Real Number System

The set $\mathbb{N}$ of Natural Numbers

We denote the set $\{1, 2, 3, \cdots\}$ of all positive integer by $\mathbb{N}$

$$\mathbb{N} = \{1, 2, 3, \cdots\}$$

Each positive integer n has a succesor, namely $(n+1)$.

Peano Axioms :

(N1) 1 belongs to $\mathbb{N}$, i.e. $1 \in \mathbb{N}$.

(N2) If n belongs to $\mathbb{N}$, then its successor $n+1 \in \mathbb{N}$.

(N3) 1 is not the successor of any element in $\mathbb{N}$.

(N4) If n and m in $\mathbb{N}$ have the same successor, then $n = m$.

(N5) A subset of $\mathbb{N}$ which contains 1 and which contains $(n+1)$ whenevere it contains n, must equal $\mathbb{N}$.

Most familier properties of $\mathbb{N}$ can be proved based on these five axioms.

The axiom (N5) is the basis of mathematical induction.

Principle of Mathematical Induction :

Let $P_1, P_2, P_3 \cdots$ be a list of statements or propositions that may or may not be true. The principle of mathematical induction asserts all the statements

$P_1, P_2, P_3, \ldots$ aree true provided

(I) P_1 is true

(II) P_{n+1} is true whenever P_n is true.

We refer the fact that P_1 is true as the basis for induction and will reefer (II) as induction step.

Example: $1+2+\cdots+n = \frac{1}{2}n(n+1)$

Solu^n P_n : $1+2+\cdots+n = \frac{1}{2}n(n+1)$

Thus P_1 asserts : $1 = \frac{1}{2}1(1+1) = 1$, So P_1 is true.

Let P_n is true i.e. we have

P_n: $1+2+\cdots+n = \frac{1}{2}n(n+1)$ true

We wish to prove P_{n+1} is true

$$1+2+\cdots+n+(n+1) = \frac{1}{2}n(n+1)+(n+1)$$

$$= \frac{1}{2}\left[n(n+1)+2(n+1)\right] = \frac{1}{2}(n+1)(n+2)$$

$$= \frac{1}{2}(n+1)\left[(n+1)+1\right]$$

$\Rightarrow$ P_{n+1} holds if P_n holds. By the principle of mathematical induction we conclude P_n is true for all n.

The set $\mathbb{Q}$ of Rational Numbers

We first leare to add and to multiply positive integers. After subtraction is introduced, the need to expand the number system to include 0 and negative integer become apparent. So we get the set of integers

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$$

After introducing division the set $\mathbb{Z}$ also becomes inadequate So the solu^n is to enlarege the world of numbers to includ all fractions. Accordingly we study the space $\mathbb{Q}$ of all rational numbers, i.e. the numbers of the form

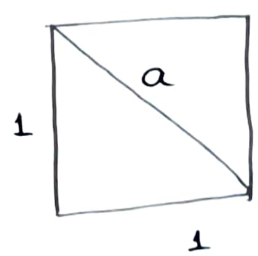
$$\frac{m}{n}, \text{ where } m, n \in \mathbb{Z}, \; n \neq 0.$$

Note: The set $\mathbb{Q}$ contains all terminating decimals.

$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0 \right\}$$

Question: Are there any other number than this set $\mathbb{Q}$

$a = \sqrt{2}$



Is $\sqrt{2}$ rational number?

We have $\mathbb{Q}$ is the set of all fractions. If we try to solve equ^n like $x^2 = 2$. In the previous example we see that there is a number which satisfies this equ^n.

Now we prove that $\sqrt{2}$ is not a rational no.

Proof: Let $\sqrt{2}$ is a rational no. So

$$\sqrt{2} = \frac{m}{n}, \quad m, n \in \mathbb{N}$$

Assume that m and n has no common factor

$\Rightarrow 2n^2 = m^2$

$\Rightarrow$ m is a even number so [$\because$ product of two odd natural number is odd]

$m = 2k$ [$\because$ m^2 is even so m is even]

$\Rightarrow 2n^2 = (2k)^2$

$\Rightarrow 2n^2 = 4k^2 \Rightarrow n^2 = 2k^2$

$\Rightarrow$ n is even $\Rightarrow n = 2p$

So 2 is a common factor of m and n which is a contradiction

$\Rightarrow \sqrt{2}$ is not rational numbers.

