Eigenvalue eigenvector

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Linear algebra- II (IC152)

Observations

- The main idea is to express $H(\mathbf{x})$ as sum of squares and hence determine the possible values that it can take.
- Note that if we replace $\mathbf{x}$ by $c\mathbf{x}$, where c is any complex number, then $H(\mathbf{x})$ simply gets multiplied by $|c|^2$ and hence one needs to study only those $\mathbf{x}$ for which $\|\mathbf{x}\| = 1$, i.e., $\mathbf{x}$ is a normalised vector.
- If $A = A^*$ (A is Hermitian) then there exists a unitary matrix U such that $U^*AU = D$ ($D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with λ_i 's the eigenvalues of the matrix A which we know are real).

Observations cont.

- So, taking $\mathbf{z} = U^*\mathbf{x}$ (i.e., choosing z_i 's as linear combination of x_j 's with coefficients coming from the entries of the matrix U^*), one gets

$$\begin{aligned} H(\mathbf{x}) = \mathbf{x}^*A\mathbf{x} &= \mathbf{z}^*U^*AU\mathbf{z} \\ &= \mathbf{z}^*D\mathbf{z} \\ &= \sum_{i=1}^{n} \lambda_i \left|z_i\right|^2 \\ &= \sum_{i=1}^{n} \lambda_i \left|\sum_{j=1}^{n} {u_{ji}}^* x_j\right|^2 . \qquad (1) \end{aligned}$$

- Thus, one knows the possible values that $H(\mathbf{x})$ can take depending on the eigenvalues of the matrix A in case A is a Hermitian matrix.

Observations cont.

- Also, for $1 \leq i \leq n$, $\sum_{j=1}^{n} {u_{ji}}^{*} x_j$ represents the principal axes of the conic that they represent in the n-dimensional space.
- Equation (1) gives one method of writing $H(\mathbf{x})$ as a sum of n absolute squares of linearly independent linear forms.
- One can easily show that there are more than one way of writing $H(\mathbf{x})$ as sum of squares.
- The question arises, “what can we say about the coefficients when $H(\mathbf{x})$ has been written as sum of absolute squares".
- This question is answered by ‘Sylvester’s law of inertia’ which we state as the next theorem.

Theorem

Every Hermitian form $H(\mathbf{x}) = \mathbf{x}^* A \mathbf{x}$ *(with* A *an Hermitian matrix) in* n *variables can be written as*

$$H(\mathbf{x}) = |y_1|^2 + |y_2|^2 + \cdots + |y_p|^2 - |y_{p+1}|^2 - \cdots - |y_r|^2$$

where $y_1, y_2, \ldots, y_r$ *are linearly independent linear forms in* $x_1, x_2, \ldots, x_n$, *and the integers* p *and* r, $0 \leq p \leq r \leq n$, *depend only on* A.

Outline of the proof

- From Equation (1), it is easily seen that $H(\mathbf{x})$ has the required form. Need to show that p and r are uniquely given by A.
- Hence, let us assume on the contrary that there exist positive integers p, q, r, s with $p > q$ such that /

$$\begin{aligned} H(\mathbf{x}) &= |y_1|^2 + |y_2|^2 + \cdots + |y_p|^2 - |y_{p+1}|^2 - \cdots - |y_r|^2 \\ &= |z_1|^2 + |z_2|^2 + \cdots + |z_q|^2 - |z_{q+1}|^2 - \cdots - |z_s|^2. \end{aligned}$$

- Since, $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ and $\mathbf{z} = (z_1, z_2, \ldots, z_n)^t$ are linear combinations of $x_1, x_2, \ldots, x_n$, we can find a matrix B such that $\mathbf{z} = B\mathbf{y}$.
- Choose $y_{p+1} = y_{p+2} = \cdots = y_r = 0$.

Outline of the proof cont.

