# Tutorial 4 (Solutions)

Linear Algebra (IC104)
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## Question 1

- Union of two subspaces need not be a subspace.
  Counterexample, Let $W_1 = \{(x, 0) : x \in \mathbb{R}\}$ and $W_2 = \{(0, y) : y \in \mathbb{R}\}$ be two subspaces of $\mathbb{R}^2$. Therefore their union include $(3, 0)$ and $(0, 8)$ both but the sum $(3, 0) + (0, 8) = (3, 8)$ is not in the union of $W_1$ and $W_2$. Hence union need not be a subspace.

- We have on $\mathbb{R}^n$
  $\alpha \bigoplus \beta = \alpha - \beta$ and $c\,\alpha = -c\,\alpha$. Then $(\mathbb{R}^n, \bigoplus, .)$ is not a vector space as it is not closed under multiplication by 1 because $1\alpha = \alpha \neq -1\alpha$.

## Question 2

We have, $S = \{(x, y, z) \in \mathbb{R}^3 \ : \ x + 2y - 3 = 0,\ 2x - y + 3 = 0\}$.
Clearly, $(0, 0, 0)$ the identity element does not belongs to $S$. Therefore it is not a subspace.

## Question 3

- (2,1,3)=0(1,1,0)+2(1,0,1)+1(0,1,1).

- (2,1,3)=2(1,0,0)+1(0,1,0)+3(0,0,1)

## Question 4

We need to determine which of the following subsets $S$ of the vector space $V$ over $F = \mathbb{R}$ are subspaces. We know that a subset $S$ of a vector space $V$ is a subspace if and only if

1. $\mathbf{0} \in S$;
2. $\forall v_1, v_2 \in S$ and $\forall \alpha, \beta \in F$, we have $\alpha v_1 + \beta v_2 \in S$

### (a)

It is given that, $S = \{(x_1, x_2, x_3) \ : \ x_1 = x_2, \ x_3 = 2x_1\}$, $V = \mathbb{R}^3$.

Here, $\mathbf{0} = (0,0,0)$ and it satisfies the above conditions, i.e., $(0,0,0) \in S$.

Let $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in S$ and $\alpha, \beta \in \mathbb{R}$. Consider, $\alpha\mathbf{x} + \beta\mathbf{y}$, i.e.,

$$\alpha\mathbf{x} + \beta\mathbf{y} = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)$$

Since, $x_1 = x_2$ and $x_3 = 2x_1$, we have

$$\alpha x_1 + \beta y_1 = \alpha x_2 + \beta y_2$$

$$\begin{aligned} \alpha x_3 + \beta y_3 &= \alpha (2x_1) + \beta (2y_1) \\ &= 2(\alpha x_1 + \beta y_1) \end{aligned}$$

Therefore, $\alpha\mathbf{x} + \beta\mathbf{y} \in S$.

Hence, $S$ is a subspace of $V$.

### (b)

It is given that, $S = \{(x_1, x_2, x_3) \ : \ x_1^2 + x_2^2 + x_3^2 = 0\}$, $V = \mathbb{R}^3$.

Here, $\mathbf{0} = (0,0,0)$ and it satisfies the above conditions, i.e., $(0,0,0) \in S$.

Note that, $\mathbf{0} = (0,0,0)$ is the only element in the set $S$. This is because for any $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_1^2 + x_2^2 + x_3^2 = 0$ implies that $x_1 = x_2 = x_3 = 0$ since if the sum of squares of 3 number is zero then all the numbers should be zero only.

Hence, $S$ is a subspace of $V$ namely the **zero space**.

## Question 5

It is given that $a_1, a_2, a_3, a_4$ are linearly independent. Therefore, whenever

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \alpha_4 a_4 = 0$$

for some scalars $\alpha_1, \alpha_2, \alpha_3$ and, $\alpha_4$ then $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.

We need to state whether the given statements are true or false.

## (a)

We need check whether $a_1 + a_2,\ a_2 + a_3,\ a_3 + a_4,\ a_4 + a_1$ are linearly independent or not.

For some scalars, $a, b, c, d$, let us consider,

$$\begin{aligned} a\,(a_1 + a_2) + b\,(a_2 + a_3) + c\,(a_3 + a_4) + d\,(a_4 + a_1) &= 0 \\ (d + a)\,a_1 + (a + b)\,a_2 + (b + c)\,a_3 + (c + d)\,a_4 &= 0 \end{aligned}$$

Therefore, $(d + a) = (a + b) = (b + c) = (c + d) = 0$, which is a system of 4 equations in 4 variables. In matrix form, it is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Which can be row reduced to get,

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, $-a = b = -c = d$ is the solution and if for some $\alpha \neq 0$, we take $-a = b = -c = d = \alpha$, then the above system of equation is satisfied, i.e. we are having a solution other than $\mathbf{0}$.

Hence, $a_1 + a_2,\ a_2 + a_3,\ a_3 + a_4,\ a_4 + a_1$ are not linearly independent.

## (b)

We need check whether $a_1 - a_2,\ a_2 - a_3,\ a_3 - a_4,\ a_4 - a_1$ are linearly independent or not.

For some scalars, $a, b, c, d$, let us consider,

$$\begin{aligned} a\,(a_1 - a_2) + b\,(a_2 - a_3) + c\,(a_3 - a_4) + d\,(a_4 - a_1) &= 0 \\ (a - d)\,a_1 + (b - a)\,a_2 + (c - b)\,a_3 + (d - c)\,a_4 &= 0 \end{aligned}$$

Therefore, $(a - d = 0) = (b - a) = (c - b) = (d - c) = 0$, i.e., $a = d;\ b = a;\ c = b;\ d = c$. Therefore, $a = b = c = d$.

