Sequence

Defn: An infinite sequence or sequence is a function whose domain is $\mathbb{N}$. A sequence f whose terms are $x_n := f(n)$ will be denoted by $x_1, x_2, \ldots$ or $\{x_n\}_{n \in \mathbb{N}}$ or $\{x_n\}_{n=1}^{\infty}$ or $\{x_n\}$.

$$x_n = f(n) : \mathbb{N} \to \mathbb{R}.$$

Example: (i) Consider a sequence $\{x_n\}_{n \in \mathbb{N}}$ where $x_n = \frac{1}{n^2}$

$x_n : \mathbb{N} \to \mathbb{R}$

$$\left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right\}.$$

(ii) consider a sequence $\{a_n\}$, $a_n = (-1)^n$, $n \geq 0$

$$\{1, -1, 1, -1, \ldots\}.$$

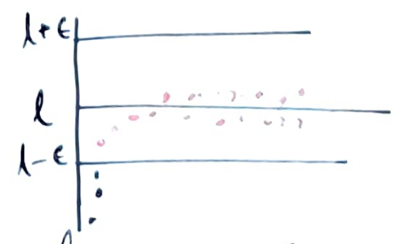
(iii) consider $\{a_n\}_{n \geq 1}$, $a_n = \sqrt[n]{n}$, $n \in \mathbb{N}$

(iv) consider $\{b_n\}_{n \geq 1}$, $b_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$.

Limit of Sequence:

Defn: A sequence $\{x_n\}$ of real numbers is said to converge to the real number l provided for each $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$

$$|x_n - l| < \epsilon.$$



If $\{x_n\}$ converges to l, we will write $\lim_{n \to \infty} x_n = l$ or $x_n \to l$. The number l is called the limit of the sequence $\{x_n\}$.

Note: The condition in the def^n of limit is an infinite number of statements, one for each positive value of ϵ.

The condition states that to each $\epsilon > 0$ there corresponds a number N with certain property namely $n \geq N$ ~~oto~~ implies $|x_n - l| < \epsilon$.

The value N depends on the value ϵ and normally N will have to be large if ϵ is small.

Theorem: Let $\{x_n\}_{n \geq 1}$ be a convergent sequence. Then it has unique limit.

Proof: Let $\lim_{n \to \infty} x_n = l_1$ and $\lim_{n \to \infty} x_n = l_2$. We will prove that $l_1 = l_2$.

Consider $\epsilon > 0$. Then by def^n of limit there exists N_1 so that $\forall\ n \geq N_1 \quad |x_n - l_1| < \epsilon/2$

and there exists N_2 so that

$$\forall\ n \geq N_2, \quad |x_n - l_2| < \epsilon/2$$

Now for $n \geq \max(N_1, N_2)$ we have

$$|l_1 - l_2| = |(l_1 - x_n) + (x_n - l_2)| \leq |x_n - l_1| + |x_n - l_2|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

So we have $\forall\, \epsilon > 0$, $|l_1 - l_2| < \epsilon \Rightarrow |l_1 - l_2| = 0$

$\Rightarrow l_1 = l_2$.

Example : (i) $\lim \frac{1}{n^2} = 0$.

Let $\epsilon > 0$ be given. then $\left|\frac{1}{n^2} - 0\right| < \epsilon$

$\Rightarrow \frac{1}{n^2} < \epsilon \Rightarrow n > \frac{1}{\sqrt{\epsilon}}$

So for a given $\epsilon > 0$, let $N = \left\lceil \frac{1}{\sqrt{\epsilon}} \right\rceil$, then $\forall\, n \geq N$,

$$\left|\frac{1}{n^2} - 0\right| < \epsilon$$

$\Rightarrow \lim \frac{1}{n^2} = 0$.

(ii) Prove $\lim\limits_{n \to \infty} \frac{3n+1}{7n-4} = \frac{3}{7}$.

For a given $\epsilon > 0$

$$\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| = \left|\frac{21n + 7 - 21n + 12}{7(7n-4)}\right| < \epsilon$$

or $\left|\frac{19}{7(7n-4)}\right| < \epsilon$

or $\frac{19}{7(7n-4)} < \epsilon$ $\quad (\because 7n - 4 > 0 \ \forall\, n \geq 1)$

or $n > \left(\frac{19}{49\epsilon} + \frac{4}{7}\right)$

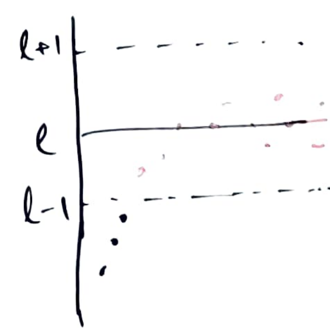
So given $\epsilon > 0$ choose $N = \left\lceil \left(\frac{19}{49\epsilon} + \frac{4}{7} \right) \right\rceil$, then $\forall\, n \geq N$

$$\left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \epsilon \implies \lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7}.$$

<u>Bounded Sequence</u>: A sequence $\{x_n\}$ of real numbers is said to be bounded if the set $\{x_n : n \in \mathbb{N}\}$ is a bounded set, that is, if there exists a constant $M > 0$ such that $|x_n| \leq M \quad \forall\, n \in \mathbb{N}$.