- Since $p > q$, Theorem 2.5.1, gives the existence of finding nonzero values of $y_1, y_2, \ldots, y_p$ such that $z_1 = z_2 = \cdots = z_q = 0$.
- Hence, we get

$$|y_1|^2 + |y_2|^2 + \cdots + |y_p|^2 = -(|z_{q+1}|^2 + \cdots + |z_s|^2).$$

- Now, this can hold only if $y_1 = y_2 = \cdots = y_p = 0$, which gives a contradiction. Hence $p = q$.
- Similarly, the case $r > s$ can be resolved.

- We complete this chapter by understanding the graph of

$$ax^2 + 2hxy + by^2 + 2fx + 2gy + c = 0$$

for $a, b, c, f, g, h \in \mathbb{R}$. We first look at the following example.

Example

- Sketch the graph of $3x^2 + 4xy + 3y^2 = 5$.
- Note that
$$3x^2 + 4xy + 3y^2 = [x, \ \ y] \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$
- The eigenpairs for $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are $(5, (1,1)^t), \ (1, (1,-1)^t)$.
- Thus,
$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$
- Let
$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} \frac{x+y}{\sqrt{2}} \\ \frac{x-y}{\sqrt{2}} \end{bmatrix}.$$

- Then

$$
\begin{aligned}
3x^2 + 4xy + 3y^2 &= [x, \quad y] \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= [x, \quad y] \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} u, & v \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\
&= 5u^2 + v^2.
\end{aligned}
$$

- Thus the given graph reduces to

$$5u^2 + v^2 = 5 \quad \text{or equivalently} \quad u^2 + \frac{v^2}{5} = 1.$$

- Therefore, the given graph represents an ellipse with the principal axes $u = 0$ and $v = 0$. That is, the principal axes are

$$y + x = 0 \text{ and } \quad x - y = 0.$$

- The eccentricity of the ellipse is $\quad e = \frac{2}{\sqrt{5}}$, the foci are at the points $S_1 = (-\sqrt{2}, \sqrt{2})$ and $S_2 = (\sqrt{2}, -\sqrt{2})$, and the equations of the directrices are $x - y = \pm\frac{5}{\sqrt{2}}$.

Definition (Associated Quadratic Form)

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be the equation of a general conic. The quadratic expression

$$ax^2 + 2hxy + by^2 = \begin{bmatrix} x, & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is called the quadratic form associated with the given conic.

- We now consider the general conic. We obtain conditions on the eigenvalues of the associated quadratic form to characterise the different conic sections in $\mathbb{R}^2$ (endowed with the standard inner product).

Theorem

Consider the general conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Prove that this conic represents

1. *an ellipse if* $ab - h^2 > 0$,
2. *a parabola if* $ab - h^2 = 0$, *and*
3. *a hyperbola if* $ab - h^2 < 0$.

- Let $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$.
- Then the associated quadratic form

$$ax^2 + 2hxy + by^2 = \begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix}.$$

- As A is a symmetric matrix, the eigenvalues λ_1, λ_2 of A are both real, the corresponding eigenvectors $\mathbf{u}_1, \mathbf{u}_2$ are orthonormal and A is unitarily diagonalisable with

$$A = \begin{bmatrix} \mathbf{u}_1^t \\ \mathbf{u}_2^t \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \tag{2}$$

- Let $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

- Then
$$ax^2 + 2hxy + by^2 = \lambda_1 u^2 + \lambda_2 v^2$$
and the equation of the conic section in the (u, v) -plane, reduces to
$$\lambda_1 u^2 + \lambda_2 v^2 + 2g_1 u + 2f_1 v + c = 0.$$
- Now, depending on the eigenvalues λ_1, λ_2, we consider different cases:

Case: 1

- $\lambda_1 = 0 = \lambda_2$. Substituting $\lambda_1 = \lambda_2 = 0$ in (6.4.2) gives $A = \mathbf{0}$.
- Thus, the given conic reduces to a straight line $2g_1u + 2f_1v + c = 0$ in the (u, v) -plane.

