Row Reduced Echelon Matrix

Definition: A matrix is called a Row reduced echelon matrix (RRE) if the following conditions are satisfied.

i) all the zero rows (if any) are at the bottom

ii) the leading coefficient of every nonzero row is equal to 1

iii) A column which contains leading nonzero entry (which is 1) of a row has all other entries equal to zero

iv) the leading entry of $(i+1)^{th}$ row (which is 1), if it exists, comes to the right of the leading entry of i^{th} row.

For example :-

$$A = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here A is in RREF while B is not!! Construct other matrices which fail the above conditions.

Let us look into an algorithm to compute RRE Form of a matrix.

The process is known as Gauss–Jordan Elimination (GJE)

We will first see through an example.

Example: Consider

$$A = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

Step 1: Apply $R_3 \longleftrightarrow R_4$, to get

$$\sim \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 4 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2: Apply $R_1 \leftrightarrow R_3$ to get

$$\sim \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 4 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 3: Apply $R_1 \to \frac{1}{2} R_1$, we get

$$\sim \begin{bmatrix} 0 & 1 & 1/2 & 0 \\ 0 & 4 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 4: Apply $R_2 \to R_2 - 4R_1$,

$$\sim \begin{bmatrix} 0 & 1 & 1/2 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 5: Apply $R_2 \to -\frac{1}{2} R_2$,

$$\sim \begin{bmatrix} 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 6: Apply $R_1 \to R_1 - \frac{1}{2} R_2$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 7: Apply $R_3 \to R_3 - 2R_2$,

$$\sim \begin{bmatrix} 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 8: Apply $R_3 \to \frac{1}{3} R_3$,

$$\sim \begin{bmatrix} 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 9: Apply $R_2 \to R_2 + R_3$ and
$R_1 \to R_1 - \frac{1}{2} R_3$,

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in RRE form.

We will write the above demonstration in the form of an algorithm below.

Step 1: Apply interchanging of rows to push the zero rows in the bottom of the matrix.

Step 2: Find the first nonzero column from left and apply the interchanging of rows to move the row which has nonzero entry in the first nonzero column, to the first row.

Step 3: Divide the first row with the leading coefficient of the row to have leading entry as 1 (Apply $R_1 \to \lambda R_1$ for suitable λ)

Step 4: Apply $R_i \to R_i + \lambda R_1$ $i > 1$ to make all the entries in the first nonzero column equal to zero.

Step 5. Ignore the first row and look for the first nonzero column, apply the interchanging of rows to send the rows which has nonzero entry in the first nonzero column (of course after ignoring first row) to the 2^{nd} row.

Step 6. Apply $R_2 \to \lambda R_2$ to make the coefficient of leading term equal to 1

Step 7. Apply $R_i \to R_i + \mu R_2$ to make other entries in the first nonzero column (count after ignoring first row) equal to zero

Step 8. Find the first nonzero column after ignoring first two rows. Repeat the process as above until there is no nonzero row.

The following Theorem talks about the uniqueness of RRE for a matrix.

Theorem: Every matrix is row equivalent to a unique row reduced echelon matrix.

Remark:- Under the equivalence relation ($A \sim B$, if A is row equivalent to B) it is easy to see that all the row equivalent matrices of A have the same RRE(A).

Applications of RREF

We will see the following three applications of RREF

i) Finding Inverse of a matrix
ii) Finding rank of a matrix
iii) Solving system of equations.

Let us assume a square matrix A. Then we have the following theorem

Theorem Let $A \in M_n(\mathbb{C})$. Then A is invertible if and only if
$$RREF(A) = I_n$$

Proof: We have to prove the result in two directions. Firstly assume that $RREF(A) = I_n$, then
$$I_n = E_1 . E_2 \cdots E_k \cdot A \text{ for some}$$
elementary matrices $E_1, E_2, - E_k$

Now, using invertibility of E_i's, we have,
$$E_1^{-1} = E_2 \cdot E_3 \cdot E_k A$$
$$\& \quad E_2^{-1} E_1^{-1} = E_3 - E_k A$$
continuing this way, we get
$$E_k^{-1} \cdots E_2^{-1} E_1^{-1} = A.$$
Moreover, $A E_1 E_2 \cdot E_k = I_n$
which implies $A^{-1} = E_1 \cdot E_2 \cdots E_k$

Conversely assume that A is invertible. Let $B = RREF(A) = E_1 E_2 \cdot E_k \cdot A$.
Note that B is invertible as E_i's and A are.
Thus B has no zero rows.
It means the leading entry of each row is 1 and there are n such rows. As B is RREF implies B is I_n.

Remark: If R is RRE and equivalent to A then A^{-1} exists $\Leftrightarrow$ R^{-1} exists.