Module - 1

LAPLACE TRANSFORM

Definition :- Changing the function from one domain to another domain without changing its actual value is called Transformation.

Mathematically, suppose $F(t)$ be a real valid function defined on $[-\infty, +\infty]$ for any Kernal or a function $K(s,t)$ such that $T[F(t)] = \int_{-\infty}^{\infty} K(s,t) \cdot F(t)\,dt = f(s)$ is called the Transformation of $F(t)$.

where T is called the Transformation operator

s is real (or) complex.

If the Kernal $K(s,t)$ is convergent, we have

$$K(s,t) = \begin{cases} e^{-st}, & t > 0 \\ 0, & t < 0 \end{cases}$$

then $T[F(t)] = \int_0^{\infty} e^{-st} F(t)\,dt = L[F(t)] = f(s)$ is called the Laplace transform of $F(t)$ in the positive time t, where

L is called the Laplace transform operator.

If $K(s,t) = e^{ist}$, then

① $\Rightarrow L[F(t)] = \int_{-\infty}^{\infty} e^{ist} F(t)\,dt = F[F(t)] = f(s)$ is called the Fourier transform of $F(t)$.

where as F is the fourier transformation operator.

Notations:- Suppose the given functions are in the capitals of alphabetics $F(t), G(t), H(t) \ldots$ then their Laplace transforms are $f(s), g(s), h(s) \ldots$

If the functions are small letters like $f(t), g(t), h(t) \ldots$ then their Laplace transforms to be in the form $\bar{f}(s), \bar{g}(s), \bar{h}(s) \ldots$

Properties of Laplace transforms:-

1. Suppose $L[F(t)] = f(s)$, $L[G(t)] = g(s)$ and then for any c_1 and c_2 such that $L[c_1 F(t) + c_2 G(t)] = c_1 L[F(t)] + c_2 L[G(t)]$.

2. If $L[F(t)] = f(s)$, then

   a) $L[e^{at} F(t)] = f(s-a)$

   b) $L[e^{-at} F(t)] = f(s+a)$

3. If $L[F(t)] = f(s)$, then $L[t^n F(t)] = (-1)^n \dfrac{d^n f(s)}{ds^n}$

4. If $L[F(t)] = f(s)$, then $L\left[\dfrac{F(t)}{t}\right] = \int_s^{\infty} f(s)\,ds$.

Standard Results of Laplace transform:-

1) Let $F(t) = 1$

wkt $L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt$

$$L[1] = \int_0^\infty e^{-st}\, 1\, dt$$

$$= \int_0^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{-1}{s}\left[e^{-st}\right]_0^\infty$$

$$= \frac{-1}{s}\left[e^{-\infty} - e^0\right]$$

$$= \frac{-1}{s}\left[0 - 1\right]$$

$$\boxed{L[1] = \frac{1}{s}}$$

2) Let $F(t) = e^{at}$

$$L[F(t)] = \int_0^\infty e^{-st} e^{at}\, dt$$

$$L[F(t)] = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left(e^{-\infty} - e^0\right)$$

$$= \frac{-1}{s-a}(0 - 1)$$

$$\boxed{L[e^{at}] = \frac{1}{s-a}}$$

3) Let $F(t) = e^{-at}$

$$\therefore L[e^{-at}] = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{-1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty$$

$$= \frac{-1}{s+a}\left[e^{-\infty} - e^{0}\right]$$

$$L[e^{-at}] = \frac{-1}{s+a}[0-1]$$

$$\boxed{L[e^{-at}] = \frac{1}{s+a}}$$

4) $$L[t^n] = \frac{n!}{s^{n+1}}, \forall n \in z^+$$

$$= \frac{\Gamma n+1}{s^{n+1}} \quad \forall\ n \in Q$$

5) Let $F(t) = \cosh at$

$$F(t) = \frac{e^{at}+e^{-at}}{2}$$

$$L[F(t)] = \frac{1}{2} L[e^{at} + e^{-at}]$$

$$= \frac{1}{2}\{L(e^{at}) + L(e^{-at})\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2}s}{s^2-a^2}$$

$$L[F(t)] = L[\cosh at] = \frac{s}{s^2-a^2}$$

III^{ly} $F(t) = \sinh at$

$$F(t) = \frac{e^{at}-e^{-at}}{2}$$

$$L[F(t)] = \frac{1}{2} L[e^{at}-e^{-at}]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2}a}{s^2-a^2}$$

$$L[F(t)] = L[\sinh at] = \frac{a}{s^2-a^2}$$

WKT $e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow e^{iat} = \cos at + i\sin at$

$\Rightarrow L[e^{ait}] = L[\cos at] + i\,L[\sin at] \rightarrow ①$

$\therefore L[e^{iat}] = L[e^{(ai)t}]$

$$= \frac{1}{s-ai} = \frac{s+ai}{(s-ai)(s+ai)}$$

$$= \frac{s+ai}{s^2-(ai)^2}$$

$$L[e^{iat}] = \frac{s+ai}{s^2+a^2}$$

$$L[\cos at] + i\,L[\sin at] = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

$$\therefore L[\cos at] = \frac{s}{s^2+a^2}$$

$$L[\sin at] = \frac{a}{s^2+a^2}$$

Find the laplace transform for the following functions:

(1) $\sin 5t \cdot \cos 2t$

(2) $\sin t \cdot \sin 2t \cdot \sin 3t$

(3) $\cos t \cdot \cos 2t \cdot \cos 3t$

(4) $(3t+4)^3 + 5^t$

(5) $t^{-5/2} + t^{5/2}$

(6) $e^{-2t} \sinh(4t)$

① Let $F(t) = \sin 5t \cdot \cos 2t$

$$F(t) = \frac{1}{2}[2\sin 5t \cdot \cos 2t]$$

$$= \frac{1}{2}[\sin(5t+2t) + \sin(5t-2t)]$$

$$= \frac{1}{2}[\sin(7t) + \sin(3t)]$$

$$\Rightarrow L[F(t)] = \frac{1}{2} L[\sin 7t + \sin 3t]$$

$$\Rightarrow f(s) = \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9}\right]$$

② Let $F(t) = \sin t \cdot \sin 2t \cdot \sin 3t$

$$= \frac{1}{2}[2\sin 2t \cdot \sin t] \sin 3t$$

$$= \frac{1}{2}[\cos(2t-t) - \cos(2t+t)\sin 3t$$

$$= \frac{1}{2}[\cos t - \cos 3t] \sin 3t$$

$$= \frac{1}{2}[\sin 3t \cdot \cos t - \sin 3t \cdot \cos 3t]$$

$$= \frac{1}{4}[2\sin 3t \cdot \cos t - 2\sin 3t \cdot \cos 3t]$$

$$= \frac{1}{4}[\sin(4t) + \sin 2t - \sin 6t]$$

$$= \frac{1}{4} L[\sin 2t + \sin 4t - \sin 6t]$$

$$= \frac{1}{4}[L(\sin 2t) + L(\sin 4t) - L(\sin 6t)]$$

$$= \frac{1}{4}\left[\frac{2}{s^2+4} + \frac{4}{s^2+16} - \frac{6}{s^2-36}\right]$$

(3) Let $F(t) = \cos t \cdot \cos 2t \cdot \cos 3t$

$$= \frac{1}{2}[2\cos 2t \quad \cos t]\cos 3t$$

$$= \frac{1}{2}[\cos 3t + \cos t]\cos 3t$$

$$= \frac{1}{2}[\cos^2 3t + \cos 3t \cdot \cos t]$$

$$= \frac{1}{2}\left[\frac{1+\cos 2(3t)}{2} + \frac{2\cos 3t \cdot \cos t}{2}\right]$$

$$\Rightarrow F(t) = \frac{1}{4}[1 + \cos 6t + \cos 4t + \cos 2t]$$

$$\Rightarrow L[F(t)] = \frac{1}{4}L[1 + \cos 6t + \cos 4t + \cos 2t]$$

$$\Rightarrow F(t) = f(s) = \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right]$$

(4) Let $F(t) = (3t+4)^3 + 5^t$

$$= (3t)^3 + 4^3 + 3(3t)^2 + 3(3t)(4)^2 + e^{\log_e(5^t)}$$

$$\Rightarrow F(t) = 27t^3 + 64 + 108t^2 + 144t + e^{t(\log 5)}$$

$$\Rightarrow L[F(t)] = 27\,L[t^3] + 64\,L[1] + 108\,L[t^2] + 144\,L[t] + L[e^{(\log 5)t}]$$

$$\Rightarrow f(s) = 27 \cdot \frac{3!}{s^4} + 64 \cdot \frac{1}{s} + 108 \cdot \frac{2!}{s^3} + 144\,\frac{1!}{s^2} + \frac{1}{s\log 5}$$

$$\Rightarrow f(s) = \frac{162}{s^4} + \frac{64}{s} + \frac{216}{s^3} + \frac{144}{s^2} + \frac{1}{s-\log 5}$$

(5) $F(t) = e^{-5t}\sinh 4t$

$$= e^{-5t}\left[\frac{e^{4t}-e^{-4t}}{2}\right]$$

$$F(t) = \frac{1}{2}[e^{9t} - e^{-bt}]$$

$$\Rightarrow L[F(t)] = \frac{1}{2} L[e^{2t} - e^{-6t}]$$

$$= \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s+6}\right]$$

⑥ Let $F(t) = t^{-5/2} + t^{5/2}$

$$L[F(t)] = L[t^{-5/2}] + L[t^{5/2}]$$

$$f(s) = \frac{\Gamma(-5/2+1)}{s^{-5/2+1}} + \frac{\Gamma(5/2+1)}{s^{5/2+1}}$$

$$= \frac{\Gamma(-3/2)}{s^{-3/2}} + \frac{\Gamma(7/2)}{s^{7/2}}$$

$$= \frac{(-3/2)\Gamma\left(\frac{-3}{2}\right)}{\left(\frac{-3}{2}\right) s^{-3/2}} + \frac{\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)}{s^{7/2}}$$

$$= \frac{\Gamma\left(-\frac{3}{2}+1\right)}{\left(-\frac{3}{2}\right) s^{-3/2}} + \frac{(5/2)\Gamma\left(\frac{5}{2}\right)}{s^{7/2}}$$

$$= \frac{\Gamma(-1/2)}{\left(\frac{-3}{2}\right) s^{-3/2}} + \frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{s^{7/2}}$$

$$= \frac{\left(-\frac{1}{2}\right)\Gamma(-1/2)}{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right) s^{-3/2}} + \frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{s^{7/2}}$$

$$= \frac{\Gamma\left(\frac{-1}{2}+1\right)}{\frac{3}{4} s^{-3/2}} + \frac{15}{8}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{7/2}}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{3}{4} s^{-3/2}} + \frac{15}{8}\frac{\Gamma(1/2)}{s^{7/2}}$$

$$\Rightarrow f(s) = \frac{4}{3}\frac{\sqrt{\pi}}{s^{-3/2}} + \frac{15}{8}\frac{\sqrt{\pi}}{s^{7/2}}$$

Find the laplace transform of the following functions:

① $e^{-2t}(2\cos 5t - \sin 5t)$

② $e^{-t}\cos^2 3t$

③ $e^{3t}\sin 5t \cdot \sin 3t$

④ $e^{-4t}\,t^{-5/2}$

① Let $F(t) = 2\cos 5t - \sin 5t$

$$L[F(t)] = 2L[\cos 5t] - L(\sin 5t)$$

$$f(s) = 2\left[\frac{s}{s^2+25}\right] - \left[\frac{5}{s^2+25}\right]$$

$$f(s) = \frac{2s-5}{s^2+25}$$

$\therefore$ WKT $L[e^{at}F(t)] = f(s-a)$

$$L[e^{-2t}F(t)] = f(s+2)$$

$$\Rightarrow L[e^{-2t}(2\cos 5t - \sin 5t)] = \frac{2(s+2)-5}{(s+2)^2+25}$$

$$= \frac{2s+4-5}{s^2+4s+4+25}$$

$$\Rightarrow L[e^{-2t}(2\cos 5t - \sin 5t)] = \frac{2s-1}{s^2+4s+29}$$

② Let $F(t) = \cos^2 3t$

$$F(t) = \frac{1+\cos 6t}{2}$$

$$L[F(t)] = \frac{1}{2}L[1+\cos 6t]$$

$$= \frac{1}{2}\{L(1) + L(\cos 6t)\}$$

$$f(s) = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

$\therefore$ WKT $L[e^{at}F(t)] = f(s-a)$

$$L[e^{-t}F(t)] = f(s+1)$$

$$L[e^{-t}\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s+1} + \frac{s+1}{(s+1)^2+36}\right]$$

(4) Let $e^{-4t} t^{-\frac{5}{2}}$

$$F(t) = t^{-5/2}$$

$$L[F(t)] = L[t^{-\frac{5}{2}}]$$

$$f(s) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{s^{-\frac{5}{2}+1}}$$

$$= \frac{\Gamma\left(-\frac{3}{2}\right)}{s^{-3/2}}$$

$$= \frac{\left(-\frac{3}{2}\right)\Gamma\left(-\frac{3}{2}\right)}{\left(-\frac{3}{2}\right)s^{-3/2}}$$

$$= \frac{\Gamma\left(-\frac{3}{2}+1\right)}{\left(-\frac{3}{2}\right)s^{-3/2}}$$

$$= \frac{\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{3}{2}\right)s^{-\frac{3}{2}}}$$

$$= \frac{\left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)s^{-3/2}}$$

$$= \frac{\left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\frac{3}{4}\, s^{-3/2}}$$

$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\frac{3}{4}s^{-3/2}}$$

$$f(s) = \frac{\sqrt{\pi}}{\frac{3}{4}s^{-3/2}}$$

$$L[e^{at}F(t)] = f(s-a)$$

$$L[e^{-4t}F(t)] = f(s+4)$$

$$L[e^{-4t}t^{-5/2}] = \frac{\sqrt{\pi}}{\frac{3}{4}(s+4)^{-3/2}} = \frac{4\sqrt{\pi}}{3(s+4)^{-3/2}}$$

(3) Let $F(t) = \sin 5t \cdot \sin 3t$

$$F(t) = \frac{1}{2}(2\sin 5t \cdot \sin 3t)$$

$$F(t) = \frac{1}{2}[\cos 2t - \cos 8t]$$

$$L[F(t)] = \frac{1}{2}L[\cos 2t - \cos 8t]$$

$$f(s) = \frac{1}{2}\left[\frac{s}{s^2+4} - \frac{s}{s^2+64}\right]$$

wkt $L[e^{at}F(t)] = f(s-a)$

$$L[e^{3t}F(t)] = f(s-3)$$

$$L[e^{3t}\sin 5t \cdot \sin 3t] =$$

$$\frac{1}{2}\left[\frac{(s-3)}{(s-3)^2+4} - \frac{(s-3)}{(s-3)^2+64}\right]$$

Find the laplace transform of the following functions:-

(1)

1. $t\cos at$
2. $t^2 \sin at$
3. $t^3 \cosh t$
4. $t^3 e^{4t} \cosh 3t$
5. $t e^{-2t} \sin 4t$

(1) Let $F(t) = \cos at$

$$L[F(t)] = L[\cos at]$$

$$f(s) = \frac{s}{s^2+a^2}$$

WKT $L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$

$$L[t F(t)] = (-1)^1 \frac{d}{ds} f(s)$$

$$L[t\cos at] = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= -\left[\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$L[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

(2) Let $F(t) = \sin at$

$$L[F(t)] = L[\sin at]$$

$$f(s) = \frac{a}{s^2+a^2}$$

WKT $L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$

$$L[t^2 F(t)] = (-1)^2 \frac{d^2}{ds^2} f(s)$$

$$L[t^2 F(t)] = \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right]$$

$$L[t^2 \sin at] = \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right]$$

$$= -2a \frac{d}{ds}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$= -2a \left[\frac{(s^2+a^2)^2(1) - s.2(s^2+a^2)(2s)}{(s^2+a^2)^4}\right]$$

$$= -2a \left[\frac{s^2+a^2-4s^2}{(s^2+a^2)^3}\right]$$

$$L[t^2 \sin at] = -2a \left[\frac{a^2-3s^2}{(s^2+a^2)^3}\right]$$

⑤ $t e^{-2t} \sin 4t$

Let $F(t) = t \sin 4t$

$$L[F(t)] = -\frac{d}{ds}\left(\frac{4}{s^2+16}\right)$$

$$f(s) = \left[\frac{(s^2+16)0 - 4(2s)}{(s^2+16)^2}\right]$$

$$= \frac{8s}{(s^2+16)^2}$$

$$L[e^{-2t} F(t)] = [f(s)]_{s \to s+2}$$

$$= \frac{8(s+2)}{[(s+2)^2+16]^2}$$

Find the Laplace transform of the following:- (7)

(1) $\dfrac{\cos at - \cos bt}{t}$

$F(t) = \cos at - \cos bt$

$L[F(t)] = L[\cos at - \cos bt]$

$$f(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$L\left[\frac{F(t)}{t}\right] = \int_0^\infty f(s)\,ds$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) \qquad \therefore \int \frac{f'(x)}{f(x)}dx = \log[f(x)]$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\lim_{s\to\infty} \log \frac{s^2(1+a^2/s^2)}{s^2(1+b^2/s^2)} - \log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\left[0 - \log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}.$$

(2) $\dfrac{2\sin t \sin 5t}{t}$

$F(t) = 2\sin t \sin 5t$

$= 2 \times \frac{1}{2}\left[\cos(4t) - \cos(6t)\right]$

$F(t) = \cos(4t) - \cos 6t$

$$L\left[\frac{F(t)}{t}\right] = \log\sqrt{\frac{s^2+36}{s^2+16}}.$$

③ $2^t + \frac{\cos 2t - \cos 3t}{t} + t\sin t$

Let $F(t) = 2^t + \frac{\cos 2t - \cos 3t}{t} + t\sin t$

Let $L[F(t)] = L\left[e^{\log 2 \cdot t}\right] + L\left[\frac{\cos 2t - \cos 3t}{t}\right] + L[t\sin t]$

$$= \frac{1}{s-\log 2} + \log\sqrt{\frac{s^2+9}{s^2+4}} - \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$L[F(t)] = \frac{1}{s-\log 2} + \log\sqrt{\frac{s^2+9}{s^2+4}} + \frac{2s}{(s^2+1)^2}$$

