Equally like Probability model for finite sample space.

Suppose that the sample space

$$\Omega = \{\omega_1, \omega_2, \dots, \omega_k\}$$

is finite (has k elements). Here $\{\omega_i\}$ are called elementary events and $\Omega = \bigcup_{i=1}^{k} \{\omega_i\}$

Suppose that

$$P(\{\omega_i\}) = \frac{1}{k}, \quad i = 1, 2, \dots k$$

(Each elementary event equally likely)

For any event $E \subseteq \Omega$, we have

$$E = \{\omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_r}\}, \text{ for some}$$

$i_1, i_2, \dots, i_r \in \{1, 2, \dots, k\}$, $1 \le r \le k$.

Then $E = \bigcup_{j=1}^{r} \{\omega_{i_j}\}$

$$P(E) = P\left(\bigcup_{j=1}^{r} \{\omega_{i_j}\}\right) = \sum_{j=1}^{r} P(\{\omega_{i_j}\}) = \sum_{j=1}^{r} \frac{1}{k} = \frac{r}{k}$$

$$= \frac{\#\text{ number of element in the event } E}{\text{total number of elements in } \Omega}.$$

"At random": In a random expt with finite sample space Ω, whenever we say that the experiment has been performed at random it means that all the outcomes in the sample space are equally likely.

Example :

Five cards are drawn at random and without replacement from ~~the~~ a deck of 52 cards. Find the prob that

(I) each card is spade (event E_1)

(II) at least one card is spade (Event E_2)

(III) Exactly three cards are king and two cards are queen (Event E_3)

(iv) Exactly two kings, two queens and one jack are drawn.

Soln

(I) $$P(E_1) = \frac{\binom{13}{5}}{\binom{52}{5}}$$

(ii) $$P(E_2) = 1 - P(E_2^c)$$

$$= 1 - P(\text{no card space}) = 1 - \frac{\binom{39}{5}}{\binom{52}{5}}$$

(iii) $$P(E_3) = \frac{\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

(iv) $$P(E_4) = \frac{\binom{4}{2}\binom{4}{2}\binom{4}{1}}{\binom{52}{5}}$$

Conditional Probability: consider a prob space $(\Omega, \mathcal{E}, P)$

Where $\Omega = \{w_1, w_2, \ldots, w_n\}$ is finite and

$$P(\{w_i\}) = \frac{1}{n}, \quad i = 1, 2, \ldots n \quad \text{(equally likeli prob. model)}$$

Then for any $A \in \mathcal{E}$.

$$P(A) = \frac{\#\text{ of cases favourable to C}}{\text{total }\#\text{ of cases}}$$

$$= \frac{|A|}{|\Omega|} = \frac{|A|}{n}.$$

Now suppose it is known a priori that event A has occured (i.e. outcome of the experiment is an element of A), where $|A| \geq 1$ so $P(A) = \frac{|A|}{n} > 0$.

Given this prior information that the event A has occured we want to define prob fun say $P(B|A)$ on the event space $\mathcal{E}$. A natural way to define is

$$P(B|A) = \frac{|A \cap B|}{|A|} = \frac{|A \cap B|/n}{|A|/n} = \frac{P(A \cap B)}{P(A)}$$

$B \in \mathcal{E}$.

Def^n: $(\Omega, \mathcal{S}, P)$ be a probability space and let $A \in \mathcal{S}$ be such that $P(A) > 0$. Then

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad B \in \mathcal{S}.$$

is called the conditional prob. of event B given the event A.

Theorem: Let $(\Omega, \mathcal{S}, P)$ be a probability space and let $A \in \mathcal{S}$ with $P(A) > 0$ be fixed. Then $P(\cdot|A) : \mathcal{S} \to \mathbb{R}$ is a prob. function (called the conditional prob. function) on $\mathcal{S}$.

