Eigenvalue eigenvector

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Linear algebra- II (IC152)

- Let V be an inner product space. Let $\mathbf{v}$ be a nonzero vector of V.
- We want to decompose an arbitrary vector $\mathbf{y}$ into the form

$$\mathbf{y} = \alpha\mathbf{v} + \mathbf{z}, \quad \text{where} \quad \mathbf{z} \in \mathbf{v}^{\perp}.$$

- Since $\mathbf{z} \perp \mathbf{v}$, we have

$$\langle \mathbf{v}, \mathbf{y} \rangle = \langle \alpha\mathbf{v}, \mathbf{v} \rangle = \alpha\langle \mathbf{v}, \mathbf{v} \rangle.$$

- This implies that

$$\alpha = \frac{\langle \mathbf{v}, \mathbf{y} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

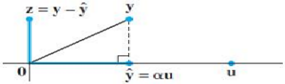

FIGURE 2
Finding α to make $\mathbf{y} - \hat{\mathbf{y}}$ orthogonal to $\mathbf{u}$.

Orthogonal Projection cont.

- We define the vector

$$\mathrm{Proj}_{\mathbf{v}}(\mathbf{y}) = \frac{\langle \mathbf{v}, \mathbf{y} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v},$$

  called the orthogonal projection of $\mathbf{y}$ along $\mathbf{v}$.
- The linear transformation $\mathrm{Proj}_{\mathbf{v}} : V \to V$ is called the orthogonal projection of V onto the direction $\mathbf{v}$.
- If $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = 1$, then we have

$$\mathrm{Proj}_{\mathbf{v}}(\mathbf{y}) = \langle \mathbf{v}, \mathbf{y} \rangle \mathbf{v},$$

Example

- The orthogonal projection of $\mathbf{y} = (6, 2, 4)$ onto $\mathbf{v} = (1, 2, 0)$ is given by

$$\begin{aligned} \mathrm{Proj}_{\mathbf{v}}(\mathbf{y}) = \frac{\langle \mathbf{v}, \mathbf{y} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v} &= \frac{10}{5}(1, 2, 0) \\ &= (2, 4, 0). \end{aligned}$$

Theorem

Let $\mathbf{v}$ *be a nonzero vector of the Euclideann-space* $\mathbb{R}^n$*. Then the orthogonal projection* $\mathrm{Proj}_{\mathbf{v}} : \mathbb{R}^n \to \mathbb{R}^n$ *is given by*

$$\mathrm{Proj}_{\mathbf{v}}(\mathbf{y}) = \frac{\mathbf{v}\mathbf{v}^T\mathbf{y}}{\langle \mathbf{v}, \mathbf{v} \rangle};$$

and the orthogonal projection $\mathrm{Proj}_{\mathbf{v}^\perp} : \mathbb{R}^n \to \mathbb{R}^n$ *is given by*

$$\mathrm{Proj}_{\mathbf{v}^\perp}(\mathbf{y}) = (I - \frac{\mathbf{v}\mathbf{v}^T}{\langle \mathbf{v}, \mathbf{v} \rangle})\mathbf{y}.$$

Example 1

- We want to find the linear mapping from $\mathbb{R}^3 \to \mathbb{R}^3$ that is a the orthogonal projection of $\mathbb{R}^3$ onto the plane $x_1 + x_2 + x_3 = 0$.
- Also we want to find the orthogonal projection of $\mathbb{R}^3$ onto the subspace $\mathbf{v}^{\perp}$, where $\mathbf{v} = [1, 1, 1]^T$.
- We find the following orthogonal projection

$$\begin{aligned}
\text{Proj}_{\mathbf{v}}(\mathbf{y}) &= \frac{\langle \mathbf{v}, \mathbf{y} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v} \\
&= \frac{y_1 + y_2 + y_3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}
\end{aligned}$$