④ $t^2 e^{-3t}\sin 2t$

Let $[F(t)] = L[t^2 e^{-3t}\sin 2t]$

$$L[\sin 2t] = \frac{2}{s^2+4}$$

$$L[t^2\sin 2t] = (-1)^2\frac{d^2}{ds^2}\left[\frac{2}{s^2+4}\right]$$

$$L[t^2\sin 2t] = \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{2}{s^2+4}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-4s}{(s^2+4)^2}\right]$$

$$= -4\frac{d}{ds}\left[\frac{s}{(s^2+4)^2}\right]$$

$$= -4\left[\frac{(s^2+4)(1) - s\cdot 2(s^2+4)(2s)}{(s^2+4)^4}\right]$$

$$= -4\left[\frac{(s^2+4) - 4s^2}{(s^2+4)^3}\right]$$

$$= -4\left[\frac{4-3s^2}{(s^2+4)^3}\right]$$

$$L[e^{-3t}t^2\sin 2t] = 4\left[\frac{3s^2-4}{(s^2+4)^3}\right]_{s\to s+3}$$

$$= 4\left[\frac{3(s+3)^2-4}{[(s+3)^2+4]^3}\right]$$

⑤ t^3+4t^2-3t+5

$$L[t^3+4t^2-3t+5] = L[t^3]+4L[t^2]-3L[t]+5L[1]$$

$$= \frac{3!}{s^4}+4\cdot\frac{2!}{s^3}-3\cdot\frac{1!}{s^2}+5\cdot\frac{1}{s}$$

$$= \frac{6}{s^4}+\frac{8}{s^3}-\frac{3}{s^2}+\frac{5}{s}$$

⑥ $\int_0^\infty t^3e^{-t}\sin t\,dt = 0$

Let $\int_0^\infty e^{-st}F(t)dt = L[F(t)] \rightarrow ①$

Let $F(t) = t^3\sin t$

$① = \int_0^\infty e^{-st}t^3\sin t\,dt = L[t^3\sin t] \rightarrow ②$

WKT $L[\sin t] = \frac{1}{s^2+1}$

$$L[t^3\sin t] = (-1)^3\frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)$$

$$= -\frac{d^2}{ds^2}\left[\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$$

$$= -\frac{d^2}{ds^2}\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$= 2\frac{d^2}{ds^2}\left[\frac{s}{(s^2+1)^2}\right]$$

$$= 2\frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{(s^2+1)^2}\right)\right]$$

$$= 2\frac{d}{ds}\left[\frac{(s^2+1)^2(1)-s.2(s^2+1).(2s)}{(s^2+4)^4}\right]$$

$$= 2\frac{d}{ds}\left[\frac{s^2+1-4s^2}{(s^2+1)^3}\right]$$

$$= 2\frac{d}{ds}\left[\frac{1-3s^2}{(s^2+1)^3}\right]$$

$$= 2\left[\frac{(s^2+1)^3(-6s)-(1-3s^2)3(s^2+1)^2(2s)}{(s^2+1)^6}\right]$$

$$= 2\left[\frac{(s^2+1)(-6s)+(-6s)(1-3s^2)}{(s^2+1)^4}\right]$$

$$= -12s\left[\frac{s^2+1+1-3s^2}{(s^2+1)^4}\right]$$

$$= -12s\left[\frac{-2s^2+2}{(s^2+1)^4}\right]$$

$$L[t^3\sin t] = 24s\left[\frac{s^2-1}{(s^2+1)^4}\right]$$

$$① \Rightarrow \int_0^\infty e^{-st}t^3\sin t\,dt = 24s\left[\frac{s^2-1}{(s^2+1)^4}\right] \rightarrow ③$$

if $s=1$

$$③ \Rightarrow \int_0^\infty e^{-t}t^3\sin t\,dt = 0$$

$$\therefore \int_0^\infty t^3e^{-t}\sin t\,dt = 0$$

③ Evaluate $\int_0^\infty te^{-3t}\cos 2t\,dt$

Let $\int_0^\infty e^{-3t}F(t)dt = L[F(t)] \rightarrow ①$

Let $F(t) = t\cos 2t$

$$① \Rightarrow \int_0^\infty te^{-3t}\cos 2t\,dt = L[t\cos 2t] \rightarrow ②$$

WKT $L[\cos 2t] = \frac{s}{s^2+4}$

$$\Rightarrow L[t\cos 2t] = (-1)\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= -\left[\frac{1.(s^2+4)-s(2s)}{(s^2+4)^2}\right]$$

$$= -\left[\frac{(s^2+4)-2s^2}{(s^2+4)^2}\right]$$

$$= -\left[\frac{-s^2+4}{(s^2+4)^2}\right]$$

$$L[t\cos 2t] = \left[\frac{s^2-4}{(s^2+4)^2}\right]$$

$$① \Rightarrow \int_0^\infty t e^{-st}\cos 2t\,dt = \left[\frac{s^2-4}{(s^2+4)^2}\right]$$

if $s=3$

$$② \Rightarrow \int_0^\infty t e^{-3t}\cos 2t = \left[\frac{9-4}{(9+4)^2}\right]$$

$$\therefore \int_0^\infty t e^{-3t}\cos 2t = \frac{5}{169}$$

<u>Laplace transform for periodic functions:-</u>

Suppose for any $T \geq 0$, the function $f(t)$ is said to be a periodic function for $f(t+T) = f(t)$.

The Laplace transform of the periodic function $f(t)$ for the period T is defined as

$$L[f(t)] = \bar{f}(s) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

① If $f(t) = \begin{cases} E, & \text{if } 0 \leq t \leq a/2 \\ -E, & \text{if } a/2 < t \leq a \end{cases}$ where $f(t+a) = f(t)$, then show that $L[f(t)] = \frac{E}{s}\tanh\left[\frac{as}{4}\right]$.

Given $f(t) = \begin{cases} E, & \text{if } 0 \leq t \leq a/2 \\ -E, & \text{if } a/2 < t \leq a \end{cases}$

$f(t+a) = f(t) \Rightarrow T = a$

WKT

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^{\infty} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^{a} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} E\, e^{-st}\,dt + \int_{a/2}^{a} (-E) e^{-st}\,dt\right]$$

$$= \frac{E}{1-e^{-as}}\left[\int_0^{a/2} e^{-st}\,dt - \int_{a/2}^{a} e^{-st}\,dt\right]$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{\frac{-1}{s}\left[e^{-st}\right]_0^{a/2} + \frac{1}{s}\left[e^{-st}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{s(1-e^{-as})}\left\{-(e^{-as/2} - e^{0}) + [e^{-as} - e^{-as/2}]\right\}$$

$$= \frac{E}{s(1-e^{-as})}\left\{-e^{-as/2} + 1 + e^{-as} - e^{-as/2}\right\}$$

$$= \frac{E}{s(1-e^{-as})}\left[1 - 2e^{-as/2} + e^{-as}\right]$$

$$= \frac{E}{s(1-e^{-as})}\left[(1)^2 - 2(1)(e^{-as/2}) + (e^{-as/2})^2\right]$$

$$= \frac{E\,(1-e^{-as/2})^2}{s[(1)^2 - (e^{-as/2})^2]}$$

$$= \frac{E\,(1-e^{-as/2})^2}{s(1-e^{-as/2})(1+e^{-as/2})}$$

$$= \frac{E(1-e^{-as/2})}{s\,(1+e^{-as/2})}$$

$$= \frac{E}{s}\,\frac{(1-e^{-as/2})\,e^{as/4}}{(1+e^{-as/2})\,e^{as/4}}$$

$$= \frac{E}{s}\,\frac{e^{as/4} - e^{-as/4}}{e^{as/4} + e^{-as/4}}$$

$\therefore L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{4}\right)$

(2) If $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a-t, & a < t \le 2a \end{cases}$ where $f(t+2a) = f(t)$ then

Show that $L[f(t)] = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$.

Given $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a-t, & a < t \le 2a \end{cases}$

$f(t+2a) = f(t) \Rightarrow T = 2a$

$$\Rightarrow L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$\Rightarrow L[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^{a} e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt\right]$$

$$\Rightarrow L[f(t)] = \frac{1}{1-e^{-2as}}\left[\int_0^{a} t e^{-st}\,dt + \int_a^{2a} (2a-t) e^{-st}\,dt\right]$$

$$\therefore \int_0^a t e^{-st}\,dt = t\int_0^a e^{-st}\,dt - \int_0^a \left[1 \int e^{-st}\,dt\right] dt$$

$$= -\frac{1}{s}\left[t e^{-st}\right]_0^a + \frac{1}{s}\int_0^a e^{-st}\,dt$$

$$= -\frac{1}{s}\left[t e^{-st}\right]_0^a - \frac{1}{s^2}\left[e^{-st}\right]_0^a$$

$$= -\frac{1}{s}\left[ae^{-as} - 0\right] - \frac{1}{s^2}\left[e^{-as} - e^0\right]$$

$$\int_0^a t e^{-st}\,dt = -\frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-as} + \frac{1}{s^2}$$

$$\Rightarrow \int_0^{2a} (2a-t) e^{-st}\,dt = (2a-t)\int_0^{2a} e^{-st}\,dt - \int_0^{2a}\left[(-1)\int e^{-st}\,dt\right] dt$$

$$= -\frac{1}{s}\left[(2a-t)e^{-st}\right]_a^{2a} - \frac{1}{s}\int_0^{2a} e^{-st}\,dt$$

$$= \frac{-1}{s}\left[(2a-t)e^{-st}\right]_a^{2a} + \frac{1}{s^2}\left[e^{-st}\right]_a^{2a}$$

$$= -\frac{1}{s}\left[0 - ae^{-as}\right] + \frac{1}{s^2}\left[e^{-2as} - e^{-as}\right]$$

$$\int_0^{2a}(2a-t)e^{-st}dt = \frac{a}{s}e^{-as} + \frac{1}{s^2}e^{-2as} - \frac{1}{s^2}e^{-as}$$

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left[-\frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-as} + \frac{1}{s^2} + \frac{a}{s}e^{-as} + \frac{1}{s^2}e^{-2as} - \frac{1}{s^2}e^{-as}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2} - \frac{2}{s^2}e^{-as} + \frac{1}{s^2}e^{-2as}\right]$$

$$= \frac{1}{s^2(1-e^{-2as})}\left[1 - 2(1)e^{-as} + (e^{-as})^2\right]$$

$$= \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1}{s^2}\,\frac{1-e^{-as}}{1+e^{-as}} \times \frac{e^{as/2}}{e^{as/2}}$$

$$= \frac{1}{s^2}\,\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}$$

$$\therefore L[f(t)] = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

③ If $f(t) = \begin{cases} E\sin(\omega t), & 0 \le t \le \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t \le \frac{2\pi}{\omega} \end{cases}$ is a periodic function of the period $\frac{2\pi}{\omega}$, then st

$$L[f(t)] = \frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi/\omega})}$$ for ω and E constants.

Given $f(t) = \begin{cases} E\sin(\omega t), & 0 \le t \le \pi/\omega \\ 0, & \pi/\omega < t \le \frac{2\pi}{\omega} \end{cases}$ and $T = \frac{2\pi}{\omega}$

$$\therefore L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} f(t)\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} f(t)\,dt \right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} E\sin\omega t\,dt$$

$$= \frac{E}{1-e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt$$

$$= \frac{E}{1-e^{-2\pi s/\omega}} \left\{ \frac{e^{-st}}{(-s)^2+\omega^2} \left[-s\sin\omega t - \omega\cos\omega t \right] \right\}_0^{\pi/\omega}$$

$$= \frac{E}{1-e^{-2\pi s/\omega}} \left\{ \left[\frac{e^{-\pi s/\omega}}{s^2+\omega^2} (0+\omega) \right] - \left[\frac{1}{s^2+\omega^2} (0-\omega) \right] \right\}$$

$$= \frac{E}{1-e^{-2\pi s/\omega}} \left[\frac{\omega e^{-\pi s/\omega}}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2} \right]$$

$$= \frac{E\omega\left(1+e^{-\pi s/\omega}\right)}{(s^2+\omega^2)(1-e^{-\pi s/\omega})(1+e^{-\pi s/\omega})}$$

$$\Rightarrow L[f(t)] = \frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$$

(4) If $f(t) = \begin{cases} E, & \text{if } 0 \le t \le a \\ -E, & \text{if } a < t \le 2a \end{cases}$ where $f(t+2a) = f(t)$, then show that $L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$.

Given $f(t) = \begin{cases} E, & \text{if } 0 \le t \le a \\ -E, & \text{if } a < t \le 2a \end{cases}$

$$f(t+2a) = f(t) \quad \therefore T = 2a$$

WKT

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left\{\int_0^a e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\int_0^a e^{-st} E\,dt + \int_a^{2a} e^{-st}(-E)\,dt\right\}$$

$$= \frac{E}{1-e^{-2as}}\left[\int_0^a e^{-st}\,dt + (-1)\int_a^{2a} e^{-st}\,dt\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\int_0^a e^{-st}\,dt - \int_a^{2a} e^{-st}\,dt\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^a - \left[\frac{e^{-st}}{-s}\right]_a^{2a}\right]$$

$$= \frac{E}{s(1-e^{-2as})}\left\{-\left[e^{-st}\right]_0^a + \left[e^{-st}\right]_a^{2a}\right\}$$

$$= \frac{E}{s(1-e^{-2as})}\left[-e^{-as} + e^0 + e^{-2as} - e^{-as}\right]$$

$$= \frac{E}{s(1-e^{-2as})}\left[-2e^{-as} + 1 + e^{-2as}\right]$$

$$= \frac{E}{s(1-e^{-2as})}\left[(1)^2 + (e^{-as})^2 - 2(1)(e^{-as})\right]$$

$$= \frac{E(1-e^{-as})^2}{s(1-e^{-2as})}$$

$$= \frac{E(1-e^{-as})^2}{s\left[(1)^2-(e^{-as})^2\right]}$$

$$= \frac{E(1-e^{-as})^2}{s\left[1+e^{-as}\right]\left[1-e^{-as}\right]}$$

$$= \frac{E(1-e^{-as})}{s(1+e^{-as})}$$

$$= \frac{E}{s}\frac{(1-e^{-as})e^{as/2}}{(1+e^{-as})e^{as/2}}$$

$$= \frac{E}{s}\frac{(e^{as/2}-e^{-as/2})}{(e^{as/2}+e^{-as/2})}$$

$$L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{2}\right).$$

(12)

⑤ If $f(t) = t, \ 0 \le t \le a/2$

$a-t, \ a/2 < t \le a$ where $f(t+a) = f(t)$ then

Show that $L[f(t)] = \frac{1}{s^2} \tanh\left(\frac{as}{4}\right)$

Given $f(t) = \begin{cases} t, & 0 \le t \le a/2 \\ a-t, & a/2 < t \le a \end{cases}$

$f(t+a) = f(t) \Rightarrow T = a$

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st} f(t)\,dt + \int_a^{a/2} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} t\, e^{-st}\,dt + \int_{a/2}^{a} e^{-st} (a-t)\,dt\right]$$

$$\int_0^{a/2} t e^{-st}\,dt = t \int_0^{a/2} e^{-st}\,dt - \int_0^{a/2} [1 \int e^{-st}\,dt]$$

$$= \frac{-1}{s}\left[t e^{-st}\right]_0^{a/2} + \int_0^{a/2} \left[\frac{e^{-st}}{-s}\right]$$

$$= \frac{-1}{s}\left[\frac{a}{2} e^{-as/2} - 0\right] - \frac{1}{s^2}\left[e^{-st}\right]_0^{a/2}$$

$$= -\frac{1}{s}\left[\frac{a}{2} e^{-as/2}\right] - \frac{1}{s^2}\left[e^{-as/2} - e^0\right]$$

$$= \frac{-a}{2s} e^{-as/2} - \frac{1}{s^2} e^{-as/2} + \frac{1}{s^2}$$

$$\therefore \int_{a/2}^{a} (a-t) e^{-st}\,dt = (a-t)\int_{a/2}^{a} e^{-st}\,dt - \int_{a/2}^{a} \left[(-1)\int e^{-st}\,dt\right]dt$$

$$= \frac{-1}{s}\left[(a-t) e^{-st}\right]_{a/2}^{a} - \frac{1}{s}\int_{a/2}^{a} e^{-st}\,dt$$

$$= \frac{-1}{s}\left[(a-t) e^{-st}\right]_{a/2}^{a} + \frac{1}{s^2}\left[e^{-st}\right]_{a/2}^{a}$$

$$= -\frac{1}{s}\left[0 - (a - a/2) e^{-\frac{as}{2}}\right] + \frac{1}{s^2}\left[e^{-as} - e^{-as/2}\right]$$

$$\int_{a/2}^{a} (a-t) e^{-st}\,dt = \frac{a}{2s} e^{-as/2} + \frac{1}{s^2} e^{-as} - \frac{e^{-as/2}}{s^2}$$

$$\therefore L[f(t)] = \frac{1}{1-e^{-as}}\left[\frac{-a}{2s}e^{-as/2} - \frac{1}{s^2}e^{-as/2} + \frac{1}{s^2} + \frac{a}{2s}e^{-as/2} + \frac{1}{s^2}e^{-as} - \frac{e^{-as/2}}{s^2}\right]$$

$$= \frac{1}{1-e^{-as}}\left[-\frac{2}{s^2}e^{-as/2} + \frac{1}{s^2} + \frac{1}{s^2}e^{-as}\right]$$

$$= \frac{1}{s^2(1-e^{-as})}\left[(1)^2 - 2(1)(e^{-as/2}) + (e^{-as/2})^2\right]$$

$$= \frac{1}{s^2(1-e^{-as})}\left[1 - e^{-as/2}\right]^2$$

$$= \frac{1}{s^2}\frac{(1-e^{-as/2})^2}{(1)^2-(e^{-as/2})^2}$$

$$= \frac{1}{s^2}\frac{(1-e^{-as/2})^2}{(1-e^{-as/2})(1+e^{-as/2})}$$

$$= \frac{1}{s^2}\frac{(1-e^{-as/2})}{(1+e^{-as/2})} \times \frac{e^{as/4}}{e^{as/4}}$$

$$= \frac{1}{s^2}\frac{(e^{as/4} - e^{-as/4})}{(e^{as/4} + e^{-as/4})}$$

$$L[f(t)] = \frac{1}{s^2}\tanh\left(\frac{as}{4}\right)$$

Unit step (or) Heaviside function :- For any $a \geq 0$ the unit step (or) Heaviside function can be defined as $u(t-a)$ (or) $H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases}$

The Laplace transform of the unit step function is $L[u(t-a)] = \frac{e^{-as}}{s}$.

1. $f(t)$ be a function, then $L[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$, where $\bar{f}(s) = L[f(t)]$.