Next we return to the equ^n $x^2 \neq 2$ $\Rightarrow$ ~~$x^2 - 2 = 0$~~. It is seen that this equ^n has a soln.

~~EX~~ (1) $\sqrt{17}$, $\sqrt[3]{6}$, $\sqrt{2+\sqrt[3]{5}}$ ~~are not rational~~

So from the above discussion we have number which are not rational. These numbers are called irrational number.

The basic algebric operations in $\mathbb{Q}$ are addition and multiflication. Given a pair a, b of rational nos the sum $(a+b)$ and ab also represent rational no. Moreover, the following properties hold

(A1). $a+(b+c) = (a+b)+c \quad \forall\ a, b, c$ (Associative)

(A2) $a+b = b+a \quad \forall\ a, b$ (commutative)

(A3) $a+0 = a \quad \forall\ a$ [additive identity]

(A4) For each a there is an element $-a$ such that

$a+(-a) = 0$ [additive inverse]

(M1) $a(bc) = (ab)c \quad \forall\ a, b, c$ (Associative)

(M2) $ab = ba \quad \forall\ a, b$. (commutative)

(M3) $a \cdot 1 = a \quad \forall\ a$ (Identity)

(M4) For each $a \neq 0$ there an element $a^{-1} \ni aa^{-1} = 1$ (Inverse

(DL) $a(b+c) = ab + ac \quad \forall\ a, b, c$. (Distributive)

A system that is more than one element satifies these nine properties is called field.

The set $\mathbb{Q}$ also has an order structure $\leq$ satisfying

(O1) Given a & b either $a \leq b$ or $b \leq a$

(O2) If $a \leq b$ & $b \leq a$, then $a = b$

(O3) If $a \leq b$ & $b \leq c$ then $a \leq c$ (Transitive law)

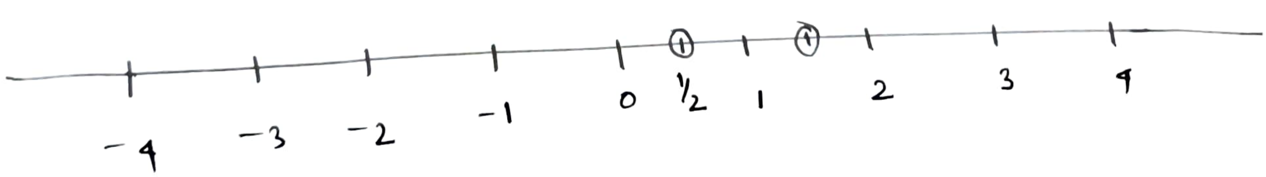
(O4) If $a \leq b$ then $a + c \leq b + c$

(O5) If $a \leq b$ & $0 \leq c$ then $ac \leq bc$.

Defn: A field with an ordering satisfying properties (O1) to (O5) is called ordered field.

The set $\mathbb{R}$ of Real Numbers.

Now to define the set of real number we have to know what exactly a real no is



$\mathbb{R}$ = Rational numbers and irrational numbers.

Real numbers, i.e. elements of $\mathbb{R}$ can be added together and multiplied together

$a, b \in \mathbb{R}$, $a+b \in \mathbb{R}$, $ab \in \mathbb{R}$.

Moreover these operations satisfies the field properties (A1) – (A4), (M1) – (M4) & (DL). The set $\mathbb{R}$ also has an ordered an order structure $\leq$ satisfies (O1) – (O5).

So $\mathbb{R}$ is an ordered field.

We will state some results for $\mathbb{R}$ that are valid in any ordered field. In particular these results would be equally valid if we restricted our attention to $\mathbb{Q}$.

<u>Theorem</u>: The following are consequences of the field properties.