- Then the orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}^{\perp}$ is given by

$$\begin{aligned}
\mathrm{Proj}_{\mathbf{v}^{\perp}}(\mathbf{y}) &= (I - \frac{\mathbf{v}\mathbf{v}^T}{\langle \mathbf{v}, \mathbf{v} \rangle})\mathbf{y} \\
&= \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}
\end{aligned}$$

- Let W be a subspace of V, and let $v_1, \cdots, v_k$ be an orthogonal basis of W.
- We want to decompose an arbitrary vector $\mathbf{y} \in V$ into the form

$$\mathbf{y} = \mathbf{w} + \mathbf{z} \text{ with } \mathbf{w} \in W \text{ and } \mathbf{z} \in W^{\perp}.$$

- Then there exist scalars $\alpha_1, \cdots, \alpha_k$ such that

$$\tilde{\mathbf{y}} = \sum_{i=1}^{k} \alpha_i v_i.$$

- Since for $1 \leq i \leq k, \mathbf{z} \perp v_i$, we have

$$\langle v_i, \mathbf{y} \rangle = \langle v_i, \alpha_1 v_1 + \cdots + \alpha_k v_k + \mathbf{z} \rangle = \alpha_i \langle v_i, v_i \rangle.$$

- Then $1 \leq i \leq k, \alpha_i = \frac{\langle v_i, \mathbf{y} \rangle}{\langle v_i, v_i \rangle}$.

- We thus define
$$\mathrm{Proj}_W(\mathbf{y}) = \sum_{i=1}^{k} \frac{\langle v_k, \mathbf{y}\rangle}{\langle v_k, v_k\rangle} v_k,$$
,called the orthogonal projection of valong W.
- The linear transformation $\mathrm{Proj}_W : V \to V$ is called the orthogonal projection of V onto W.

Theorem

Let V be an n dimensional Inner product space. Let W be a basis $\mathcal{B} = \{v_1, \cdots, v_k\}$. Then for any $\mathbf{v} \in V$,

$$\mathrm{Proj}_W(\mathbf{y}) = \sum_{i=1}^{k} \frac{\langle v_k, \mathbf{y} \rangle}{\langle v_k, v_k \rangle} v_k, \mathrm{Proj}_{W^\perp}(\mathbf{y}) = \mathbf{y} - \mathrm{Proj}_W(\mathbf{y}).$$

In particular, if $\mathcal{B}$ is an orthonormal basis of W, then

$$\mathrm{Proj}_W(\mathbf{y}) = \sum_{i=1}^{k} \langle v_k, \mathbf{y} \rangle v_k.$$

Theorem

Let W be a subspace of $\mathbb{R}^n$. Let $U = \begin{bmatrix} u_1 & \cdots & u_k \end{bmatrix}$ be an $n \times k$ matrix, whose columns form an orthonormal basis of W. Then the $\operatorname{Proj}_{\mathbf{v}} : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\operatorname{Proj}_W(\mathbf{y}) = UU^T\mathbf{y}.$$

Example 1

- We want to find orthogonal projection

$$\mathrm{Proj}_W(\mathbf{y}) : \mathbb{R}^3 \to \mathbb{R}^3,$$

where W is the plane $x + y + z = 0$.

- The two vectors $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ form an orthogonal basis of W.
- Then

$$\begin{aligned}
\mathrm{Proj}_W(\mathbf{y}) &= \sum_{i=1}^{2} \frac{\langle v_k, \mathbf{y} \rangle}{\langle v_k, v_k \rangle} v_k \\
&= \frac{y_1 - y_2}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{y_1 + y_2 - 2y_3}{6} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}
\end{aligned}$$

Example 2

- We want to find the matrix of the orthogonal projection

$$\mathrm{Proj}_W(\mathbf{y}) : \mathbb{R}^3 \to \mathbb{R}^3,$$

where

$$W = \quad \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

- The vectors $u_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, u_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ form an orthogonal basis of W.

Example 2 cont.

