

System of Linear Equations

Let us consider $F (= \mathbb{R} \text{ or } \mathbb{C})$ a underlying field. Look at the following scalar linear equation in one variable

$$ax = b, \quad a, b \in \mathbb{R} \quad \text{--- (1)}$$

Let us look at the following three cases

- i) The equation (1) has UNIQUE solution if $a \neq 0$
- ii) The equation (1) has INFINITELY MANY solutions if $a = 0$ & $b = 0$
- iii) The equation has NO solution if $a = 0$ & $b \neq 0$

Let us look at the case of
system of two equations (linear)
in two variables. We begin with
the following examples

$$\begin{cases} 2x + y = 3 \\ x + 2y = 1 \end{cases}$$

The system has the solution

$$x = 5/3, y = -1/3$$

This solution is UNIQUE.

Consider the case

$$\begin{cases} 2x + y = 4 \\ 4x + 2y = 2 \end{cases}$$

The system has NO
solutions as the lines are
parallel to each other

$$(2/4 = 1/2)$$

Finally consider

$$\begin{cases} 2x + y = 4 \\ 4x + 2y = 8 \end{cases}$$

The system has infinitely many
solutions as the lines are
coinciding to each other as

$$(2/4 = 1/2 = 4/8)$$

Thus a system of two linear
equations has unique solution,
infinitely many solutions or no
solution. The solutions are
the intersection points of the
lines under consideration.

Let us formulate this discussion
for the following general system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases} \quad \begin{cases} a_i, b_i, c_i \in \mathbb{R} \\ (a_i, b_i) \neq (0, 0) \\ \text{for } i=1, 2 \end{cases}$$

We may encounter either of
the following three cases

I) UNIQUE solution if

$$a_1b_2 - a_2b_1 \neq 0$$

II) Infinitely many solutions if

$$a_1b_2 - b_1a_2 = 0 \text{ \& \& } b_1c_2 - b_2c_1 = 0$$

III) No solution if

$$a_1b_2 - b_1a_2 = 0 \text{ \& \& } b_1c_2 - b_2c_1 \neq 0$$

Let us consider the general case of a system of m -linear equations in n -variables x_1, x_2, \dots, x_n as

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \right\}$$

where $a_{ij} \in \mathbb{R}, b_i \in \mathbb{R}$ for all $1 \leq i \leq m$ & $1 \leq j \leq n$.

Let us denote by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Then above system can be written as

$$Ax = b, \text{ where}$$

$$x = (x_1, x_2, \dots, x_n)^T \text{ and } b = (b_1, b_2, \dots, b_m)^T$$

Definition (i) The system $Ax = b$, is homogeneous, if $b = 0$ otherwise system is called non-homogeneous.

The matrix A is called coefficient matrix and the block matrix

$$[A \ b]$$

is called as augmented matrix of the linear system.

Note that $[A \ b]$ is formed by attaching the column vector

$$b = (b_1, b_2, \dots, b_m)^T \text{ to the}$$

matrix A in $(n+1)^{\text{th}}$ column. Thus

$$[A, b] \text{ is a matrix of size } m \times (n+1).$$

* One should be careful while writing the system $Ax = b$, as the unknowns should be arranged in a uniform way.

iii) A solution of the equation $Ax=b$ is a vector of size $n \times 1$ satisfying $Ay=b$.

iv) The collection of all solutions of the system is called solution set of the system.

v) The system $Ax=b$ is called consistent if it has at least one solution otherwise it is inconsistent.

For example $x+y=1$, $2x+2y=3$ is inconsistent while

$x+y=1$, $2x+y=6$ is consistent.