(i) $a + c = b + c \Rightarrow a = b$ (ii) $a \cdot 0 = 0 \ \forall a$

(iii) $(-a)b = -(ab) \ \forall a, b$ (iv) $(-a)(-b) = ab \ \forall a, b$.

(v) $ac = bc,\ c \neq 0 \Rightarrow a = b$ (iv) $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$;

for $a, b, c \in \mathbb{R}$.

<u>Theorem</u>: The following are consequences of the properties of an ordered field.

(i) If $a \leq b$, then $-b \leq -a$ (ii) If $a \leq b$, $c \leq 0$, then $bc \leq ac$

(iii) $0 \leq a$ and $0 \leq b$ then $0 \leq ab$ (iv) $0 \leq a^2 \ \forall a$ (v) $0 < 1$

(vi) If $0 < a$ then $0 < a^{-1}$ (vii) $0 < a < b$ then $0 < b^{-1} < a^{-1}$

for $a, b, c \in \mathbb{R}$.

Note: $a < b$ means $a \leq b$ & $a \neq b$.

Intervals: Consider real numbers a and b where $a < b$.

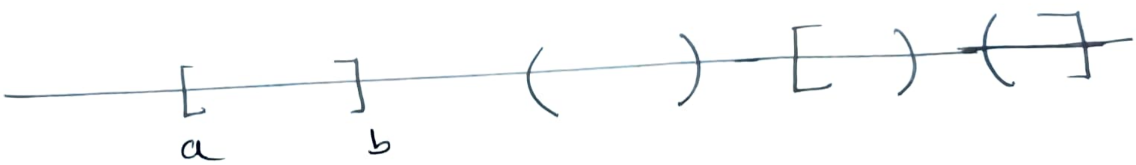
$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $(a,b) = \{x \in \mathbb{R} : a < x < b\}$

$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$, $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$

$[a,b] \rightarrow$ closed intervals

$(a,b) \rightarrow$ open intervals

Other two are called half-open or semi closed intervals



The completeness Axiom

Def^n: Let $S \subset \mathbb{R}$ and S is nonempty

(I) If S contains a largest element s_0, i.e. $s_0 \in S$ and $s \leq s_0 \; \forall s \in S$, then we call s_0 the maximum of S, and write $s_0 = \max S$.

(II) If S contains a smallest element, we call the smallest element the minimum of S and write $\min S$.

Example: (1) Every finite nonempty subset of $\mathbb{R}$ has a maximum and a minimum.

(ii) $\max[a,b] = b$, $\min[a,b] = a$. The set (a,b) has no maximum or minimum since a, b not belongs to (a,b). The set $[a,b)$ has minimum that is a but no maximum.

(iii) The set $\mathbb{Z}$ and $\mathbb{Q}$ have no maximum or minimum. The set $\mathbb{N}$ has no ~~minimum~~ maximum but $\min \mathbb{N} = 1$.

Defn: Let $S \subset \mathbb{R}$ and $S \neq \phi$

(a) If a real number M satisfies $s \leq M \ \forall s \in S$ then M is called an upper bound of S and the set S is said to be bounded above.

(b) If a real number m satifies $m \leq s \ \forall s \in S$ then m is called lower bound of S and the set S is said to be bounded below.

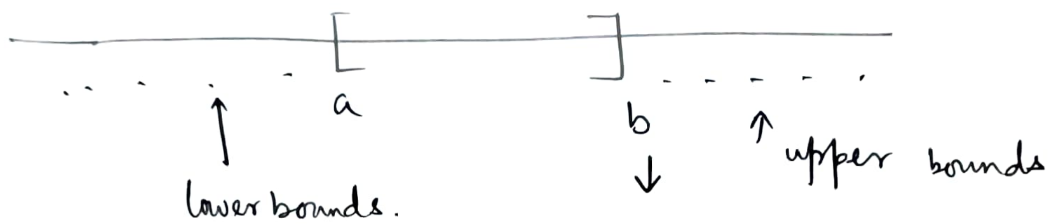
(c) The set S is said to be bounded if it is bounded above and bounded below. Thus S is bdd if there exists real numbers m and $M \ni$

$$m \leq s \leq M \quad \forall s \in S \quad \text{i.e.} \quad S \subseteq [m, M]$$

(d) Unbounded if it is not bdd.