- Then the standard matrix of Proj_W is the product

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

- Alternatively, the matrix can be found by computing the orthogonal projection:

$$\begin{aligned} \mathrm{Proj}_W(\mathbf{y}) &= \frac{y_1 + y_2 + y_3}{6} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{y_1 - y_2}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} -5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \end{aligned}$$

Observation

- The projection map P_W depends on the complementary subspace W_0.
- Observe that for a fixed subspace W, there are infinitely many choices for the complementary subspace W_0.
- It will be shown later that if V is an inner product space with inner product, $\langle \, , \, \rangle$, then the subspace W_0 is unique if we put an additional condition that $W_0 = \{\mathbf{v} \in V \ : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}$.
- We now prove some basic properties about projection maps.

Complementary subspaces

Theorem

Let W and W_0 be complementary subspaces of a vector space V. Let $P_W : V \longrightarrow V$ be a projection operator of V onto W along W_0. Then

1. *the null space of P_W,*
$$\mathcal{N}(P_W) = \{\mathbf{v} \in V : P_W(\mathbf{v}) = \mathbf{0}\} = W_0.$$
2. *the range space of P_W,*
$$\mathcal{R}(P_W) = \{P_W(\mathbf{v}) : \mathbf{v} \in V\} = W.$$
3. $P_W^2 = P_W$. *The condition $P_W^2 = P_W$ is equivalent to*
$$P_W(I - P_W) = \mathbf{0} = (I - P_W)P_W.$$

Outline of the proof

- We only prove the first part of the theorem.
- Let $\mathbf{w}_0 \in W_0$. Then $\mathbf{w}_0 = \mathbf{0} + \mathbf{w}_0$ for $\mathbf{0} \in W$.
- So, by definition, $P(\mathbf{w}_0) = \mathbf{0}$. Hence, $W_0 \subset \mathcal{N}(P_W)$.
- Also, for any $\mathbf{v} \in V$, let $P_W(\mathbf{v}) = \mathbf{0}$ with $\mathbf{v} = \mathbf{w} + \mathbf{w}_0$ for some $\mathbf{w}_0 \in W_0$ and $\mathbf{w} \in W$.
- Then by definition $\mathbf{0} = P_W(\mathbf{v}) = \mathbf{w}$. That is, $\mathbf{w} = \mathbf{0}$ and $\mathbf{v} = \mathbf{w}_0$. Thus, $\mathbf{v} \in W_0$.
- Hence $\mathcal{N}(P_W) = W_0$.

- The next result uses the Gram-Schmidt orthogonalisation process to get the complementary subspace in such a way that the vectors in different subspaces are orthogonal.

Definition (Orthogonal Subspace of a Set)

Let V be an inner product space. Let S be a non-empty subset of V. We define

$$S^{\perp} = \{\mathbf{v} \in V \ : \langle \mathbf{v}, \mathbf{s} \rangle = 0 \text{ for all } \mathbf{s} \in S\}.$$

Theorem

Let S be a subset of a finite dimensional inner product space V, with inner product $\langle\ ,\ \rangle$. Then

1. $S^{\perp}$ *is a subspace of* V.
2. *Let S be equal to a subspace W . Then the subspaces W and $W^{\perp}$ are complementary. Moreover, if $\mathbf{w} \in W$ and $\mathbf{u} \in W^{\perp}$, then $\langle \mathbf{u}, \mathbf{w} \rangle = 0$ and $V = W + W^{\perp}$.*

Outline of the proof

- We leave the prove of the first part as an exercise.
- The prove of the second part is as follows:
- Let $\dim(V) = n$ and $\dim(W) = k$.
- Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ be a basis of W. By Gram-Schmidt orthogonalisation process, we get an orthonormal basis, say, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of W. Then, for any $\mathbf{v} \in V$,

$$\mathbf{v} - \sum_{i=1}^{k} \langle \mathbf{v}, \mathbf{v}_i \rangle \mathbf{v}_i \in W^{\perp}.$$