Case 2

- $\lambda_1 = 0, \lambda_2 \neq 0$. In this case, the equation of the conic reduces to

$$\lambda_2(v + d_1)^2 = d_2 u + d_3 \quad \text{for some} \quad d_1, d_2, d_3 \in \mathbb{R}.$$

1. If $d_2 = d_3 = 0$, then in the (u, v) -plane, we get the pair of coincident lines $v = -d_1$.
2. If $d_2 = 0,\ d_3 \neq 0$.
3. If $\lambda_2 \cdot d_3 > 0$, then we get a pair of parallel lines $v = -d_1 \pm \sqrt{\frac{d_3}{\lambda_2}}$.
4. If $\lambda_2 \cdot d_3 < 0$, the solution set corresponding to the given conic is an empty set.
5. If $d_2 \neq 0$. Then the given equation is of the form $Y^2 = 4aX$ for some translates $X = x + \alpha$ and $Y = y + \beta$ and thus represents a parabola.
6. Also, observe that $\lambda_1 = 0$ implies that the $\det(A) = 0$. That is, $ab - h^2 = \det(A) = 0$.

Case: 3

- $\lambda_1 > 0$ and $\lambda_2 < 0$. Let $\lambda_2 = -\alpha_2$.
- Then the equation of the conic can be rewritten as

$$\lambda_1(u + d_1)^2 - \alpha_2(v + d_2)^2 = d_3 \quad \text{for some} \quad d_1, d_2, d_3 \in \mathbb{R}.$$

- In this case, we have the following subcases:
  Subcase 1:
  1. suppose $d_3 = 0$. Then the equation of the conic reduces to

$$\lambda_1(u + d_1)^2 - \alpha_2(v + d_2)^2 = 0.$$

  2. The terms on the left can be written as product of two factors as $\lambda_1, \alpha_2 > 0$.
  3. Thus, in this case, the given equation represents a pair of intersecting straight lines in the (u, v) -plane.

Subcase 2

- suppose $d_3 \neq 0$. As $d_3 \neq 0$, we can assume $d_3 > 0$. So, the equation of the conic reduces to

$$\frac{\lambda_1(u+d_1)^2}{d_3} - \frac{\alpha_2(v+d_2)^2}{d_3} = 1.$$

- This equation represents a hyperbola in the (u, v) -plane, with principal axes

$$u + d_1 = 0 \text{ and } \quad v + d_2 = 0.$$

- As $\lambda_1\lambda_2 < 0$, we have

$$ab - h^2 = \det(A) = \lambda_1\lambda_2 < 0.$$

Case 4

- $\lambda_1, \lambda_2 > 0$. In this case, the equation of the conic can be rewritten as

$$\lambda_1(u + d_1)^2 + \lambda_2(v + d_2)^2 = d_3, \quad \text{for some} \quad d_1, d_2, d_3 \in \mathbb{R}.$$

  We now consider the following Subcases:
  Subcase: 1: suppose $d_3 = 0$. Then the equation of the ellipse reduces to a pair of perpendicular lines $u + d_1 = 0$ and $v + d_2 = 0$ in the (u, v) -plane.
  Subcase: 2: suppose $d_3 < 0$. Then there is no solution for the given equation. Hence, we do not get any real ellipse in the (u, v) -plane.
  Subcase: 3: suppose $d_3 > 0$. In this case, the equation of the conic reduces to

$$\frac{\lambda_1(u + d_1)^2}{d_3} + \frac{\alpha_2(v + d_2)^2}{d_3} = 1.$$

Case 4 cont.

- This equation represents an ellipse in the (u, v) -plane, with principal axes

$$u + d_1 = 0 \text{ and } \quad v + d_2 = 0.$$

- Also, the condition $\lambda_1\lambda_2 > 0$ implies that

$$ab - h^2 = \det(A) = \lambda_1\lambda_2 > 0.$$

Thank You