Infinite Series

Defn: Let $\{a_n\}$ be any infinite sequence of real numbers. Then $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ is called an infinite series.

Sequence of partial sum:

We define sequence of partial sum $\{S_n\}$ as

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$\vdots$$
$$S_n = a_1 + a_2 + \cdots\cdots + a_n$$
$$\vdots$$

Defn: Let $\sum_{n=1}^{\infty} a_n$ be any infinite series and let $\{S_n\}$ be the sequence of partial sum

(i) If $S_n \to l$, $l \in \mathbb{R}$ then we say $\sum_{n=1}^{\infty} a_n$ converges ~~and~~ to l and we write $\sum_{n=1}^{\infty} a_n = l$

(ii) If $S_n \to +\infty$ (or $-\infty$), then we say that $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$ or $(-\infty)$ and we write $\sum_{n=1}^{\infty} a_n = +\infty$ (or $-\infty$).

(iii) If $\{S_n\}$ oscillates, then we say that $\sum_{n=1}^{\infty} a_n$ oscillates.

Ex Let $x_{2n-1} = \frac{1}{n}$ and $x_{2n} = \frac{-1}{n}$. Then find the seq^n of partial sums.

Sol^n: Here the series is

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + - + \cdots$$

$$S_1 = 1, \quad S_2 = 1 - 1 = 0, \quad S_3 = 1 - 1 + \frac{1}{2} = \frac{1}{2}$$

$$S_4 = 1 + 1 + \frac{1}{2} - \frac{1}{2} = 0.$$

$$S_{2n-1} = \frac{1}{n}, \quad S_{2n} = 0.$$

Discussion: For any infinite series $\sum_{n=1}^{\infty} a_n$ we have two sequence $\{S_n\}$ and $\{a_n\}$. We know

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n.$$

Now from $\{S_n\}$ we can find a_n

$$a_1 = S_1, \quad a_2 = S_2 - S_1, \quad a_3 = S_3 - S_2$$

So $a_n = S_n - S_{n-1}$. Now for $n = 1$ we get $S_{n-1} = S_0$. So we define $S_0 = 0$.

Then we can write

$$a_n = S_n - S_{n-1} \quad \forall\, n \geq 1.$$

Theorem : If $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$

Let $S_n = \begin{cases} 0 & \text{if } n = 0 \\ \sum_{i=1}^{n} a_i & \text{if } n = 1, 2, 3 \ldots \end{cases}$

Given $\sum_{n=1}^{\infty} a_n$ converges i.e. $S_n \to l$ for some $l \in \mathbb{R}$

Now $a_n = S_n - S_{n-1}$

Since $S_n \to l$ as $n \to \infty$ so $S_{n-1} \to l$ as $n \to \infty$

$\therefore a_n \to l - l = 0$

Note: The converse of the above theorem is not ture. That is $a_n \to 0$ does not imply $\sum_{n=1}^{\infty} a_n$ converges.

Suppose A and B are two condition such that

$A \Rightarrow B$

Then we say "A is sufficient for B". Also we can say "B is necessary for A"

So we can say $a_n \to 0$ as $n \to \infty$ is a necessary condition for $\sum_{n=1}^{\infty} a_n$ is convergence.

So if $a_n \not\to 0$ then $\sum_{n=1}^{\infty} a_n$ not convergent.

Ex: Examine the convergence of the series

$$\frac{1}{3} + \left(\frac{1}{3}\right)^{1+\frac{1}{2}} + \left(\frac{1}{3}\right)^{1+\frac{1}{2}+\frac{1}{4}} + \cdots$$

[If $\lim x_n = l$ and $a>0$ then $\lim a^{x_n} = a^l$.]

Here $a_n = \left(\frac{1}{3}\right)^{r_n}$, where

$$r_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} \to \frac{1}{1-\frac{1}{2}}$$

$$= 2.$$

So $a_n \to \left(\frac{1}{3}\right)^2 = \frac{1}{9} \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges to ∞.

Since it is a series of positive terms.

Example: Discuss the convergence of

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

Here $a_n = \frac{1}{n(n+1)}$

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} \to 1 \text{ as } n \to \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = 1.$$

Example: Consider the series $1-1+1-1\cdots\cdots$

In this case $a_n = (-1)^{n+1}$

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$= 1-1+1-1\cdots + (-1)^{n+1} = \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

So $\{S_n\}$ does not converge $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ not convergent.