- So, $V \subset W + W^{\perp}$. Also, for any $\mathbf{v} \in W \cap W^{\perp}$, by definition of $W^{\perp}$, $\;0 = \langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2$. So, $\mathbf{v} = \mathbf{0}$.
- That is,

$$W \cap W^{\perp} = \{\mathbf{0}\}.$$

Definition (Orthogonal Projection)

Let W be a subspace of a finite dimensional inner product space V, with inner product $\langle \ , \ \rangle$. Let $W^{\perp}$ be the orthogonal complement of W in V. Define $P_W : V \longrightarrow V$ by

$$P_W(\mathbf{v}) = \mathbf{w} \text{ where } \mathbf{v} = \mathbf{w} + \mathbf{u} \text{ with } \mathbf{w} \in W, \text{ and } \mathbf{u} \in W^{\perp}.$$

Then P_W is called the orthogonal projection of V onto W along $W^{\perp}$.

Definition (Self-Adjoint Transformation/Operator)

Let V be an inner product space with inner product $\langle \ , \ \rangle$. A linear transformation $T : V \longrightarrow V$ is called a self-adjoint operator if $\langle T(\mathbf{v}), \mathbf{u} \rangle = \langle \mathbf{v}, T(\mathbf{u}) \rangle$ for every $\mathbf{u}, \mathbf{v} \in V$.

Example 1

- Let A be an $n \times n$ real symmetric matrix.
- That is, $A^t = A$. Then show that the linear transformation $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $T_A(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}^t \in \mathbb{R}^n$ is a self-adjoint operator.
- By definition, for every $\mathbf{x}^t, \mathbf{y}^t \in \mathbb{R}^n$,

$$\langle T_A(\mathbf{x}), \mathbf{y} \rangle = (\mathbf{y})^t A x = (\mathbf{y})^t A^t x = (A\mathbf{y})^t \mathbf{x} = \langle \mathbf{x}, T_A(\mathbf{y}) \rangle.$$

- Hence, the result follows.

Example 2

- Let A be an $n \times n$ Hermitian matrix, that is, $A^* = A$.
- Check that the linear transformation $T_A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ defined by

$$T_A(\mathbf{z}) = A\mathbf{z}$$

for every $\mathbf{z}^t \in \mathbb{C}^n$ is a self-adjoint operator.

Observations

- By above Proposition, the map P_W defined above is a linear transformation.
- $$P_W^2 = P_W, \ (I - P_W)P_W = \mathbf{0} = P_W(I - P_W).$$
- Let $\mathbf{u}, \mathbf{v} \in V$ with $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ for some $\mathbf{u}_1, \mathbf{v}_1 \in W$ and $\mathbf{u}_2, \mathbf{v}_2 \in W^{\perp}$.
- Then we know that $\langle \mathbf{u}_i, \mathbf{v}_j \rangle = 0$ whenever $1 \leq i \neq j \leq 2$.
- Therefore, for every $\mathbf{u}, \mathbf{v} \in V$,
$$\langle P_W(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{u}_1, \mathbf{v} \rangle = \langle \mathbf{u}_1, \mathbf{v}_1 \rangle = \langle \mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1 \rangle = \langle \mathbf{u}, P_W(\mathbf{v}) \rangle.$$
- Thus, the orthogonal projection operator is a self-adjoint operator.

Observations, cont.

- Let $\mathbf{v} \in V$ and $\mathbf{w} \in W$. Then $P_W(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w} \in W$. Therefore, using above observations, we get

$$\begin{aligned}
\langle \mathbf{v} - P_W(\mathbf{v}), \mathbf{w} \rangle &= \langle \big(I - P_W\big)(\mathbf{v}), P_W(\mathbf{w}) \rangle \\
&= \langle P_W\big(I - P_W\big)(\mathbf{v}), \mathbf{w} \rangle \\
&= \langle \mathbf{0}(\mathbf{v}), \mathbf{w} \rangle \\
&= \langle \mathbf{0}, \mathbf{w} \rangle \\
&= 0
\end{aligned}$$

  for every $\mathbf{w} \in W$.