<u>Theorem</u>: convergent sequence are bounded

<u>Proof</u>: Let $\{x_n\}$ be a convergent sequence and let $\lim x_n = l$. Then for $\epsilon = 1$, $\exists\, N \in \mathbb{N}$ so that $\forall\, n \geq N$, $|x_n - l| < 1$.



So $\forall\, n \geq N$, $l-1 < x_n < l+1$

$$\implies |x_n| < \max\{|l-1|, |l+1|\}$$

Now take $M = \max\{|l-1|, |l+1|, |x_1|, |x_2| \cdots |x_N|\}$

Then $|x_n| \leq M \quad \forall\, n \in \mathbb{N}$

This proves the result.

Limit theorems: Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences that converges to x and y respectevely.

(I) $K \in \mathbb{R}$, $\lim (Kx_n) = Kx$

(II) $\lim (x_n + y_n) = (x+y) = \lim x_n + \lim y_n$

(III) $\lim (x_n y_n) = xy = (\lim x_n)(\lim y_n)$

(IV) If $\{x_n\}$ is sequence of non zero reals and if $x \neq 0$ then $\left\{\frac{1}{x_n}\right\}$ converges to $\frac{1}{x}$

(V) If $x_n \neq 0$ $\forall n$ and $x \neq 0$, then $\left\{\frac{y_n}{x_n}\right\}$ converges to y/x, i.e. $\lim \frac{y_n}{x_n} = \frac{y}{x} = \frac{\lim y_n}{\lim x_n}$

Proof (I) Assume $K \neq 0$, since for $K=0$ the result is obvious. We have to established for a given $\epsilon > 0$

$$\left[\exists N \in \mathbb{N},\ s.t.\ \forall n \geq N,\ |Kx_n - Kx| < \epsilon \quad \text{i.e. } |K|\,|x_n - x| < \epsilon.\right]$$

Now it is given that $\lim x_n = x$. So $\exists N \in \mathbb{N}$, s.t

$$\forall n \geq N \quad |x_n - x| < \frac{\epsilon}{|K|}$$

$$\Rightarrow \forall n \geq N, \quad |K|\,|x_n - x| < \epsilon$$

This proves the result.

(II) Home Work:

(iii) consider

$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy|$$

$$\leq |x_n y_n - x_n y| + |x_n y - xy|$$

$$= |x_n| \, |y_n - y| + |y| \, |x_n - x|$$

Given then $\{x_n\}$ is convergent so it is bdd. Then $\exists$ $M > 0$

$\ni$ $|x_n| < M$ $\forall$ $n \in \mathbb{N}$.

Again $\lim x_n = x$, $\lim y_n = y$.

So for $\epsilon > 0$ $\exists$ $N_1 \in \mathbb{N}$ $\ni$ $\forall$ $n \geq N_1$,

$$|y_n - y| < \frac{\epsilon}{2M}.$$

Again $\exists$ $N_2 \in \mathbb{N}$ $\ni$ $\forall$ $n \geq N_2$,

$$|x_n - x| < \frac{\epsilon}{2(|y|+1)}$$

[Here we have taken $\frac{\epsilon}{2(|y|+1)}$ instade of $\frac{\epsilon}{2|y|}$ because y could be zero].

Now take $N = \max\{N_1, N_2\}$, then $\forall$ $n \geq N$

$$|x_n y_n - xy| \leq |x_n| \, |y_n - y| + |y| \, |x_n - x|$$

$$\leq M \cdot \frac{\epsilon}{2M} + |y| \frac{\epsilon}{2(|y|+1)} < \frac{\epsilon}{2} + \epsilon/2 = \epsilon.$$

This proves the result.

Proof: (IV) & (V) $\rightarrow$ H.W.

Sandwich Theorem: Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be three sequence of real numbers and there is a natural number m such that $\forall\, n \geq m$,

$$u_n < v_n < w_n.$$

If $\lim u_n = \lim w_n = l$, then $\{v_n\}$ is convergent and ~~limit is~~ $\lim v_n = l$.

*** **Proof:** Given that $\lim u_n = \lim w_n = l$. So for $\epsilon > 0$

$\exists\; N_1 \in \mathbb{N}$, such that $\forall\, n > N_2$

$$|u_n - l| < \epsilon \quad \text{i.e.} \quad l - \epsilon < u_n < l + \epsilon$$

and $\exists\; N_2 \in \mathbb{N}$ s.t. $\forall\, n \geq N_2$

$$|w_n - l| < \epsilon, \quad \text{i.e.} \quad l - \epsilon < w_n < l + \epsilon$$

Let $N_3 = \max\{N_1, N_2\}$. Then $\forall\, n \geq N_3$ we have

$$l - \epsilon < u_n < l + \epsilon \quad \text{and} \quad l - \epsilon < w_n < l + \epsilon$$

Now choose $N = \max\{m, N_3\}$ then $\forall\, n \geq N$

$$l - \epsilon < u_n < v_n < w_n < l + \epsilon$$

$\Rightarrow \forall\, n \geq N$ we have

$$|v_n - l| < \epsilon$$

This proves $\{v_n\}$ convergent and $\lim v_n = l$.

