① $(n+1)! > 2^n$, $n \geq 2$

Take $n=2$, then $(2+1)! > 2^2$

Let the inequality hold for $k \geq 2$ then $(k+1)! > 2^k$

Now $(k+2)! = (k+1)!\,(k+2) > 2^k\,(k+2)$

$> 2^k \cdot 2 = 2^{k+1}$ $[\because 2+k > 2]$

$\Rightarrow$ Hence by the principle of mathematical induction the inequality holds for $\forall\ n \geq 2$

② $P(n)$: $1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$

$n=1$: $1 = \frac{1}{6}\cdot 1\cdot 2\cdot 3 = 1$.

So $P(1)$ is true. Let $P(k)$ is true. i.e.

$1^2+2^2+3^2+\cdots+k^2 = \frac{1}{6}k(k+1)(2k+1)$

Now $1^2+2^2+3^2+\cdots+k^2+(k+1)^2 = \frac{1}{6}k(k+1)(2k+1)+(k+1)^2$

$= \frac{1}{6}(k+1)\left[k(2k+1)+6(k+1)\right]$

$= \frac{1}{6}(k+1)\left[2k^2+3k+4k+6\right]$

$= \frac{1}{6}(k+1)(2k+3)(k+2)$

$= \frac{1}{6}(k+1)\left(2(k+1)+1\right)\left((k+1)+1\right)$

so the $P(k+1)$ is true.

$\Rightarrow$ $P(n)$ is true for $\forall n \in \mathbb{N}$.

(3) Similar.

(4) $a < b \Rightarrow a+a < a+b \Rightarrow a < \frac{a+b}{2}$

$a < b \Rightarrow a+b < b+b \Rightarrow \frac{a+b}{2} < b$

$\Rightarrow a < \frac{a+b}{2} < b$

Suppose a is the least positive real then $a > 0$.

$\Rightarrow 0 < \frac{a}{2} < a$ (by above).

$\because \frac{a}{2} \in \mathbb{R}$ and $\frac{a}{2} > 0$, $\frac{a}{2} < a$

$\Rightarrow a$ is not least positive real.

$\Rightarrow$ There is no least positive real.

(5) Let $r \in \mathbb{Q}$, $x \in \mathbb{R} \setminus \mathbb{Q}$ but $q = r+x \in \mathbb{Q} \Rightarrow x = q - r \in \mathbb{Q}$

which is a contradiction to $x \in \mathbb{R} \setminus \mathbb{Q}$.

$\Rightarrow q = r+x \in \mathbb{R} \setminus \mathbb{Q}$.

$||^{ly}$ if $rx = q \in \mathbb{Q}$, $r \neq 0$, $\Rightarrow x = \frac{q}{r} \in \mathbb{Q} \Rightarrow \Leftarrow$

$\Rightarrow rx \in \mathbb{R} \setminus \mathbb{Q}$.

(6) $a, b \in \mathbb{R}$ and $a < b$ is given, Now $\sqrt{2}a, \sqrt{2}b \in \mathbb{R}$.

Then by density property of rational $\exists\, r \in \mathbb{Q}$ such that

$$\sqrt{2}a < r < \sqrt{2}b \Rightarrow a < \frac{r}{\sqrt{2}} < b$$

Take $\xi = \frac{r}{\sqrt{2}}$ which is irrational.

(7) $S = \left\{ \frac{5}{n} : n \in \mathbb{N} \right\}$. We will prove that $\inf S = 0$

we have $\forall\, n \in \mathbb{N}$, $5/n > 0$. So '0' is a lower bound of S.

By Archimedean property $\exists\, n \in \mathbb{N}$, for $\epsilon > 0$

$$n\epsilon > 5 \Rightarrow \frac{5}{n} < \epsilon$$

choose $x = 5/n$

So $\forall\, \epsilon > 0$ $\exists\, x \in S$, $x < 0 + \epsilon$ $\Rightarrow$ 0 is infinum.

~~(8) $S \subseteq \mathbb{R}$, $S \neq \phi$ and S is bdd. So lub and glb exists. We have $\text{glb}\, S \le s \quad \forall\, s \in S$~~

~~$s \le \text{glb}\, S \quad \forall\, s \in S$.~~

~~Let $s \in S$ then~~

(8) $S \subseteq \mathbb{R}$, $S \neq \phi$ and S is bdd. Let $s \in S$

then $s \geq \inf S$ and $s \le \sup S$

$$\Rightarrow \inf S \le s \le \sup S$$

$$\Rightarrow \inf S \le \sup S$$

If $\sup S = \inf S = A$ then $A \le s \le A \Rightarrow A = s$

So S has only one element A.