Note: (I) Any real number which is larger than M is also an upper bound of S

(II) Any real number which is smaller than m is a lower bound of S.



Defn: Let S be a non empty subset of $\mathbb{R}$.
We say that S has a least upper bound in $\mathbb{R}$ if $\exists\ M \in \mathbb{R} \ni$

(I) M is an upper bound of S i.e. $s \leq M\ \forall\ s \in S$

(II) Nothing smaller than M is an upper bound for S, that is if $M' < M \Rightarrow \exists\ s \in S \ni M' < s$.

If such M exists it is called least upper bound, or lub or supremum of A.

$$M = \text{lub } A \text{ or } \sup A.$$

Defn: Let $S \subset \mathbb{R}$ and $S \neq \phi$. We say that S has a greatest lower bound in $\mathbb{R}$ if there exists an element $m \in \mathbb{R}$ such that

ⓐ m is a lower bound for S, i.e. $m \le s \ \forall s \in S$.

ⓑ If m' is ~~a~~ such that $m < m'$ then $\exists\, s \in S$ $\ni s < m'$.

In this $m = \text{glb } S$ or $m = \inf S$.

<u>Example</u> (i) If a set S has a maximum then $\max S = \sup S$

$\|^{ly}$ $\min S = \inf S$.

(ii) If $a, b \in \mathbb{R}$, $a < b$ then

$$\sup[a,b] = \sup(a,b) = \sup[a,b) = \sup(a,b] = b.$$

(iii) If $A = \{r \in \mathbb{Q} : 0 \le r \le \sqrt{2}\}$, then $\sup A = \sqrt{2}$ and $\inf A = 0$.

(iv) $D = \{x \in \mathbb{R} : x^2 < 10\}$ is the open interval $(-\sqrt{10}, \sqrt{10})$. Thus it is bounded above and below but it has no maximum and minimum.

$\inf D = -\sqrt{10}$, $\sup D = \sqrt{10}$.

Define sets

$$A = \{r \in \mathbb{Q} : r^2 < 2\} \quad B = \{r \in \mathbb{Q} : r^2 > 2\}$$

i) Does $\exists$ a largest element of A in $\mathbb{Q}$

ii) Does $\exists$ a smallest element of B in $\mathbb{Q}$.

It can be shown that the A and B has no largest and smallest element respectively

This shows that the set of real number has got certain gaps, if we dont assume irrational number to fill these gaps.

Completeness Axiom:

Every nonempty subset S of $\mathbb{R}$ that is bdd above has a least upper bound. In other words, $\sup S$ exists and is a real number.

Note: So $\mathbb{R}$ is a complete ordered field. For the present course $\mathbb{R}$ is the only one complete ordered field.

Remark: (1) In other words $\mathbb{R}$ is a set with two operations (addition and multiplication) satisfying field axiom (A1) – (A4), (M1) – (M4) and DL, the order axioms (O1) - (O5) and the completeness axioms.

Ex: The completeness axiom does not hold for $\mathbb{Q}$.

Theorem: Let A be a nonempty sub set of $\mathbb{R}$. Write

$$-A = \{-x : x \in A\}$$

for the set of negative of the elements of A.

Let $c \in \mathbb{R}$, then

(i) c is an upper bound for $A \Leftrightarrow -c$ is a lower bound for $-A$.

(ii) c is a lower bound for $A \Leftrightarrow -c$ is an upper bound for $-A$

(iii) If A has a least upper bound, then $-A$ has a greatest lower bound and ~~$\sup(-A) = -\inf(A)$~~ $\inf(-A) = -\sup(A)$

(iv) If A has a glb then $-A$ has a lub and $\sup(-A) = -\inf(A)$.