2. If $f(t) = \begin{cases} f_1(t), & \text{for } t < a \\ f_2(t), & \text{for } t \geq a \end{cases}$, then the Heaviside form of $f(t)$ can be written as $f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a)$ and its Laplace transform is

$L[f(t)] = L[f_1(t)] + L\{[f_2(t) - f_1(t)]u(t-a)\}$

3. If $f(t) = \begin{cases} f_1(t), & \text{for } 0 \le t < a \\ f_2(t), & \text{for } a \le t < b \\ f_3(t), & \text{for } t \ge L \end{cases}$ then, the Heaviside form

of $f(t)$ can be written as

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$$

$$L f(t) = L[f_1(t)] + L\{[f_2(t) - f_1(t)]u(t-a)\} + L\{[f_3(t) - f_2(t)]u(t-b)\}.$$

① Find the Laplace transform for the following Heaviside function:-

① $(t^2-1)u(t-1)$

② $\sin t \cdot u(t-\pi)$

③ $(2t^2-t+1)u(t-1)$.

① Let $f(t-1) = t^2-1$

$$f(t) = (t+1)^2-1$$

$$f(t) = t^2+2t+1-1$$

$$f(t) = t^2+2t$$

$$L(f(t)) = L[t^2] + 2L[t]$$

$$\bar{f}(s) = \frac{2}{s^3} + \frac{2}{s^2}$$

WKT

$$\Rightarrow L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$$

$$\Rightarrow L[f(t-1)u(t-1)] = e^{-s}\bar{f}(s)$$

$$\Rightarrow L[(t^2-1)u(t-1)] = e^{-s}\left[\frac{2}{s^3} + \frac{2}{s^2}\right]$$

② $\sin t \cdot u(t-\pi)$

Let $f(t-\pi) = \sin t$

$$f(t) = \sin(t+\pi)$$

$$f(t) = -\sin t$$

$$L[f(t)] = -L[\sin t]$$

$$\Rightarrow \bar{f}(s) = \frac{-1}{s^2+1}.$$

WKT $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$$\Rightarrow L[f(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{f}(s)$$

$$\Rightarrow L[\sin t \cdot u(t-\pi)] = \frac{-e^{-\pi s}}{s^2+1}$$

③ $(2t^2-t+1)u(t-1)$

$f(t-1) = 2t^2-t+1$

$f(t) = 2(t+1)^2-(t+1)+1$

$= 2[t^2+1+2t]-t$

$f(t) = 2t^2+3t+2$

$L[f(t)] = 2L[t^2]+3L[t]+2L[1]$

$= 2\cdot\frac{2!}{s^3}+3\frac{1!}{s^2}+\frac{2}{s}$

$L[f(t)] = \frac{4}{s^3}+\frac{3}{s^2}+\frac{2}{s}$

$\Rightarrow L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$\Rightarrow L[f(t-1)u(t-1)] = e^{-s}\bar{f}(s)$

$\Rightarrow L[(2t^2-t+1)u(t-1)] = e^{-s}\left(\frac{4}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right)$

① Express $f(t) = \begin{cases} \sin t, & \text{for } t < \pi \\ \pi - t, & \text{for } t \geq \pi \end{cases}$ to unit step function, hence find its Laplace transform.

Given $f(t) = \begin{cases} \sin t, & \text{for } t < \pi \\ \pi - t, & \text{for } t \geq \pi \end{cases}$

$\therefore f(t) = \sin t + [(\pi-t)-\sin t]u(t-\pi)$

$\Rightarrow L[f(t)] = L[\sin t] + L\{[(\pi-t)-\sin t]u(t-\pi)\} \rightarrow ①$

$\Rightarrow g(t-\pi) = (\pi-t)-\sin t$

$\Rightarrow g(t) = \pi-(t+\pi)-\sin(t+\pi)$

$\Rightarrow g(t) = \sin t - t$

$L[g(t)] = L[\sin t] - L[t]$

$\Rightarrow \bar{g}(s) = \frac{1}{s^2+1} - \frac{1}{s^2}$

$\therefore$ WKT $L[g(t-a)u(t-a)] = e^{-as}\bar{g}(s)$

$\Rightarrow L[g(t-\pi)u(t-\pi)] = e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2}\right)$

$\Rightarrow \bar{f}(s) = \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2}\right)$

(2) Express $f(t) = \begin{cases} \sin t, & \text{for } 0 \le t < \pi/2 \\ \cos t, & \text{for } t \ge \pi/2 \end{cases}$ to unit step function, hence find its Laplace transform.

Given $f(t) = \begin{cases} \sin t, & \text{for } 0 \le t < \pi/2 \\ \cos t, & \text{for } t \ge \pi/2 \end{cases}$

$\Rightarrow f(t) = \sin t + [\cos t - \sin t]u(t-\pi/2) \rightarrow (1)$

$\Rightarrow L[f(t)] = L[\sin t] + L\{[\cos t - \sin t]u(t-\pi/2)\} \rightarrow (2)$

$\Rightarrow g(t-\pi/2) = \cos t - \sin t$

$\Rightarrow g(t) = \cos(t+\pi/2) - \sin(t+\pi/2)$

$\Rightarrow g(t) = \sin t - \cos t$

$\Rightarrow L[g(t)] = -L[\sin t] - L[\cos t]$

$\Rightarrow \bar{g}(s) = \frac{-1}{s^2+1} - \frac{s}{s^2+1}$

$\Rightarrow \bar{g}(s) = -\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$

WKT

$\Rightarrow L[g(t-a)u(t-a)] = e^{-as}\bar{g}(s)$

$\Rightarrow L[g(t-\pi/2)u(t-\pi/2)] = e^{-\pi s/2}\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$

$(2) \Rightarrow \frac{1}{s^2+1} - e^{-\pi/2}\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$

③ Express $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi \\ \sin 3t, & t \ge 2\pi \end{cases}$ to unit step function, hence find its Laplace transform.

$\Rightarrow f(t) = \sin t + (\sin 2t - \sin t)u(t-\pi) + (\sin 3t - \sin 2t)u(t-2\pi)$

$\Rightarrow L[f(t)] = L[\sin t] + L[(\sin 2t - \sin t)u(t-\pi)] + L[(\sin 3t - \sin 2t)u(t-2\pi)]$ $\rightarrow ①$

$\Rightarrow g(t-\pi) = \sin 2t - \sin t$

$\Rightarrow g(t) = \sin(2t + 2\pi) - \sin(t + \pi)$

$\Rightarrow g(t) = \sin 2t + \sin t$

$\Rightarrow L[g(t)] = L[\sin 2t] + L[\sin t]$

$\Rightarrow \bar{g}(s) = \dfrac{2}{s^2+4} + \dfrac{1}{s^2+1}$

$\therefore L[g(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{g}(s)$

$\Rightarrow L[g(t-\pi)u(t-\pi)] = e^{-\pi s}\left[\dfrac{2}{s^2+4} + \dfrac{1}{s^2+1}\right]$

$\Rightarrow h(t-2\pi) = \sin 3t - \sin 2t$

$\Rightarrow h(t) = \sin(3t + 6\pi) - \sin(2t + 4\pi)$

$\Rightarrow h(t) = \sin 3t - \sin 2t$

$\Rightarrow L[h(t)] = L[\sin 3t] - L[\sin 2t] = \dfrac{3}{s^2+9} - \dfrac{2}{s^2+4}$

$\therefore L[h(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\bar{h}(s)$

$\Rightarrow L[h(t-2\pi)u(t-2\pi)] = e^{-2\pi}\left[\dfrac{3}{s^2+9} - \dfrac{2}{s^2+4}\right]$

① $\Rightarrow \bar{f}(s) = \dfrac{1}{s^2+1} + e^{-\pi s}\left[\dfrac{2}{s^2+4} + \dfrac{1}{s^2+1}\right] + e^{-2\pi s}\left[\dfrac{3}{s^2+9} - \dfrac{2}{s^2+4}\right]$ //

④ Express $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & 1 \le t < 2 \\ t^2, & t \ge 2 \end{cases}$ to unit step function, hence find its Laplace transform.

Given $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & 1 \le t < 2 \\ t^2, & t \ge 2 \end{cases}$

$\Rightarrow f(t) = 1 + (t-1)u(t-1) + (t^2-t)u(t-2)$

$\Rightarrow L[f(t)] = L[1] + L[(t-1)u(t-1)] + L[(t^2-t)u(t-2)] \rightarrow (1)$

$\Rightarrow$ let $g(t-1) = t-1$

$\Rightarrow g(t) = t$

$\Rightarrow L[g(t)] = L[t]$

$\Rightarrow \bar{g}(s) = \frac{1}{s^2}$

$\therefore L[g(t-1)u(t-1)] = e^{-s}\bar{g}(s) = \frac{e^{-s}}{s^2}$

$\Rightarrow$ Let $h(t-2) = t^2-t$

$\Rightarrow h(t) = (t+2)^2 - (t+2)$

$\Rightarrow h(t) = t^2 + 4t + 4 - t - 2$

$\Rightarrow h(t) = t^2 + 3t + 2$

$\Rightarrow L[h(t)] = L[t^2] + 3L[t] + 2L[1]$

$\Rightarrow \bar{h}(s) = \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}$

$\therefore L[h(t-2)u(t-2)] = e^{-2s}\bar{h}(s) = e^{-2s}\left[\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right]$

$(1) \Rightarrow \therefore \bar{f}(s) = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)$

(5) $f(t) = \begin{cases} t^2, & 0 \le t < 2 \\ 4t, & 2 \le t < 4 \\ 8, & t \ge 4 \end{cases}$ to unit step function hence find its Laplace transform.

$\Rightarrow f(t) = t^2 + (4t - t^2)u(t-2) + (8-4t)u(t-4)$

$L[f(t)] = L[t^2] + L[(4t - t^2)u(t-2)] + L[(8-4t)u(t-4)]$

let $g(t-2) = 4t - t^2$

$\Rightarrow g(t) = 4(t+2) - (t+2)^2$

$= 4t + 8 - (t^2 + 4t + 4)$

$= 4t + 8 - t^2 - 4t - 4$

$g(t) = 4 - t^2$

$$L[g(t)] = L[4] - L[t^2]$$

$$\bar{g}(s) = \frac{4}{s} - \frac{2}{s^3}$$

$$\therefore L[g(t-2)u(t-2)] = e^{-2s}\bar{g}(s)$$

$$= e^{-2s}\left[\frac{2}{s^3} - \frac{4}{s}\right]$$

$$\Rightarrow h(t-4) = 8 - 4t$$

$$\Rightarrow h(t) = 8 - 4(t+4)$$

$$= 8 - 4t + 16$$

$$\Rightarrow h(t) = -8 - 4t$$

$$\Rightarrow L[h(t)] = -L\,8(1) - 4L[t]$$

$$\Rightarrow L[h(t)] = -\frac{8}{s} - \frac{4}{s^2}$$

$$\Rightarrow \bar{h}(s) = -\left(\frac{4}{s^2} + \frac{8}{s}\right)$$

$$\therefore L[h(t-4)u(t-4)] = e^{-4s}\bar{h}(s)$$

$$= -e^{-4s}\left(\frac{4}{s^2} + \frac{8}{s}\right)$$

$$\therefore (1) \Rightarrow \bar{f}(s) = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} - \frac{4}{s}\right) - e^{-4s}\left(\frac{4}{s^2} + \frac{8}{s}\right)$$

6) Express $f(t) = \begin{cases} \cos t, & 0 \le t < \pi \\ \cos 2t, & \pi \le t < 2\pi \\ \cos 3t, & t \ge 2\pi \end{cases}$ to unit step function.

hence find its Laplace transform.

$$\Rightarrow \text{Given } f(t) = \begin{cases} \cos t, & 0 \le t < \pi \\ \cos 2t, & \pi \le t < 2\pi \\ \cos 3t, & t \ge 2\pi \end{cases}$$

$$\Rightarrow f(t) = \cos t + [(\cos 2t - \cos t)u(t-\pi)] + [(\cos 3t - \cos 2t)u(t-2\pi)] \quad \text{—(1)}$$

$$\Rightarrow L[f(t)] = L[\cos t] + L[(\cos 2t - \cos t)u(t-\pi)] + L[(\cos 3t - \cos 2t)u(t-2\pi)]$$

$$\Rightarrow g(t-\pi) = \cos 2t - \cos t$$

$$\Rightarrow g(t) = \cos 2(t+\pi) - \cos(t+\pi)$$

$$\Rightarrow g(t) = \cos(2t+2\pi) - \cos(t+\pi)$$

$$\Rightarrow g(t) = \cos 2t + \cos t$$

$\Rightarrow L[g(t)] = L[\cos(2t)] + L[\cos t]$

$\Rightarrow \bar{g}(s) = \frac{s}{s^2+4} + \frac{s}{s^2+1}$

$\therefore L[g(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{g}(s)$

$= e^{-\pi s}\left[\frac{s}{s^2+4} + \frac{s}{s^2+1}\right]$

$\Rightarrow$ Let $h(t-2\pi) = \cos 3t - \cos 2t$

$\Rightarrow h(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi)$

$\Rightarrow h(t) = \cos 3t - \cos 2t$

$\Rightarrow L[h(t)] = L[\cos 3t] - L[\cos 2t]$

$\Rightarrow \bar{h}(s) = \frac{s}{s^2+9} - \frac{s}{s^2+4}$

$\Rightarrow L[h(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\bar{h}(s)$

$= e^{-2\pi s}\left[\frac{s}{s^2+9} - \frac{s}{s^2+4}\right]$

(1) $\Rightarrow \bar{f}(s) = \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{s}{s^2+4} + \frac{s}{s^2+1}\right] + e^{-2\pi s}\left[\frac{s}{s^2+9} - \frac{s}{s^2+4}\right]$

Inverse Laplace Transform:- Suppose $f(s)$ be the Laplace transform of $F(t)$, then the inverse Laplace transform of $f(s)$ can be defined as $L^{-1}[f(s)] = F(t)$, where L^{-1} is called the inverse Laplace transform.

Some Important Results:-

1. $L^{-1}\left[\frac{1}{s}\right] = 1$
2. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
3. $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
4. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
5. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$
6. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$
7. $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$
8. $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}\ \forall\ n = 0, 1, 2, \ldots$
9. $L^{-1}[f(s+a)] = e^{-at}L^{-1}[f(s)] = e^{-at}F(t)$
10. $L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)] = e^{at}F(t)$
11. $L^{-1}[f^{n}(s)] = (-1)^n t^n F(t)$

Find the Laplace transform for the following:-

1. $\frac{1}{s(s+1)(s+2)}$
2. $\frac{4s+5}{(s+1)^2(s+2)}$
3. $\frac{s+2}{s^2-4s+3}$
4. $\frac{4s+5}{(s-1)^2(s+2)}$
5. $\frac{1}{(s-1)(s^2+1)}$
6. $\frac{3s+2}{s^2-s-2}$

① $\frac{1}{s(s+1)(s+2)}$

Let $f(s) = \frac{1}{s(s+1)(s+2)}$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \rightarrow ①$$

$$1 = A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1) \rightarrow ②$$

when $s = 0$

② $\Rightarrow 1 = A(1)(2)$

$1 = 2A$

$A = 1/2$

when $s = -1$

② $\Rightarrow 1 = B(-1)(-1+2)$

$1 = -B$

$B = -1$

when $s = -2$

$$② \Rightarrow 1 = C(-2)(-2+1)$$

$$1 = -2C(-1)$$

$$1 = 2C$$

$$C = 1/2$$

$$\therefore ① \Rightarrow f(s) = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\,\frac{1}{s+2}$$

$$\Rightarrow L^{-1}[f(s)] = \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right]$$

$$\Rightarrow F(t) = \frac{1}{2}(1) - e^{-t} + \frac{1}{2}e^{-2t}$$

$$\Rightarrow F(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

② $\dfrac{4s+5}{(s+1)^2(s+2)}$

Let $f(s) = \dfrac{1}{(s+1)^2(s+2)}$

$$\frac{4s+5}{(s+1)^2(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+2)} \rightarrow ①$$

when $s = -1$

$② \Rightarrow 1 = B(-1+2)$

$B = 1$

when $s = -2$

$② \Rightarrow -3 = C(-2+1)^2$

$C = -3$

By comparing s^2 coefficient on both sides we have $A + C = 0$

$$A = -C$$

$$\boxed{A = 3}$$

$$\therefore ① \Rightarrow f(s) = \frac{3}{s+1} + \frac{1}{(s+1)^2} - \frac{3}{s+2}$$

$$L^{-1}[f(s)] = 3L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2}\right] - 3L^{-1}\left[\frac{1}{s+2}\right]$$

$$= 3e^{-t}L^{-1}\left[\frac{1}{s}\right] + e^{-t}L^{-1}\left[\frac{1}{s^2}\right] - 3e^{-2t}L^{-1}\left[\frac{1}{s}\right]$$

$F(t) = 3e^{-t} + te^{-t} - 3e^{-2t}$.

