Tutorial-2 Solun

(1) We will show by induction that $\forall n \in \mathbb{N}$, $x_{n+1} > x_n$

For $n=1$, $x_2 = \sqrt{x_1 + 6} = \sqrt{\sqrt{6} + 6} > \sqrt{6} = x_1$

Assume the result is true for $n = 1, 2, \ldots m-1$. We will show $x_{m+1} > x_m$

i.e. $\sqrt{x_m + 6} > \sqrt{x_{m-1} + 6}$

i.e. $x_m + 6 > x_{m-1} + 6$ i.e. $x_m > x_{m-1}$

Which is true by induction hypothesis

So the result is proved by induction

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(2) Given $0 < x_1 < 1$, Now we will prove $0 < x_n < 1$ $\forall n \in \mathbb{N}$

For $n=1$ given $0 < x_1 < 1$. Let the result is true for k i.e. $0 < x_k < 1$.

$x_{k+1} = x_k(2 - x_k) > 0$ $\because$ $0 < x_k < 1$

Again $1 - x_k(2 - x_k) = 1 - 2x_k + x_k^2 = (1 - x_k)^2 > 0$

$\Rightarrow$ $x_k(2 - x_k) < 1$

$\Rightarrow$ $0 < x_{k+1} < 1$

So by induction $0 < x_n < 1$ $\forall n \in \mathbb{N} \Rightarrow \{x_n\}$ is bdd.

consider $x_{n+1} - x_n = x_n(2-x_n) - x_n$

$= x_n(1-x_n) > 0 \quad \because \quad 0 < x_n < 1$

$\Rightarrow \quad x_{n+1} > x_n \quad \Rightarrow \quad \{x_n\}$ is increasing.

$\Rightarrow \{x_n\}$ bdd and increasing $\Rightarrow \{x_n\}$ convergent.

Let $\lim\limits_{n\to\infty} x_n = l \in \mathbb{R}$.

Then $\lim x_{n+1} = \lim x_n(2-x_n)$

$\Rightarrow \quad l = l(2-l) \Rightarrow l = 0$ or 1.

$\because \quad x_1 > 0$ and $\{x_n\}$ increasing so $l \neq 0$,

Here $l = 1$.

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(3) $\alpha = \dfrac{1+\sqrt{29}}{2}$ be the positive root of the eqn

$x^2 - x - 7 = 0$. We show that $x_n \le \alpha \quad \forall n \in \mathbb{N}$

$n=1$: $x_1 = \sqrt{7}$. We will show $x_1 = \sqrt{7} \le \alpha$

i.e. $2\sqrt{7} - 1 \le \sqrt{29}$ i.e. $\sqrt{28} - 1 \le \sqrt{29}$

which is obvious.

Let $x_k \le \alpha$ we show that $x_{k+1} \le \alpha$

i.e. $\sqrt{7 + x_k} \le \alpha$ i.e. $7 + x_k \le \alpha^2 = \alpha + 7$

Which is true since ~~$x_n \leq \alpha$~~ $x_K \leq \alpha$.

$\Rightarrow$ $x_{K+1} \leq \alpha$. So by induction $x_n \leq \alpha$ $\forall$ $n \in \mathbb{N}$

Now we show $x_{n+1} > x_n$ $\forall$ $n \in \mathbb{N}$.

$n=1$: $x_2 = \sqrt{7+x_1} = \sqrt{7+\sqrt{7}} \geq \sqrt{7} = x_1$ Since $\sqrt{7} \geq 0$

Let the result is assume for $K \geq 1$. We will show for $K+1$

$$x_{K+2} - x_{K+1} = \sqrt{7+x_{K+1}} - \sqrt{7+x_K} \geq 0$$

$\because$ ~~$\sqrt{\ }$~~ $x_{K+1} \geq x_K$

$\Rightarrow$ $x_{K+2} \geq x_{K+1}$. So by induction $x_{n+1} \geq x_n$ $\forall$ $n \in \mathbb{N}$.