But then for any $\alpha \neq 0$ and $a = b = c = d = \alpha$, the above equation always holds.

Hence, $a_1 - a_2,\ a_2 - a_3,\ a_3 - a_4,\ a_4 - a_1$ are not linearly independent.

## (c)

We need check whether $a_1 + a_2,\ a_2 + a_3,\ a_3 + a_4,\ a_4 - a_1$ are linearly independent or not.

For some scalars, $a, b, c, d$, let us consider,

$$\begin{aligned} a\left(a_1 + a_2\right) + b\left(a_2 + a_3\right) + c\left(a_3 + a_4\right) + d\left(a_4 - a_1\right) &= 0 \\ \left(a - d\right) a_1 + \left(a + b\right) a_2 + \left(b + c\right) a_3 + \left(c + d\right) a_4 &= 0 \end{aligned}$$

Therefore, $(a - d) = (a + b) = (b + c) = (c + d) = 0$, which is a system of 4 equations in 4 variables. In matrix form, it is

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Which can be row reduced to get,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, $a = b = c = d = 0$ is the only solution.

Hence, $a_1 + a_2,\ a_2 + a_3,\ a_3 + a_4,\ a_4 + a_1$ are linearly independent.

## (d)

We need check whether $a_1 + a_2,\ a_2 + a_3,\ a_3 - a_4,\ a_4 - a_1$ are linearly independent or not.

For some scalars, $a, b, c, d$, let us consider,

$$\begin{aligned} a\left(a_1 + a_2\right) + b\left(a_2 + a_3\right) + c\left(a_3 - a_4\right) + d\left(a_4 - a_1\right) &= 0 \\ \left(a - d\right) a_1 + \left(a + b\right) a_2 + \left(b + c\right) a_3 + \left(d - c\right) a_4 &= 0 \end{aligned}$$

Therefore, $(a-d) = (a+b) = (b+c) = (d-c) = 0$, which is a system of 4 equations in 4 variables. In matrix form, it is

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Which can be row reduced to get,

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Therefore, $a = -b = c = d$ is the solution and if for some $\alpha \neq 0$, we take $a = -b = c = d = \alpha$, then the above system of equation is satisfied, i.e. we are having a solution other than $\mathbf{0}$.

Hence, $a_1 + a_2,\ a_2 + a_3,\ a_3 + a_4,\ a_4 + a_1$ are not linearly independent.

# Question 6.

Since any three linearly independent vectors form a basis of $\mathbb{R}^3$. So we need to check whether the vectors $a_1 = (1,1,0), a_2 = (1,0,1), a_3 = (0,1,1)$ form linearly independent set.
Consider the linear combination
$x_1(1,1,0) + x_2(1,0,1) + x_3(0,1,1) = 0$.
This is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Vectors are linearly independent if and only if det is non-zero.

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2 \neq 0.$$

Hence $a_1, a_2, a_3$ are linearly independent and form a basis for $\mathbb{R}^3$.

## Question 7.

Consider the linear combination
$x_1c_1a_1 + x_2c_2a_2 + \cdots + x_nc_na_n = 0.$

Let $x_ic_i = y_i$, for $i = 1, 2, \cdots, n$. Then
$y_1a_1 + y_2a_2 + \cdots + y_na_n = 0$, since $\{a_1, a_2, \cdots, a_n\}$ is a basis, so $y_i = 0$, for $i = 1, 2, \cdots, n$.

This implies $x_ic_i = 0$, but $c_i \neq 0$. for $i = 1, 2, \cdots, n$
Therefore $x_i = 0$. Hence $\{c_1a_1, c_2a_2, \cdots, c_na_n\}$ is linearly independent and forms basis of $V$.

## Question 8

Given that $a_1, a_2, a_3$ be linearly independent then there exist $c_1, c_2, c_3$ such that

$$c_1a_1 + c_2a_2 + c_3a_3 = 0$$

where $c_1 = c_2 = c_3 = 0$. Now we have to find the value of $k$ such that $a_2 - a_1, ka_3 - a_2, a_1 - a_3$ are linearly independent. Thus

$$\begin{aligned} c_1(a_2 - a_1) + c_2(ka_3 - a_2) + c_3(a_1 - a_3) &= 0 \\ c_1a_2 - c_1a_1 + kc_2a_3 - c_2a_2 + c_3a_1 - c_3a_3 &= 0 \\ a_1(c_3 - c_1) + a_2(c_1 - c_2) + a_3(kc_2 - c_3) &= 0 \end{aligned}$$

Then we have

$$c_3 - c_1 = 0 \tag{1}$$
$$c_1 - c_2 = 0 \tag{2}$$
$$kc_2 - c_3 = 0. \tag{3}$$

After solving (1),(2) we get $c_1 = c_2 = c_3$ and putting in (3), we get

$$\begin{aligned} kc_3 - c_3 &= 0 \\ c_3(k - 1) &= 0. \end{aligned}$$

So when $k \neq 1$, we get $c_3 = 0$ and hence $c_1 = c_2 = c_3 = 0$. Consequently, when $k \neq 1$ the vectors $a_2 - a_1, ka_3 - a_2, a_1 - a_3$ are linearly independent.

# Question 9

**Part 1:** Verify the subspace test, i.e.,

(a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$.

(b) Let $A = \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} \in W$ and $B = \begin{pmatrix} a_2 & b_2 \\ b_2 & c_2 \end{pmatrix} \in W$ then $A + kB \in W$ and $\forall k \in \mathbb{R}$(field).

**Part 2:** Take an arbitrary matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in W$. Then,

$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Thus $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ span $W$.

Next we show this spanning set is linearly independent. Writing

$a_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Thus, $a_1 = a_2 = a_3 = 0$, hence the set $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is linearly independent and consequently form a basis for $W$.