③ $\dfrac{1}{(s-1)(s^2+1)}$

Let $f(s) = \dfrac{1}{(s-1)(s^2+1)}$

$$\Rightarrow \frac{1}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+1} \rightarrow ①$$

$$\Rightarrow 1 = A(s^2+1) + (Bs+C)(s-1) \rightarrow ②$$

$$\Rightarrow 1 = As^2 + A + Bs^2 - Bs + Cs - C$$

$$\Rightarrow 1 = (A+B)s^2 + (C-B)s + (A-C) \rightarrow ③$$

when $s = 1$

② $\Rightarrow 1 = A(1+1)$

$1 = 2A$

$A = \frac{1}{2}$

By comparing s^2 coefficient on both sides we have

$A + B = 0$

$B = -A$

$B = \frac{-1}{2}$

By comparing coefficient of s

$C - B = 0$

$C = B$

$C = \frac{-1}{2}$

$$\frac{1}{(s-1)(s^2+1)} = \frac{1}{2(s-1)} + \frac{(-\frac{1}{2})s}{(s^2+1)} - \frac{1}{2}\frac{1}{(s^2+1)}$$

$$= \frac{1}{2(s-1)} - \frac{s}{2(s^2+1)} - \frac{1}{2}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{2}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}e^{t} - \frac{1}{2}\cos t - \frac{1}{2}e^{-t}\,t$$

④ $\dfrac{s+2}{s^2-4s+13}$

Let $f(s) = \dfrac{s+2}{s^2 - 2(s)(2) + 2^2 - 2^2 + 13}$

$f(s) = \dfrac{s+2}{(s-2)^2 - 4 + 13}$

$f(s) = \dfrac{s+2}{(s-2)^2 + 9}$

$f(s) = \dfrac{(s-2)+2+2}{(s-2)^2+9}$

$f(s) = \dfrac{(s-2)+4}{(s-2)^2+9}$

$L^{-1}f(s) = L^{-1}\left[\dfrac{(s-2)+4}{(s-2)^2+9}\right]$

$= e^{2t} L^{-1}\left[\dfrac{s+4}{s^2+9}\right]$

$= e^{2t} L^{-1}\left[\dfrac{s}{s^2+9} + \dfrac{4}{s^2+9}\right]$

$= e^{2t} L^{-1}\left[\dfrac{s}{s^2+9} + \dfrac{4}{s^2+9}\right]$

$= e^{2t}\left\{L^{-1}\left[\dfrac{s}{s^2+9}\right] + 4L^{-1}\left[\dfrac{1}{s^2+9}\right]\right\}$

$\Rightarrow L^{-1}f(s) = e^{2t}\left[\cos 3t + \dfrac{4}{3}\sin 3t\right]$

⑤ $\dfrac{4s+5}{(s-1)^2(s+2)}$

$f(s) = \dfrac{1}{(s-1)^2(s+2)}$

$\dfrac{4s+5}{(s-1)^2(s+2)} = \dfrac{A}{(s-1)} + \dfrac{B}{(s-1)^2} + \dfrac{C}{(s+2)} \rightarrow$ ①

$4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$

when $s = 1$

$9 = 3B \Rightarrow B = 9/3 \Rightarrow B = 3$

when $s = -2$

$4(-2)+6 = C(-3)^2$

$-2 = 9C$

$C = -\frac{2}{9}$

By comparing s^2 coefficient on both side we have

$A + C = 0$

$A = -C$

$A = -\frac{1}{3}$

$\therefore ① \Rightarrow f(s) = \frac{1}{3(s-1)} + \frac{3}{(s-1)^2} - \frac{1}{3(s+2)}$

$\Rightarrow L^{-1}[f(s)] = \frac{1}{3} L^{-1}\left[\frac{1}{(s-1)}\right] + 3 L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3} L^{-1}\left[\frac{1}{(s+2)}\right]$

$= \frac{1}{3} e^{t} L^{-1}\left[\frac{1}{s}\right] + 3e^{t} L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{3} e^{-2t} L^{-1}\left[\frac{1}{s}\right]$

$\Rightarrow F(t) = \frac{1}{3} e^{t} + 3e^{t} t - \frac{1}{3} e^{-2t}$

⑥ $\frac{3s+2}{s^2-s-2}$

Let $f(s) = \frac{3s+2}{s^2-s-2}$

$= \frac{3s+2}{s^2+s-2s-2}$

$= \frac{3s+2}{s(s+1)-2(s+1)}$

$f(s) = \frac{3s+2}{(s-2)(s+1)}$

$\therefore \frac{3s+2}{(s-2)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s+1)} \rightarrow ①$

$3s+2 = A(s+1) + B(s-2) \rightarrow ②$

when $s = 2$

$② \Rightarrow 8 = A(3)$

$A = \frac{8}{3}$

when $s = -1$

$② \Rightarrow -1 = B(-3)$

$B = \frac{1}{3}$

$① \Rightarrow f(s) = \frac{8}{3} \frac{1}{s-2} + \frac{1}{3} \frac{1}{s+1}$

$$L^{-1}[f(s)] = \frac{8}{3}L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]$$

$$F(t) = \frac{8}{3}e^{2t} + \frac{1}{3}e^{-t}$$

(3) Find the inverse laplace transform of $\frac{2s^2+5s-4}{s^3+s^2-2s}$

$$\Rightarrow f(s) = \frac{2s^2+5s-4}{s(s^2+s-2)}$$

$$= \frac{2s^2+5s-4}{s(s^2-s+2s-2)}$$

$$= \frac{2s^2+5s-4}{s[s(s-1)+2(s-1)]}$$

$$\Rightarrow f(s) = \frac{2s^2+5s-4}{s(s-1)(s+2)}$$

$$\therefore \frac{2s^2+5s-4}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s+2)} \rightarrow (1)$$

$$\Rightarrow 2s^2+5s-4 = A(s-1)(s+2) + B(s)(s+2) + C(s)(s-1) \rightarrow (2)$$

when $s=1$

(2) $\Rightarrow 2+5-4 = B(3)$

$3 = B(3)$

$B = 1$

when $s=-2$

(2) $\Rightarrow 8-10-4 = C(-2)(-2-1)$

$-6 = C(6)$

$C = -1$

(2) $\Rightarrow 2 = A+B+C$

$2-A = B+C$

$-A = B+C-2$

$-A = 1-1-2$

$A = 2$

$$(1) \Rightarrow f(s) = 2\cdot\frac{1}{s} + \frac{1}{(s-1)} - \frac{1}{(s+2)}$$

$$L^{-1}[f(s)] = 2L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$

$$F(t) = 2 + e^{t} - e^{-2t}$$

Fi Find the inverse laplace transform for the following:- (20)

① $\log\left|\frac{s^2+1}{s(s+1)}\right|$ ② $\log\left|\frac{s(s+\pi)}{(s^2+25)(s-7)}\right|$

③ $\log\left|\frac{s+a}{s+b}\right|$ ④ $\tan^{-1}\left(\frac{2}{s^2}\right)$

① $\log\left|\frac{s+a}{s+b}\right|$

let $f(s) = \log\left|\frac{s+a}{s+b}\right|$

$f(s) = \log(s+a) - \log(s+b) \rightarrow ①$

diff. w.r.t s

$① \Rightarrow f'(s) = \frac{1}{s+a} - \frac{1}{s+b}$

$\Rightarrow L^{-1}[f'(s)] = L^{-1}\left(\frac{1}{s+a}\right) - L^{-1}\left(\frac{1}{s+b}\right)$

$\Rightarrow (-1)^1 t^1 F(t) = e^{-at} - e^{-bt}$

$\Rightarrow -t\, F(t) = e^{-at} - e^{-bt}$

$\Rightarrow F(t) = \frac{e^{-at} - e^{-bt}}{-t}$

$\Rightarrow F(t) = \frac{e^{-bt} - e^{-at}}{t}$

③ $\log\left|\frac{s^2+1}{s(s+1)}\right|$

let $f(s) = \log\left|\frac{s^2+1}{s(s+1)}\right|$

$\Rightarrow f(s) = \log|s^2+1| - \log|s(s+1)|$

$\Rightarrow f(s) = \log(s^2+1) - \log s - \log(s+1) \rightarrow ①$

Diff. ① wrt 's'

$① \Rightarrow f'(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$

$\Rightarrow L^{-1}[f'(s)] = 2\left[L^{-1}\left(\frac{s}{s^2+1}\right) - L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right)\right]$

$\Rightarrow (-1)^1 t^1 F(t) = 2\cos t - 1 - e^{-t}$

$$\Rightarrow F(t) = \frac{1+e^{-t}-2\cos t}{t}$$

③ $\log\left|\frac{s(s+5)}{(s^2+25)(s-7)}\right|$

Let $f(s) = \log\left|\frac{s(s+5)}{(s^2+25)(s-7)}\right|$

$$\Rightarrow f(s) = \log|s(s+5)| - \log|(s^2+25)(s-7)|$$

$$\Rightarrow f(s) = \log s + \log(s+5) - \log(s^2+25) - \log(s-7) \rightarrow ①$$

diff. ① wrt 's'

$$① \Rightarrow f'(s) = \frac{1}{s} + \frac{1}{s+5} - \frac{2s}{s^2+25} - \frac{1}{s-7}$$

$$\Rightarrow L^{-1}[f'(s)] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s+5}\right] - 2L^{-1}\left[\frac{s}{s^2+25}\right] - L^{-1}\left[\frac{1}{s-7}\right]$$

$$\Rightarrow -t\,F(t) = 1 + e^{-5t} - 2\cos 5t - e^{7t}$$

$$\Rightarrow F(t) = \frac{2\cos 5t + e^{7t} - e^{-5t} - 1}{t}$$

④ $\tan^{-1}\left(\frac{2}{s^2}\right)$

Let $f(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

Diff. wrt 's'

$$\Rightarrow f'(s) = \frac{1}{1+\left(\frac{2}{s^2}\right)^2}\frac{d}{ds}\left(\frac{2}{s^2}\right)$$

$$\Rightarrow f'(s) = \frac{1}{1+\frac{4}{s^4}}\left(\frac{-4}{s^3}\right)$$

$$\Rightarrow f'(s) = \frac{s^4}{s^4+4}\left(\frac{-4}{s^3}\right)$$

$$\Rightarrow f'(s) = \frac{-4s}{s^4+4}$$

$$\Rightarrow f'(s) = \frac{-4s}{(s^2)^2+(2)^2}$$

$$f'(s) = \frac{-4s}{(s^2+2)^2 - 2(s^2)(2)}$$

$$= \frac{-4s}{(s^2+2)^2 - 4s^2}$$

$$= \frac{-4s}{(s^2+2)^2 - (2s)^2}$$

$$= \frac{-4s}{(s^2+2s+2)(s^2-2s+2)}$$

$$= \frac{-4s}{(s^2+2s+2)(s^2-2s+2)}$$

$$= \frac{(s^2-2s+2)-(s^2+2s+2)}{(s^2+2s+2)(s^2-2s+2)}$$

$$\Rightarrow f'(s) = \frac{1}{s^2+2s+2} - \frac{1}{s^2-2s+2}$$

$$= \frac{1}{s^2+2(s)(1)+1+1} - \frac{1}{s^2-2(s)(1)+1+1}$$

$$\Rightarrow f'(s) = \frac{1}{(s+1)^2+1} - \frac{1}{(s-1)^2+1}$$

$$\Rightarrow L^{-1}[f'(s)] = L^{-1}\left[\frac{1}{(s+1)^2+1}\right] - L^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$\Rightarrow -t\,F(t) = e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] - e^{t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\Rightarrow -t\,F(t) = e^{-t}\sin t - e^{t}\sin t$$

$$\Rightarrow F(t) = \sin t\left[\frac{e^{t}-e^{-t}}{t}\right]$$

Laplace transform for convolution of two functions :-

Suppose $f(t)$ and $g(t)$ be the two functions and let $\bar{f}(s)$ and $\bar{g}(s)$ be the laplace transforms of $f(t)$ and $g(t)$. Then the convolution of $f(t)$ and $g(t)$ can be defined as $f(t)*g(t) = \int_{u=0}^{t} f(u)g(t-u)du = \int_{u=0}^{t} f(t-u)g(u)du$

$$\therefore L[f(t)*g(t)] = L\left[\int_{u=0}^{t} f(u)g(t-u)du\right] = L\left[\int_{u=0}^{t} f(t-u)g(u)du\right] = \bar{f}(s).\bar{g}(s)$$

$$\therefore L^{-1}[\bar{f}(s).\bar{g}(s)] = f(t)*g(t) = \int_{u=0}^{t} f(u)g(t-u)du = \int_{u=0}^{t} f(t-u)g(u)du$$

Find the following by using laplace transform with the convolution theorem :-

1. $\frac{s}{(s-1)(s^2+4)}$
2. $\frac{1}{(s-1)(s^2+1)}$
3. $\frac{1}{s(s^2+a^2)}$
4. $\frac{s}{(s^2+a^2)^2}$
5. $\frac{s^2}{(s^2+a^2)^2}$
6. $\frac{1}{(s^2+a^2)(s^2+b^2)}$
7. $\frac{s}{(s^2+a^2)(s^2+b^2)}$
8. $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

① $\frac{s}{(s-1)(s^2+4)}$

let $\bar{f}(s).\bar{g}(s) = \frac{s}{(s-1)(s^2+4)} = \frac{1}{(s-1)} \cdot \frac{s}{s^2+4}$

$\therefore \bar{f}(s) = \frac{1}{s-1}$; $\bar{g}(s) = \frac{s}{s^2+4}$

$\Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s-1}\right] = e^t = f(t)$

$\Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t = g(t)$

WKT $L^{-1}[\bar{f}(s).\bar{g}(s)] = f(t)*g(t)$

$$= \int_{u=0}^{t} f(t-u)g(u)du$$

$$= \int_{u=0}^{t} e^{t-u} \cos 2u \, du$$

$$= \int_{u=0}^{t} e^{t} . e^{-u} \cos 2u \, du$$

$$= e^{t} \int_{u=0}^{t} e^{-u} \cos 2u \, du$$

$$= e^{t} \left\{ \frac{e^{u}}{(-1)^2+2^2} \left[-\cos 2u + 2 \sin 2u \right] \right\}_0^t$$

$$= e^{t} \left\{ \frac{e^{-u}}{5} \left[2 \sin 2u - \cos 2u \right] \right\}_0^t$$

$$= e^{t} \left\{ \frac{e^{-t}}{5} \left[2 \sin 2t - \cos 2t \right] - \frac{1}{5} (0-1) \right\}$$

$$= e^{t} \left\{ \frac{e^{-t}}{5} \left[2 \sin 2t - \cos 2t \right] + \frac{1}{5} \right\}$$

(2) $\dfrac{1}{s(s^2+a^2)}$

Let $\bar{f}(s) \cdot \bar{g}(s) = \dfrac{1}{s(s^2+a^2)} = \dfrac{1}{s} \cdot \dfrac{1}{(s^2+a^2)}$

$\therefore \bar{f}(s) = \dfrac{1}{s} \Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{1}{s}\right] = 1 = f(t)$

$\bar{g}(s) = \dfrac{1}{s^2+a^2} \Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{\sin at}{a} = g(t)$

$$\therefore L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t) = \int_{u=0}^{t} f(t-u) \, g(u) \, du$$

$$= \int_{u=0}^{t} 1 \cdot \frac{1}{a} \sin au \, du$$

$$= \frac{1}{a} \int_{u=0}^{t} \sin au \, du$$

$$= \frac{1}{a} \left[\frac{-\cos au}{a} \right]_0^t$$

$$= \frac{-1}{a^2} \left[\cos au \right]_0^t$$

$$= \frac{-1}{a^2} \left[\cos at - 1 \right]$$

$$= \frac{1 - \cos at}{a^2}$$

③ $\frac{s}{(s^2+a^2)^2}$

Let $\bar{f}(s)\cdot\bar{g}(s) = \frac{s}{(s^2+a^2)^2} = \frac{1}{(s^2+a^2)}\cdot\frac{s}{(s^2+a^2)}$

$\therefore \bar{f}(s) = \frac{1}{(s^2+a^2)} \Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at = f(t)$

$\bar{g}(s) = \frac{s}{(s^2+a^2)} \Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = g(t)$

$\therefore L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t)*g(t) = \int_{u=0}^{t} f(t-u)g(u)\,du$

$= \frac{1}{a}\int_{u=0}^{t}\sin(at-au)\cos au\,du$

$= \frac{1}{2a}\int_{u=0}^{t} 2\sin(at-au)\cos au\,du$

$= \frac{1}{2a}\int_{u=0}^{t}[\sin(at-au+au) + \sin(at-au-au)]\,du$

$= \frac{1}{2a}\int_{u=0}^{t}[\sin at + \sin(at-2au)]\,du$

$= \frac{1}{2a}\left\{\sin at\int_{u=0}^{t} 1\cdot du + \int_{u=0}^{t}\sin(at-2au)\,du\right\}$

$= \frac{1}{2a}\left[u\sin at - \frac{\cos(at-2au)}{-2a}\right]_{u=0}^{t}$

$= \frac{1}{2a}\left[u\sin at + \frac{1}{2a}\cos(at-2au)\right]_{u=0}^{t}$

$= \frac{1}{2a}\left\{\left[t\sin at + \frac{1}{2a}\cos at\right] - \left[0 + \frac{1}{2a}\cos at\right]\right\}$

$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}\cos at - \frac{1}{2a}\cos at\right]$

$= \frac{1}{2a}t\sin at$

4) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

Let $\bar{f}(s)\cdot\bar{g}(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{s}{(s^2+a^2)}\cdot\frac{s}{(s^2+b^2)}$

$\therefore \bar{f}(s) = \frac{s}{s^2+a^2} \Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = f(t)$

$\bar{g}(s) = \frac{s}{s^2+b^2} \Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt = g(t)$

$\therefore L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t)*g(t) = \int_{u=0}^{t} f(t-u)\,g(u)\,du$

$= \int_{u=0}^{t} \cos(at-au)\cos bu\,du$

$= \frac{1}{2}\int_{u=0}^{t} 2\cdot\cos(at-au)\cos bu\,du$

$= \frac{1}{2}\int_{u=0}^{t} [\cos(at-au+bu)+\cos(at-au-bu)]\,du$

$= \frac{1}{2}\int_{u=0}^{t} \{\cos[at+(b-a)u]+\cos[at-(b+a)u]\}\,du$

$= \frac{1}{2}\left\{\frac{\sin[at+(b-a)u]}{b-a} + \frac{\cos[at-(b+a)u]}{b+a}\right\}_{u=0}^{t}$

$= \frac{1}{2}\left\{\left[\frac{\sin bt}{b-a} + \frac{\sin bt}{b-a}\right] - \left[\frac{\sin at}{b-a} - \frac{\sin at}{b+a}\right]\right\}$

$= \frac{1}{2}\left\{\left[\frac{1}{b-a}+\frac{1}{b+a}\right]\sin bt - \left[\frac{1}{b-a}-\frac{1}{b+a}\right]\sin at\right\}$

$= \frac{1}{2}\left\{\frac{2b}{b^2-a^2}\sin bt - \frac{2a}{b^2-a^2}\sin at\right\}$

$= \frac{1}{b^2-a^2}[b\sin bt - a\sin at]$

⑤ $\frac{1}{(s-1)(s^2+1)}$

Let $\bar{f}(s)\cdot\bar{g}(s) = \frac{1}{(s-1)(s^2+1)} = \frac{1}{(s-1)}\cdot\frac{1}{(s^2+1)}$

$\therefore \bar{f}(s) = \frac{1}{(s-1)} \Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s-1}\right] = e^t = f(t)$

$\bar{g}(s) = \frac{1}{(s^2+1)} \Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t = g(t)$

$\therefore L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t) * g(t) = \int_{u=0}^{t} f(t-u)g(u)\,du$