Example (i) Prove that $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p>0$.

Solⁿ: Let $\epsilon>0$,

$$\left|\frac{1}{n^p}-0\right| < \epsilon \Rightarrow \frac{1}{n^p} < \epsilon \Rightarrow \epsilon > \frac{1}{n^p}$$

So $n > \left(\frac{1}{\epsilon}\right)^{1/p}$

Take $N = \left\lceil \left(\frac{1}{\epsilon}\right)^{1/p} \right\rceil$, then

$$\left|\frac{1}{n^p} - 0\right| < \epsilon \quad \forall\, n \geq N$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{1}{n^p} = 0$$

(ii) $\lim_{n\to\infty} a^n = 0$ if $|a|<1$

Solⁿ For $a=0$ the result is obvious. Since $|a|<1$ we can write $|a| = \frac{1}{1+b}$ where $b>0$

By Binomial theorem $(1+b)^n \geq 1+nb > nb$.

So for $\epsilon>0$

$$|a^n-0| = |a|^n = \frac{1}{(1+b)^n} < \frac{1}{nb} < \epsilon$$

For $\epsilon>0$ take $N = \left\lceil \frac{1}{b\epsilon} \right\rceil$. Then $\forall\, n \geq N$

$$|a^n - 0| < \frac{1}{nb} < \epsilon.$$

Example: $\lim n^{1/n} = 1$

Define $S_n = n^{1/n} - 1 \Rightarrow S_n \geq 0 \quad \forall n.$ So it is suffices to show $\lim S_n = 0$

Now $$n = (S_n+1)^n = 1 + nS_n + \frac{n(n-1)}{2} S_n^2 + \dots + S_n^n$$
$$> \frac{1}{2} n(n-1) S_n^2$$

$$\Rightarrow S_n^2 < \frac{2}{n-1} \quad \forall n > 1.$$

or $$|S_n| < \sqrt{\frac{2}{n-1}} \quad \forall n > 1.$$

Let $\epsilon > 0$. Then $|S_n| = |n^{1/n} - 1| < \sqrt{\frac{2}{n-1}} < \epsilon$

$$\Rightarrow n > 1 + \frac{2}{\epsilon^2}$$

Let $N = \left\lceil 1 + \frac{2}{\epsilon^2} \right\rceil$. Then for $\epsilon > 0$, $\forall n \geq N$,

$$|n^{1/n} - 1| < \epsilon$$

$$\Rightarrow \lim n^{1/n} = 1.$$

Example $\lim a^{1/n} = 1$ if $a > 0$.

Case-1 $a > 1$. Then $a^{1/n} > 1$. Take $x_n = a^{1/n} - 1 > 0$

Now $a = (1+x_n)^n > 1 + nx_n \quad \forall n > 1$

$\Rightarrow x_n < \frac{a-1}{n}$. so $|x_n| < \left|\frac{a-1}{n}\right| = \frac{a-1}{n} < \epsilon$

$$n > \frac{a-1}{\epsilon},$$

for $\epsilon > 0$, take $N = \left\lceil \frac{a-1}{\epsilon} \right\rceil$. then $\forall\, n \geq N$

$$|a^{1/n} - 1| < \epsilon$$

$\Rightarrow \lim a^{1/n} = 1$, $a > 1$.

Case-2

Now consider $0 < a < 1$. Then $\frac{1}{a} > 1$.

So $\lim \left(\frac{1}{a}\right)^{1/n} = 1$ by case 1.

Now $\frac{1}{a^{1/n}} \longrightarrow 1 \neq 0 \Rightarrow a^{1/n} \longrightarrow 1$. $\left[\because \text{ if } x_n \to l \neq 0,\ \frac{1}{x_n} \to \frac{1}{l}, \text{ where } x_n \neq 0\ \forall n\right]$

Case-3 $a = 1$ is trivial.

Ex: Find $\lim\limits_{n \to \infty} \frac{n-5}{n^2+7}$

Solⁿ $\frac{n-5}{n^2+7} = \frac{\frac{1}{n} - 5/n^2}{1 + 7/n^2}$

Now $\frac{1}{n} \to 0$, $5/n^2 \to 0$ as $n \to \infty$

& $7/n^2 \longrightarrow$ as $n \to \infty$

So $\left(\frac{1}{n} - \frac{5}{n^2}\right) \to 0$ as $n \to \infty$

$1 + 7/n^2 \longrightarrow 1$ as $n \to \infty$

$\Rightarrow \lim \left(\frac{n-5}{n^2+7}\right) = 0$.

~~Some examples~~:

(1) Prove that $\lim x_n = \frac{1}{4}$, where $x_n = \dfrac{n^3+6n^2+7}{4n^3+3n-4}$

$$x_n = \frac{1+\frac{6}{n}+7/n^3}{4+\frac{3}{n^2}-\frac{4}{n^3}},$$

We have $\frac{6}{n} \to 0$, $7/n^3 \to 0 \Rightarrow$

$$1+6/n+7/n^3 \to 1 \quad \text{as } n \to \infty$$

$$\text{lly} \quad 4+\frac{3}{n^2}-\frac{4}{n^3} \longrightarrow 4 \quad \text{as } n \to \infty$$

So $\lim x_n = \frac{1}{4}$.