Hence $\{x_n\}$ is bdd above and increasing $\Rightarrow$ Convergent. Let $\lim x_n = l$.

$\Rightarrow$ $l = \sqrt{7+l}$ $\Rightarrow$ $l^2 - l - 7 = 0$

Since $\forall$ n $x_n \geq 0$. hence $l \geq 0$.

$\Rightarrow$ l must be the (+) ve root of $x^2 - x - 7 = 0$

(4) Given $\lim x_{2n} = l$ & $\lim x_{2n-1} = l$

So for $\epsilon > 0$ $\exists$ $N_1, N_2 \in \mathbb{N}$ s.t. $\forall$ $n \geq N_1$

$$|x_{2n} - l| < \epsilon$$

& $\forall$ $n \geq N_2$, $|x_{2n-1} - l| < \epsilon$

Let $N = \max\{N_1, N_2\}$. Then $\forall$ $n \geq N$ we have

$$l - \epsilon < x_{2n} < l + \epsilon \quad \& \quad l - \epsilon < x_{2n-1} < l + \epsilon$$

$\Rightarrow$ $\forall$ $n \geq 2N-1$ $\quad l - \epsilon < x_n < l + \epsilon$

$\Rightarrow$ $\lim x_n = l$

(5) Let $\{x_n\}$ be a monotone increasing sequ^n of real numbers

Then $m > n$ we have $x_m > x_n$.

Let $\{x_{n_k}\}$ be subsequence of $\{x_n\}$. Then $\{n_k\}$ is a strictly increasing sequ^n of natural number

Thus $\quad n_{k+1} > n_k \Rightarrow x_{n_{k+1}} > x_{n_k} \quad \forall$ k

$\Rightarrow$ $\{x_{n_k}\}$ is a monotone increasing

$||^{ly}$ for decreasing.

(6) Let $\{x_n\}$ be a monotone increasing and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = l$.

Since $\{x_n\}$ is monotone increasing so $\{x_{n_k}\}$.

Since $\{x_{n_k}\}$ convergent so bdd above.

We claim $\{x_n\}$ is bdd above. if not

let $\{x_n\}$ is not bdd above then for $M>0$ $\exists$ $N \in \mathbb{N}$ $\ni$ $x_n > M$ $\forall$ $n \geq N$.

Since $\{n_k\}$ is strictly increasing sequence of natural number $\exists$ $K_0 \in \mathbb{N}$ such that $\forall$ $K > K_0$

$n_k > N$ $\Rightarrow$ $x_{n_k} > M$ $\forall$ $K > K_0$.

Since M is arbitrary so $x_{n_k} \to \infty$ $\Rightarrow \Leftarrow$.

So $\{x_n\}$ is bdd $\Rightarrow$ $\{x_n\}$ converges to l.

(7) Let $x_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$

We want to show $\{x_n\}$ is a cauchy sequⁿ i.e.

$\forall$ $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ $\forall$ $m, n \geq N$ $|x_n - x_m| < \epsilon$.

for any $n > m$

$$|x_n - x_m| = \frac{1}{(m+1)!} + \cdots + \frac{1}{n!}$$

Now we know $k! = 2 \times 3 \times \cdots (k-1) \times k$

$$k! = 2 \times 3 \times \cdots (k-1) \times k > \underbrace{2 \cdot 2 \cdots 2}_{k-1}$$

$$\Rightarrow k! > 2^{k-1} \Rightarrow \frac{1}{k!} < \frac{1}{2^{k-1}}$$

So $$|x_n - x_m| = \frac{1}{(m+1)!} + \cdots + \frac{1}{n!}$$

$$\leq \frac{1}{2^{m}} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^m}\left[1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m-1}}\right] = \frac{1}{2^m} \frac{1-\left(\frac{1}{2}\right)^{n-m}}{1-\frac{1}{2}}$$

