Department of Mathematics
Indian Institute of Technology Bhilai
IC104: Linear Algebra-I
Tutorial Sheet 3: Linear Transformation

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1. Verify that the functions, $T : F^3 \to F^3$ defined as below are linear transformations. Also write a basis of the range and null space of each linear transformation.

   (a) $T(x, y, z) = (x + y + z, x - y + z, x + z)$.

   (b) $T(x, y, z) = (x - y + 2z, 2x + y, -x - 2y = 2z)$

2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator with $T \neq 0$ and $T^2 = 0$. Prove that there exists a vector $x \in \mathbb{R}^n$ such that the set $\{x, T(x)\}$ is linearly independent.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (2x + 3y + 4z, x + y + z, x + y + 3z)$. Find the value of α for which there exists a vector $x \in \mathbb{R}^3$ such that $T(x) = (9, 3, \alpha)$.

4. Let $B \in M_{2\times2}(\mathbb{R})$. Now, define a map $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ by $T(A) = BA - AB$, for all $A \in M_{2\times2}(\mathbb{R})$. Determine if T is a linear transformation. If yes, find the range and null space of T.

5. Find a linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1, -1, 1) = (1, 2)$ and $T(-1, 1, 2) = (1, 0)$. Is it unique linear transformation of this type?

6. Let V be $n-$dimensional vector space over the field F. Let T be a linear transformation from V into V such that the range and null spaces are identical. Prove that n is even. Also give an example of such linear transformation.

7. Let V be finite dimensional vector space and $T : V \to V$. Suppose that rank (T^2)=rank (T), then prove that range and null space of T have only 0 vector in common.

8. Find the rank of the following matrix by exhibiting a basis of column space.

$$\begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

9. Suppose $m > n$. Justify the following statements:

   (a) There is no one to one (injective) linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$

   (b) There is no onto (surjective) linear transformation from $\mathbb{R}^n$to $\mathbb{R}^m$

10. Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ such that $T(1, 0, 0) = (1, 0, i)$, $T(0, 1, 0) = (0, 1, 1)$ and $T(0, 0, 1) = (i, 1, 0)$. Is T invertible?

11. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x, y, z) = (3x, x - y, 2x + y + z)$. Is T invertible? If yes find the inverse of T. Moreover, show that $(T^2 - I)(T - 3I) = 0$.

12. Let T be the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by

$$T(x, y, z) = (x + y, 2y - x)$$

Let $\mathcal{B} = \{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$ and $\mathcal{B}' = \{(1, 1), (2, 3))\}$ be the ordered bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Then what is the matrix of T relative to the pair $\mathcal{B}, \mathcal{B}'$?

13. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by as $T(x, y) = (-y, x)$. Then prove the following

    (a) What is the matrix of T in the standard basis of $\mathbb{R}^2$?

    (b) What is the matrix of T in the ordered basis $\mathcal{B} = \{(1, 2), (1, -1)\}$ of $\mathbb{R}^2$?

    (c) For every real number c the transformation $(T - cI) : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible.

    (d) If $\mathcal{B}$ is any ordered basis of $\mathbb{R}^2$ and $[T]_{\mathcal{B}} = A$, then $A_{21}.A_{12} \neq 0$.

14. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined as

$$T(x, y, z) = 3x + z, -2x + y, -x + 2y + 4z$$

    (a) What is the matrix of T in the standard basis of $\mathbb{R}^3$?

    (b) What is the matrix of T in the ordered basis $\mathcal{B} = \{(1, 0, 1), (-1, 2, 1), (2, 1, 1)\}$

    (c) Prove that T is invertible. Determine T^{-1} also.