Tutorial 4: Calculus I (IC153)

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1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that $|f(x)| \leq |g(x)|$ for all $x \in \mathbb{R}$. If g is continuous at 0 and $g(0) = 0$, then show that f is continuous.

2. Prove that the function $f(x) = \begin{cases} 4x + 2 & \text{if } x < 1 \\ 5x^2 & \text{if } x \geq 1 \end{cases}$ is not continuous at 1.

3. If $f : [0, 1] \to [0, 1]$ is continuous, then there exists $c \in [0, 1]$ such that $f(c) = c$

4. Let $f : [0, 2] \to \mathbb{R}$ continuous such that $f(0) = f(2)$. Then there exists $x_1, x_2 \in [0, 2]$ such that $x_1 - x_2 = 1$ and $f(x_1) = f(x_2)$.

5. Prove that the function $f(x) = 1 - x^{2/3}$ for all $x \in \mathbb{R}$ has no local maximum or local minimum at any non zero x. Also prove show that f has local maximum at $x = 0$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(x) = x^2 + 5,\ x \in \mathbb{Q}$. The find $f(\sqrt{2})$.

7. prove that between any two real roots of the equation $e^x \cos x + 1 = 0$ has at least one real root of the equation $e^x \sin x + 1 = 0$

8. If $f'(x)$ exists on $[0, 1]$ then show by Cauchy's MVT that the equation $3x^2[f(1) - f(0)] = f'(x)$ has at least one solution in $(0, 1)$.

9. Prove that if $a_0, a_1, a_2, \cdots, a_n$ are real number such that $\frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{2} + a_n = 0$. Then there exists at least one number x between 0 and 1 such that $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots a_n = 0$.

10. Let $p(x)$ be a polynomial in x and α is a real number then prove that there exists a root of $p'(x) + \alpha p(x) = 0$ between any pair of roots of $p(x)$.

11. If $f''(x) > 0$ on $[a, b]$. Prove that $f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}[f(x_1) + f(x_2)]$ for any two points x_1, x_2 in $[a, b]$.