Def^n: (i) A sequence $\{x_n\}$ is said to diverge to $+\infty$ if $\forall\, M > 0$ there is a number N such that $\forall\, n \geq N$

$$x_n > M.$$

We write $\lim x_n = +\infty$

(ii) Similarly we write seq^n $\{x_n\}$ is diverge to $-\infty$ if $\forall\, M < 0$ there is a number $N \in \mathbb{N} \ni \forall\, n \geq N$

$$x_n < M.$$

$$\lim x_n = -\infty.$$

Def^n: The sequence which are not convergent & divergent called oscillating seq^n.

Note: Many sequences do not have limits $+\infty$ or $-\infty$ even if they are unbounded.

Ex: $x_n = (-1)^n n$.

Ex: $\lim (\sqrt{n} + 7) = \infty$

Sol^n

We need to consider an arbitrary $M > 0$ and show that there exists N (depend on M) $\ni$ $\forall\, n \geq N$

$$\sqrt{n} + 7 > M.$$

Now $\sqrt{n} + 7 > M \Rightarrow \sqrt{n} > (M - 7)$ or $n > (M-7)^2$

Let $M > 0$ and $N = (M-7)^2$ then $\forall\, n \geq N$

we have $\sqrt{n} + 7 > M$.

$$\Rightarrow \lim (\sqrt{n} + 7) = \infty$$

Ex: $\lim n^2 = -\infty$, $\lim (-n) = -\infty$, $\lim 2^n = \infty$.

Theorem: Let $\{S_n\}$ and $\{t_n\}$ be sequences such that $\lim S_n = +\infty$ and $\lim t_n > 0$ [$\lim t_n < \infty$ or $\lim t_n = +\infty$].

Then $\lim S_n t_n = +\infty$.

Ex: Prove that $\lim \frac{n^2+3}{n+1} = +\infty$

$$x_n = \frac{n^2+3}{n+1} = \frac{n+\frac{3}{n}}{1+\frac{1}{n}} = s_n t_n,$$

As $n \to \infty$ we have.

$$s_n = n + \frac{3}{n} \to \infty, \quad t = \frac{1}{1+1/n} \to 1.$$

$$\Rightarrow \quad x_n \to +\infty \quad \text{as } n \to \infty$$

Theorem: For sequence $\{x_n\}$ of positive real numbers, we have $\lim x_n = +\infty$ iff $\lim \left(\frac{1}{x_n}\right) = 0$.

Monotone Sequence:

Def^n: (I) A sequence $\{x_n\}$ of real numbers is called an increasing sequ^n if $x_n \le x_{n+1}$ $\forall n$.

(II) $\{x_n\}$ is called decreasing if $x_n \geq x_{n+1}$ $\forall n$.

(iii) A sequence that is increasing or decreasing will be called a monotone sequence.

Def (Bdd above): A sequence $\{x_n\}$ is called bdd above if $\exists M \in \mathbb{R} \ni x_n \le M$ $\forall n$.

(Bdd below): If $\exists M \in \mathbb{R} \ni M \le x_n$ $\forall n$ then we say the sequ^n is bdd below.

Ex: (i) $x_n = 2^n$, $n \geq 1$ Here $x_{n+1} \geq x_n$ $\forall n \in \mathbb{N}$.

(ii) $x_n = \frac{1}{n}$, $n \geq 1$, $x_{n+1} \leq x_n$ $\forall n \in \mathbb{N}$.

(iii) $x_n = (-2)^n$ is neither ~~monotone~~ increasing nor decreasing.

Theorem: A increasing sequ^n, if bdd above is convergent and it converges to the lub.

Proof: Let $\{x_n\}$ be an increasing sequ^n bdd above. Then the set $\{x_n : n \in \mathbb{N}\}$ has a least upper bound say M. (i) So $x_n \leq M$ $\forall n \in \mathbb{N}$.

(ii) For a given $\epsilon > 0$ $\exists$ $N \ni x_N > M - \epsilon$.

Again $\{x_n\}$ is increasing so $x_N \leq x_n$ $\forall n \geq N$

$\Rightarrow$ $M - \epsilon < x_n < M < M + \epsilon$ $\forall n \geq N$

$\Rightarrow$ $|x_n - M| < \epsilon$ $\forall n \geq N$.

$\Rightarrow$ $\lim_{n \to \infty} x_n = M$.

Theorem: A decreasing sequence, if bdd below is convergent and it converges to glb.

Proof: H.W.

Theorem: (i) If a sequence $\{x_n\}$ is an unbounded increasing then

$$\lim x_n = +\infty$$

(ii) If $\{x_n\}$ is unbounded and decreasing then limit $x_n = -\infty$

Corollary: If $\{x_n\}$ is a monotone sequence, then the sequence either converges, diverges to $+\infty$ or diverges to $-\infty$.

Note: Thus limit of a monotone sequence always meaningful.