$= \int_{u=0}^{t} e^{(t-u)}\sin u\,du$

$= \int_{u=0}^{t} e^t\cdot e^{-u}\sin u\,du$

$= e^t\int_{u=0}^{t} e^{-u}\sin u\,du$

$= e^t\left\{\frac{e^{-u}}{(-1)^2+1^2}[-\sin u - \cos u]\right\}_0^t$

$= e^t\left\{\frac{e^{-t}}{2}[-\sin t - \cos t] - \frac{1}{2}(-1)\right\}$

$= e^t\left\{\frac{e^{-t}}{2}[-\sin t - \cos t] + \frac{1}{2}\right\}$

⑥ $\frac{s^2}{(s^2+a^2)^2}$

Let $\bar{f}(s)\cdot\bar{g}(s) = \frac{s}{(s^2+a^2)}\cdot\frac{s}{(s^2+a^2)}$

$\bar{f}(s) = \frac{s}{s^2+a^2} \Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = f(t)$

$\bar{g}(s) = \frac{s}{s^2+a^2} \Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = g(t)$

$\therefore L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t) * g(t) = \int_{u=0}^{t} f(t-u)\,g(u)\,du$

$= \int_{u=0}^{t} \cos(at-au)\cos au\,du$

(91.)

$$= \frac{1}{2}\int_{u=0}^{t} 2\cos(at-au)\cos au \; du$$

$$= \frac{1}{2}\int_{u=0}^{t} [\cos(at-au+au) + \cos(at-au-au)]du$$

$$= \frac{1}{2}\int_{u=0}^{t} [\cos at + \cos(at-2au)]du$$

$$= \frac{1}{2}\left\{ \cos at \int_{u=0}^{t} 1 \cdot du + \int_{u=0}^{t} \cos(at-2au)du \right\}$$

$$= \frac{1}{2}\left[u\cos at + \frac{\sin(at-2au)}{-2a}\right]_{u=0}^{t}$$

$$= \frac{1}{2}\left[u\cos at - \frac{1}{2a}\sin(at-2au)\right]_{u=0}^{t}$$

$$= \frac{1}{2}\left\{ \left[t\cos at - \frac{1}{2a}\sin(at-2at)\right] - \frac{1}{2a}\sin(0)\right\}$$

$$= \frac{1}{2}\left\{ t\cos at + \frac{1}{2a}\sin at \right\}$$

$$= \frac{t\cos at}{2} + \frac{1}{4a}\sin at$$

(7) $\frac{1}{(s^2+a^2)(s^2+b^2)}$

Let $\bar{f}(s) \cdot \bar{g}(s) = \frac{1}{(s^2+a^2)} \cdot \frac{1}{(s^2+b^2)}$

$\bar{f}(s) = \frac{1}{(s^2+a^2)} \Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \sin at$

$\bar{g}(s) = \frac{1}{(s^2+b^2)} \Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s^2+b^2}\right] = \sin bt$

$\therefore L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t) = \int_{u=0}^{t} f(t-u)\, g(u)\, du$

$$= \int_{u=0}^{t} \sin(at-au) \cdot \sin bu \, du$$

$$= \frac{1}{2ab}\int_{u=0}^{t} 2\sin(at-au) \cdot \sin bu \; du$$

$$= \frac{1}{2ab}\int_{u=0}^{t} [\cos(at-au-bu) - \cos(at-au+bu)]du$$

$$= \frac{1}{2ab}\int_{u=0}^{t}\left[\cos(at-u(a+b)) - \cos(at-u(a-b))\right]du$$

$$= \frac{1}{2ab}\left[\frac{\sin at-u(a+b)}{-(a+b)} - \frac{\sin at-u(a-b)}{-(a-b)}\right]_0^t$$

$$= \frac{1}{2ab}\left[\frac{\sin at-u(a-b)}{a-b} - \frac{\sin at-u(a+b)}{a+b}\right]_0^t$$

$$= \frac{1}{2ab}\left[\frac{\sin at-at+bt}{a-b} - \frac{\sin at-at-bt}{a+b}\right] - \left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b}\right]$$

$$= \frac{1}{2ab}\left\{\left[\frac{\sin bt}{a-b} - \frac{\sin(-bt)}{a+b}\right] - \left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b}\right]\right\}$$

$$= \frac{1}{2ab}\left\{\left[\frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right] - \left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b}\right]\right\}$$

$$= \frac{1}{2ab}\left\{\sin bt\left[\frac{1}{a-b}+\frac{1}{a+b}\right] - \sin at\left[\frac{1}{a-b}-\frac{1}{a+b}\right]\right\}$$

$$= \frac{1}{2ab}\left\{\sin bt\left[\frac{2a}{a^2-b^2}\right] - \sin at\left[\frac{2b}{a^2-b^2}\right]\right\}$$

$$= \frac{1}{2ab}\left\{\frac{\sin bt}{b} - \frac{\sin at}{a}\right\}$$

(8) $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

$$\bar{f}(s)\cdot\bar{g}(s) = \frac{1}{(s^2+a^2)}\cdot\frac{s}{(s^2+b^2)}$$

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin ab = f(t)$$

$$L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos bt = g(t)$$

WKT

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t) * g(t)$$

$$= \int_{u=0}^{t} f(t-u)\,g(u)\,du$$

$$= \int_{u=0}^{t} \frac{1}{a} \sin(at - au) \cos bu \, du$$

$$= \frac{1}{2a} \int_{u=0}^{t} 2 \sin(at - au) \cos bu \, du$$

$$= \frac{1}{2a} \int_{u=0}^{t} \left[\sin(at - au + bu) + \sin(at - au - bu)\right] du$$

$$= \frac{1}{2a} \int_{u=0}^{t} \sin\{at + u(b-a)\} + \sin\left[at - u(a+b)\right] du$$

$$= \frac{1}{2a} \left[\frac{-\cos at - u(a-b)}{-(a-b)} - \frac{\cos(at - u(a+b))}{-(a+b)}\right]_0^t$$

$$= \frac{1}{2a} \left\{\left[\frac{\cos at - u(a+b)}{a+b} - \frac{\cos at - u(a-b)}{-a-b}\right]_0^t - \left[\frac{\cos at}{a+b} + \frac{\cos at}{a-b}\right]\right\}$$

$$= \frac{1}{2a} \left\{\left[\frac{\cos(-bt)}{a+b} + \frac{\cos bt}{a-b}\right] - \left[\frac{\cos at}{a+b} + \frac{\cos at}{a-b}\right]\right\}$$

$$= \frac{1}{2a} \left\{\cos bt \left[\frac{1}{a+b} + \frac{1}{a-b}\right] - \cos at \left[\frac{1}{a+b} + \frac{1}{a-b}\right]\right\}$$

$$= \frac{1}{2a} \left\{\cos bt \left[\frac{a-b+a+b}{a^2-b^2}\right] - \cos at \left[\frac{a-b+a+b}{a^2-b^2}\right]\right\}$$

$$= \frac{1}{2a} \left\{\cos bt \, \frac{2a}{a^2-b^2} - \cos at \, \frac{2a}{a^2-b^2}\right\}$$

$$= \frac{2a}{2a(a^2-b^2)} \left[\cos bt - \cos at\right]$$

$$= \frac{1}{a^2-b^2} \left[\cos bt - \cos at\right]$$

Applications of Laplace transform:-

Working rule:-

Step 1:- Express the given differential eq. with the notations $y(t), y'(t), y''(t)\ldots$.

Step 2:- Apply Laplace transform on both sides and substitute $L[y(t)] = \bar{y}(s)$

$$L[y'(t)] = s\bar{y}(s) - y(0)$$

$$L[y''(t)] = s^2\bar{y}(s) - sy(0) - y'(0)$$

$$L[y'''(t)] = s^3\bar{y}(s) - s^2y(0) - sy'(0) - y''(0).$$

for the initial conditions $y(0), y'(0), y''(0)$.

Step 3:- Write the function of the variable 's'.

Step 4:- Take the inverse Laplace transform on the both side and hence find $y(t)$.

① Using Laplace transform $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}$ where $y(0)=0$ $y'(0)=0$.

Given $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}$ $y(0)=0$ $y'(0)=0$

$$\Rightarrow y''(t) + 4y'(t) + 4y(t) = e^{-t}$$

$$\Rightarrow L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = L[e^{-t}]$$

$$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 4[s\bar{y}(s) - y(0)] + 4\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow s^2\bar{y}(s) + 4s\bar{y}(s) + 4\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow (s^2 + 4s + 4)\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow (s+2)^2\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow \bar{y}(s) = \frac{1}{(s+1)(s+2)^2} \rightarrow ①$$

$$\Rightarrow \frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1) \rightarrow (2)$$

when $s=-1$

$(2) \Rightarrow 1 = A(1)^2$

$A = 1$

when $s=-2$

$(2) \Rightarrow 1 = C(-2+1)$

$C = -1$

$A+B=0$

$B=-A$

$B=-1$

$$\therefore (1) \Rightarrow \bar{y}(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$\Rightarrow L^{-1}[\bar{y}(s)] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$\Rightarrow y(t) = e^{-t} - e^{-2t} - t\,e^{-2t}$$

(2) Solve the differential equation $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-t}$ with the initial conditions $y(0)=0$, $y'(0)=1$.

Given $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-t}$

$$\Rightarrow y''(t) + 3y'(t) + 2y(t) = e^{-t}$$

$$\Rightarrow L[y''(t)] + 3L[y'(t)] + 2L[y(t)] = L[e^{-t}]$$

$$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 3[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow s^2\bar{y}(s) - 1 + 3s\bar{y}(s) + 2\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow (s^2+3s+2)\bar{y}(s) = \frac{1}{s+1} + 1$$

$$\Rightarrow (s+1)(s+2)\bar{y}(s) = \frac{1}{s+1} + 1$$

$$= \frac{s+2}{s+1}$$

$$\Rightarrow \bar{y}(s) = \frac{s+2}{(s+1)^2(s+2)}$$

$$\Rightarrow \bar{y}(s) = \frac{1}{(s+1)^2}$$

$$\Rightarrow L^{-1}[y(s)] = L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$\Rightarrow y(t) = e^{-t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$\Rightarrow y(t) = t e^{-t}$$

(3) Using laplace transform solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$

given $y(0) = 0$ $\frac{dy}{dt} = 0$ where $t = 0$.

Given $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$ $y(0) = 0$ $y'(0) = 1$

$$\Rightarrow y''(t) + 2y'(t) - 3y(t) = \sin t$$

$$\Rightarrow L[y''(t)] + 2L[y'(t)] - 3L[y(t)] = L[\sin t]$$

$$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 2[s\bar{y}(s) - y(0)] - 3\bar{y}(s) = \frac{1}{s^2+1}$$

$$\Rightarrow (s^2 + 2s - 3)\bar{y}(s) = \frac{1}{s^2+1}$$

$$\Rightarrow (s-1)(s+3)\bar{y}(s) = \frac{1}{s^2+1}$$

$$\Rightarrow \bar{y}(s) = \frac{1}{(s-1)(s+3)(s^2+1)}$$

$$\Rightarrow \frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{(s-1)} + \frac{B}{(s+3)} + \frac{Cs+D}{(s^2+1)}$$

$$\Rightarrow 1 = A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3) \rightarrow (2)$$

$$\Rightarrow 1 = A(s^3 + s + 3s^2 + 3) + B(s^3 + s - s^2 - 1) + (Cs + D)(s^2 + 2s - 3)$$

$$\Rightarrow 1 = A(s^3 + 3s^2 + s + 3) + B(s^3 - s^2 + s - 1) + Cs^3 + 2Cs^2 - 3Cs + Ds^2 + 2Ds - 3D$$

$$\Rightarrow 1 = (A + B + C)s^3 + (3A - B + 2C + D)s^2 + (A + B - 3C + 2D)s + (3A - B - 3D) \rightarrow (3)$$

when $s = 1$

(2) $\Rightarrow 1 = 8A$

$A = 1/8$

when $s = -3$

$1 = -40B$

$B = \frac{-1}{40}$

$A + B + C = 0$

$C = -A - B$

$C = \frac{-1}{10}$

$3A - B - 3D = 1$

$3D - 3A - B = 1$

$3D = \frac{3}{8} + \frac{1}{40} - 1$

$D = -\frac{3}{5}$

$\therefore \bar{y}(s) = \frac{1}{8}\frac{1}{s-1} - \frac{1}{40}\frac{1}{s+3} - \frac{1}{10}\frac{s}{s^2+1} - \frac{1}{5}\frac{1}{s^2+1}$

$\Rightarrow y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t$

4) Solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-t}$ given $y(0) = y'(0) = 1$

Given $y''(t) + 4y'(t) + 3y(t) = e^{-t}$

$\therefore L[y''(t)] + 4L[y'(t)] + 3L[y(t)] = L[e^{-t}]$

$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 4[s\bar{y}(s) - y(0)] + 3\bar{y}(s) = \frac{1}{s+1}$

$\Rightarrow s^2\bar{y}(s) - s - 1 + 4s\bar{y}(s) - 4 + 3\bar{y}(s) = \frac{1}{s+1}$

$\Rightarrow (s^2+4s+3)\bar{y}(s) - (s+5) = \frac{1}{s+1}$

$\Rightarrow (s^2+4s+3)\bar{y}(s) = \frac{1}{s+1} + (s+5)$

$\Rightarrow (s+1)(s+3)\bar{y}(s) = \frac{1+(s+1)(s+5)}{s+1}$

$\Rightarrow (s+1)(s+3)\bar{y}(s) = \frac{s^2+6s+6}{s+1}$

$\therefore \bar{y}(s) = \frac{s^2+6s+6}{(s+1)^2(s+3)} \rightarrow ①$

$\therefore \frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+3)}$

$s^2+6s+6 = A(s+1)(s+3) + B(s+3) + C(s+1)^2 \rightarrow ②$

when $s = -1$

② $\Rightarrow 1 = 2B$

$B = \frac{1}{2}$

when $s = -3$

$-3 = 4C$

$C = -3/4$

$A + C = 1$

$A = 1 - C$

$A = 7/4$

(25)

$$① \Rightarrow \bar{y}(s) = \frac{7}{4}\cdot\frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+1)^2} - \frac{3}{4}\frac{1}{s+3}$$

$$\Rightarrow L^{-1}[\bar{y}(s)] = \frac{7}{4}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right] - \frac{3}{4}L^{-1}\left[\frac{1}{s+3}\right]$$

$$\Rightarrow y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}t - \frac{3}{4}e^{-3t}$$

⑤ By using laplace transform $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^{2t}$

with $x(0) = 0$, $\frac{dx}{dt}(0) = -1$.

Given $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^{2t}$ $x(0) = 0$ $x'(0) = -1$

$$\Rightarrow x''(t) - 2x'(t) + x(t) = e^{2t}$$

$$\Rightarrow L[x''(t)] - 2L[x'(t)] + L[x(t)] = L[e^{2t}]$$

$$\Rightarrow [s^2\bar{x}(s) - s\,x(0) - x'(0)] - 2[s\bar{x}(s) - x(0)] + \bar{x}(s) = \frac{1}{s-2}$$

$$\Rightarrow s^2\bar{x}(s) - 0 + 1 - 2s\,\bar{x}(s) + 0 + \bar{x}(s) = \frac{1}{s-2}$$

$$\Rightarrow (s^2 - 2s + 1)\bar{x}(s) = \frac{1}{s-2} - 1$$

$$\Rightarrow (s-1)^2\bar{x}(s) = \frac{1-s+2}{(s-2)}$$

$$\Rightarrow (s-1)^2\bar{x}(s) = \frac{3-s}{(s-2)}$$

$$\Rightarrow \bar{x}(s) = \frac{3-s}{(s-1)^2(s-2)} \rightarrow ①$$

$$\therefore \frac{3-s}{(s-1)^2(s-2)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-2)}$$

$$3s = A(s-1) + B(s-2) + C(s-1)^2$$

when $s=1$ when $s=2$ and $A + C = 0$

② ⇒ $2 = -B$ ② ⇒ $1 = C$ $A = -C$

$B = -2$ $C = 1$ $A = 1$

$$① \Rightarrow \bar{x}(s) = \frac{-1}{s-1} - \frac{2}{(s-1)^2} + \frac{1}{s-2}$$

$$\Rightarrow L^{-1}[\bar{x}(s)] = L^{-1}\left[\frac{1}{s-2}\right] - L^{-1}\left[\frac{1}{(s-1)}\right] - 2L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$\Rightarrow x(t) = e^{2t} - e^{t} - 2te^{t}$$

⑥ Solve the equation $y'' - 3y' + 2y = e^{3t}$, $y(0) = 1$, $y'(0) = 0$ using Laplace transform technique.

$\Rightarrow$ Given $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$

$$\Rightarrow y''(t) - 3y'(t) + 2y(t) = e^{3t}$$

$$\Rightarrow L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = Le^{3t}$$

$$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] - 3s\bar{y}(s) + 3y(0) + 2\bar{y}(s) = \frac{1}{s-3}$$

$$\Rightarrow s^2\bar{y}(s) - s - 3s\bar{y}(s) + 3 + 2\bar{y}(s) = \frac{1}{s-3}$$

$$\Rightarrow (s^2 - 3s + 2)\bar{y}(s) - s + 3 = \frac{1}{s-3}$$

$$\Rightarrow (s^2 - 3s + 2)\bar{y}(s) - (s-3) = \frac{1}{s-3}$$

$$\Rightarrow (s^2 - 3s + 2)\bar{y}(s) = \frac{1}{(s-3)} + (s-3)$$

$$\Rightarrow (s-1)(s-2)\bar{y}(s) = \frac{1 + (s-3)^2}{(s-3)}$$

$$\Rightarrow \bar{y}(s) = \frac{1 + s^2 + 9 - 3 \times 2s}{(s-1)(s-2)(s-3)}$$

$$\Rightarrow \bar{y}(s) = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} \rightarrow ①$$

$$\therefore \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$\Rightarrow s^2 - 6s + 10 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) \rightarrow ②$$

$\Rightarrow$ when $s = 1$

① $\Rightarrow 1 - 6 + 10 = A(-1)(-2)$

$\Rightarrow 5 = 2A$

$\Rightarrow A = 5/2$

when $s = 2$

② $\Rightarrow 4 - 12 + 10 = B(2-1)(2-3)$

$\Rightarrow 2 = B(-1)$

$\Rightarrow B = -1/2$

when $s = 3$

② $\Rightarrow 9 - 18 + 10 = C(3-1)(3-2)$

$1 = C(2)$

$C = \frac{1}{2}$

① $\Rightarrow \bar{y}(s) = \frac{5}{2}\frac{1}{s+1} + \left(\frac{-1}{2}\right)\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}$

$\Rightarrow L^{-1}[\bar{y}(s)] = \frac{5}{2}L^{-1}\left[\frac{1}{s+1}\right] + \left(\frac{-1}{2}\right)L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s-3}\right]$

$y(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{2t} + \frac{1}{2}e^{3t}$ //

⑦ solve the equation $y'' + 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$ using laplace transform technique.