Proof: clearly (i) $P(B|A) = \dfrac{P(A \cap B)}{P(A)} \geq 0 \quad \forall B \in \mathcal{S}$

(ii) $P(\Omega|A) = \dfrac{P(A)}{P(A)} = 1.$

Let $\{B_n\}_{n \geq 1}$ be a sequence of disjoint events in $\mathcal{S}$. Then $P\left(\bigcup_{n=1}^{\infty} B_n \,\middle|\, A\right) = \dfrac{P\left(\bigcup_{n=1}^{\infty} (B_n \cap A)\right)}{P(A)}$

Since $B_i \cap B_j = \phi$, $i \neq j$ then

$$(B_i \cap A) \cap (B_j \cap A) = (B_i \cap B_j) \cap A = \phi, \quad i \neq j$$

So $$P\left(\bigcup_{n=1}^{\infty} B_n \middle| A\right) = \frac{\sum P(B_n \cap A)}{P(A)} = \sum \frac{P(B_n \cap A)}{P(A)}$$

$$= \sum P(B_n | A)$$

So $P(\cdot | A)$ is a prob. fun^n on $\mathcal{S}$. for fixed $A \in \mathcal{S}$

with $P(A) > 0$.

Ex: Five cards are drawn at random from a deck of 52 cards. Define events

B : All spade

A : at least 4 spade.

Find $P(B|A)$

Soln: $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)} \quad (\because B \subseteq A)$$

$$= \frac{\binom{13}{5} / \binom{52}{5}}{\left[\binom{13}{4}\binom{39}{1} + \binom{13}{5}\right] / \binom{52}{5}} = 0.441$$

Multiplication law :

(I) $P(A \cap B) = P(A)\, P(B|A)$ if $P(A) > 0$

(II) $P(A \cap B \cap C) = P(A \cap B)\, P(C|A \cap B)$

$= P(A)\, P(B|A)\, P(C|A \cap B)$

Provided $P(A \cap B) > 0$ [Which ensures $P(A) > 0$ $\because$ $A \cap B \subseteq P(A)$]

(III) ~~By~~ Using principle of mathematical induction we have

$$P\left(\bigcap_{i=1}^{n} C_i\right) = P(C_1)\, P(C_2|C_1)\, P(C_3|C_1 \cap C_2) \cdots P(C_n|C_1 \cap \cdots \cap C_{n-1})$$

provided $P(C_1 \cap C_2 \cap \cdots \cap C_{n-1}) > 0$

Theorem of Total probability :

Let $\{E_\alpha : \alpha \in \Delta\}$ be a countable collection of mutually exclusive and exhaustive events ($E_\alpha \cap E_\beta = \phi$, $\alpha \neq \beta$ and $P\left(\bigcup_{\alpha \in \Delta} E_\alpha\right) = 1$). Then for any $E \in \mathcal{S}$

$$P(E) = \sum_{\alpha \in \Delta} P(E \cap E_\alpha) = \sum_{\substack{\alpha \in \Delta \\ P(E_\alpha) > 0}} P(E|E_\alpha)\, P(E_\alpha)$$

Proof: Since $P\left(\bigcup_{\alpha \in \Delta} E_\alpha\right) = 1$, we have

$$P(E) = P\left(E \cap \left(\bigcup_{\alpha \in \Delta} E_\alpha\right)\right) = P\left(\bigcup_{\alpha \in \Delta} (E \cap E_\alpha)\right)$$

$$= \sum_{\alpha \in \Delta} P(E \cap E_\alpha) = \sum_{\substack{\alpha \in \Delta \\ P(E_\alpha) > 0}} P(E \cap E_\alpha) \quad \left[\because P(E_\alpha) = 0 \Rightarrow P(E \cap E_\alpha) = 0\right]$$

$$= \sum_{\substack{\alpha \in \Delta \\ P(E_\alpha) > 0}} P(E \mid E_\alpha)\, P(E_\alpha).$$

Theorem (Bayes Theorem): Let $\{E_\alpha : \alpha \in \Delta\}$ be a countable collection of mutually exclusive and exhaustive events and let E be any event $P(E) > 0$. Then for $j \in \Delta$ with $P(E_j) > 0$

$$P(E_j \mid E) = \frac{P(E_j)\, P(E \mid E_j)}{\sum_{\substack{\alpha \in \Delta \\ P(E_\alpha) > 0}} P(E_\alpha)\, P(E \mid E_\alpha)}.$$

Proof: For $j \in \Delta$, $P(E_j \mid E) = \dfrac{P(E_j \cap E)}{P(E)}$

$$= \frac{P(E_j)\, P(E \mid E_j)}{\sum_{\alpha \in \Delta,\, P(E_\alpha) > 0} P(E_\alpha)\, P(E \mid E_\alpha)}. \quad \left[\text{using thm of total prob}\right]$$

Remark: (a) suppose that occurence of any of the mutually exclusive and exhaustive events $\{E_\alpha : \alpha \in \Delta\}$ (where Δ is a countable set) may cause the occurence of an event E. Given that the event E has occured (i.e. given the ~~event~~ effect), Baye's theorem provides the conditional probability that the event (effect) is caused by occurence of event E_j, $j \in \Delta$.

(b) In Baye's theorem $\{P(E_j) : j \in \Delta\}$ are called prior probabilities and $\{P(E_j | E) : j \in \Delta\}$ are called posterior probabilities.

Independent events: Let $\{E_j : j \in \Delta\}$ be a collection of events

(1) Events $\{E_j : j \in \Delta\}$ are said to be pairwise independent if for any pair of events E_α and E_β $(\alpha, \beta \in \Delta, \alpha \neq \beta)$ in the collection $\{E_j : j \in \Delta\}$ we have

$$P(E_\alpha \cap E_\beta) = P(E_\alpha) P(E_\beta)$$

(ii) Events $\{E_1, E_2, \ldots, E_n\}$ are said to be independent if for any subcollection $\{E_{\alpha_1}, \ldots, E_{\alpha_k}\}$ of $\{E_1, \ldots, E_n\}$ $(k = 1, 2, \ldots n)$

$$P\left(\bigcap_{j=1}^{k} E_{\alpha_j}\right) = \prod_{j=1}^{k} P(E_{\alpha_j})$$

(iii) Let $\Delta \subseteq \mathbb{R}$ be an arbitrary index set so that $\{E_\alpha : \alpha \in \Delta\}$ is an arbitrary collection of events. Events $\{E_\alpha : \alpha \in \Delta\}$ are said to be independent if any finite subcollection of events in $\{E_\alpha : \alpha \in \Delta\}$ forms a collection of independent events.

Theorem: Let $E_1, E_2, \ldots$ be a collection of independent events. Then $P\left(\bigcap_{k=1}^{\infty} A_k\right) = \prod_{k=1}^{\infty} P(A_k)$.

Remark: (1) To verify that n events $E_1, E_2, \ldots, E_n$ are independent one must verify

$$\binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n - n - 1$$

conditions.

For an example to conclude that three events E_1, E_2, E_3 are independent the following 4 $(2^3 - 3 - 1)$ conditions must be verified:

$$P(E_1 \cap E_2) = P(E_1) P(E_2); \quad P(E_1 \cap E_3) = P(E_1) P(E_3)$$

$$P(E_2 \cap E_3) = P(E_2) P(E_3), \quad P(E_1 \cap E_2 \cap E_3) = P(E_1) \cap P(E_2) \cap P(E_3).$$

(ii) If E_1 and E_2 are independent events $(P(E_1) > 0, \; P(E_2) > 0)$, then

$$P(E_1 | E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(E_1) P(E_2)}{P(E_2)} = P(E_2)$$

Example: Consider the probability space $(\Omega, \mathcal{S}, P)$ with $\Omega = \{1, 2, 3, 4\}$ and $P(\{i\}) = \frac{1}{4}$, $i = 1, 2, 3, 4$. Let $A = \{1, 4\}$, $B = \{2, 4\}$, $C = \{3, 4\}$. Then show that A, B and C are pair wise independent but not independent.

Solu^n: $P(A) = P(B) = P(C) = \frac{1}{2}$

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = P(\{4\}) = 1/4$$

~~Thus P(A∩B) P(A) P(B) = P(A) P(C)~~ =

Thus $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$ &

$P(B \cap C) = P(B)P(C)$

$\Rightarrow$ A, B, C are pairwise independent.

However

$$P(A \cap B \cap C) = P(\{4\}) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)P(C)$$

$\Rightarrow$ A, B, C are not independent. although they are pairwise independent.