Let us begin with the following observation regarding elementary row operations. Let $A \in M_{m\times n}(\mathbb{R})$ and e be any elementary row operation (either of the three operations defined in lecture-1) then effect of this elementary operation satisfies

$$e(A) = e(I)\cdot A \quad - (1)$$

where $I \in M_{m\times m}(\mathbb{R})$ is an identity matrix.

Next we see the following definition

<u>Definition</u> :- A matrix $E \in M_{n\times n}(\mathbb{R} \text{ or } \mathbb{C})$ is called elementary matrix if it is obtained by $I \in M_n(\mathbb{R} \text{ or } \mathbb{C})$ identity matrix after applying exactly one elementary row operation, i.e. $E = e(I) \quad - (2)$

From (1) and (2) it is clear that applying row elementary operation on a matrix A is equal to multiplying (left multiplication) elementary matrix (corresponding to same row elementary operation) with the matrix itself. i.e

$$e(A) = E\cdot A$$

where $E = e(I)$

Let us verify it with the help of the following example

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Consider $e = R_1 \leftrightarrow R_2$

Then $e(I) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $e(A) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

Is $e(I)\cdot A = e(A)$?

One can verify other elementary row operations in a similar way as

$e = R_2 \mapsto \lambda R_2$, $\lambda \neq 0$, then

$$e(I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad e(A) = \begin{bmatrix} 1 & 0 & 3 \\ 2\lambda & \lambda & \lambda \\ 3 & 2 & 1 \end{bmatrix}$$

Now we need to check if

$$e(I).A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 2\lambda & \lambda & \lambda \\ 3 & 2 & 1 \end{bmatrix} \quad !!$$

Also you can try with $e = R_3 \to R_3 + \lambda R_2$

Let us look into the following question

<u>Ques</u>: ~~Is~~ Are the following matrices elementary matrix?

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Now let us look back into our elementary row operations. It is easy to see that every elementary row operation is invertible. See the chart below.

E.R. operation	Inverse
$R_i \longleftrightarrow R_j$	$R_i \longleftrightarrow R_j$ (self)
$R_i \to \lambda R_i, \lambda \neq 0$	$R_i \to \frac{1}{\lambda} R_i$
$R_i \to R_i + \lambda R_j$	$R_i \to R_i - \lambda R_j$

Can we say similar thing for elementary matrices !!

Answer is YES. We have the following Theorem.

<u>Theorem</u>: Every elementary matrix is invertible and the inverse is the elementary matrix of the same type.

Indeed, if $e = R_i \leftrightarrow R_j$ then applying e twice on I, we get back the identity matrix. Thus if $E = e(I)$ then $E^2 = E \cdot E = I$ implies E is inverse of itself.

Next if $e = R_i \to \lambda R_i$ then if $E = e(I)$, it is very evident that the inverse of E is an elementary matrix formed by applying elementary row operation $R_i \to \frac{1}{\lambda} R_i$ on I. (Here $\lambda \neq 0$)

Finally if $e = R_i \to R_i + \lambda R_j$ then for $E = e(I)$, it is easy to verify that the inverse of E is the elementary matrix formed by the direct application of row-elementary operation $R_i \to R_i - \lambda R_j$ on I.

Now we see the following proposition which relates two row-equivalent matrices via elementary matrices.

<u>Proposition</u> :- Let A and B be two row-equivalent matrices then

there exist elementary matrices $E_1, E_2, \ldots E_k$ such that

$A = E_1 \cdot E_2 \cdots E_k B$

<u>Proof</u>: A proof of this proposition follows by above discussion and the definition of row equivalent matrices.

Now we go back and see the last result of lecture 1 (row equivalent systems have same solution set).

We have two systems

$$Ax = b \ \& \ Cx = d,$$ then

by the previous proposition, as $[A\ b]$ and $[C, d]$ are row equivalent, there exist elementary matrices $E_1, E_2, \cdots E_k$ such that

$$E_1 . E_2 \cdots E_k [A\ b] = [C, d]$$

Let us denote $E = E_1 . E_2 . E_3 \cdots E_k$ for writing convenience, then

$$E[A\ b] = [C, d]$$

which implies

$$EA = C \ \& \ Eb = d$$

Now as E is invertible (product of invertible (elementary) matrices), we get $A = E^{-1}C$ and $b = E^{-1}d$

Now we would like to show the following. If y solves $Ax = b$ then y also solves $Cx = d$ and if z solves $Cx = d$ then z solves $Ax = b$ also.

First assume that y solves

$$Ax = b \Rightarrow Ay = b$$

Now compute Cy which is equal to EAy ie

$$Cy = EAy = Eb = d$$

Conversely if z satisfies $Cz = d$, then $Az = E^{-1}Cz = E^{-1}d = b$

Hence both the systems have common solution set.

Rk: Two systems are not row equivalent to each other if they have different solution set.