$\Rightarrow$ Given $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$

$\Rightarrow y''(t) + 3y'(t) + 2y(t) = 0$

$\Rightarrow L[y''(t)] + 3L[y'(t)] + 2L[y(t)] = 0$

$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 3[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = 0$

$\Rightarrow s^2\bar{y}(s) - sy(0) - y'(0) + 3s\bar{y}(s) - 3 + 2\bar{y}(s) = 0$

$\Rightarrow s^2\bar{y}(s) - s + 3s\bar{y}(s) - 3 + 2\bar{y}(s) = 0$

$\Rightarrow (s^2 + 3s + 2)\bar{y}(s) - (s+3) = 0$

$\Rightarrow (s+1)(s+2)\bar{y}(s) = (s+3)$

$\Rightarrow \bar{y}(s) = \frac{(s+3)}{(s+1)(s+2)} \rightarrow ①$

$\frac{(s+3)}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$

$s + 3 = A(s+2) + B(s+1) \rightarrow ②$

when $s = -2$

② $\Rightarrow 1 = -B$

$B = -1$

when $s = -1$

② $\Rightarrow 2 = A$

$A = 2$

$$\Rightarrow \bar{y}(s) = \frac{2}{(s+1)} + \frac{(-1)}{(s+2)}$$

$$L^{-1}[\bar{y}(s)] = 2\,L^{-1}\left[\frac{1}{s+1}\right] - 1\,L^{-1}\left[\frac{1}{s+2}\right]$$

$$y(t) = 2e^{-t} - e^{-2t}$$

⑧ Solve the equation $x'' - 2x' + x = e^{2t}$, $x(0) = 1$, $\frac{dx(0)}{dt} = -1$ using laplace transform technique.

$\Rightarrow$ Given $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^{2t}$ $x(0) = 1$ $x'(0) = -1$

$$\Rightarrow x''(t) - 2x'(t) + x(t) = e^{2t}$$

$$\Rightarrow L[x''(t)] - 2L[x'(t)] + L[x(t)] = L[e^{2t}]$$

$$\Rightarrow [s^2\bar{x}(s) - s\,x(0) - x'(0)] - 2[s\,\bar{x}(s) - x(0)] + \bar{x}(s) = \frac{1}{(s-2)}$$

$$\Rightarrow s^2\bar{x}(s) - s + 1 - 2s\,\bar{x}(s) + 2 + \bar{x}(s) = \frac{1}{(s-2)}$$

$$\Rightarrow (s^2 - 2s + 1)\bar{x}(s) - s + 3 = \frac{1}{(s-2)}$$

$$\Rightarrow (s^2 - 2s + 1)\bar{x}(s) - (s-3) = \frac{1}{(s-2)}$$

$$\Rightarrow (s-1)^2(\bar{x}(s)) = \frac{1}{(s-2)} + (s-3)$$

$$\Rightarrow (s-1)^2(\bar{x}(s)) = \frac{1 + (s-3)(s-2)}{(s-2)}$$

$$\Rightarrow (s-1)^2(\bar{x}(s)) = \frac{1 + s^2 - 2s - 3s + 6}{(s-2)}$$

$$\Rightarrow (s-1)^2(\bar{x}(s)) = \frac{s^2 - 5s + 7}{(s-2)}$$

$$\bar{x}(s) = \frac{s^2 - 5s + 7}{(s-2)(s-1)^2} \rightarrow ①$$

$$\frac{s^2 - 5s + 7}{(s-2)(s-1)^2} = \frac{A}{(s-2)} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}$$

$$s^2 - 4s + 5 = A(s-1)^2 + B(s-1)(s-2) + C(s-2) \rightarrow (3)$$

when $s = 2$

$(3) \Rightarrow 4 - 8 + 5 = A(2-1)^2$

$1 = A$

$A = 1$

when $s = 1$

$(3) \Rightarrow 1 - 4 + 5 = C(1-2)$

$2 = -C$

$C = -2$

$1 = A + B$

$B = 1 - A$

$= 1 - 1$

$\boxed{B = 0}$

$$(2) \Rightarrow Y(s) = \frac{1}{(s-2)} - \frac{2}{(s-1)^2}$$

$$\Rightarrow L^{-1}[Y(s)] = L^{-1}\left[\frac{1}{s-2}\right] - 2L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$\Rightarrow y(t) = e^{2t} - 2e^t t$$

Module - 2

FOURIER SERIES

Introduction:-

Expressing the function $f(x)$ in the combinations of constants and trignometric ratios in any other interval $(0,\pi)$, $(0,2\pi)$... is called the Fourier series.

$$\therefore f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \dots + a_n\cos nx +$$
$$\dots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \dots + b_n\sin nx + \dots$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\cos nx + b_n\sin nx)$$

where a_0 is called the constant.

a_n, b_n are called fourier coefficients.

Fourier Series Expansion of $f(x)$ over the period 2π:-

The Fourier series Expansion of $f(x)$ over the interval $(c, c+2\pi)$ can be denoted as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\cos nx + b_n\sin nx) \rightarrow (1)$$

where $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

This is also called direchlet's Property.

If $c=0$, then the constant and the fourier coefficients of $f(x)$ can be evaluated as $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

If $c=-\pi$, then the fourier series coefficients of $f(x)$ in the interval $(-\pi,\pi)$ can be evaluated as

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Note:-

① Suppose $f(x)$ is the continuous function in the interval $[0,2\pi]$ then $f(x)$ is an even function when $f(2\pi-x)=f(x)$ and $f(x)$ is an add function when $f(2\pi-x)=-f(x)$.

② If $f(x)$ is a discontinuous function over the interval $[0,2\pi]$ $f(x)= \begin{cases} \phi(x), & 0\le x\le \pi \\ \psi(x), & \pi\le x<2\pi \end{cases}$, then

$f(x)$ is an even function when $\phi(2\pi-x)=\psi(x)$ and $f(x)$ is an odd function when $\phi(2\pi-x)=-\psi(x)$.

③ Suppose $f(x)$ is the continuous function in the interval $[-\pi,\pi]$, then

$f(x)$ is an even function when $f(-x)=f(x)$ and $f(x)$ is an odd function when $f(-x)=-f(x)$.

④ Suppose $f(x)$ is a discontinuous function in the interval $[-\pi,\pi]$, that is $f(x)= \begin{cases} \phi(x), & -\pi\le x<0 \\ \psi(x), & 0\le x<\pi \end{cases}$, then

$f(x)$ is an even function when $\phi(-x)=\psi(x)$ and

$f(x)$ is an even function when $\phi(-x) = \psi(x)$ and $f(x)$ is an odd function when $\phi(-x) = -\psi(x)$.

Even and odd functions:-

WKT the fourier series expansion of $f(x)$ over the period 2π is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$, then

1. If $f(x)$ is an even function, then $b_n = 0$, hence the fourier series of $f(x)$ can be expanded as

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ is also called the half-range cosine series in the $[0,\pi]$ for which

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

2. If $f(x)$ is an odd function, then a_0 and a_n values will be zero, hence the fourier series expansion of $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ is also called the fourier half-range sin series in the $[0,\pi]$ for which $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

Important Results:-

1. $\sin(n\pi) = 0$, $\sin(2n\pi) = 0$, $\forall\, n \in \mathbb{Z}$
2. $\cos(n\pi) = (-1)^n$, $\cos(2n\pi) = 1$, $\forall\, n \in \mathbb{Z}$
3. $\sin\left(n+\frac{1}{2}\right)\pi = (-1)^n$, $\forall\, n \in \mathbb{Z}$
4. $\cos\left(n+\frac{1}{2}\right)\pi = 0$, $\forall\, n \in \mathbb{Z}$
5. $\sin\left(\frac{n\pi}{2}\right) = \begin{cases} (-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$
6. $\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2}, & \text{if } n \text{ is even} \end{cases}$

① Obtain the fourier series of $f(x) = \frac{\pi - x}{2}$ in $[0, 2\pi]$ and hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Given: $f(x) = \frac{\pi - x}{2}$, $x \in [0, 2\pi]$

$$\therefore f(2\pi - x) = \frac{\pi - (2\pi - x)}{2}$$

$$= \frac{\pi - 2\pi + x}{2}$$

$$= \frac{-\pi + x}{2}$$

$$\therefore f(2\pi - x) = -f(x)$$

$\therefore f(x)$ is an odd function in $[0, 2\pi]$

$\therefore a_0 = 0$ & $a_n = 0$

$\therefore$ The fourier series of

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow ①$$

when $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi - x}{2}\right) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (\pi - x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left\{[(\pi - x)\int_0^{\pi} \sin nx \, dx - \int_0^{\pi} (-1)\left[\int \sin nx \, dx\right] dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[(\pi - x)\cos nx\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi} \cos nx \, dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[(\pi - x)\cos nx\right]_0^{\pi} - \frac{1}{n^2}\left[\sin nx\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}[0 - \pi] - \frac{1}{n^2}[0 - 0]\right\}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\right]$$

$$\Rightarrow \boxed{b_n = \frac{1}{n}}$$

$$① \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

$$\Rightarrow \frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \rightarrow ②$$

$$② \Rightarrow \frac{\pi - \pi/2}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{1}\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\sin(\pi) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{4}\sin(2\pi) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \dots$$

$$\Rightarrow \frac{\pi}{4} = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} - \frac{1}{7} + \dots$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

② Obtain a series of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $[0, 2\pi]$, Hence deduce that 1. $\frac{\pi^2}{6} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

2. $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \dots$

$$\Rightarrow f(x) = \left(\frac{\pi - x}{2}\right)^2, \quad x \in (0, 2\pi)$$

$$\Rightarrow f(x) = \left(\frac{x - \pi}{2}\right)^2 = \frac{1}{4}(x - \pi)^2$$

$$\therefore f(2\pi - x) = \frac{1}{4}(2\pi - x - \pi)^2$$

$$= \frac{1}{4}(\pi - x)^2$$

$$= \frac{1}{4}(x - \pi)^2$$

$$\therefore f(2\pi - x) = f(x)$$

$\therefore f(x)$ is an even function.

$$\Rightarrow b_n = 0$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{4}(x - \pi)^2\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} (x^2 - 2\pi x + \pi^2)\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3} - \frac{2\pi x^2}{2} + \pi^2 x\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^3}{3} - \pi^3 + \pi^3\right]$$

$$= \frac{1}{2\pi} \times \frac{\pi^3}{3}$$

$$\therefore \boxed{a_0 = \frac{\pi^2}{6}}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^\pi \frac{1}{4}(x-\pi)^2 \cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^\pi (x-\pi)^2 \cos nx\, dx$$

$$= \frac{1}{2\pi}\left\{(x-\pi)^2\int_0^\pi \cos nx\, dx - \int_0^\pi \left[2(x-\pi)\int \cos nx\, dx\right]dx\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{1}{n}\left[(x-\pi)^2 \sin nx\right]_0^\pi - \frac{2}{n}\int_0^\pi \left[(x-\pi)\sin nx\, dx\right]\right\}$$

$$= \frac{1}{2\pi}\left[0 - \frac{2}{n}\int_0^\pi (x-\pi)\sin nx\, dx\right]$$

$$= -\frac{1}{n\pi}\int_0^\pi (x-\pi)\sin nx\, dx$$

$$= -\frac{1}{n\pi}\left\{(x-\pi)\int_0^\pi \sin nx\, dx - \int_0^\pi \left[1\cdot\int \sin nx\, dx\right]dx\right\}$$

$$= -\frac{1}{n\pi}\left\{-\frac{1}{n}\left[(x-\pi)\cos nx\right]_0^\pi + \left[\frac{1}{n^2}\sin nx\right]_0^\pi\right\}$$

$$= \frac{1}{n\pi}\left\{-\frac{1}{n}[0+\pi] + \frac{1}{n^2}(0)\right\}$$

$$= \frac{1}{n\pi}\left[\frac{-\pi}{n}\right]$$

$$\boxed{a_n = \frac{1}{n^2}}$$

$$① \Rightarrow \frac{1}{4}(x-\pi)^2 = \frac{\pi^2/6}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$\Rightarrow \frac{1}{4}(x-\pi)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx \rightarrow ②$$

when $x = 0$

$$② \Rightarrow \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2} \rightarrow ①$$

$$① \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{3\pi^2 - \pi^2}{12} = \frac{\pi^2}{6}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \frac{\pi^2}{6}$$

ii) when $x = \pi$

$$(2) \Rightarrow 0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$\Rightarrow -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \dots \frac{-\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \dots \frac{\pi^2}{12}$$

(3) Find the fourier series expansion of $f(x) = (\pi - x)^2$ in $[0, 2\pi]$.

Given: $f(x) = (\pi - x)^2, \ x \in [0, 2\pi]$

$$\Rightarrow f(x) = (x - \pi)^2$$

$$\therefore f(2\pi - x) = (2\pi - x - \pi)^2$$

$$= (\pi - x)^2$$

$$\therefore f(2\pi - x) = (x - \pi)^2$$

$\therefore f(x)$ is an even function.

$$\Rightarrow b_n = 0$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow (1)$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (x - \pi)^2\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (x^2 + \pi^2 - 2x\pi)\,dx$$

$$= \frac{2}{\pi} \left(\frac{x^3}{3} + \pi^2 x - \frac{2x^2\pi}{2} \right)_0^{\pi}$$

$$= \frac{2}{\pi} \left(\frac{\pi^3}{3} \right) = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ \cos nx.\ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (x-\pi)^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left\{ (x-\pi)^2 \int_0^{\pi} \cos nx\, dx - \int_0^{\pi} [2(x-\pi)\int \cos nx\, dx]dx \right\}$$

$$= \frac{2}{\pi}\left\{ \left[(x-\pi)^2 \sin nx\right]_0^{\pi} \frac{1}{n} - \frac{2}{n} \int (x-\pi)\sin nx\, dx \right\}$$

$$= \frac{2}{\pi}\left\{ -0 + \frac{2}{n^2}\left[(x-\pi)\cos nx\right]_0^{\pi} \right.$$

$$= \frac{2}{\pi}\left\{ -\frac{2}{n^2}(-\pi) \right\}$$

$$\boxed{a_n = \frac{4}{n^2}}$$

$$① \Rightarrow (x-\pi)^2 = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

$$(x-\pi)^2 = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx \longrightarrow ②$$

when $x = 0$

$$② \Rightarrow \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{4}{n^2} = \pi^2 - \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad \frac{\pi^2}{6}$$

when $x = \pi$

$$② \Rightarrow 0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4}{n^2} = \frac{-\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$\Rightarrow -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \dots \quad \frac{-\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots \frac{\pi^2}{12}$$

(4) obtain the fourier series of $f(x) = \begin{cases} x, & 0 \le x < \pi \\ 2\pi - x, & \pi \le x < 2\pi \end{cases}$ in $[0, 2\pi]$.

Given: $f(x) = \begin{cases} x, & 0 \le x < \pi \\ 2\pi - x, & \pi \le x < 2\pi, \ x \in [0, 2\pi] \end{cases}$

hence $\phi(x) = x \quad \psi(x) = 2\pi - x$

Let $\phi(2\pi - x) = 2\pi - x = \psi(x)$

$\therefore f(x)$ is an even function.

$\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow (1)$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x\,dx$$

$$a_0 = \frac{2}{\pi} \left[\frac{x^2}{2}\right]_0^{\pi}$$

$$a_0 = \frac{1}{\pi} (\pi^2 - 0)$$

$$\therefore \boxed{a_0 = \pi}$$

$$\therefore a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx\,dx$$

$$a_n = \frac{2}{\pi} \left\{ x \int_0^{\pi} \cos nx\,dx - \int_0^{\pi} [1 \cdot \int \cos nx\,dx]\,dx \right.$$

$$a_n = \frac{2}{\pi} \left\{ \frac{1}{n} (x \sin nx)_0^{\pi} + \frac{1}{n^2} (\cos nx)_0^{\pi} \right\}$$

$$a_n = \frac{2}{\pi} \left\{ \frac{1}{n} (0 - 0) + \frac{1}{n^2} (\cos n\pi - 1) \right\}$$

$$a_n = \frac{2}{\pi n^2} [(-1)^n - 1]$$

$$a_n = \begin{cases} \frac{-4}{\pi n^2}, & \forall n = 1, 3, 5 \ldots \\ 0, & \forall n = 2, 4, 6 \ldots \end{cases}$$

$$① \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{\pi n^2} \cos nx$$

⑤ Expand the function $f(x) = x(2\pi - x)$ in fourier series over the limits $[0, 2\pi]$ hence deduce $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Given $f(x) = x(2\pi - x)$, $x \in [0, 2\pi]$

$$\Rightarrow f(2\pi - x) = (2\pi - x)(2\pi - 2\pi + x)$$

$$= x(2\pi - x)$$

$\therefore f(x)$ is even function.