$$= \frac{1}{2^{m-1}}\left[1-\left(\frac{1}{2}\right)^{n-m}\right]$$

$$\leq \frac{1}{2^{m-1}}$$

Now if we ~~che~~ show $\frac{1}{2^{m-1}} < \epsilon$ Then proved

$\because \frac{1}{2^n} \to 0$ as $n \to \infty$ then $\exists\, N \in \mathbb{N} \ni$

$$\frac{1}{2^{N-1}} < \epsilon.$$

Now choose this N and take $m, n \geq N$

then we have $\frac{1}{2^{m-1}} \leq \frac{1}{2^{N-1}}$

So $|x_n - x_m| \leq \frac{1}{2^{m-1}} \leq \frac{1}{2^{N-1}} < \epsilon$ [Here $n > m$ with out loss of generealits]

(8) The negation of the defn of cauchy sequn is

$\exists\ \epsilon > 0 \quad \forall\ N \in \mathbb{N} \quad \exists\ m, n \geq N, \ |x_m - x_n| \geq \epsilon.$

Now take $N \in \mathbb{N}$. choose $m = N$, $n = 2N$

$$|x_n - x_m| = \frac{1}{N+1} + \cdots + \frac{1}{2N}$$

$$\geq \frac{1}{2N} + \cdots + \frac{1}{2N} = 1/2 \quad \left[\because \text{each term} > \frac{1}{2N}\right.$$

Take $\epsilon = 1/2$

Then for this $\epsilon > 0$ $\forall\ N \in \mathbb{N}$ we take $m = N$, $n = 2N$

get $|x_n - x_m| \geq \epsilon \Rightarrow \{x_n\}$ is not cauchy.

(9) (a) Given that $|x_{n+1} - x_n| \leq \alpha^n \quad \forall\ \alpha \in (0,1)$

Now for all $m, n \in \mathbb{N}$ with $m > n$ we have

$$|x_m - x_n| = |x_n - x_{n+1} + x_{n+1} - \cdots - x_{m-1} + x_{m-1} - x_m|$$

$$\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m|$$

$$\leq \alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1} = \alpha^n \left[1 + \alpha + \alpha^2 + \cdots + \alpha^{m-n-1}\right]$$

$$= \alpha^n \frac{1 - \alpha^{m-n}}{1 - \alpha} = \frac{\alpha^n}{1-\alpha}\left(1 - \alpha^{m-n}\right)$$

$$< \frac{\alpha^n}{1-\alpha} \quad [\because 0 < \alpha < 1].$$

Since $\alpha \in (0,1)$ so $\alpha^n \to 0$ as $n \to \infty$. $\forall\ \epsilon > 0\ \exists\ N \in \mathbb{N} \ni$

$$\frac{\alpha^n}{1-\alpha} < \epsilon \quad \forall\ n \geq N.$$

Hence for all $m, n \geq N$ we have

$$|x_m - x_n| < \frac{\alpha^n}{1-\alpha} < \epsilon \Rightarrow \{x_n\} \text{ is a Cauchy seq}^n.$$

(6) Let $0 < \alpha < 1$ and $\{x_n\}$ be a seq^n satisfy

$$|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|$$

$$\leq \alpha \alpha |x_n - x_{n-1}|$$

$$\leq \alpha^3 |x_{n-1} - x_{n-2}| \cdots \leq \alpha^n |x_2 - x_1|$$

For all $m, n \in \mathbb{N}$ with $m > n$ we have

$$|x_m - x_n| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m|$$

$$\leq \alpha^{n-1}|x_2 - x_1| + \alpha^n |x_2 - x_1| + \alpha^{n+1}|x_2 - x_1| + \cdots + \alpha^{m-2}|x_2 - x_1|$$

$$= \alpha^{n-1}(1 + \alpha + \alpha^2 + \cdots + \alpha^{m-n-1}) |x_2 - x_1|$$

$$= \alpha^{n-1} \frac{1 - \alpha^{m-n}}{1-\alpha} |x_2 - x_1|$$

$$= \frac{\alpha^{n-1}}{1-\alpha}(1 - \alpha^{m-n}) |x_2 - x_1| < \frac{\alpha^{n-1}}{1-\alpha} |x_2 - x_1| \quad \because 0<\alpha<1$$