Example (Geometric Series): Let us consider the series

$$1 + r + r^2 + r^3 + \cdots\cdots$$

Now we consider the following cases.

(I) $|r|<1$, $\quad a_n = r^n$

$$S_n = 1 + r + r^2 + \cdots + r^n$$

$$= \frac{1+r^{n+1}}{1-r}.$$

Now $r^{n+1} \to 0$ as $n \to \infty$ $\because |r|<1$

$\Rightarrow \lim S_n = \dfrac{1}{1-r}$. So $1 + r + r^2 + \cdots = \dfrac{1}{1-r}$ if $|r|<1$.

(II) $r = 1$. Then $S_n = n+1 \longrightarrow \infty$ as $n \to \infty$

So $\sum\limits_{n=0}^{\infty} r^n \longrightarrow \infty$ as $n \to \infty$.

Case-4: ~~and~~ $r > 1$ then $r^{n+1} \longrightarrow \infty$

$$\Rightarrow S_n \to \infty \quad \Rightarrow \sum_{n=0}^{\infty} a_n = \infty.$$

Case-5: $r \leq -1$. Let $r = -q$, $q \geq 1$.

Then $\{r^{n+1}\} = \{-q, q^2, -q^3, \dots\}$

So $r^{n+1} = (-q)^{n+1}$ oscillates.

So $\{S_n\}$ oscillates $\Rightarrow \sum a_n$ not convergent

Discussion (Tail is more important than head)

We are taking $\lim_{n\to\infty} S_n$. Since $n \to \infty$ so we do not care about terms in the begining of S_n. Hence the terms in the begining of the series $\sum a_n$ are not that much important.

If we change few terms in the begining of $\sum a_n$, there will be no change in the behaviour of the series ~~do not change~~.

For example let

$a_1 + a_2 + a_3 + \dots$ be a series

Let $\{S_n\}$ be the sequence of partial sum.

Suppose we ~~add~~ change 1st two terms b_1 and b_2

$$b_1 + b_2 + a_3 + a_4 + a_5 + \cdots\cdots$$

Now consider the sequence of partial sum

$$T_1 = b_1 = (b_1 - a_1) + S_1$$

$$T_2 = b_1 + b_2 = (b - a_1) + (b_2 - a_2) + S_2$$

$$T_3 = b_1 + b_2 + a_3 = (b_1 - a_1) + (b_2 - a_2) + S_3$$

$$T_4 = b_1 + b_2 + a_3 + a_4 = (b_1 - a_1) + (b_2 - a_2) + S_4$$

$$\vdots$$

Now if we dont consider T_1 and T_2

$$T_n = (b_1 - a_1) + (b_2 - a_2) + S_n, \quad n \geq 3.$$

So $\{T_n\}$ & $\{S_n\}$ have the same behaviour

If T_n converge $\Leftrightarrow$ S_n converge

T_n diverge $\Leftrightarrow$ S_n diverge.

T_n oscillate $\Leftrightarrow$ S_n oscillate.

$$\lim_{n\to\infty} T_n = (b_1 - a_1) + (b_2 - a_2) + \lim_{n\to\infty} S_n$$

(So the limit are not same)

So if we add, remove or change finite number of terms in the begining of the series then the behaviour of $\sum a_n$ does not change. So we can write $\sum_{n=1}^{\infty} a_n$ in place of $\sum a_n$

~~So we can write~~ Which means

$$\sum_{n = \text{something (some natural number)}}^{\infty} a_n$$

Cauchy Criterian for Series

A series ~~$\sum a_n$~~ $\sum a_n$ is converges iff $\forall \epsilon > 0$ there exists $N \in \mathbb{N}$, for all $m, n \geq N$ $(m < n)$

$$|a_{m+1} + a_{m+2} + \dots + a_n| < \epsilon$$

We know that $\sum a_n$ converges if $\{S_n\}$ converges. Also we know that $\{S_n\}$ converges $\Leftrightarrow$ $\{S_n\}$ is a cauchy sequence i.e. $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ $\forall m, n \geq N$ $|S_n - S_m| < \epsilon$

Now we take $m < n$ then

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall m, n \geq N$$
$$|a_{m+1} + a_{m+2} + \dots + a_n| < \epsilon.$$

p-series: $\sum \frac{1}{n^p} = \infty$ if $p \leq 1$

$< \infty$ if $p > 1$

i.e. $\sum \frac{1}{n^p}$ is convergent if $p > 1$ and diverge to ∞ if $p \leq 1$.

