Eigenvalue eigenvector

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Linear algebra- II (IC152)

Theorem

Let $A = [a_{ij}]$ *be an* $n \times n$ *matrix with eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$, *not necessarily distinct. Then* $\det(A) = \prod_{i=1}^{n} \lambda_i$ *and* *tr*$(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$.

Outline of the proof

- Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the n eigenvalues of A, by definition,

$$\det(A - \lambda I_n) = p(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n). \tag{1}$$

- Equation (1) is an identity in λ as polynomials. Therefore, by substituting $\lambda = 0$ in (1), we get

$$\det(A) = (-1)^n(-1)^n \prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n} \lambda_i.$$

- Also,

$$\det(A-\lambda I_n) = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$
$$= a_0 - \lambda a_1 + \cdots + (-1)^{n-1}\lambda^{n-1}a_{n-1} + (-1)^n\lambda^n \qquad (2)$$

  for some $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}$.
- Note that a_{n-1}, the coefficient of $(-1)^{n-1}\lambda^{n-1}$, comes from the product

$$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda).$$

- So,

$$a_{n-1} = \sum_{i=1}^{n} a_{ii} = \mathrm{tr}(A)$$

  by definition of trace.

- From (1) and (2), we get

$$a_0-\lambda a_1+\cdots+(-1)^{n-1}\lambda^{n-1}a_{n-1}+(-1)^n\lambda^n = (-1)^n(\lambda-\lambda_1)\cdots(\lambda-\lambda_n).$$

- Therefore, comparing the coefficient of $(-1)^{n-1}\lambda^{n-1}$, we have

$$\text{tr}(A) = a_{n-1} = (-1)\{(-1)\sum_{i=1}^{n}\lambda_i\} = \sum_{i=1}^{n}\lambda_i.$$

- Let A be an $n \times n$ matrix. Then in the proof of the above theorem, we observed that the characteristic equation $\det(A - \lambda I) = 0$ is a polynomial equation of degree n in λ.
- Also, for some numbers $a_0, a_1, \ldots, a_{n-1} \in \mathbb{F}$, it has the form

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0 = 0.$$

- Note that, in the expression $\det(A - \lambda I) = 0, \;\; \lambda$ is an element of $\mathbb{F}$. Thus, we can only substitute λ by elements of $\mathbb{F}$.
- It turns out that the expression

$$A^n + a_{n-1}A^{n-1} + \cdots a_1 A + a_0 I = \mathbf{0}$$

holds true as a matrix identity.
- This is a celebrated theorem called the Cayley Hamilton Theorem.

- Let us consider a 2×2 matrix $A = \begin{bmatrix} x^2+x+1 & x^3+2x \\ 3x^3+x & 4x^2+3 \end{bmatrix}$ whose elements are real polynomials in x.
- A can be expressed as the polynomial in x

$$\begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} x^3 + \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} x^2 + \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},$$

  whose co-efficients are real matrices of order 2×2.
- Such a polynomial is said to be matrix polynomial.
- The degree of the matrix polynomial is the degree of the constituent polynomial of highest degree appearing in the matrix A.
- In general, if A is an $n \times n$ matrix whose elements are real(complex) polynomials of x, then A can be expressed as a matrix polynomial whose co-efficients are real(complex) matrices of order $n \times n$.

- Two matrix polynomials $F(x)$ and $G(x)$ whose co-efficients are matrices of the same order over the same field are said to be equal if they have the same degree and the co-efficients of likes powers of x be equal matrices.
- Let $F(x) = \sum_{k=0}^{n} A_k x^k$ and $G(x) = \sum_{k=0}^{m} B_k x^k$ be two matrix polynomials whose co-efficients are square matrices of the same order over the same field.

Sum and Product of two matrix polynomials

- The sum of $F(x) + G(x)$ is defined as

$$F(x) + G(x) = \begin{cases} \sum_{k=0}^{m} (A_k + B_k)x^k + A_{m+1}x^{m+1} + \cdots + A_n x^n & \text{if } m < n \\ \sum_{k=0}^{n} (A_k + B_k)x^k + B_{n+1}x^{n+1} + \cdots + B_m x^m & \text{if } n < m \\ \sum_{k=0}^{m} (A_k + B_k)x^k & \text{if } n = m. \end{cases}$$

- The product of $F(x)G(x)$ is defined as

$$F(x)G(x) = \sum_{k=0}^{m+n} C_k x^k,$$

  where $C_k = \sum_{i=0}^{k} A_i B_{k-i}, k = 1, \cdots, m+n.$
- Observe that $F(x)G(x) \neq G(x)F(x)$, because matrix multiplication is not commutative.

