Let us see some examples of bases.

Let $V = \mathbb{R}^n$, be a vector space over $\mathbb{R}$, then the set of following vectors forms a basis for V.

$B = \{e_1, e_2, e_3, \ldots e_n\}$, where for $1 \le i \le n$

$e_i = (0, 0, \ldots 1, 0, 0, \ldots 0)$ with 1 at ith place.

This is called standard basis for $\mathbb{R}^n$ and thus dimension of $\mathbb{R}^n$ is n.

Another example is, Let V be the vector space of all matrices of size $m \times n$ over $\mathbb{R}$ i.e.

$V = M_{m\times n}(\mathbb{R})$, then the following set is a basis for V,

$B = \{E_{11}, E_{12}, E_{13} \ldots E_{1n}, E_{21}, E_{22} \ldots$
$E_{2n}, \ldots E_{m1}, E_{m2} \ldots E_{mn}\}$

where $E_{ij} \in M_{m\times n}(\mathbb{R})$ with i^{th} row and j^{th} column entry as 1 and rest are zero. A typical example for $m = 3$, $n = 2$ can be is

$$E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Is it true that every vector space has a basis (finite or infinite)?

Answer is YES.

Ques: Let V be the vector space of polynomials of order $\leq R$ over $\mathbb{R}$, find the basis of V.

Next, we see some results related with basis and dimension of vector space. We will not give proofs unless or untill it is unavoidable.

Theorem: Let V be a vector space of finite dimension. Then any two bases of V have the same (finite) no of elements.

We draw the following observations from above theorem, Let $\dim V = n$

a) any subset of vector space V containing more than 'n' elements is linearly dependent.

b) no subset of V which contains less than n elements can span V.

Ques: What will be the dimension of trivial subspace $\{0\}$

The next result is a lemma which will help us to prove that every linearly independent subset is a part of some basis of vector space.

Lemma: Let S be a linearly independent subset of vector space V. Assume that $\beta \in V$ does not belong to the span of S. Then ~~the~~ set $S \cup \{\beta\}$ is a linearly independent subset of V.

A proof of this lemma is very easy to write. Let $S=\{\alpha_1,\alpha_2\dots\alpha_k\}$ then if we want to prove that
$S\cup\{\beta\}=\{\alpha_1,\alpha_2,\dots\alpha_k,\beta\}$ is linearly independent, we must consider an arbitrary linear combination of these vectors as
$$c_1\alpha_1+c_2\alpha_2+\dots c_k\alpha_k+d.\beta=0$$
and then we should come out with the conclusion that
$$c_i=0\ \forall\ i=1,2,\dots k\ \&\ d=0$$
Note that if $d=0$, then we are left with
$$c_1\alpha_1+c_2\alpha_2+\dots c_k\alpha_k+0.\beta=0$$
$$\Rightarrow\ c_1\alpha_1+c_2\alpha_2+\dots c_k\alpha_k=0$$
$\Rightarrow c_i=0\ \forall\ i=1,2,\dots k$ as S is linearly independent.

Assume that $d\neq0$, then you can write
$$\beta=-\frac{c_1}{d}\alpha_1-\frac{c_2}{d}\alpha_2+\dots-\frac{c_k}{d}\alpha_k$$
which will contradict the fact that β does not belong to the span of S.

Hence $d=0$ and thus $S\cup\{\beta\}$ is linearly independent.

Theorem: If $W\subseteq V$ is a subspace of a finite dimensional vector space V, every linearly independent subset of W is finite and is part of basis of W.

Proof: Theorem says that, in other words, any linearly independent subset of W can be extended to a basis of W.

The process to extend it to the basis of W follows from previous lemma. Let S_0 is any linearly independent subset of W. If S_0 spans W then S_0 is a basis for W and we are done. If S_0 fails to span W then we can find an element $\beta_1 \in W$ s.t. $\beta_1 \notin \operatorname{span}\{S\} = \langle S \rangle$ and then using previous lemma,

$S_0 \cup \{\beta_1\}$ is a linearly independent subset of W.

Next if $S_0 \cup \{\beta_1\}$ spans W then we stop otherwise we find

$$\beta_2 \in W \text{ s.t. } \beta_2 \notin \operatorname{span}\{S \cup \{\beta_1\}\}$$

and therefore invoking the previous lemma, $S \cup \{\beta_1\} \cup \{\beta_2\}$ is a linearly independent subset of W. This process will keep on going untill we reach the basis of W. It will take only finitly many steps as and linearly independent subset in W (which is also linearly independent in V) can not have more than $\dim V$ element. Thus if $\dim V = n$, then the process will not take more than n steps.

Let us see some corollaries

<u>Corollary</u>: If W is a proper subspace of V then $\dim W < \dim V$.

Let $\alpha \neq 0$ belongs to W, then we can find a basis of W containing α which contains no more than $\dim V$ elements and therefore $\dim W \leq \dim V$. Since W is a proper subspace of V, implies $\exists\, \beta \in V$ which does not belong to W and therefore if we take union of β with the basis of W we get a linearly independent subset of V containing more than $\dim W$ element. As we know that if V has dimension n then we can not find a linearly independent subset of V having more than n elements. Thus $\dim V > \dim W$.

<u>Corollary</u>: In a finite dimensional vector space V, every non-empty linearly independent set is a part of basis.

<u>Corollary</u>: Let $A \in M_{n \times n}(\mathbb{F})$ and row vectors of A form a linearly independent set of vectors in $\mathbb{F}^n$. Then A is invertible.

We finish this lecture with the following theorem.

Theorem: If W_1 and W_2 are finite dimensional subspaces of vector space V, then W_1+W_2 is finite dimensional and

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1+W_2)$$

Proof: As $W_1 \cap W_2$ is a subspace of W_1 and thus has a basis (say) $\{\alpha_1, \alpha_2, \ldots \alpha_k\}$ of less than $\dim W_1$ elements which is a part of basis of W_1 (say)
$\{\alpha_1, \alpha_2, \ldots \alpha_k, \beta_1, \ldots \beta_m\}$
and also $W_1 \cap W_2$ is a subspace of W_2 and thus a basis of $W_1 \cap W_2$ will be a part of W_2 (say)
$\{\alpha_1, \alpha_2, \ldots \alpha_k, \gamma_1, \gamma_2, \ldots \gamma_n\}$

Note that the subspace $W_1 + W_2$ is spanned by
$\{\alpha_1, \alpha_2, \ldots \alpha_k, \beta_1, \beta_2, \ldots \beta_m, \gamma_1, \gamma_2, \ldots \gamma_n\}$

Our claim is these vectors are linearly independent and hence form a basis for W_1+W_2 (Try to prove this claim)

Now compare the dimensions of these subspaces as

$$\begin{aligned}\dim W_1 + \dim W_2 &= (k+m)+(k+n)\\ &= k+(m+k+n)\\ &= \dim(W_1\cap W_2)+\dim(W_1+W_2)\end{aligned}$$