Series of positive terms: If all the terms of the series $\sum a_n$ is positive then the sequⁿ $\{S_n\}$ is a monotone increasing because

$$S_n - S_{n-1} = a_n > 0.$$

Theorem: If $a_n > 0$ then either $\sum a_n$ converges or it diverge to ∞. Accordingly we write $\sum a_n < \infty$ or $\sum a_n = \infty$.

Example: (I) $\sum \frac{1}{n}$ is divergent $\because p = 1$

(II) $\sum \frac{1}{n\sqrt{n}} = \sum \frac{1}{n^{3/2}}$ convergent $p = 3/2$

Compareison Test:

Inequality form: Let $\sum a_n$ and $\sum b_n$ be two series such that $\exists\, N \in \mathbb{N}$ $\forall\, n \geq N$ $a_n > b_n > 0$, then

1) $\sum a_n$ converges $\Rightarrow$ $\sum b_n$ converges

II) $\sum b_n$ diverge to ∞ $\Rightarrow$ $\sum a_n = \infty$.

EX. Check the convergence of $\sum \frac{3}{n^2+10}$

Solu^n:

$$\frac{3}{n^2+10} \le \frac{3}{n^2}$$

Now we have $\sum \frac{1}{n^2}$ converges

so $$\sum \frac{3}{n^2+10} \le \sum \frac{3}{n^2} = 3 \sum \frac{1}{n^2} < \infty$$

EX: Check the convergence of $\sum \frac{1}{\sqrt{n}-3/2}$

Now $$\frac{1}{\sqrt{n}-3/2} \geq \frac{1}{\sqrt{n}}$$

$$\Rightarrow \sum \frac{1}{\sqrt{n}-3/2} \geq \sum \frac{1}{\sqrt{n}} = \infty$$

$$\Rightarrow \sum \frac{1}{\sqrt{n}-3/2} = \infty.$$

Comparison Test (limit form):

Let $\sum a_n$ and $\sum b_n$ two series of positive terms.

and $\lim_{n\to\infty} \frac{a_n}{b_n} = l$,

(a) If $0 < l < \infty$ then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

(b) If $l = 0$, $\sum a_n$ converges if $\sum b_n$ converge

(c) If $l = \infty$ then $\sum a_n$ diverges if $\sum b_n$ diverge.

Ex: Show that $\sum \frac{3n+2}{n^3+5n+6} < \infty$

solⁿ

Let $a_n = \frac{3n+2}{n^3+5n+6}$ and $b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{n^2(3n+2)}{n^3+5n+6} = \frac{3+\frac{2}{n}}{1+5/n^2+6/n^3} \to \frac{3+0}{1+0+0} = 3 < \infty$$

Now $\sum \frac{1}{n^2} < \infty \Rightarrow \sum a_n < \infty$.

Ex Show that $\sum \frac{n+2}{n^2+15n+9} = \infty$

solⁿ

$a_n = \frac{n+2}{n^2+15n+9}$, $b_n = 1/n$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$.

Since $\sum b_n = \infty \Rightarrow \sum a_n = \infty$.

Root test: Let $\{a_n\}$ be a sequence of positive numbers then

(I) $\lim \sqrt[n]{a_n} < 1 \Rightarrow \sum a_n < \infty$

(II) $\lim \sqrt[n]{a_n} > 1 \Rightarrow \sum a_n = \infty$

(III) $\lim \sqrt[n]{a_n} = 1$ then $\sum a_n$ may converge or diverge.

Ratio Test

Let $\sum a_n$ be an infinite series of positive terms then

(I) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \sum a_n < \infty$

(II) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \Rightarrow \sum a_n = \infty$

(III) If $\lim \frac{a_{n+1}}{a_n} = 1$ then $\sum a_n$ may converge or diverge.

Ex: $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots$

$a_n = \frac{2n-1}{n!}$. $\frac{a_{n+1}}{a_n} = \frac{2n+1}{(n+1)(2n-1)}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1$

$\Rightarrow$ By ratio test $\sum a_n$ ~~is~~ convergent

Ex. Examine the convergence of the series

$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \quad , x > 0$

Soln: $a_n = \frac{x^n}{n}$

$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} = \frac{nx}{n+1}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x$. So $\sum a_n$ converge if $x < 1$

diverge $x > 1$

$x = 1, \quad \sum a_n = \sum \frac{1}{n}$ diverge.