In our last lecture, we discussed about matrix representation of a linear transformation.

Let us see the following discussion.

Let $T: V \to W$ be a linear transformation. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ be an ordered basis of V and $\mathcal{B}' = \{\beta_1, \beta_2, \dots \beta_m\}$ be an ordered basis for W.

Consider the $\{T\alpha_1, T\alpha_2, \dots T\alpha_n\}$ n-vectors in W, then T can be uniquely determined by $T\alpha_j$, $j = 1, 2, \dots n$. As for $\alpha \in V$,

$$T\alpha = T(c_1\alpha_1 + c_2\alpha_2 + \dots c_n\alpha_n)$$
$$= c_1 T\alpha_1 + c_2 T\alpha_2 + \dots c_n T\alpha_n$$

As $T\alpha_j$ is a vector in W and hence can be uniquely written as

$$T\alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i, \quad A_{ij} \in F \text{ for } \forall j = 1, 2, \dots n$$

and thus $[T\alpha_j]_{\mathcal{B}'} = [A_{1j}, A_{2j} \dots A_{mj}]^T$

Now for any $\alpha \in V$, $\exists$ $c_1, c_2, \dots c_n \in F$ s.t.

$$c_1\alpha_1 + c_2\alpha_2 + \dots c_n\alpha_n = \alpha$$

and thus

$$T\alpha = T(c_1\alpha_1 + c_2\alpha_2 + \dots c_n\alpha_n)$$
$$= \sum_{j=1}^{n} c_j T\alpha_j = \sum_{j=1}^{n} c_j \left(\sum_{i=1}^{m} A_{ij}\beta_i\right)$$
$$= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} A_{ij}c_j\right)\beta_i$$

$$[T\alpha]_{\mathcal{B}'} = A[c_1, c_2 \dots c_n]^T$$

Thus, as $[c_1, c_2, \dots c_n]^T = [\alpha]_{\mathcal{B}}$, we get

$$[T\alpha]_{\mathcal{B}'} = A\,[\alpha]_{\mathcal{B}}$$

Let $T: V \to W$ be a linear transformation with A as it's matrix relative to an ordered basis pair $\mathcal{B}$ & $\mathcal{B}'$ respectively.

Let U be another linear transformation from V into W and B be the matrix of U, i.e. $[U]_{\mathcal{B}}^{\mathcal{B}'} = B$.

Then we would like to see what will happen to the matrix of linear transformation
$cT+U: V \to W$.

Note that, if A_j & B_j represents the j^{th} columns of A & B respectively, then

$$c A_j + B_j = c[T\alpha_j]_{\mathcal{B}'} + [U\alpha_j]_{\mathcal{B}'}$$
$$= [cT\alpha_j + U\alpha_j]_{\mathcal{B}'}$$
$$= [(cT+U)\alpha_j]_{\mathcal{B}'}$$

Thus $[cT+U]_{\mathcal{B}}^{\mathcal{B}'} = c[T]_{\mathcal{B}}^{\mathcal{B}'} + [U]_{\mathcal{B}}^{\mathcal{B}'}$

Now we would like to see what happen to the matrix representations when we compose two linear transformations?

Let $T: V \to W$, $U: W \to Z$ be two linear transformations. Let $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ be the ordered bases of V, W and Z respectively. Let

$\mathcal{B} = \{\alpha_1, \alpha_2, \dots \alpha_n\}$

$\mathcal{B}' = \{\beta_1, \beta_2, \dots \beta_m\}$

$\mathcal{B}'' = \{\gamma_1, \gamma_2, \dots \gamma_p\}$

be the corresponding bases

We claim that if $A = [T]_{\mathcal{B}}^{\mathcal{B}'}$, $B = [U]_{\mathcal{B}'}^{\mathcal{B}''}$
then $[UT]_{\mathcal{B}}^{\mathcal{B}''} = BA$

when $V = W = Z$, then

$$[UT]_{\mathcal{B}} = [U]_{\mathcal{B}} [T]_{\mathcal{B}}$$

Note that $[I]_{\mathcal{B}} = I$ (identity transformation has matrix representation as I). Let T be invertible linear transformation, then $\exists\, U : V \to V$ s.t

$UT = TU = I$.

Now if $\mathcal{B}$ is an ordered basis of V then $[UT]_{\mathcal{B}} = [I]_{\mathcal{B}} = I = [U]_{\mathcal{B}} [T]_{\mathcal{B}}$

which implies

$$[T]_{\mathcal{B}}^{-1} = [U]_{\mathcal{B}} = [T^{-1}]_{\mathcal{B}}$$

Next, we would like to see the effect of change of basis on matrix representations.

Let $T : V \to V$ be a linear transformation and $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ & $\mathcal{B}' = \{\alpha_1', \alpha_2', \ldots \alpha_n'\}$ are two ordered bases of V

We denote $[T]_{\mathcal{B}}$ & $[T]_{\mathcal{B}'}$ as the matrices relatives to ordered bases $\mathcal{B}$ & $\mathcal{B}'$ respectively.

Recall that, for any $\alpha \in V$

$$[\alpha]_{\mathcal{B}} = P [\alpha]_{\mathcal{B}'} \quad \text{— (1)}$$

defines the effect of change of basis on coordinate matrix of $\alpha \in V$.

Moreover we know

$$[T\alpha]_{\mathcal{B}} = [T]_{\mathcal{B}} [\alpha]_{\mathcal{B}} \quad \text{— (2)}$$

Use α as $T\alpha$ in (1) to get

$$[T\alpha]_{\mathcal{B}} = P [T\alpha]_{\mathcal{B}'} \quad \text{— (3)}$$

On combining all these three equations, we get

$$[T\alpha]_{\mathcal{B}} = [T]_{\mathcal{B}} P [\alpha]_{\mathcal{B}'}$$

$$\Rightarrow P[T\alpha]_{\mathcal{B}'} = [T]_{\mathcal{B}} P [\alpha]_{\mathcal{B}'}$$

$$\Rightarrow P[T]_{\mathcal{B}'}[\alpha]_{\mathcal{B}'} = [T]_{\mathcal{B}} P [\alpha]_{\mathcal{B}'}$$

$$\Rightarrow P^{-1} P [T]_{\mathcal{B}'}[\alpha]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}} P [\alpha]_{\mathcal{B}'}$$

$$\Rightarrow [T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}} P.$$

The above discussion in the light of the definition of similar matrices (A & B are called similar matrices if there exists an invertible matrix Q such that

$$A = Q^{-1} B Q)$$

implies that the change of basis produces similar matrix representations of linear transformations.

Let us see the example as we did earlier also

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$T(x,y) = (x,0)$, the matrix representation of T in the ordered basis $\{(1,0),(0,1)\}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Now take another ordered basis on $\mathbb{R}^2$ as $\mathcal{B}' = \{(1,1),(2,1)\}$ then

$$[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}} P$$

where $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. !!