Tutorial Solution

1. The associated symmetric matrix is $A = \begin{pmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ -5 & -2 & 10 \end{pmatrix}$. Apply row and column operation to obtain the normal form of A and check that $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the normal form. Thus the normal form of the given quadratic form is $x^2 + y^2 + z^2$. Thus, the quadratic form is positive definite.

2. The associated symmetric matrix is $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$. Apply row and column operation to obtain the normal form of A and check that $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is the normal form. Thus the normal form of the given quadratic form is $x^2 + y^2 - z^2$. Thus, the quadratic form is indefinite.

3. The real quadratic form corresponding to the matrix B^tB is $X^t(B^tB)X$, where X is an $n \times 1$ real matrix. Now,

$$X^t(B^tB)X = (BX)^t(BX) = Y^tY, \text{ where } Y = BX.$$

Let

$$X = (x_1\ x_2\ \ldots x_n)^t \text{ and } Y = (y_1\ y_2\ \ldots y_n)^t, \text{ then } Y^tY = y_1^2 + \cdots + y_n^2.$$

The quadratic form Y^tY assumes real values greater than or equal to zero. So the quadratic form $X^t(B^tB)X$ is either positive definite or positive semi-definite and therefore the matrix B^tB is either positive definite or positive semi-definite . Since Y is a real $n \times 1$ matrix, $Y^tY = 0$ occurs only when $Y = O$, i.e., when $BX = O$.

If B be non-singular, $BX = 0$ occurs only when $X = O$. If B be singular, $BX = 0$ holds for $X = O$ and also for some $X \neq O$. Therefore the quadratic form is positive definite if B be non-singular and positive semi-definite if B be singular. Hence the matrix B^tB is positive definite or positive semi-definite according as B is non-singular or singular.

4. Let λ be an eigenvalue of A. Then $\det(A-\lambda I_n)=0$. Therefore, there exist non-null solutions of the homogeneous system $(A-\lambda I_n)X=O$. Let X_1 be one such solution. Then

$$(A-\lambda I_n)X_1=O, \text{ that is } AX_1=\lambda X_1.$$

Taking transpose of the conjugate of the above, we get

$$(\bar{X}_1)^t(\bar{A})^t=\bar{\lambda}(\bar{X}_1)^t \implies (\bar{X}_1)^t A=\bar{\lambda}(\bar{X}_1)^t, \text{ since } A^t=A=\bar{A}^t.$$

Multiplying by X_1 from the right, we have

$$(\bar{X}_1)^t\lambda X_1=\bar{\lambda}(\bar{X}_1)^t X_1 \implies (\lambda-\bar{\lambda})(\bar{X}_1)^t X_1=0.$$

But $(\bar{X}_1)^t X_1\neq 0$, since X_1 is non-null. It follows that $\lambda=\bar{\lambda}$ and therefore λ is purely real.

7. Let $A=\begin{pmatrix}1 & -3\\ -3 & 1\end{pmatrix}$, $B=(-4 \quad -4)$, $X=\begin{pmatrix}x\\ y\end{pmatrix}$. Then the equation takes the form

$$X^tAX+BX+12I_1=O.$$

The eigenvalues of A are 4, -2. The eigenvectors corresponding to the eigenvalues 4 and -2 are

$$c\begin{pmatrix}1\\ -1\end{pmatrix},\ c\neq 0;\ d\begin{pmatrix}1\\ 1\end{pmatrix},\ d\neq 0 \text{ respectively.}$$

The orthonormal set of eigenvectors is

$$\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}\right\}.$$

Let $P=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$. Then P is an orthogonal matrix. P^tAP is a diagonal matrix whose eigenvalues are same as those of A. So $P^tAP=\begin{pmatrix}4 & 0\\ 0 & -2\end{pmatrix}$. Also $BP=(0 \ -4\sqrt{2})$.

By the orthogonal transformation $X=PX'$ where $X'=\begin{pmatrix}x'\\ y'\end{pmatrix}$, the equation transforms to

$$4(x')^2-2(y')^2-4\sqrt{2}y'+12=0 \implies 4(x')^2-2(y'+\sqrt{2})^2=-16.$$

Let us apply the translation $x'' = x',\ y'' = y' + \sqrt{2}$. The equation transforms to

$$4(x'')^2 - 2(y'') = -16.$$

The canonical form is $2x^2 - y^2 = -8$. The equation represents a hyperbola.