Example

- Let $F(x) = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 0 \end{smallmatrix}\right) x + \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$. We will show that $F(x)\text{adj}F(x) = \det F(x) I_2$.
- $F(x)\text{adj}F(x) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} x^2 + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -(2x^2+1)I_2$
- This shows that $F(x)\text{adj}F(x) = \det F(x) I_2$.

Example

Here we will discuss few examples related to the Cayley-Hamilton Theorem.

1. For a 1×1 matrix $A = (a_{11})$, the characteristic polynomial is given by $p(\lambda) = \lambda - a$, and so $p(A) = (a) - a_{11} = 0$ is trivial.
2. For a generic 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
    1. The characteristic polynomial is given by
    $$p(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc),$$
    2. So, the Cayley-Hamilton theorem states that
    $$p(A) = A^2 - (a + d)A + (ad - bc)I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$
    .
3. Does the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ satisfies the equation (3)?

Theorem (Cayley Hamilton Theorem)

Let A be a square matrix of order n and

$$c_0x^n + c_1x^{n-1} + \cdots + c_n$$

be the characteristic polynomial of A. Then A satisfies its characteristic equation. That is,

$$c_0A^n + c_1A^{n-1} + \cdots + c_nI_n = \mathbf{0}$$

holds true as a matrix identity.

Outline of proof

- Let A be an $n \times n$ matrix. Then

$$\det(A - xI_n) = c_0x^n + c_1x^{n-1} + \cdots + c_n.$$

- $A - xI_n$ is a matrix polynomial in x of degree 1 and $\operatorname{adj}(A - xI_n)$ is a matrix polynomial in x of degree $(n-1)$, since each element of $\operatorname{adj}(A - xI_n)$, that is, a cofactor of an element of the matrix $A - xI_n$, is a polynomial in x of degree at most $(n-1)$.
- Let $\operatorname{adj}(A - xI_n) = B_0x^{n-1} + B_1x^{n-2} + \cdots + B_{n-1}$, where each B_i is an $n \times n$ matrix.
- $(A - xI_n)\operatorname{adj}(A - xI_n) = [\det(A - xI_n)]I_n$ gives

$$\begin{aligned} A(B_0x^{n-1} + B_1x^{n-2} + \cdots + B_{n-1}) - (B_0x^n + B_1x^{n-1} + \cdots + B_{n-1}x) \\ = c_0I_nx^n + c_1I_nx^{n-1} + \cdots + c_nI_n. \end{aligned}$$

- Equating coefficents of like powers of x, we have

- $-B_0 = c_0 I_n, AB_0 - B_1 = c_1 I_n, \cdots, AB_{n-2} - B_{n-1} = c_{n-1} I_n, AB_{n-1} = c_n I_n.$
- Pre-multiplying the relation by $A^n, \cdots, A, I_n$ respectively and adding, we have

$$c_0 A^n + c_1 A^{n-1} + \cdots + c_n I_n = \mathbf{0}$$

This completes the proof.

Example

We will use the Cayley Hamilton theorem to find the inverse of a matrix A, where $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$.

1. The characteristic equation of A is $\det(A - \lambda I) = \lambda^2 - 7\lambda + 7 = 0$.
2. By Cayley-Hamilton theorem we get $A^2 - 7A + 7I_2 = O$.
3. Solving further we get $-\frac{1}{7}A(A - 7I_2) = I_2$.
4. This gives $A^{-1} = -\frac{1}{7}(A - 7I_2) = \frac{1}{7}\begin{pmatrix} 5 & -1 \\ -3 & 2 \end{pmatrix}$.

Example

We will use the Cayley Hamilton theorem to find A^{50}, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

1. The characteristic equation of A is $\det(A - \lambda I) = \lambda^2 - 2\lambda + 1 = 0$.
2. By Cayley-Hamilton theorem, $A^2 - 2A + I_2 = O$ or , $A^2 - A = A - I_2$.
3. Therefore $A^3 - A^2 = A^2 - A = A - I_2, ..., A^{50} - A^{49} = A - I_2$.
4. Adding, we have $A^{50} = 50A - 49I_2 = \begin{pmatrix} 1 & 50 \\ 0 & 1 \end{pmatrix}$.