Example (1) Find $\lim \dfrac{\sin(n^2 - 34n + 5)}{n^2+1}$ (Application of Sandwich theorem)

Soln:

$$x_n = \frac{\sin(n^2 - 34n + 5)}{n^2+1}$$

Take $a_n = -\dfrac{1}{n^2+1} \to 0$ as $n \to \infty$

$b_n = +\dfrac{1}{n^2+1} \to 0$ as $n \to \infty$

Also we have

$$a_n \le x_n \le b_n \quad \forall n.$$

$$\Rightarrow \quad x_n \to 0 \text{ as } n \to \infty.$$

Subsequence

Defn: $\{x_{n_k}\}_k$ is called a subsequence of $\{x_n\}_n$ if $n_k \in \mathbb{N}$ are such that

$$n_1 < n_2 < n_3 < \cdots$$

Note: $\{x_{n_k}\}$ is a subsequence then it is required $n_k \to \infty$

Ex: $\{x_n\}$ is a sequence then $(x_2, x_4, x_6, \ldots)$ or $(x_1, x_4, x_9, x_{16}, \ldots)$ are subsequences of $\{x_n\}$.

Remember that subsequence is nothing but a sequence. So all the concept which is applicable for sequence, these are applicable for subsequence also.

There some properties which are also true for subsequence also and some are not true for subsequence.

Theorem: Every subsequence of a convergent sequence is convergent and have the same limit.

Result: If a sequence is bdd then all its subsequences are also bdd.

Ex: Show that the sequence $\{x_n\}_n$ defined by

$$x_n = \left(1-\frac{1}{n}\right) \sin \frac{n\pi}{2} \quad (n = 1, 2, \dots)$$

has a convergent subsequence but the sequence is not converge

Solⁿ

n	1	2	3	4	5	6	7	8	9
$\sin \frac{n\pi}{2}$	1	0	-1	0	1	0	-1	0	1

So we can see that even terms are zero

$$x_{2n} = \left(1-\frac{1}{2n}\right) \sin\left(\frac{2n\pi}{2}\right) = 0$$

Thus $\{x_{2n}\}$ is a convergent subsequence.

Now if we see the pattern of $\sin \frac{n\pi}{2}$, $(4n+1)^{th}$ terms are 1.

So consider $x_{4n+1} = \left(1+\frac{1}{4n+1}\right) \sin \frac{(4n+1)\pi}{2} = \left(1+\frac{1}{4n+1}\right)$

$\longrightarrow 1$ as $n \to \infty$.

This shows that $\{x_n\}$ is not convergent because have found two ~~diff~~ subsequens Which have different limits.

Remark: This example shows that if a sequence does not converge but it may have convergent subsequence.

Bolzano - Weierstras theorem : Everey bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Cauchy sequence :

Defn : A sequence $\{x_n\}$ of real numbers is called a Cauchy sequence if $\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \forall m, n \geq N$

$$|x_n - x_m| < \epsilon$$

Example : Show that $\left\{\frac{n}{n+1}\right\}_n$ is a cauchy sequence.

Solun : $x_n = \frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1}$

Take any $\epsilon > 0$

$$|x_n - x_m| = \left|\frac{1}{n+1} - \frac{1}{m+1}\right| \leq \left|\frac{1}{n+1}\right| + \left|\frac{1}{m+1}\right|$$

Note : If we take $N \in \mathbb{N}$ and take $m, n \geq N$ then

$$|x_n - x_m| \leq \left|\frac{1}{n+1}\right| + \left|\frac{1}{m+1}\right| \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N}$$

If we how that for given $\epsilon > 0$ $\exists N \in \mathbb{N} \ni \frac{2}{N} < \epsilon$

then proved

By Archimedean property $\exists N \in \mathbb{N} \ni N\epsilon > 2 \Rightarrow \frac{2}{N} < \epsilon$

Now choose this N ($\exists N$).

Now take $m, n \geq N$ then

$$|x_n - x_m| = \left|\frac{1}{n+1} - \frac{1}{m+1}\right| \leq \left|\frac{1}{m+1}\right| + \left|\frac{1}{n+1}\right| \leq \frac{2}{N} < \epsilon$$

So $\forall\ \epsilon > 0\ \exists\ N,\ \forall\ m, n > N$

$$|x_n - x_m| < \epsilon$$

$\Rightarrow \{x_n\}$ is a cauchy sequence.

<u>Result -1</u>: Every cauchy sequence is bounded.

<u>Proof</u>: Let $\{x_n\}$ be a cauchy sequence.

Take $\epsilon = 1$, $\exists\ N \in \mathbb{N}$. $m, n \geq N$

$$|x_n - x_m| < 1.$$

[The above statment says that difference between two element of the sequence always less than 1, $\forall\ n \geq N$.]

$\Rightarrow\ \forall\ n \geq N$, $x_n \in (x_N - 1, x_N + 1)$.

Hence take $M = \max\{|x_1|, |x_2|, \dots |x_{N-1}|, |x_N - 1|, |x_N + 1|\}$

So $\quad |x_n| < M \quad \forall\ n$.

$\Rightarrow \{x_n\}$ is bdd.

Result-2: Every convergent sequence is cauchy.

