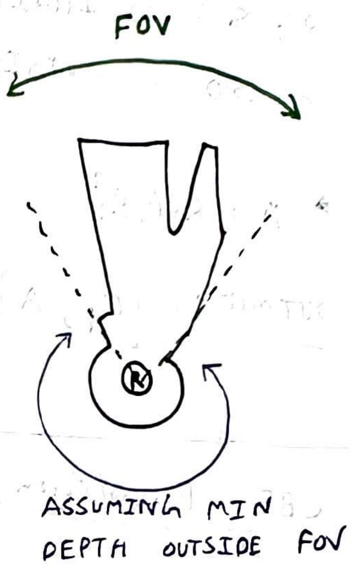
16/01/24

FORMULATION WITH ELLIPSES APROXIMATING THE UNSAFE REGION

1. → The depth image is first downsampled & a slice is taken



→ The size is around ~~20,0~~ (20,)

→ Each point at an angle $\theta_i \in [0, 20]$ is fitted with an ellipse defined below

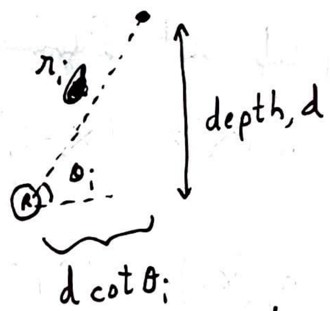
2. Ellipse Generation

→ INPUT: $[\theta_i, d]$; # i represents i^{th} point ; $i <$ len(depth_1D.shape)

ALGO:

$x_i = d_i \cot\theta_i \quad \in \mathbb{R}$

$r_i = \sqrt{x_i^2 + d_i^2 \cot^2\theta_i} \quad \in \mathbb{R}$

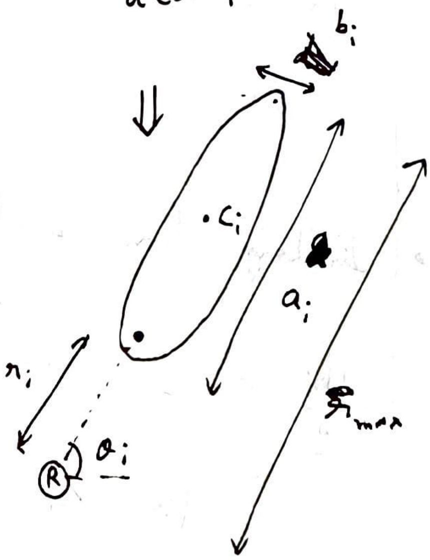


Ellipse parameters

$b_i = 0.1$ #(Semi-~~major~~ minor axis length) $\in \mathbb{R}$

$a_i = \dfrac{r_{max} - r_i}{2} + d_s \quad \in \mathbb{R}$

$C_i = \left(\dfrac{r_{max} + r_i}{2}\right) \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix} \quad \in \mathbb{R}^2$



$R_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix} \in \mathbb{R}^{2\times2}$; $P_i = \begin{bmatrix} 1/a_i^2 & 0 \\ 0 & 1/b_i^2 \end{bmatrix} \in \mathbb{R}^{2\times2}$

$$\boxed{\sigma_i = (X - c_i)^T R_i P_i R_i^T (X - c_i) - 1}$$

c_i - Ellipse Center
R_i - Rotation Matrix
P_i - Scaling Matrix (a & b)
X - Point $(x, y) \in \mathbb{R}^2$

$\sigma_i < 0 \Rightarrow$ INSIDE Ellipse
$\sigma_i > 0 \Rightarrow$ OUTSIDE Ellipse

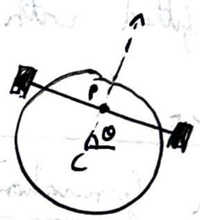
★ $A = R_i P_i R_i^T$

<u>OUTPUT</u>: $[c_i, A]$ OR $[c_{x,i}, c_{y,i}, a_i, b_i, \theta_i]$

③ CBF Formulation

$$\boxed{\begin{aligned} x_p &= x_c + l\cos\theta_c \\ y_p &= y_c + l\sin\theta_c \\ \theta_p &= \theta_c \end{aligned}}$$

$X_c = \begin{bmatrix} x_c \\ y_c \\ \theta_c \end{bmatrix}$; $X_p = \begin{bmatrix} x_p \\ y_p \\ \theta_p \end{bmatrix}$



→ Safe Set Definition

★ $h_i(X_c) = (X_c - c_i)^T A (X_c - c_i) - 1$; $h_i : \mathbb{R}^2 \to \mathbb{R}$

$$h_i(X_p) = \begin{bmatrix} x_p - l\cos\theta_p - c_x & y_p - l\sin\theta_p - c_{i,y} \end{bmatrix} A \begin{bmatrix} x_p - l\cos\theta_p - c_{i,x} \\ y_p - l\sin\theta_p - c_{i,y} \end{bmatrix} - 1$$