Let us look at the case of homogeneous system, $Ax=0$

Note the following points

i) $x=0$ always solves $Ax=0$, irrespective of choice of A .

ii) If x_1, x_2 are two solutions of $Ax=b$, then $x_1 - x_2$ solves $Ax=0$.

iii) If x_1 is a solution of $Ax=0$, then αx is also a solution of $Ax=0$
 $\forall \alpha \in \mathbb{F}$

In general, if x_1, x_2, \dots, x_k are solutions of $Ax=0$, then any linear combination

$\sum_{i=1}^k \alpha_i x_i$ solves $Ax=0$

$\forall \alpha_i \in \mathbb{F}$.

Ques: We are discussing the cases of solutions being unique, or no solution or infinitely many solutions.

Is it possible for a system

$Ax=b$, $A \in M_{m \times n}(\mathbb{R})$ to have two or finitely many solutions?

Elementary Row Operations

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Let us look at the solution process of the following system.

Example:

$$2y + z = 2, 2x + 3z = 5, x + 2y + 2z = 4$$

Step 1: Rewriting the above system into

$Ax = b$ form, we get

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 2 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, b = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 0 & 3 & 5 \\ 1 & 2 & 2 & 4 \end{bmatrix} = [A \ b]$$

Step 2: Interchange 1st and 3rd equation to get

$$\begin{aligned} x + 2y + 2z &= 4 \\ 2x + 3z &= 5 \\ 2y + z &= 2 \end{aligned} \quad , \quad B_1 = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 0 & 3 & 5 \\ 0 & 2 & 1 & 2 \end{bmatrix}$$

Step 3: Interchange 2nd and 3rd equation to get

$$\begin{aligned} x + 2y + 2z &= 4 \\ 2y + z &= 2 \\ 2x + 3z &= 5 \end{aligned} \quad B_2 = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 1 & 2 \\ 2 & 0 & 3 & 5 \end{bmatrix}$$

Step 4: Multiply equation 2 by $\frac{1}{2}$ to get

$$\begin{aligned} x + 2y + 2z &= 4 \\ y + \frac{1}{2}z &= 1 \\ 2x + 3z &= 5 \end{aligned} \quad , \quad B_3 = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & \frac{1}{2} & 1 \\ 2 & 0 & 3 & 5 \end{bmatrix}$$

Step 5: Replace 3rd equation by 3rd equation - 2 times 1st equation

$$\begin{aligned} x + 2y + 2z &= 4 \\ y + \frac{1}{2}z &= 1 \\ -4y - z &= -3 \end{aligned} \quad B_4 = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & -4 & -1 & -3 \end{bmatrix}$$

Step 6: Replace equation 3rd by equation 3 plus 4 times equation 2

$$\begin{aligned} x + 2y + 2z &= 4 \\ y + \frac{1}{2}z &= 1 \\ z &= 1 \end{aligned} \quad B_5 = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus the last equation provides $z=1$. and from equation 2, on substituting $z=1$, we get $y=1/2$ and from equation 1, we get $x=4-2 \times 1/2 - 2 \times 1 = 1$.

Thus, $(1, 1/2, 1)$ solves the system uniquely.

Inspired from the above operations applied on the system which resulted into row operations on the augmented matrix, we define the following -

Definition: Let $A \in M_{m \times n}(\mathbb{F})$. Then the elementary row operations are as follows

- 1) interchanging i^{th} and j^{th} row. $(R_i \leftrightarrow R_j)$
- 2) multiplying k^{th} row by $0 \neq \lambda \in \mathbb{F}$ $(R_k \rightarrow \lambda R_k)$
- 3) replace i^{th} row by i^{th} row plus λ times j^{th} row for $\lambda \in \mathbb{F}$ $(R_i \rightarrow R_i + \lambda R_j)$

Definition: Two matrices are called row-equivalent if one can be obtained via elementary row operations on other.

Definition: The linear systems $Ax=b$ & $Cx=d$ are row equivalent if $[A \ b]$ is row equivalent to $[C \ d]$.

We conclude with the following theorem on row equivalent systems

Theorem: Let $Ax=b$ & $Cx=d$ be two row equivalent linear systems then they have the same solution set.