- In particular,

$$\langle \mathbf{v} - P_W(\mathbf{v}), P_W(\mathbf{v}) - \mathbf{w} \rangle = 0.$$

Observations, cont.

- Thus,
$$\langle \mathbf{v} - P_W(\mathbf{v}), P_W(\mathbf{v}) - \mathbf{w}' \rangle = 0,$$
for every $\mathbf{w}' \in W$.
- Hence, for any $\mathbf{v} \in V$ and $\mathbf{w} \in W$, we have
$$\begin{aligned} \|\mathbf{v} - \mathbf{w}\|^2 &= \|\mathbf{v} - P_W(\mathbf{v}) + P_W(\mathbf{v}) - \mathbf{w}\|^2 \\ &= \|\mathbf{v} - P_W(\mathbf{v})\|^2 + \|P_W(\mathbf{v}) - \mathbf{w}\|^2 \\ &+ 2\langle \mathbf{v} - P_W(\mathbf{v}), P_W(\mathbf{v}) - \mathbf{w} \rangle \\ &= \|\mathbf{v} - P_W(\mathbf{v})\|^2 + \|P_W(\mathbf{v}) - \mathbf{w}\|^2. \end{aligned}$$
- Therefore,we have
$$\|\mathbf{v} - \mathbf{w}\| \geq \|\mathbf{v} - P_W(\mathbf{v})\|$$
- The equality holds if and only if $\mathbf{w} = P_W(\mathbf{v})$.

Observations, cont.

- Since $P_W(\mathbf{v}) \in W$, we see that

$$d(\mathbf{v}, W) = \inf \{\|\mathbf{v} - \mathbf{w}\| \ : \mathbf{w} \in W\} = \|\mathbf{v} - P_W(\mathbf{v})\|.$$

- That is, $P_W(\mathbf{v})$ is the vector nearest to $\mathbf{v} \in W$.
- This can also be stated as: the vector $P_W(\mathbf{v})$ solves the following minimisation problem:

$$\inf_{\mathbf{w} \in W} \|\mathbf{v} - \mathbf{w}\| = \|\mathbf{v} - P_W(\mathbf{v})\|.$$

Matrix orthogonal Projection: Motivation

- The minimization problem stated above arises in lot of applications.
- So, it will be very helpful if the matrix of the orthogonal projection can be obtained under a given basis.
- To this end, let W be a k -dimensional subspace of $\mathbb{R}^n$ with $W^\perp$ as its orthogonal complement.
- Let $P_W : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the orthogonal projection of $\mathbb{R}^n$ onto W . Suppose, we are given an orthonormal basis $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ of W.
- Under the assumption that $\mathcal{B}$ is known, we explicitly give the matrix of P_W with respect to an extended ordered basis of $\mathbb{R}^n$.

Matrix orthogonal Projection: Motivation

- Let us extend the given ordered orthonormal basis $\mathcal{B}$ of W to get an orthonormal ordered basis $\mathcal{B}_1 = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1} \ldots, \mathbf{v}_n)$ of $\mathbb{R}^n$.
- Then, for any

$$\mathbf{v} \in \mathbb{R}^n,\ \mathbf{v} = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{v}_i \rangle \mathbf{v}_i.$$

- Thus, by definition,

$$P_W(\mathbf{v}) = \sum_{i=1}^{k} \langle \mathbf{v}, \mathbf{v}_i \rangle \mathbf{v}_i.$$

- Let $A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k]$.
- Consider the standard orthogonal ordered basis

$$\mathcal{B}_2 = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$$

of $\mathbb{R}^n$.