$\because \alpha \in (0,1)$ so $\alpha^{n-1} \to 0$ as $n \to \infty$. So $\forall \epsilon > 0$

$\exists N \in \mathbb{N} \ni \forall n \geq N$, $\frac{\alpha^{n-1}}{1-\alpha} |x_2 - x_1| < \epsilon$

~~So $\forall \epsilon > 0$ ...~~ so $\forall \epsilon > 0 \ \exists N \in \mathbb{N}$, $\forall m, n \geq N$

$$|x_m - x_n| < \frac{\alpha^{n-1}}{1-\alpha} |x_2 - x_1| < \epsilon$$

$\Rightarrow \{x_n\}$ is Cauchy.

(10) Given $x_1 = 1$, $x_{n+1} = \dfrac{1}{x_n + 2}$ $\forall\, n \in \mathbb{N}$. For all $n \in \mathbb{N}$

~~consider~~ we have

$$|x_{n+2} - x_{n+1}| = \left| \frac{1}{x_{n+1}+2} - \frac{1}{x_n+2} \right| = \frac{|x_{n+1} - x_n|}{|x_{n+1}+2|\,|x_n+2|}$$

Now $x_1 = 1 > 0$. Let $x_k > 0$. $x_{k+1} = \dfrac{1}{x_k+2} > 0$

So $x_n > 0$ $\forall\, n \in \mathbb{N}$ (By induction).

$|x_{n+1}+2| = x_{n+1}+2 > 2$ $\because$ $x_{n+1} > 0$

$|x_n+2| = x_n + 2 > 2$

$\Rightarrow \dfrac{1}{|x_{n+1}+2|} \cdot \dfrac{1}{|x_n+2|} < 1/4$.

Hence $|x_{n+2} - x_{n+1}| < \frac{1}{4}|x_{n+1} - x_n|$ $\Rightarrow \{x_n\}$ is Cauchy since $0 < \frac{1}{4} < 1$

Since the sequence convergent so $\lim x_n = l > 0$

$\because$ $x_n > 0$ $\forall n$.

Hence $\lim x_{n+1} = \lim \dfrac{1}{x_n+2} \Rightarrow l = \dfrac{1}{l+2} \Rightarrow l^2 + 2l - 1 = 0$

$l = -1 \pm \sqrt{2}$ $\because$ $l > 0$. so $l = (\sqrt{2} - 1)$.

(11) $\{1, \frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, 1, \ldots\}$

$x_{2n-1} = 1 \to 1$, $x_{2n} = \dfrac{n}{n+1} = \dfrac{n+1-1}{n+1} = 1 - \dfrac{1}{n+1} \to 1$

$\Rightarrow$ ~~$\{x_n\} \to 1$~~ $x_n \to 1$.

(12)

(i) $\dfrac{2n^2 - 3n}{3n^2+5n+3} = \dfrac{2 - \frac{3}{n}}{3 + 5/n + 3/n^2} \rightarrow \dfrac{2}{3}$ [By limit Theorem

(ii) $\sqrt{n+1} - \sqrt{n} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} = \dfrac{\frac{1}{\sqrt{n}}}{\sqrt{1+\frac{1}{n}} + 1} \rightarrow 0$

As $\frac{1}{\sqrt{n}} \rightarrow 0$, $\sqrt{1+\frac{1}{n}} + 1 \rightarrow 1$.