$\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow ①$$

$$\Rightarrow a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(2\pi - x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} [2\pi x - x^2]\,dx$$

$$= \frac{2}{\pi}\left[\pi x^2 - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\pi^3 - \frac{\pi^3}{3}\right)$$

$$= \frac{2}{\pi}\left(\frac{2\pi^3}{3}\right)$$

$$\boxed{a_0 = \frac{4\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (2\pi x - x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(2\pi x - x^2)\int_0^{\pi} \cos nx\,dx - \int_0^{\pi}\left[(2\pi - 2x)\int \cos nx\,dx\right]dx \right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n}(2\pi x - x^2)\sin nx\right]_0^{\pi} - \frac{2}{n^2}(\cos nx)_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{1}{n}\int_0^{\pi}[(2\pi - 2x)\sin nx]\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n}\int_0^{\pi}(\pi - x)\sin nx\right]dx$$

$$= \frac{2}{\pi}\left[\frac{2}{n}(\pi - x)\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^2}\pi\right]$$

$$\boxed{a_n = -\frac{4}{n^2}}$$

$$(1) \Rightarrow x(2\pi - x) = \frac{4\pi^2/3}{2} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2}\right)\cos nx$$

$$x(2\pi - x) = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx \rightarrow (2)$$

when $x = 0$

$$(2) \Rightarrow \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{12}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \frac{\pi^2}{6}$$

when $x = \pi$

$$(2) \Rightarrow \pi^2 = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi$$

$$\sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi = \frac{2\pi^2}{3} - \pi^2$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots \quad \frac{-\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \frac{\pi^2}{12}$$

⑥ Find the fourier series of $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x, & 0 \le x < \pi. \end{cases}$

Given: $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x, & 0 \le x < \pi \end{cases}$

$\therefore f(x)$: $y(x)$ either even no. (or) odd.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \rightarrow ①$$

$$\therefore a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x) dx + \int_{0}^{\pi} f(x) dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} x\, dx \right]$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$\Rightarrow a_0 = \frac{1}{\pi} \left(\frac{\pi^2}{2} \right)$$

$$\Rightarrow \boxed{a_0 = \frac{\pi}{2}}$$

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x) \cos nx\, dx + \int_{0}^{\pi} f(x) \cos nx\, dx \right]$$

$$= \frac{1}{\pi} \int_{0}^{\pi} x \cos nx\, dx$$

$$= \frac{1}{\pi} \left[x \int_{0}^{\pi} \cos nx\, dx - \int_{0}^{\pi} \left[1 . \int \cos nx\, dx \right] dx \right]$$

$$= \frac{1}{\pi} \left\{ \frac{1}{n} (x \sin nx)_0^{\pi} + \frac{1}{n^2} (\cos nx)_0^{\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{1}{n} (0-0) + \frac{1}{n^2} (\cos n\pi - 1) \right\}$$

$$\Rightarrow a_n = \frac{1}{\pi} \left\{ \frac{1}{n^2} (-1)^n - 1 \right\}$$

$$\Rightarrow b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{x\int_{0}^{\pi}\sin nx\,dx - \int_{0}^{\pi}\left[1\cdot\int \sin nx\,dx\right]dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}(x\cos nx)_{0}^{\pi} + \frac{1}{n^2}(\sin nx)_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}(\pi\cos n\pi - 0) + \frac{1}{n^2}(0-0)\right\}$$

$$\Rightarrow b_n = \frac{-\pi\cos n\pi}{n\pi}$$

$$\Rightarrow b_n = -\frac{(-1)^n}{n}$$

$$\Rightarrow b_n = \frac{(-1)^{n+1}}{n}$$

$$① \Rightarrow f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2}\left((-1)^n - 1\right)\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$$

⑦ obtain the fourier series of $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

hence deduce the series

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots$$

Given $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}, & -0 \le x \le \pi \end{cases}$

$$\therefore \phi(x) = 1 + \frac{2x}{\pi}, \quad \psi(x) = 1 - \frac{2x}{\pi}$$

$$\phi(-x) = 1 - \frac{2x}{\pi} = \psi(x)$$

$\therefore f(x)$ is an even function

$$\Rightarrow b_n = 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx \rightarrow ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi}$$

$$= \frac{2}{\pi}(0-0)$$

$$\boxed{a_0 = 0}$$

$$\therefore\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\left(1 - \frac{2x}{\pi}\right)\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\left[\left(\frac{-2}{\pi}\right)\int \cos nx\,dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}\left(1 - \frac{2x}{\pi}\right)\sin nx - \int_0^{\pi}\frac{2}{n^2\pi}\left(\cos nx\right)_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}(0-0) - \frac{2}{n^2\pi}(\cos n\pi - 1)\right\}$$

$$= \frac{4}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

$$a_n = \frac{4}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$$\therefore\ f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1-(-1)^n\right]\cos nx$$

$$\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases} \rightarrow (2)$$

when $x = 0$

$$(2) \Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = 1$$

$$\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = \frac{\pi^2}{4}$$

$$\frac{2}{1^2} + 0 + \frac{2}{3^2} + 0 + \frac{2}{5^2} + 0 + \dots\ \frac{\pi^2}{4}$$

$$2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+0+\right]\cdots \frac{\pi^2}{4}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}-\cdots \frac{\pi^2}{8}$$

(5) Obtain the fourier series $f(x)=|x|$ in the interval $[-\pi,\pi]$ and hence deduce the series $\frac{1}{1^2}+\frac{1}{3^2}+\cdots+\frac{\pi^2}{8}$.

Given $f(x)=|x|=\begin{cases}-x, & -\pi\le x\le 0\\ x, & 0\le x\le\pi\end{cases}$

$\therefore \phi(x)=-x \qquad \psi(x)=x$

$\phi(-x)=-(-x)=x=\psi(x)$

$$\therefore f(x)=\frac{a_0}{2}+\sum_{n=0}^{\infty}a_n\cos nx \rightarrow (1)$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x\,dx$$

$$=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$\boxed{a_0=\pi}$$

$$\therefore a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx$$

$$=\frac{2}{\pi}\left\{x\int_0^{\pi}\cos nx\,dx-\int_0^{\pi}\left[1\cdot\int\cos nx\,dx\right]dx\right\}$$

$$=\frac{2}{\pi}\left\{\frac{1}{n}\left[x\sin nx\right]_0^{\pi}+\frac{1}{n^2}\left(\cos nx\right)_0^{\pi}\right\}$$

$$=\frac{2}{\pi n^2}\left[\cos n\pi-1\right]$$

$$\therefore a_n=\frac{2}{\pi n^2}\left[(-1)^n-1\right]$$

$$\therefore (1)\Rightarrow f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[(-1)^n-1\right]\cos nx$$

$$\frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{n^2}\right)\cos nx = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases} \rightarrow (2)$$

when $x = 0$

$$(2) \Rightarrow \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2} = 0$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2} = \frac{-\pi^2}{4}$$

$$\frac{-2}{1^2} + 0 - \frac{2}{3^2} + 0 - \frac{2}{5^2} + \dots \quad \frac{-\pi^2}{4}$$

$$-2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right] = \frac{-\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

(9) Find the fourier series of $f(x) = \sqrt{1-\cos x}$ in the interval $[-\pi, \pi]$ (or) $[0, 2\pi]$ and hence deduce

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}.$$

Let $f(x) = \sqrt{1-\cos x} = \sqrt{2\sin^2\left(\frac{x}{2}\right)} = \sqrt{2}\sin\left(\frac{x}{2}\right), x \in [-\pi, \pi]$

$$\therefore f(-x) = \sqrt{1-\cos(-x)}$$

$$= \sqrt{1-\cos x}$$

$$= f(x)$$

$\therefore f(x)$ is an even function.

$b_n = 0.$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow (1)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sqrt{2}\sin\left(\frac{x}{2}\right)dx$$

$$= \frac{2\sqrt{2}}{\pi}\left[-\frac{\cos(x/2)}{1/2}\right]_0^{\pi}$$

$\therefore f(-x) = \sqrt{1-\cos(-x)} = \sqrt{1-\cos x} = f(x)$

$\therefore f(x)$ is an even function.

$\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow (1)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\sin\left(\frac{x}{2}\right)dx$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{-\cos(x/2)}{1/2}\right]_0^{\pi}$$

$$= -\frac{4\sqrt{2}}{\pi}\left[\cos\left(\frac{x}{2}\right)\right]_0^{\pi}$$

$$= -\frac{4\sqrt{2}}{\pi}\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= -\frac{4\sqrt{2}}{\pi}[0-1]$$

$$\therefore \boxed{a_0 = \frac{4\sqrt{2}}{\pi}}$$

$$\therefore a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sqrt{2}\sin(x/2)\cos nx\,dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi} 2\cos nx \sin\left(\frac{x}{2}\right)dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(nx+\frac{x}{2}\right) - \sin\left(nx - \frac{x}{2}\right)\right]dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)\right]dx$$

$$= \frac{\sqrt{2}}{\pi}\left[-\frac{\cos(n+1/2)x}{n+1/2} + \frac{\cos(n-1/2)x}{n-1/2}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left\{(0-0)-\left[-\frac{1}{n+1/2}+\frac{1}{n-1/2}\right]\right\}$$

$$= \frac{-\sqrt{2}}{\pi}\left[\frac{1}{n-1/2}-\frac{1}{n+1/2}\right]$$

$$= \frac{-\sqrt{2}}{\pi}\left[\frac{1}{\frac{2n-1}{2}}-\frac{1}{\frac{2n+1}{2}}\right]$$

$$= \frac{-2\sqrt{2}}{\pi}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]$$

$$= -2\sqrt{2}\left[\frac{2n+1-2n+1}{4n^2-1}\right]$$

$$\therefore a_n = -\frac{4\sqrt{2}}{\pi}\left[\frac{1}{4n^2-1}\right]$$

$$① \Rightarrow \sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx$$

$x = 0$

$$② \Rightarrow 0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{2\sqrt{2}}{\pi}$$

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{2\sqrt{2}}{\pi}\cdot\frac{\pi}{4\sqrt{2}} = \frac{1}{2}$$

⑩ Find the fourier series $f(x) = x - x^2,\ x \in [-\pi, \pi]$

Given $f(x) = x - x^2,\ x \in [-\pi, \pi]$

$\therefore f(x)$ is neither even nor odd.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx] \rightarrow ①$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx$$

$$= \frac{1}{\pi}\left(\frac{x^2}{2}-\frac{x^3}{3}\right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi^2}{2} - \frac{\pi^3}{3}\right] - \left[\frac{\pi^2}{2} + \frac{\pi^3}{3}\right]\right\}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi^3}{3}\right]$$

$$\boxed{a_0 = \frac{-2}{3}\pi^2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{(x-x^2)\int_{-\pi}^{\pi}\cos nx - \int_{-\pi}^{\pi}(1-2x)\int\cos nx\,dx\right\}dx$$

$$= \frac{1}{\pi}\left[(1-2x)\int_{-\pi}^{\pi}\sin nx\,dx + \frac{(-2x)}{n}\int_{-\pi}^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[(1-2x)\,[\sin x]_{-\pi}^{\pi} - \left[\frac{(1-2x)\cos nx}{n^2}\right]_{-\pi}^{\pi}\right.\right.$$

$$= -\frac{1}{n\pi}\left\{-\frac{1}{n}\left[(1-2\pi)\cos n\pi - (1+2\pi)\cos n\pi\right] - 0\right\}$$

$$= \frac{1}{n^2\pi}\left\{(1-2\pi-1-2\pi)\cos n\pi\right\}$$

$$= \frac{1}{n^2\pi^2}\left[-4\pi\cos n\pi\right]$$

$$= \frac{-4}{n^2}\cos n\pi$$

$$= \frac{-4}{n^2}(-1)^n$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^{n+1}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\left[(x-x^2)\int_{-\pi}^{\pi}\sin nx\,dx - \int_{-\pi}^{\pi}(1-2x)\int\sin nx\,dx\right]dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[(x-x^2)\cos nx\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}(1-2x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[(\pi-\pi^2)\cos n\pi - (-\pi-\pi^2)\cos n\pi\right] + \frac{1}{n}\int_{-\pi}^{\pi}(1-2x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}(2\pi\cos n\pi) + \frac{1}{n}I_1\right\}$$

$$b_n = \frac{1}{\pi}\left[-\frac{2\pi\cos n\pi}{n} + \frac{1}{n}I_1\right]$$

$$\therefore I_1 = \int_{-\pi}^{\pi}(1-2x)\cos nx\,dx$$

$$= \frac{1}{n}\left[(1-2x)\sin nx\right]_{-\pi}^{\pi} - \frac{2}{n^2}(\cos nx)_{-\pi}^{\pi}$$

$$\Rightarrow I_1 = 0$$

$$\therefore b_n = \frac{-2\pi\cos n\pi}{n\pi}$$

$$= -\frac{2}{n}\cos n\pi$$

$$= \frac{-2}{n}(-1)^n$$

$$\therefore b_n = \frac{2}{n}(-1)^{n+1}$$

$$① \Rightarrow (x-x^2) = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^{n+1}\cos nx + \frac{2}{n}(-1)^{n+1}\sin nx\right] \rightarrow ②$$

when $x=0$

$$② \Rightarrow 0 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}$$

$$4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{3}$$

$$4\left[\frac{1}{1^2} + -\frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right] = \frac{\pi^2}{3}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

(1) Find the fourier series $f(x) = x^2$ $[0,\pi]$ or $f(x) = x^2$ $[-\pi,\pi]$. (1)

Given $f(x) = x^2,\ x \in [0,\pi]$

WKT

The fourier half-range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \rightarrow (1)$$

$$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$\boxed{a_0 = \frac{2}{3}\pi^2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left\{x^2\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\left[2x\int \cos nx\,dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}\left[x^2 \sin nx\right]_0^{\pi} - \frac{2}{n}\int_0^{\pi}\left[x \sin nx\right]dx\right\}$$

$$= \frac{-4}{n\pi}\int_0^{\pi} x \sin nx\,dx$$

$$= \frac{-4}{n\pi}\left\{x\int_0^{\pi}\sin nx\,dx - \int_0^{\pi}\left[1.\int \sin nx\,dx\right]dx\right\}$$

$$= \frac{-4}{n\pi}\left\{-\frac{1}{n}\left[x \cos nx\right]_0^{\pi} + \frac{1}{n^2}\left(\sin nx\right)_0^{\pi}\right\}$$

$$= \frac{-4}{n\pi}\left\{-\frac{1}{n}\left[\pi \cos n\pi - 0\right]\right\}$$

$$= \frac{4}{n^2\pi}\left[\pi(-1)^n\right]$$

$$a_n = \frac{4}{n^2}(-1)^n$$

$\therefore$ ① $\Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx.$

⑫ Find the fourier half range cosine & sin series of $f(x) = x(\pi - x)$ in the $[0,\pi]$.

Given $f(x) = x(\pi - x) = \pi x - x^2, \; x \in [0,\pi]$

$\therefore$ WKT

The fourier-half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow ①$$

$$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{6}\right]$$

$$a_0 = \frac{2\pi^2}{6} = \frac{\pi^2}{3}$$

$$\therefore a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{(\pi x - x^2)\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\left[(\pi - 2x)\int \cos nx\,dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi}\left[(\pi x - x^2)\sin nx\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}(\pi - 2x)\sin nx\,dx\right\}$$

$$= -\frac{2}{n\pi}\int_0^{\pi}(\pi - 2x)\sin nx\,dx$$

$$= -\frac{2}{n\pi}\left\{(\pi - 2x)\int_0^{\pi}\sin nx\,dx - \int_0^{\pi}\left[(-2)\int \sin nx\,dx\right]dx\right\}$$

$$= -\frac{2}{n\pi}\left\{-\frac{1}{n}\left[(\pi - 2x)\cos nx\right]_0^{\pi} - \frac{2}{n^2}\left[\sin nx\right]_0^{\pi}\right\}$$

$$= -\frac{2}{n\pi}\left\{-\frac{1}{n}\left[-\pi\cos n\pi - \pi\right] - \frac{2}{n^2}(0 - 0)\right\}$$

$$= -\frac{2}{n\pi}\left\{\frac{\pi}{n}(1+\cos n\pi)\right\}$$

$$a_n = \frac{-2}{n^2}\left[1+(-1)^n\right]$$

$$① \Rightarrow x(\pi - x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\left[\frac{1+(-1)^n}{n^2}\right]\cos nx.$$

WK T

The fourier half range sin series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow ②$$

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\left\{(\pi x - x^2)\int_0^{\pi}\sin nx\,dx - \int_0^{\pi}\left[(\pi - 2x)\int \sin x\,dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{1}{n}\left[(\pi x - x^2)\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}(\pi - 2x)\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi - 2x)\cos nx\,dx$$

$$= \frac{2}{n\pi}\left\{(\pi - 2x)\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\left[(-2)\int \cos nx\,dx\right]dx\right\}$$

$$= \frac{2}{n\pi}\left\{\frac{1}{n}\left[(\pi - 2x)\sin nx\,dx\right]_0^{\pi} - \frac{2}{n^2}\left[\cos nx\right]_0^{\pi}\right\}$$

$$= \frac{2}{n\pi}\left\{\frac{1}{n}(0-0) - \frac{2}{n^2}\left[\cos n\pi - \cos 0\right]\right\}$$

$$= -\frac{4}{n^3\pi}\left[(-1)^n - 1\right]$$

$$\therefore b_n = \frac{4}{n^3\pi}\left[1-(-1)^n\right]\sin nx.$$

(13) Expand the function $f(x) = x\sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$. Deduce that

$$\frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \frac{1}{7\cdot9} + \cdots = \frac{\pi-2}{4}.$$

For the given function $f(x) = x\sin x$, over the interval $[-\pi,\pi]$ which is an even function.

$$f(x) = x\sin x$$

$$f(-x) = (-x)\sin(-x) = x\sin x = f(x)$$

Hence, $b_n = 0$

$$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$

$$= \frac{2}{\pi}\left[x(-\cos x) - (1)(-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\cos\pi\right]$$

$$\boxed{a_0 = 2}$$

$$\therefore a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\}\right]$$

$$\cos(n\pi - \pi) = \cos(\pi - n\pi) = -\cos n\pi$$

$$\cos(n\pi + \pi) = \cos(\pi + n\pi) = -\cos n\pi$$

$$\sin(n+1)\pi = \sin(\pi + n\pi) = -\sin n\pi = 0$$

$$\sin(n-1)\pi = -\sin(\pi - n\pi) = -\sin n\pi = 0$$

$$= \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}$$

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$$= \frac{-\cos n\pi}{n-1} - \frac{\cos n\pi}{n+1}$$

$$a_n = \cos n\pi \left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \cos n\pi \left[\frac{n-1-n-1}{n^2-1}\right] \quad n \neq 1$$

$$a_n = \frac{-2(-1)^n}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1} \quad n \neq 1$$

when $n=1$ in ① we get

$$a_1 = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx = \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\cos 2\pi + 0 - 0 - 0\right] = -\frac{1}{2}$$

Therefore, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$x \sin x = \frac{2}{2} + \left(\frac{-1}{2}\right)\cos x + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n^2-1} \cos nx$$

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\left[\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \frac{\cos 5x}{5^2-1} + \dots\right]$$

Put $x = \frac{\pi}{2}$ we get

$$\frac{\pi}{2} = 1 - 2\left[\frac{-1}{2^2-1} + \frac{1}{4^2-1} - \frac{1}{6^2-1} + \dots\right]$$

$$\frac{\pi}{2} - 1 = 2\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} + \dots\right)$$

$$\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots$$

Fourier series having the period $2l$:-

The Fourier series of $f(x) = [c, c+2l]$, over the period $2l$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Note :-

① If $c=0$, we can define $f(x)$ in the interval $[0,2l]$

for $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

② If $c=-\pi$, we can define $f(x)$ in the interval $[-\pi,\pi]$

for $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

③ If $f(x)$ is an even function in the intervals $[0,2l]$ or $[-l,l]$, then $b_n=0$, we have

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$ and it also called the

Fourier half-range cosine series in $[0, l]$.