(9) (a) By assumption any $t \in T$ is an upper bound of S and any $s \in S$ is a lower bound of T. Since S and T are non-empty, we conclude that S is bdd above and T is bdd below.

(b) Fix $t \in T$. Since $s \le t$ $\forall s \in S \Rightarrow \sup S \le t$

Now $\sup S \le t$ $\forall t \in T$

$\Rightarrow \sup S \le \inf T$

(c) $S = [0,1]$, $T = [1,2]$. $S \cap T = \{1\}$

$\sup S = \inf T$

(d) $S = [0,1)$, $T = (1,2]$, $S \cap T = \phi$,

$\sup S = \inf T = 1$.

(10) Let $S = \{r \in \mathbb{Q} : r < a\}$. Then for any $r \in S$, $r < a$

$\Rightarrow \sup S \le a$. If $\sup S \ne a$ then $\sup S < a$.

Then by denseness of $\mathbb{Q}$, there is $r \in \mathbb{Q}$ such that $\sup S < r < a$. Now since $r < a \Rightarrow r \in S$

$\Rightarrow r \le \sup S$ which is contradiction to $\sup S < r$

$\Rightarrow \sup S = a$.

(11) (a) Let $\epsilon > 0$. consider

$$\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} < \epsilon \Rightarrow n > \frac{1}{\epsilon}$$

Take $N = \left\lceil \frac{1}{\epsilon} \right\rceil$

Then for any $\epsilon > 0$, $\forall\ n \geq N$, $\left|\frac{(-1)^n}{n} - 0\right| < \epsilon$

$$\Rightarrow \lim_{n \to \infty} \frac{(-1)^n}{n} = 0.$$

(b) similar

(c) Let $\epsilon > 0$. consider

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \epsilon$$

$$\Rightarrow \left|\frac{6n-3-6n-4}{(3n+2)3}\right| < \epsilon \Rightarrow \frac{7}{3(3n+2)} < \epsilon$$

$$\Rightarrow n > \left(\frac{7}{9\epsilon} - \frac{2}{3}\right)$$

Take $N = \left\lceil \left(\frac{7}{9\epsilon} - \frac{2}{3}\right) \right\rceil$

So for any $\epsilon > 0$, $\forall\ n \geq N$, $\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \epsilon$

$$\Rightarrow \lim_{n \to \infty} \frac{2n-1}{3n+2} = 2/3$$

(12) $\{x_n\}$ is a bdd sequence so $\exists\ M > 0 \ni$

$$|x_n| \leq M \quad \forall\ n \in \mathbb{N}.$$

Again $\{y_n\}$ converges to zero. So for any $\epsilon > 0\ \exists\ N \in \mathbb{N}$

$\ni \forall\ n \geq N, \quad |y_n - 0| < \frac{\epsilon}{M}.$

Now $\forall\ n \geq N$

$$|x_n y_n - 0| < M \cdot \epsilon/M = \epsilon$$

$$\Rightarrow \lim_{n \to \infty} x_n y_n = 0.$$

(13) Given that $\lim a_n = a$. So for any $\epsilon > 0\ \exists\ N \in \mathbb{N}$

$\forall\ n \geq N, \quad |a_n - a| < \epsilon$

Now choose same N as above and $\epsilon > 0$

$$\big|\, |a_n| - |a| \,\big| \leq |a_n - a| < \epsilon$$

So for any $\epsilon > 0$, $\exists\ N \in \mathbb{N}$, $\forall\ n \geq N$,

$$\big|\, |a_n| - |a| \,\big| < \epsilon$$

$$\Rightarrow \lim |a_n| = |a|$$

The converse of this statement is not true. For an example consider $(-1)^n$ $a_n = -1$ and hence $a = 1$

$|a_n| = 1 \not\Rightarrow$ $|a_n| = 1 \to 1 = a$. But $\{a_n\}$ does not converge to a.

The converse is if $|a_n| \to |a|$ then $a_n \to a$.

This will be true when $l = -l$ i.e. $l = 0$.

(14) Consider $a_n = (-1)^n$, So $|a_n| \le 1$. The terms of the sequence are

$\{-1, 1, -1, \ldots\}$, which is not convergent.

~~So the sequence does not converge.~~

(15) and (16) applications of limit theorems.