Proof: Let $\{x_n\}$ be a convergent sequence and let $\lim_{n\to\infty} x_n = l$. Then for an $\epsilon > 0$ $\exists N \in \mathbb{N}, \forall n \geq N$

$$|x_n - l| < \epsilon/2$$

Now choose this N and take any $m, n \geq N$, then

$$|x_m - x_n| = |x_m - l - (x_n - l)|$$
$$\leq |x_m - l| + |x_n - l| < \epsilon/2 + \epsilon/2 = \epsilon$$

ϵ is is taken arbitrary.

So $\forall \epsilon > 0 \ \exists N \in \mathbb{N}, \forall m, n \geq N, \quad |x_m - x_n| < \epsilon$

$\Rightarrow \{x_n\}$ is cauchy.

Result: Every cauchy sequence is convergent.

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Proof: Given that $\{x_n\}$ is cauchy. We want to prove $\{x_n\}$ is convergent.

Target: $\exists\, l \in \mathbb{R}, \forall \epsilon > 0 \ \exists N \in \mathbb{N}, \forall n \geq N$

$$|x_n - l| < \epsilon$$

Since $\{x_n\}$ is cauchy so $\{x_n\}$ is bounded. Then by Bolzano-Weierstrass theorem $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ say.

choose $l = \lim_{k \to \infty} x_{n_k}$.

Then for any $\epsilon > 0$ $\exists$ $K \in \mathbb{N}$ $\forall$ $k \geq K$ $|x_{n_k} - l| < \epsilon/2$

Also $\{x_n\}$ is cauchy so $\exists$ $N_1 \in \mathbb{N}$ $\forall$ $m, n \geq N_1$

$$|x_m - x_n| < \epsilon/2.$$

Now $\{a_{n_k}\}_k$ is a subsequence $\therefore n_k \to \infty$

$\therefore$ $\exists$ $K_1 \in \mathbb{N}$, $n_{K_1} \geq \max\{N_1, n_K\}$

Take $N = n_{K_1}$. Now with this choice of N we can see that $|x_N - l| < \epsilon/2$

Because $n_{K_1} > n_K \Rightarrow K_1 \geq K$ ($\because$ $\{x_{n_k}\}$ is a subsequence so $x_{n_{K_1}}$ will arrise after x_{n_K})

so $|x_{n_{K_1}} - l| < \epsilon/2$ i.e. $|x_N - l| < \epsilon$

Take any $n \geq N$ then $|x_N - l| < \epsilon/2$ and $|x_n - x_N| < \epsilon/2$

$$\therefore \quad |x_n - l| = |(x_n - x_N) + (x_N - l)|$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon$$

This proves the result.

Cauchy's General Principle of convergence

A sequence of real number is convergent iff it is a cauchy sequence.

Proof: Already proved.

Example: Show that the sequence $\{x_n\}_n$ where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not convergent.

soln: We will prove $\{x_n\}$ is not cauchy.

That is we will show that $\exists\ \epsilon > 0\ \forall N \in \mathbb{N}$

$\exists\ m, n \geq N,\quad |x_m - x_n| \geq \epsilon.$

Take any $N \in \mathbb{N}$, choose $m = N$, $n = 2N$

$$|x_n - x_m| = \frac{1}{N+1} + \cdots + \frac{1}{2N}$$

$$\geq \frac{1}{2N} + \cdots + \frac{1}{2N} = 1/2 = \epsilon.$$

So for $\epsilon = 1/2$, $m = N$, $n = 2N$

$$|x_n - x_m| \geq 1/2$$

$\Rightarrow \{x_n\}$ is not cauchy $\Rightarrow \{x_n\}$ is not convergent.

Example: The sequence $\{(1+\frac{1}{n})^n\}$ is a montonic increasing sequ^n and bdd above.

solu^n. Consider $(n+1)$ positive numbers $\underbrace{1+\frac{1}{n}, 1+\frac{1}{n}, \dots, 1+\frac{1}{n}}_{n \text{ times}}, 1$

Apply AM > GM $\Rightarrow$

$$\frac{n(1+\frac{1}{n})+1}{n+1} > \sqrt[1+n]{1\cdot(1+\frac{1}{n})\cdots(1+\frac{1}{n})}$$

$$= \sqrt[1+n]{(1+\frac{1}{n})^n}$$

$$= \left(1+\frac{1}{n}\right)^{\frac{n}{1+n}}$$

$$\Rightarrow \quad \frac{n+1+1}{n+1} > \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$$

$$\Rightarrow \left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n \Rightarrow x_{n+1} > x_n \quad \forall n \in \mathbb{N}$$

$\Rightarrow \{x_n\}$ is monotone increasing.

Now $x_n = \left(1+\frac{1}{n}\right)^n = 1+1+\frac{n(n-1)}{2!}\frac{1}{n^2}+\cdots+\frac{n(n-1)\cdots 2\cdot 1}{n!}\frac{1}{n^n}$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\frac{2}{n}\cdot\frac{1}{n}$$

$$< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \quad \forall n \geq 2$$

we have $n! > 2^{n-1} \quad \forall n > 2$

So $1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$ for $n>2$.

$$= 1+2\left(1-\left(\frac{1}{2}\right)^n\right) < 3 \quad \forall n \in \mathbb{N}.$$

$\Rightarrow x_n < 3 \quad \forall n \in \mathbb{N} \Rightarrow \{x_n\}$ is bdd above.

