\}$.

$\Rightarrow$ $f(x)$ not diff^n at $x \in \mathbb{R} \setminus \{0\}$.

Now for a given $\epsilon > 0$ Consider

$$\left| \frac{f(x) - f(0)}{x - 0} \right| \leq \frac{|x^2 - 0|}{|x|} = |x| \qquad \begin{array}{l} \because x \in \mathbb{R} \setminus \mathbb{Q} \\ \quad f(x) = 0 \\ x \in \mathbb{Q} = x^2 \end{array}$$

$$< \epsilon$$

Take $\delta = \epsilon$. Then $|x| < \delta \Rightarrow \left| \frac{f(x) - f(0)}{x - 0} \right| < \epsilon$

$$\Rightarrow \lim_{x \to 0} f(x) = 0.$$

Defn (Local maxima): Let $f: I \to \mathbb{R}$, I be an interval. A point $x_0 \in I$ is a local maximum of f if there a $\delta > 0$ such that $f(x) \leq f(x_0)$ when ever $x \in I \cap (x_0 - \delta, x_0 + \delta)$

Local minima: A point $y_0 \in I$ is local minimum of f if there a $\delta > 0$ such that $f(x) \geq f(y_0)$ when ever $x \in I \cap (y_0 - \delta, y_0 + \delta)$.

Theorem: Suppose $f: [a,b] \to \mathbb{R}$ and suppose f has either a local maximum or a local minimum at x_0 and if f is differentiable at x_0. Then $f'(x_0) = 0$.

Remark: The previous theorem is not valid if x_0 is a arb. for example, if we consider the fun $f: [0,1] \to \mathbb{R}$ such that $f(x) = x$. Then the maximum of f at $x = 1$ but $f'(x) = 1$ $\forall x \in [0,1]$.

The following result is an application of previous Theorem

Theorem (Roll's Theorem): Let f be a continuous function on $[a,b]$, f is differentiable on (a,b) and satisfies $f(a) = f(b)$. Then there exists a point $x \in (a,b)$ such that $f'(c) = 0$.

Proof: Since f is continuous on on $[a,b]$ so f is bounded on $[a,b]$ and also there exists x_0, y_0 in $[a,b]$ s.t

$$f(x_0) \le f(x) \le f(y_0).$$

Now if x_0 and y_0 are both endpoints of $[a,b]$ then f is a constant function ($\because f(a)=f(b)$). $\Rightarrow f'(x)=0$ $\forall x \in (a,b)$. Otherwise f assume either a maximum or a minimum at point x in (a,b), in which case $f'(x)=0$

Example: Let f and g be functions continuous and differentiable on (a,b) and let $f(a)=f(b)=0$. Prove that there is a point $c \in (a,b)$ such that $g'(c)f(c)+f'(c)=0$.

Soln: Define $h(x) = f(x)e^{g(x)}$
then $h(x)$ is continuous on $[a,b]$ & differentiable on (a,b). Also $h(a)=0=h(b)$. So by Roll's theorem there exists $c \in (a,b)$ s.t. $h'(c)=0$
Now

$$h'(x) = [f'(x) + f(x)g'(x)]e^{g(x)}$$

So $h'(c)=0 \Rightarrow f'(c)+f(c)g'(c)=0 \quad [\because e^{g(c)} \neq 0]$

Geometrical Interpretation: