Linear Transformation :-

Definition: Let V & W be two vector spaces over field F, A linear transformation from V into W is a function T from V into W such that

$$T(c\alpha+\beta) = cT\alpha + T\beta$$

$\forall\ \alpha, \beta \in V$ & $c \in F$.

Some trivial examples of linear transformation are Identity transformation, i.e. $I(\alpha) = \alpha\ \forall\ \alpha \in V$. thus $I: V \to V$ satisfies $I(c\alpha+\beta) = c\alpha+\beta = cI\alpha + I\beta$

and zero transformation, i.e.

$$O\alpha = 0, \quad O: V \to \{0\}$$

Remark: 1. A linear transformation is a "linear map" passing through origin, as $T: V \to W$,

$$T(0) = T(0+0) = T(0) + T(0)$$

As $T(0) \in W$, $\exists\ -T(0)$ s.t.

$$0 = T(0) - T(0) = T(0) + T(0) - T(0) = T(0) + 0 = T(0)$$

Question Can we have two different fields for V and W while defining linear transformation from V to W. ?

Let us see some examples of linear transformation

1. $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (y,x)$

Check $T(c(x,y)+(x',y'))$

$= T(cx+x', cy+y')$

$= (cy+y', cx+x')$

$= (cy, cx) + (y', x')$

$= c(y,x) + (y', x')$

$= cT(x,y) + T(x',y')$

2. $T(x_1,x_2) = (1+x_1, x_2)$, $T: \mathbb{R}^2 \to \mathbb{R}^2$

Note that $T(0,0) = (1,0) \neq (0,0)$

Hence T is not a linear transformation here.

3. $T(x_1,x_2) = (x_1 - x_2, 0)$

Verify $T(c(x_1,x_2)+(y_1,y_2))$

$= T(cx_1+y_1, cx_2+y_2)$

$= (cx_1+y_1-cx_2-y_2, 0)$

$= (c(x_1-x_2)+(y_1-y_2), 0)$

$= (c(x_1-x_2), 0) + (y_1-y_2, 0)$

$= c(x_1-x_2, 0) + (y_1-y_2, 0)$

$= cT(x_1,x_2) + T(y_1,y_2)$

The next theorem talks about if $T: V \to W$ and V is finite dimensional vector space, then how many images one requires to determine the linear transformation T.

Theorem: Let V be a finite dimensional vector space over the field F and let $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field and let $\beta_1, \beta_2, \dots \beta_n$ be any vectors in W. Then there is precisely one linear transformation $T: V \to W$ such that

$$T\alpha_j = \beta_j \quad j = 1, 2, \dots n$$

Proof: The proof involves a construction of a linear transformation $T: V \to W$ satisfying $T\alpha_j = \beta_j$ $j = 1, 2, \dots n$ uniquely.

We start with the definition as follows

Let $\alpha \in V$, then $\exists\ x_1, x_2, \dots x_n \in F$ such that $\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots x_n\alpha_n$

Define $T\alpha = x_1\beta_1 + x_2\beta_2 + \dots x_n\beta_n$

Then we can verify the following three steps

a) T is a linear transformation

b) $T\alpha_j = \beta_j$

c) If $U\alpha_j = \beta_j$ then $U = T$.

Let us verify first that T is linear.

For that consider $\beta = y_1\alpha_1 + y_2\alpha_2 \cdot + y_n\alpha_n \in V$ & $c \in F$

then $T(c\alpha + \beta) = T(c(x_1\alpha_1 + x_2\alpha_2 + \dots x_n\alpha_n) + (y_1\alpha_1 + \dots y_n\alpha_n))$

$= T(cx_1\alpha_1 + y_1\alpha_1, cx_2\alpha_2 + y_2\alpha_2, \dots cx_n\alpha_n + y_n\alpha_n)$

$= T((cx_1 + y_1)\alpha_1, (cx_2 + y_2)\alpha_2, \dots (cx_n + y_n)\alpha_n)$

$= (cx_1 + y_1)\beta_1 + (cx_2 + y_2)\beta_2 + \dots (cx_n + y_n)\beta_n$

$= (cx_1\beta_1 + y_1\beta_1) + (cx_2\beta_2 + y_2\beta_2) + \dots (cx_n\beta_n + y_n\beta_n)$

$= c(x_1\beta_1 + x_2\beta_2 + \dots x_n\beta_n) + (y_1\beta_1 + y_2\beta_2 + \dots y_n\beta_n)$

$= cT\alpha + T\beta$

Next we check, if $T\alpha_j = \beta_j$. As

$\alpha_j = 0\alpha_1 + 0\alpha_2 + \cdots 1.\alpha_j + \cdots 0.\alpha_n$

$T\alpha_j = 0\beta_1 + 0\beta_2 \cdots 1.\beta_j + \cdots 0.\beta_n$

$T\alpha_j = \beta_j$

Finally we ensure that such T satisfying $T\alpha_j = \beta_j$ is unique.

Let if there exists $U: V \to W$ satisfying $U\alpha_j = \beta_j$. Then let us see action of U on any arbitrary element $\alpha \in V$.

As $\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots x_n\alpha_n$

$$U\alpha = U(x_1\alpha_1 + x_2\alpha_2 + \cdots x_n\alpha_n)$$

$$= x_1 U\alpha_1 + x_2 U\alpha_2 + \cdots x_n U\alpha_n$$

$$= x_1\beta_1 + x_2\beta_2 + \cdots x_n\beta_n$$

$$U\alpha = T\alpha \quad \forall \alpha \in V$$

Remark:- Let me put a situation of a function from a set $A = \{x_1, x_2, \cdots x_n\}$ to a set B. Then how one can determine $f: A \to B$? The answer is, if you know the action of f on each of $x_1, x_2, \cdots x_n$, then f will be determined completely. The situation here is compromised with a infinite set V but advantageous with T being linear. Any element of V can uniquely be uniquely determined by the finitely many basis elements by determining co-ordinate matrix. In this way linear transformation can be determined uniquely by knowing the action of on basis elements.

Let us see the example

Example: Let $\alpha_1 = (1,-1)$ $\alpha_2 = (0,1)$ be vectors of $\mathbb{R}^2$, then $\{\alpha_1, \alpha_2\}$ forms a basis of $\mathbb{R}^2$. Let $(2,3,0), (1,1,0)$ be vectors in $\mathbb{R}^3$ then there exists a unique linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ satisfying $T(1,-1) = (2,3,0)$ & $T(0,1) = (1,1,0)$.

For, take $\alpha \in \mathbb{R}^2$ as $\alpha = (x, y)$ then we will find co-ordinates of α w.r.t $\{\alpha_1, \alpha_2\}$ as

$$(x,y) = c_1(1,-1) + c_2(0,1)$$

$$\Rightarrow c_1 = x, \quad -c_1 + c_2 = y \Rightarrow c_2 = x+y$$

Hence any $(x,y) \in \mathbb{R}^2$ can be uniquely written as

$$(x,y) = x(1,-1) + (x+y)(0,1)$$

Now define

$$T(x,y) = x(2,3,0) + (x+y)(1,1,0)$$

$$= (2x + x + y, 3x + x + y, 0)$$

$$T(x,y) = (3x+y, 4x+y, 0)$$

Check if $T(1,-1) = (2,3,0)$ and $T(0,1) = (1,1,0)$ or not?

Let us see the range of a linear transformation $T: V \to W$.

Let $R = \{\beta \in W : \beta = T\alpha \text{ for some } \alpha \in V\}$ then it is easy to see that R is a subspace of W. For, take $\beta_1, \beta_2 \in R$ and $c \in F$, then check if $c\beta_1 + \beta_2 \in R$?

In fact yes, as $\exists\, \alpha_1$ s.t. $\beta_1 = T\alpha_1$ & α_2 s.t. $\beta_2 = T\alpha_2$. Therefore $T(c\alpha_1 + \alpha_2) = cT\alpha_1 + T\alpha_2 = c\beta_1 + \beta_2$ for $c\alpha_1 + \alpha_2 \in V$.

Another interesting subset is of V.
Let us define null space of T as
$N = \{\alpha \in V : T\alpha = 0\}$. Then one can see N is a subspace of N.
As $0 \in N$ implies N is non-empty.
Now take $\alpha, \beta \in N$ and $c \in F$ then
$T\alpha = 0$, $T\beta = 0$
Now $T(c\alpha + \beta) = cT\alpha + T\beta = c.0 + 0 = 0$
implies $c\alpha + \beta \in N$.

Definition Let $T: V \to W$ be a linear Transformation
1. The dimension of range space of T is called rank of T
2. The dimension of null space of T is called nullity of T.

The following theorem is very important result of linear algebra in the context of linear transformations.

Theorem: Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose that V is finite dimensional, then

$$\text{rank}(T) + \text{nullity}(T) = \dim V.$$

Proof: Let N denotes the null space of T and R denotes the range space of T.
Assume $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$ be a basis of N, i.e. $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$ is a linearly independent subset of V and hence can be extended to the basis of V

Let $\{\alpha_{k+1}, \alpha_{k+2}, \dots \alpha_n\} \in V$ be such that $\{\alpha_1, \alpha_2, \dots \alpha_k, \alpha_{k+1}, \dots \alpha_n\}$ forms a basis of V. For proving the result it will be sufficient to show that $\{T\alpha_{k+1}, \dots T\alpha_n\}$ forms a basis for R.
(note that $T\alpha_1 = T\alpha_2 = \dots T\alpha_k = 0$ will not be the part of the basis of R)
It is clear that $\{T\alpha_{k+1}, T\alpha_{k+2} \dots T\alpha_n\}$ span the Range space R. Hence we need to show linear independence only. For, consider the arbitrary linear combination

$$\sum_{i=k+1}^{n} c_i T\alpha_i = 0$$

$$\Rightarrow \sum_{i=k+1}^{n} T(c_i\alpha_i) = 0$$

$$\Rightarrow T\left(\sum_{i=k+1}^{n} c_i\alpha_i\right) = 0$$

Hence $\sum_{i=k+1}^{n} c_i\alpha_i \in N$ and hence can be written in the linear combination of vectors $\{\alpha_1, \alpha_2, \dots \alpha_k\}$
i.e.

$$\sum_{i=k+1}^{n} c_i\alpha_i = \sum_{i=1}^{k} c_i\alpha_i$$

$$\Rightarrow \sum_{i=1}^{n} c_i\alpha_i = 0 \Rightarrow c_i = 0 \;\forall\, i = 1, 2, \dots k, k+1, \dots n$$

as $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ are linearly independent.

Hence $\dim V = n = k + (n-k)$

$\Rightarrow \dim V = \dim N + \dim R$