Again $x_1 = 2$, $\quad 2 = x_1 < x_2 < \cdots < 3 \Rightarrow 2 \leq x_n < 3 \quad \forall n \in \mathbb{N}$

The sequence is monotone increasing and bdd above so it is convergent and the limit of the sequence is denoted by e.

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.}$$

<u>EX</u> Prove that $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)=1$

Let $u_n = \frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}$

We have $\frac{1}{\sqrt{n^2+2}} < \frac{1}{\sqrt{n^2+1}}$, $\frac{1}{\sqrt{n^2+3}} < \frac{1}{\sqrt{n^2+1}}$, ...

So $u_n < \frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+1}}+\cdots+\frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}}$

Again $\frac{1}{\sqrt{n^2+1}} > \frac{1}{\sqrt{n^2+1}}$, $\frac{1}{\sqrt{n^2+2}} > \frac{1}{\sqrt{n^2+n}}$, ...

So $\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+n}}+\cdots+\frac{1}{\sqrt{n^2+n}} < u_n$

$\Rightarrow \frac{n}{\sqrt{n^2+n}} < u_n < \frac{n}{\sqrt{n^2+1}}$

$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}} = 1$

$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}} = 1 \Rightarrow \lim_{n\to\infty} u_n = 1$ [By Sandwich theorem]

Solved Examples

Ex 1 Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ $\forall\, n \geq 1$ converges to 2

Solu^n: The sequence is

$$\left\{ \sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \ldots \right\}$$

Here $u_n > 0 \quad \forall\, n \in \mathbb{N}$.

we have $u_2 > u_1$, consequently $u_3 > u_2$, $u_4 > u_3 \ldots$ and therefore $\{u_n\}$ is monotone increasing i.e. $u_{n+1} > u_n$

Again $u_{n+1}^2 = 2u_n$

$$\Rightarrow 2u_n > u_{n+1}^2 > u_n^2 \Rightarrow u_n < 2 \quad [\because u_n > 0]$$

$\Rightarrow \{u_n\}$ is bdd. above therefore convergent.

Let $\lim u_n = l$, then By def^n $u_{n+1}^2 = 2u_n \quad \forall\, n \in \mathbb{N}$

as $n \to \infty$ we have $l^2 = 2l \Rightarrow l = 0$ or 2

Since $\{u_n\}$ is monotone increasing and

$u_1 = \sqrt{2} > 1 \Rightarrow l = 2$.

So $\lim\limits_{n \to \infty} u_n = 2$.

Ex 2 Give examples of two non convergent sequence such that their product is convergent.

Solu^n: Let $x_n = (-1)^n$ and $y_n = (-1)^n \cdot 2$

$x_n y_n = 2$ which is a constant sequence and hence convergent.

(Ex 3) Let $a_{n+1} = a_n^2$ and $0 < a_1 < 1$. Prove that $\{a_n\}$ is monotone decreasing.

Solu^n: $a_{n+1} - a_n = a_n^2 - a_n = a_n(a_n - 1)$

Given that $0 < a_1 < 1$. Let $0 < a_n < 1 \Rightarrow 0 < a_n^2 < 1$

$\Rightarrow 0 < a_{n+1} < 1$.

So, $0 < a_n < 1 \quad \forall n \in \mathbb{N}. \Rightarrow a_{n+1} - a_n < 0$

$\Rightarrow a_{n+1} < a_n \quad \forall n$

$\Rightarrow \{a_n\}$ is monotone decreasing.

(Ex 4) Prove that the sequence $\{x_n\}$ defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

for all $n \geq 1$, $\alpha > 0$, $x_1 > \sqrt{\alpha}$ is a convergent sequence. Find the limit.

Solu^n: Given $x_1 \geq \sqrt{\alpha} > 0$

Assume $x_n > 0$ for sum $n \geq 1$

Then $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) > 0$

So by induction we have $\forall n \in \mathbb{N}$, $x_n > 0$

Now we will show that $x_{n+1} \geq \sqrt{\alpha}$, let $x_n \geq \sqrt{\alpha} > 0$

$$x_{n+1} = \frac{\left(x_n + \frac{x_n}{\alpha}\right)}{2} \geq \sqrt{x_n \cdot \cancel{x_n} \frac{\alpha}{x_n}} = \sqrt{\alpha} \qquad (AM \geq GM)$$

So $x_{n+1} \geq \sqrt{\alpha}$.

Consider $x_{n+1} \leq x_n$

$$\Rightarrow \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \leq x_n \Rightarrow \frac{1}{2}\left(\frac{\alpha}{x_n}\right) \leq \frac{1}{2} x_n$$

$\Rightarrow \frac{\alpha}{x_n} \leq x_n$ which is true because $x_n \geq \sqrt{\alpha}$.

So $\{x_n\}$ is decreasing and bdd below show convergent. Let $x_n \longrightarrow l$. Sin $\{x_n\}$ is a sequ^n of nonnegative real number so $l \geq 0$.

So taking $n \to \infty$ we get

$$l = \frac{1}{2}\left(l + \frac{\alpha}{l}\right) \Rightarrow l = \sqrt{\alpha}.$$

Limit Superior and Limit Inferior

Let $\{x_n\}$ be a bdd sequ^n. Let α is lub and β glb of $\{x_n\}$. Then $\beta \le x_n \le \alpha \quad \forall n$.