Equivalent Def^n of lub and glb

LUB: Let $S \subseteq \mathbb{R}$ be any set and $M \in \mathbb{R}$. We say that M is supremum or lub of S if

(1) $x \le M \quad \forall x \in S$

(11) For all $\epsilon > 0 \ \exists x \in S, \ x > M - \epsilon$

GLB: Let $S \subseteq \mathbb{R}$ and $m \in \mathbb{R}$. We say m is glb of S if

(i) $m \le x \quad \forall x \in S$

(ii) $\forall \epsilon > 0 \ \exists x \in S \quad x < m + \epsilon$.

Applications of LUB axioms:

Result-1: If A is nonempty subset of $\mathbb{R}$ that is bounded below, then A has a greatest lower bound, namely

$$\inf A = -\sup(-A).$$

Proof: The set A is bdd below then the set

$$-A = \{-a : a \in A\}$$ is nonempty and bdd above

So by lub axiom it has a least upper bound. Then

$-(-A)$ has a greatest lower bound [By Theorem-(II) Page-13]

$\Rightarrow A$ has a greatest lower bound.

So $\inf(A) = \inf(-(-A)) = -\sup(-A)$.

Result-2: The set of natural number $\mathbb{N}$ has no upper bound.

Proof: We will prove by contradiction. Let if possible $\mathbb{N}$ be bdd above.

Now $\mathbb{N} \neq \phi \because 1 \in \mathbb{N}$.

By completeness axiom $\mathbb{N}$ has supremum $L \in \mathbb{R}$ (say) $\Rightarrow \exists\, n \in \mathbb{N},\ n > L-1$. $(\epsilon = 1)$

Because from def^n of supremum we have $\forall\ \epsilon > 0\ \exists\ n \in \mathbb{N}$

$n > L - \epsilon$

Here we have taken $\epsilon = 1$.

Now $n \in \mathbb{N}$ then $n+1 \in \mathbb{N}$

So we get $n+1 > L$ which is contradiction to L is upper bound

This contradiction proves that $\mathbb{N}$ must be unbounded above.

Archimedean Property: For each pair of elements $x, y \in \mathbb{R}$ with $x > 0$, there exists a positive integer n such that $nx > y$, i.e. $\forall\ x > 0\ \forall\ y \in \mathbb{R}\ \exists\ n \in \mathbb{N},\ nx > y$.

Proof: Let $x, y \in \mathbb{R}$. If $y \leq 0$ then $1 \cdot x > y$, $n = 1$.

We assume $y > 0$, we want to show

$$\forall\ x > 0,\ \forall\ y > 0\ \exists\ n \in \mathbb{N}\ \ nx > y.$$

i.e

$$\forall\ x > 0,\ \forall\ y > 0\ \exists\ n \in \mathbb{N}.\ n > y/x.$$

We will prove it by contradiction. Let if possible this statement be false then $\exists\ x > 0\ \exists\ y > 0$ s.t.

$$\forall\ n \in \mathbb{N},\ \ n \leq \frac{y}{x}$$

$\Rightarrow \mathbb{N}$ is bounded by y/x which is contradiction.

&

So our assumption is wrong. This shows that

$$\forall\, x > 0,\ \forall\, y > 0\ \exists\, n \in \mathbb{N},\ nx > y.$$

Result: For each real number x there exists a unique integer n such that $n \leq x < n+1$.

Density of the Rationals

Result: If x and y are real number such that $x < y$ then there exists a rational number r such that

$x < r < y$.

Absolute value: The absolute value of a number $a \in \mathbb{R}$ is the number

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0. \end{cases}$$

Properties: (i) $|a| \geq 0 \ \forall\, a \in \mathbb{R}$

(ii) $|ab| = |a|\,|b|$

(iii) $|a+b| \leq |a| + |b|$.

Theorem. Let $x, y, a \in \mathbb{R}$

(i) $x < y + \epsilon \ \forall\, \epsilon > 0 \iff x \leq y$

(ii) $x > y - \epsilon \ \forall\, \epsilon > 0 \iff x \geq y$

(iii) $|a| < \epsilon \ \forall\, \epsilon > 0 \iff a = 0$.