→ Finding time derivative of h_i

$$\frac{dh_i}{dt} = [X_c - c_i]^T (A + A^T) \dot{X}_c$$

$$= [X_c - c_i]^T (A + A^T) \begin{bmatrix} \dot{x}_p + l\sin\theta_p \dot{\theta} \\ \dot{y}_p - l\cos\theta_p \dot{\theta}_p \end{bmatrix}$$

$$= [X_c - c_i]^T (A + A^T) \begin{bmatrix} 1 & 0 & l\sin\theta_p \\ 0 & 1 & -l\cos\theta_p \end{bmatrix} \begin{bmatrix} \dot{x}_p \\ \dot{y}_p \\ \dot{\theta}_p \end{bmatrix}$$

$$\frac{dh_i}{dt} = \left[x_p - l\cos\theta_p - c_{i,x} \,,\, y_p - l\sin\theta_p - c_{i,y}\right](A + A^T)\begin{bmatrix} 1 & 0 & l\sin\theta_p \\ 0 & 1 & -l\cos\theta_p \end{bmatrix}\begin{bmatrix} \dot{x}_p \\ \dot{y}_p \\ \dot{\theta}_p \end{bmatrix}$$

④ MODEL DYNAMICS (UNICYCLE) & DEFINITIONS

$$\begin{bmatrix} \dot{x}_p \\ \dot{y}_p \\ \dot{\theta}_p \end{bmatrix} = \begin{bmatrix} \cos\theta_p & 0 \\ \sin\theta_p & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{B(\theta_p)}_{\in \mathbb{R}^{3\times 2}} \underbrace{U}_{\in \mathbb{R}^{2\times 1}}$$

$\star$ $$B(\theta_p) = \begin{bmatrix} \cos\theta_p & 0 \\ \sin\theta_p & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

$\star$ $$a_i(x_p) = \left[x_p - l\cos\theta_p - c_{i,x} \,,\, y_p - l\sin\theta_p - c_{i,y}\right](A + A^T)\begin{bmatrix} 1 & 0 & l\sin\theta_p \\ 0 & 1 & -l\cos\theta_p \end{bmatrix}$$

$$\Rightarrow \frac{dh_i}{dt} = a_i(x_p)\, B(\theta_p)\, U$$

⑥ OPTIMISATION PROBLEM (CONTROL SYNTHESIS)

$$U^* = \operatorname{argmin} \left(U - U_{ref}\right)^T Q \left(U - U_{ref}\right)$$

$$\text{s.t} \quad \frac{dh_i(x_p)}{dt} \geq -\gamma h_i(x_p) \quad \forall\ i \in [0, P]$$

7 OPTIMISATION (Implementation)

$$U - U_{ref} = \tilde{U}$$

$$\Rightarrow \quad \frac{dh_i}{dt} \geq -\gamma h_i$$

$$a_i(x_p)\, B(\theta_p)\left(\tilde{U} + U_{ref}\right) \geq -\gamma h_i(x_p)$$

$$a_i(x_p)\, B(\theta_p)\, \tilde{U} + a_i(x_p)\, B(\theta_p)\, U_{ref} \geq -\gamma h_i(x_p)$$

$$-a_i(x_p)\, B(\theta_p)\, \tilde{U} \leq \gamma h_i(x_p) + a_i(x_p)\, B(\theta_p)\, U_{ref}$$

$\Rightarrow$ G_{opt} :

$$\begin{bmatrix} -a_1(x_p)\, B(\theta_p) \\ -a_2(x_p)\, B(\theta_p) \\ \vdots \\ -a_p(x_p)\, B(\theta_p) \end{bmatrix} \in \mathbb{R}^{p \times n_u}$$

number of ellipses (p)

n_u → number of controls

$\Rightarrow$ h_{opt} :

$$\begin{bmatrix} \gamma h_1(x_p) + a_1(x_p)\, B(\theta_p)\, U_{ref} \\ \gamma h_2(x_p) + a_2(x_p)\, B(\theta_p)\, U_{ref} \\ \vdots \\ \gamma h_p(x_p) + a_p(x_p)\, B(\theta_p)\, U_{ref} \end{bmatrix} \in \mathbb{R}^{p \times 1}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \tilde{U}^* &= \underset{\tilde{U}}{\text{argmin}} \; \tilde{U}^T Q \tilde{U} \\ & \qquad G_{opt}\, \tilde{U} < h_{opt} \\ U^* &= \tilde{U}^* + U_{ref} \end{aligned}}$$