- Therefore, if $\mathbf{v}_i = \sum_{j=1}^{n} a_{ji}\mathbf{e}_j$, for $1 \leq i \leq k$, then

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix}, [\mathbf{v}]_{\mathcal{B}_2} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}\langle \mathbf{v}, \mathbf{v}_i \rangle \\ \sum_{i=1}^{n} a_{2i}\langle \mathbf{v}, \mathbf{v}_i \rangle \\ \vdots \\ \sum_{i=1}^{n} a_{ni}\langle \mathbf{v}, \mathbf{v}_i \rangle \end{bmatrix}$$

and

$$[P_W(\mathbf{v})]_{\mathcal{B}_2} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}\langle \mathbf{v}, \mathbf{v}_i \rangle \\ \sum_{i=1}^{n} a_{2i}\langle \mathbf{v}, \mathbf{v}_i \rangle \\ \vdots \\ \sum_{i=1}^{n} a_{ni}\langle \mathbf{v}, \mathbf{v}_i \rangle \end{bmatrix}.$$

- Then we have, $A^t A = I_k$.

- That is, for $1 \leq i, j \leq k$,

$$\sum_{s=1}^{n} a_{si}a_{sj} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{1}$$

- Thus, using the associativity of matrix product and , we get

$$(AA^t)(\mathbf{v})A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{n} a_{1i}\langle \mathbf{v}, \mathbf{v}_i \rangle \\ \sum_{i=1}^{n} a_{2i}\langle \mathbf{v}, \mathbf{v}_i \rangle \\ \vdots \\ \sum_{i=1}^{n} a_{ni}\langle \mathbf{v}, \mathbf{v}_i \rangle \end{bmatrix}$$

$$= A \begin{bmatrix} \sum\limits_{s=1}^{n} a_{s1} \left(\sum_{i=1}^{n} a_{si}\langle \mathbf{v}, \mathbf{v}_i \rangle \right) \\ \sum\limits_{s=1}^{n} a_{s2} \left(\sum_{i=1}^{n} a_{si}\langle \mathbf{v}, \mathbf{v}_i \rangle \right) \\ \vdots \\ \sum\limits_{s=1}^{n} a_{sk} \left(\sum_{i=1}^{n} a_{si}\langle \mathbf{v}, \mathbf{v}_i \rangle \right) \end{bmatrix}$$

$$
(AA^t)(\mathbf{v})A = A \begin{bmatrix} \sum_{s=1}^{n} \left(\sum_{i=1}^{n} a_{s1} a_{si} \langle \mathbf{v}, \mathbf{v}_i \rangle \right) \\ \sum_{s=1}^{n} \left(\sum_{i=1}^{n} a_{s2} a_{si} \langle \mathbf{v}, \mathbf{v}_i \rangle \right) \\ \vdots \\ \sum_{s=1}^{n} \left(\sum_{i=1}^{n} a_{sk} a_{si} \langle \mathbf{v}, \mathbf{v}_i \rangle \right) \end{bmatrix}
$$

$$
= A \begin{bmatrix} \langle \mathbf{v}, \mathbf{v}_i \rangle \\ \langle \mathbf{v}, \mathbf{v}_i \rangle \\ \vdots \\ \langle \mathbf{v}, \mathbf{v}_i \rangle \end{bmatrix}
$$

$$
= \begin{bmatrix} \sum_{i=1}^{n} a_{1i} \langle \mathbf{v}, \mathbf{v}_i \rangle \\ \sum_{i=1}^{n} a_{2i} \langle \mathbf{v}, \mathbf{v}_i \rangle \\ \vdots \\ \sum_{i=1}^{n} a_{ni} \langle \mathbf{v}, \mathbf{v}_i \rangle \end{bmatrix}
$$

$$
= [P_W(\mathbf{v})]_{\mathcal{B}}.
$$

- Thus $P_W[\mathcal{B}_2, \mathcal{B}_2] = AA^t$.
- Thus, we have proved the following theorem.

Thank You