(iii) ~~Case~~ Here as $n \to \infty$, $n^3+1 \to \infty$. Let $M > 1$

$n^3 + 1 > M \Rightarrow n > (M-1)^{1/3}$, $N = \lceil (M-1)^{1/3} \rceil$

then $x_n > M \;\forall\; n \geq N \Rightarrow x_n \to \infty$.

(iv) $x_n = (2^n + 3^n)^{1/n}$

$3^n < 2^n + 3^n < 3^n + 3^n$

$\Rightarrow 3^n < 2^n + 3^n < 2\, 3^n$

$\Rightarrow 3 < (2^n + 3^n)^{1/n} < 2^{1/n}\, 3$.

Now $2^{1/n} \to 1$ as $n \to \infty$. So By Sandwich theorem $x_n \to 3$ as $n \to \infty$.

(v) Similar.

(vi) Apply Sandwich theorem

Hint $0 < x_n < \frac{n}{(n+1)^2} \quad \forall n \in \mathbb{N}$

$$\frac{n}{(n+1)^2} = \frac{\frac{1}{n}}{\left(1+\frac{1}{n}\right)^2} \to 0 \quad \text{as } n \to \infty.$$

So $x_n \to 0$

(vii)
$$x_n = \frac{3n^2 + \sin n - 4}{2n^2 - 3} = \frac{3 + \frac{\sin n}{n^2} - \frac{4}{n^2}}{2 - 3/n^2}$$

as $n \to \infty$, $\frac{\sin n}{n^2} \to 0$, $\frac{4}{n^2} \to 0$, $\frac{3}{n^2} \to 0$

So $x_n \to \frac{3}{2}$ as $n \to \infty$.

(13) Let $l \in \mathbb{R}$. Consider $u_n = l - \frac{1}{n}$, $v_n = l + \frac{1}{n}$

$u_n \to l$, $v_n \to l$

By density of rationals we have, $x_n \in \mathbb{Q} \ni$

$$l - \frac{1}{n} < x_n < l + \frac{1}{n} \quad \forall n \in \mathbb{N}$$

Now by Sandwich theorem $x_n \to l$.

2nd part Similar.

(14) (i) $x_n = \dfrac{3^n}{3^n+1}$, $x_{n+1} = \dfrac{3^{n+1}}{3^{n+1}+1}$

$$\frac{x_{n+1}}{x_n} = \frac{3^{n+1}}{3^{n+1}+1} \times \frac{3^n+1}{3^n} = \frac{3^{n+1}+3}{3^{n+1}+1} > 1$$

$x_{n+1} > x_n \Rightarrow \{x_n\}$ increasing.

(ii), (iii), (iv) similar

(v) $\cos \pi/3 = 1/2$, $\cos \frac{3\pi}{3} = -1$, $\cos \frac{6\pi}{3} = 1$

So neither increasing nor decreasing.

(15)(i) $x_n = (-1)^n \dfrac{n+1}{n} = (-1)^n\left(1+\frac{1}{n}\right)$

$x_{2n} = \left(1+\frac{1}{2n}\right) \to 1$

$x_{2n-1} = -\left(1+\frac{1}{2n-1}\right) \to -1$

$\limsup x_n = 1$
$\liminf x_n = -1$

(ii), (iii), (iv) Try yourself.

(v)
$$x_n = \begin{cases} (-1)^{n/2} \dfrac{n}{n+1}, & n \text{ is even} \\ \dfrac{n^2-1}{2n^2+1}, & n \text{ is odd} \end{cases}$$

$x_{4n} = \dfrac{4n}{4n+1} \to 1$, $x_{4n-1} = \dfrac{(4n)^2-1}{2(4n)^2+1} \to \dfrac{1}{2}$

$x_{4n-2} = -\dfrac{4n-2}{(4n-2)+1} \to -1$, $x_{4n-3} \to \dfrac{1}{2}$

So $\limsup x_n = 1$
$\liminf = -1$