We know a bdd sequ^n may not be a convergent sequence.

Ex: $x_n = (-1)^{n+1}$, $\alpha = 1$, $\beta - 1$.

Define

$$\alpha_1 = \text{lub}\{x_1, x_2 \cdots\} \qquad \beta_1 = \text{glb}\{x_1, x_2 \cdots\}$$

$$\alpha_2 = \text{lub}\{x_2, x_3 \cdots\} \qquad \beta_2 = \text{glb}\{x_2, x_3, \cdots\}$$

$$\vdots \qquad\qquad \vdots$$

$$\alpha_k = \text{lub}\{x_k, x_{k+1} \cdots\} \qquad \beta_k = \text{glb}\{x_k, x_{k+1} \cdots\}$$

So $\alpha_k = \text{lub}\{x_n : n \geqslant k\}$, $\beta_k = \text{glb}\{x_n : n \geqslant k\}$

So (i) $\alpha_k \geqslant \alpha_{k+1} \quad \forall k$ then $\{\alpha_k\}$ is a decreasing sequ^n.

(ii) $\beta_k \le \beta_{k+1} \quad \forall k$ then $\{\beta_k\}$ is an increasing sequ^n.

Also $\beta \le \beta_k \le \alpha_k \le \alpha \quad \forall k$

So $\{\alpha_k\}$ and $\{\beta_k\}$ are bdd.

$\Rightarrow$ $\{\alpha_k\}$ and $\{\beta_k\}$ aree convergent sequ^n.

Here $\{\alpha_k\}$ is decreasing and bdd so convergent

$\lim \alpha_k$ is called limit superior of $\{x_n\}$

$$\lim \alpha_k = \inf_k \alpha_k$$

$\{\beta_k\}$ is increasing and bdd so convergent

$\lim \beta_k$ is called limit inferior of $\{x_n\}$

$$\lim \beta_k = \sup_k \beta_k$$

$$x^* = \limsup_{n \to \infty} x_n = \lim_{k \to \infty} \alpha_k = \lim_{k \to \infty} \sup_{n \geq k} x_n$$

$$= \inf_k \sup_{n \geq k} x_n$$

$$x_* = \liminf_{n \to \infty} x_n = \lim_{k \to \infty} \beta_k = \sup_k \beta_k = \sup_k \inf_{n \geq k} x_n$$

So for a given bdd sequ^n $\{x_n\}$ may or may not convergent but it has limsup or liminf.

Sometimes x^* denoted as $x^* = \overline{\lim}\, x_n$

and x_* denoted as $x_* = \underline{\lim}\, x_n$.

Theorem: If $x_* = x^*$, then $\{x_n\}$ converges to the common value. Conversely if $\lim_{n\to\infty} x_n = l$ then $l = x^* = x_*$.

Theorem: If $\{x_n\}$ is a bdd sequence then it has a subsequence converging to $\limsup x_n$ and a subsequence converging to $\liminf x_n$.

Defn (Subsequential limit): A real number that is the limit of a subsequence of $\{x_n\}$ is called subsequential limit of $\{x_n\}$.

Equivalent definition of limsup & liminf:

Let $\{x_n\}$ be a bdd sequence. Let S be the set of all subsequential limits. Then limsup and lim inf defined as

$$x^* = \limsup x_n = \sup S = \max S.$$

$$x_* = \liminf x_n = \inf S = \min S.$$

respectively.

Ex: Let $\{x_n\}$ be a seqn defined as $x_n = \left(1-\frac{1}{n^2}\right) \sin \frac{n\pi}{2}$.

Find two subsequences of $\{x_n\}$ one of which converges to limsup and other converges to liminf.

Soln.

$$x_n = \left(1-\frac{1}{n^2}\right) \sin \frac{n\pi}{2}$$

$$x_{2k} = \left(1-\frac{1}{4k^2}\right) \sin \frac{2k\pi}{2} = 0 \to 0 \text{ as } k \to \infty$$

$$x_{4k+1} = \left(1-\frac{1}{(4k+1)^2}\right) \sin \frac{(4k+1)\pi}{2}$$

$$= 1 - \frac{1}{(4k+1)^2} \to 1 \text{ as } k \to \infty$$

$$x_{4k+3} = \left(1+\frac{1}{(4k+3)^2}\right) \sin \frac{(4k+3)\pi}{2}$$

$$= -\left(1+\frac{1}{(4k+3)^2}\right) \to -1. \text{ as } k \to \infty$$

Since every positive integer is either of the form $2k$ or $(4k+1)$ or $(4k+3)$. So these are the possible subsequential limits.

$S = \{-1, 0, 1\}$

$\sup S = 1$

$\inf S = -1$

Thus $\{x_{4k+1}\}$ is a subsequ^n converging to limsup

$\{x_{4k+3}\}$ is a subsequ^n converging to liminf.

<u>Ex</u>: Find lim sup x_n and liminf x_n, where

$$x_n = \left(2\cos\frac{n\pi}{2}\right)^{(-1)^{n+1}}, \quad n \in \mathbb{N}.$$

Solⁿ: H.W.