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

④ If $f(x)$ is an odd function in $[0, 2l]$ or $[-l, l]$, then $a_0 = 0$, $a_n = 0$, we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$ and it is also called the Fourier half-range sin series in $[0, l]$.

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

⑤ If $f(x)$ is continuous in $[0, 2l]$, then $f(x)$ is even when $f(2l-x) = f(x)$, $f(x)$ is odd then, $f(2l-x) = -f(x)$.

⑥ $f(x) = \begin{cases} \phi(x), & 0 \le x \le l \\ \psi(x), & l \le x \le 2l \end{cases}$, then

$\phi(2l-x) = \psi(x)$ is even function

$\phi(2l-x) = -\psi(x)$ is odd function

⑦ If $f(x)$ is continuous $[-l, l]$, then $f(x)$ is even $f(-x) = f(x)$ and odd when $f(-x) = -f(x)$.

⑧ $f(x) = \begin{cases} \phi(x), & -l \le x \le 0 \\ \psi(x), & 0 \le x \le l \end{cases}$ then

$\phi(-x) = \psi(x)$ is even function

$\phi(-x) = -\psi(x)$ is odd function.

① Find the Fourier series of $f(x) = x(2-x)$ in the interval $(0,2)$ hence deduce $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Given $f(x) = x(2-x)$, $x \in (0,2)$

$l = 1$

$\therefore f(2-x) = (2-x)[2-(2-x)]$

$\Rightarrow f(2-x) = (2-x)[2-2+x]$

$\Rightarrow f(2-x) = x(2-x) = f(x)$

$\therefore$ f(x) is an even function.

$\Rightarrow b_n = 0$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \rightarrow ①$

$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$= \frac{2}{l}\int_0^l x(2-x)\,dx$

$= 2\left[x^2 - \frac{x^3}{3}\right]_0^{l=1}$

$= 2\left[1 - \frac{1}{3}\right]$

$= 2\left[\frac{2}{3}\right]$

$\boxed{a_0 = \frac{4}{3}}$

$\therefore a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$= 2\int_0^1 f(x)\cos(n\pi x)\,dx$

$= 2\int_0^1 (2x-x^2)\cos(n\pi x)\,dx$

$= 2\left\{(2x-x^2)\int_0^1 \cos(n\pi x)\,dx - \int_0^1\left[(2-2x)\int\cos(n\pi x)\,dx\right]dx\right\}$

$= 2\left\{\frac{1}{n\pi}\left[(2x-x^2)\sin(n\pi x)\right]_0^1 - \frac{2}{n\pi}\int_0^1 (1-x)\sin(n\pi x)\,dx\right\}$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \to ①$

$\therefore a_0 = \frac{2}{l}\int_0^l f(x)\,dx.$

$= \frac{2}{1}\int_0^1 \pi x\,dx.$

$= 2\pi \int_0^1 x\,dx$

$= 2\pi \left[\frac{x^2}{2}\right]_0^1$

$= 2\pi \left[\frac{1}{2} - 0\right]$

$\boxed{a_0 = \pi}$

$\therefore a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$

$\therefore a_n = \frac{2}{l}\int_0^1 f(x)\cos(n\pi x)\,dx.$

$= 2\int_0^1 \pi x \cos(n\pi x)\,dx.$

$= 2\pi \int_0^1 x \cos(n\pi x)\,dx.$

$= 2\pi \left\{ x \int_0^1 \cos(n\pi x)\,dx - \int_0^1 \left[1 \cdot \int \cos(n\pi x)\,dx\right]dx \right\}$

$= 2\pi \left\{ \frac{1}{n\pi}\left[x \sin(n\pi x)\right]_0^1 + \frac{1}{n^2\pi^2}\left[\cos(n\pi x)\right]_0^1 \right\}$

$= \frac{2\pi}{n^2\pi^2}\left[\cos n\pi - 1\right]$

$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$

$\therefore ① \Rightarrow \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos(n\pi x) = \begin{cases} \pi x, & 0 \le x < 1 \\ \pi(2-x), & 1 \le x < 2 \end{cases} \to ⑤$

when $x = 0$

$⑤ \Rightarrow \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2} = 0$

$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2} = \frac{-\pi}{2}$

$$= -\frac{4}{n\pi}\int_0^1 (1-x)\sin(n\pi x)\,dx$$

$$= \frac{4}{n\pi}\int_0^1 (x-1)\sin(n\pi x)\,dx$$

$$= \frac{4}{n\pi}\left\{(x-1)\int_0^1 \sin(n\pi x)\,dx - \int_0^1\left[1\cdot\int \sin(n\pi x)\,dx\right]dx\right\}$$

$$= \frac{4}{n\pi}\left\{-\frac{1}{n\pi}\left[(x-1)\cos(n\pi x)\right]_0^1 + \frac{1}{n^2\pi^2}\left[\sin(n\pi x)\right]_0^1\right\}$$

$$= \frac{-4}{n^2\pi^2}[0+1]$$

$$\boxed{a_n = \frac{-4}{n^2\pi^2}}$$

$$\therefore ① \Rightarrow x(2-x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(n\pi x) \rightarrow ②$$

when $x=0$

$$0 = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{3}\times\frac{\pi^2}{4}$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}.$$

② Find the fourier series of $f(x) = \begin{cases}\pi x, & 0\le x<1\\ \pi(2-x), & 1\le x<2\end{cases}$

Given $f(x) = \begin{cases}\pi x, & 0\le x<1\\ \pi(2-x), & 1\le x<2\end{cases}$

$$f(x) = \begin{cases}\pi x, & 0\le x<l\\ \pi(2-x), & l\le x<2l\end{cases} \qquad 2l = 1 \quad \therefore l = 1$$

$\therefore \phi(x) = \pi x \qquad \psi(x) = \pi(2-x)$

$\therefore \phi(2-x) = \pi(2-x) = \psi(x)$

$\therefore f(x)$ is an even function. $\Rightarrow b_n = 0$

$$\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} = -\frac{\pi^2}{4}$$

$$\Rightarrow -\frac{\pi^2}{4} = \frac{(-2)}{1^2} + \frac{(-2)}{3^2} + \frac{(-2)}{5^2} + \cdots$$

$$\Rightarrow -\frac{\pi^2}{4} = (-2)\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

③ Find the fourier series of $f(x) = \begin{cases} l+x, & -l \le x \le 0 \\ l-x, & 0 \le x \le l \end{cases}$

Given $f(x) = \begin{cases} l+x, & -l \le x \le 0 \\ l-x, & 0 \le x \le l \end{cases}$

$\therefore \phi(x) = l+x \qquad \psi(x) = l-x$

$\therefore \phi(-x) = l-x = \psi(x).$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \rightarrow ①$$

$$\therefore a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$= \frac{2}{l}\int_0^l (l-x)dx$$

$$= \frac{2}{l}\left[lx - \frac{x^2}{2}\right]_0^l$$

$$= \frac{2}{l}\left[l^2 - \frac{l^2}{2}\right]$$

$$= \frac{2}{l}\left[\frac{l^2}{2}\right]$$

$$\boxed{a_0 = l}$$

$$\therefore a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (l-x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left\{(l-x)\int_0^l \cos\left(\frac{n\pi x}{l}\right)dx - \int_0^l \left[(-1)\int \cos\left(\frac{n\pi x}{l}\right)dx\right]dx\right\}$$

$$= \frac{2}{l}\left\{\frac{l}{n\pi}\left[(l-x)\sin\left(\frac{n\pi x}{l}\right)\right]_0^l - \frac{l^2}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{l}\right)\right]_0^l\right\}$$

$$= \frac{2}{l}\left\{0 - \frac{l^2}{n^2\pi^2}(\cos n\pi - 1)\right\}$$

$$= \frac{2l^2}{l n^2\pi^2}\left((-1)^n - 1\right)$$

$$= \frac{-2l}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$\boxed{a_n = \frac{2l}{n^2\pi^2}\left(1-(-1)^n\right)}$$

$$① \Rightarrow \frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos\left(\frac{n\pi x}{l}\right) = \begin{cases} l+x, & -l \le x \le 0 \\ l-x, & 0 \le x \le l \end{cases} \rightarrow ②$$

when $x=0$

$$② \Rightarrow \frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = l$$

$$\Rightarrow \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = l - \frac{l}{2} = \frac{l}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = \frac{l}{2}\times\frac{\pi^2}{2l} = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \cdots = \frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

④ Find the fourier series of $f(x) = |x|$ in the interval $(-l, l)$.

$$f(x) = |x| = \begin{cases} -x, & -l \le x \le 0 \\ x, & 0 \le x \le l \end{cases}$$

$$\phi(x) = -x \qquad \psi(x) = x$$

$$\phi(-x) = -(-x) = x = \psi(x) \Rightarrow b_n = 0.$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) \rightarrow ①$$

$$\therefore a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l x\,dx$$

$$= \frac{2}{l}\left(\frac{x^2}{2}\right)_0^l$$

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$$= \frac{2}{l}\left(\frac{l^2}{2}\right)$$

$$\boxed{a_0 = l}$$

$$\therefore a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left\{x\int_0^l \cos\left(\frac{n\pi x}{l}\right) - \int_0^l\left[1 \cdot \int\cos\left(\frac{n\pi x}{l}\right)dx\right]dx\right\}$$

$$= \frac{2}{l}\left\{\frac{x}{n\pi}\left[x\sin\left(\frac{n\pi x}{l}\right)\right]_0^l + \frac{l^2}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{l}\right)\right]_0^l\right.$$

$$= \frac{2l^2}{ln^2\pi^2}\left[\cos n\pi - 1\right]$$

$$a_n = \frac{2l^2}{ln^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\therefore ① \Rightarrow f(x) = \frac{l}{2} + \sum_{n=1}^{\infty}\frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\frac{l}{2} + \frac{2l}{n^2\pi^2}\sum_{n=1}^{\infty}\left[(-1)^n - 1\right]\cos\left(\frac{n\pi x}{l}\right) = \begin{cases} -x, & -l \le x \le 0 \\ x, & 0 \le x \le l \end{cases}$$

where $x = 0$

$$\frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right] = 0$$

$$\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right] = -\frac{l}{2} \times \frac{\pi^2}{2l}$$

$$\left[\frac{-2}{1^2}\right] + \left[\frac{-2}{3^2}\right] + \left[\frac{-2}{5^2}\right] + \cdots = \frac{-\pi^2}{4}$$

$$-2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{-\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

5) Find the fourier series of $f(x) = \begin{cases} 1+\frac{4x}{3}, & -\frac{3}{2} \le x \le 0 \\ 1-\frac{4x}{3}, & 0 \le x \le \frac{3}{2} \end{cases}$

and hence deduce that the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

$$f(x) = \begin{cases} 1+\frac{4x}{3}, & -\frac{3}{2} \le x \le 0 \\ 1-\frac{4x}{3}, & 0 \le x \le \frac{3}{2} \end{cases} = \begin{cases} \phi(x), & -l \le x \le 0 \\ \psi(x), & 0 \le x \le l \end{cases}$$

$\therefore \phi(x) = 1 + \frac{4x}{3} \qquad \psi(x) = 1 - \frac{4x}{3} \qquad l = \frac{3}{2}$

$\therefore \phi(-x) = 1 - \frac{4x}{3} = \psi(x)$

$\therefore f(x)$ is an even function. $\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \rightarrow ①$$

$$\therefore a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{4}{3}\int_0^{3/2} f(x)\,dx$$

$$= \frac{4}{3}\int_0^{3/2} \left(1 - \frac{4x}{3}\right)dx$$

$$= \frac{4}{3}\left[x - \frac{2x^2}{3}\right]_0^{3/2}$$

$$= \frac{4}{3}\left[\frac{3}{2} - \frac{2}{3}\left(\frac{3}{2}\right)^2\right]$$

$$= \frac{4}{3}\left[\frac{3}{2} - \frac{3}{2}\right]$$

$$\Rightarrow \boxed{a_0 = 0}$$

$$\therefore a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{4}{3}\int_0^{3/2} \left(1 - \frac{4x}{3}\right)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\left\{\left(1 - \frac{4x}{3}\right)\int_0^{3/2} \cos\left(\frac{2n\pi x}{3}\right)dx - \int_0^{3/2}\left[\left(-\frac{4}{3}\right)\int \cos\left(\frac{2n\pi x}{3}\right)dx\right]dx\right\}$$

(15)

$$= \frac{4}{3}\left\{\frac{3}{2n\pi}\left[\left(1-\frac{4x}{3}\right)\sin\left(\frac{2n\pi x}{3}\right)\right]_0^{3/2} - \frac{4}{3}\,\frac{3^2}{4n^2\pi^2}\left[\cos\left(\frac{2n\pi x}{3}\right)\right]_0^{3/2}\right\}$$

$$= \frac{4}{3}\left\{0 - \frac{3}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]\right\}$$

$$= \frac{-4}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$= \frac{-4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\therefore a_n = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

$$\therefore (1) \Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2}\right)\cos\left(\frac{2n\pi x}{3}\right) = \begin{cases} 1+\frac{4x}{3}, & -\frac{3}{2}\le x\le 0 \\ 1-\frac{4x}{3}, & 0\le x\le 3/2 \end{cases} \rightarrow (2)$$

when $x = 0$

$$(2) \Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = 1$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{2}{1^2}+\frac{2}{3^2}+\frac{2}{5^2}+\dots = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots = \frac{\pi^2}{8}$$

(6) Find the half range cosine series of $f(x) = x(l-x)$ in the interval $[0,l]$.

Given: $f(x) = x(l-x)$, $x \in [0,l]$

WKT The fourier half range cosine series in $[0,l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \rightarrow (1)$$

$$\therefore a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\,dx$$

$$= \frac{2}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^3}{2} - \frac{l^3}{3}\right]$$

$$a_0 = \frac{2}{l} \times \frac{l^3}{6}$$

$$\boxed{a_0 = \frac{l^2}{3}}$$

$$\therefore a_n = \frac{2}{l}\int_0^l a_n \cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left\{(lx-x^2)\int_0^l \cos\left(\frac{n\pi x}{l}\right)dx - \int_0^l\left[(l-2x)\int\cos\left(\frac{n\pi x}{l}\right)dx\right]dx\right\}$$

$$= \frac{2}{l}\left\{\frac{l}{n\pi}(0-0) - \frac{l}{n\pi}\int_0^l (l-2x)\sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{-2}{n\pi}\left\{(l-2x)\int_0^l \sin\left(\frac{n\pi x}{l}\right)dx - \int_0^l\left[(-2)\int\sin\left(\frac{n\pi x}{l}\right)dx\right]dx\right\}$$

$$= \frac{-2}{n\pi}\left\{-\frac{l}{n\pi}\left[(l-2x)\cos\left(\frac{n\pi x}{l}\right)\right]_0^l - \frac{2l^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi x}{l}\right)\right)_0^l\right\}$$

$$= \frac{-2}{n\pi}\left\{-\frac{l}{n\pi}\left[-l\cos n\pi - l\right] - 0\right\}$$

$$a_n = \frac{-2l^2}{n^2\pi^2}\left[(-1)^n + 1\right]$$

$$\therefore f(x) = \frac{l^2}{6} + \left(\frac{-2l^2}{\pi^2}\right)\sum_{n=1}^{\infty}\left(\frac{(-1)^n+1}{n^2}\right)\cos\left(\frac{n\pi x}{l}\right)$$

(7) Find the half-range cosine series $f(x) = (x-1)^2$ in the interval $[0,1]$.

Given: $f(x) = (x-1)^2$, $x \in [0,1]$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \rightarrow (1)$$

$$\therefore a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^1 (x-1)^2\,dx$$

$$= 2\left[\frac{(x-1)^3}{3}\right]_0^1$$

$$\boxed{a_0 = \frac{2}{3}}$$

$$\therefore a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 2\int_0^1 f(x)\cos(n\pi x)dx$$

$$= 2\int_0^1 (x-1)^2\cos(n\pi x)dx$$

$$= 2\left\{(x-1)^2\int_0^1 \cos(n\pi x)dx - \int_0^1\left[2(x-1)\int\cos(n\pi x)dx\right]dx\right\}$$

$$= 2\left\{\frac{1}{n\pi}(x-1)^2\left[\sin(n\pi x)\right]_0^1 + \frac{2}{n^2\pi^2}\left[\cos(n\pi x)\,2(x-1)\right]_0^1\right\}$$

$$a_n = \frac{4}{n^2\pi^2}\left\{\left[(x-1)\cos(n\pi x)\right] + 0\right\} = \frac{4}{n^2\pi^2}\left[0+1\right] = \frac{4}{n^2\pi^2}$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(n\pi x)$$

Harmonic Fourier Series:-

(1) Let $y = f(x)$ be a periodic function of the period 2π, then the fourier series of $f(x)$ in the harmonics can be expressed as $f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x) + \dots$

where $\frac{a_0}{2}$ is called the constant term, (a_1, b_1) are called the first harmonic coefficients, (a_2, b_2) are called the second harmonic coefficients and which will be evaluated as

$$a_0 = \frac{2}{N}\sum y \qquad a_2 = \frac{2}{N}\sum y\cos 2x$$

$$a_1 = \frac{2}{N}\sum y\cos x \qquad b_2 = \frac{2}{N}\sum y\sin 2x$$

$$b_1 = \frac{2}{N}\sum y\sin x$$

Generally, $a_n = \frac{2}{N} \Sigma y \cos nx$, $\forall n = 1, 2, 3, \ldots$

$$b_n = \frac{2}{N} \Sigma y \sin nx, \quad \forall n = 1, 2, 3 \ldots$$

where N is the number of terms given in the table which should always be even number.

② Suppose $y = f(x)$ be a periodic function over the period $2l$, then $f(x) = \frac{a_0}{2} + \left[a_1 \cos\left(\frac{\pi x}{l}\right) + b_1 \sin\left(\frac{\pi x}{l}\right)\right] + \left[a_2 \cos\left(\frac{2\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right)\right] + \ldots$, where $a_0 = \frac{2}{N} \Sigma y$ $\quad \therefore \theta = \frac{\pi x}{l}$

$$a_1 = \frac{2}{N} \Sigma y \cos\theta$$

$$a_2 = \frac{2}{N} \Sigma y \sin\theta$$

$$a_n = \frac{2}{N} \Sigma y \cos n\theta$$

$$b_n = \frac{2}{N} \Sigma y \sin n\theta.$$

① The following value of function y gives the displacement in inches of a certain machine part for rotation x of a flywheel. Expand y in terms of fourier series upto the second harmonic.

rotations	x	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$	π
displacement	y	0	9.2	14.4	17.8	17.3	11.7	0

Sol: The fourier series of $y = f(x)$ upto the second harmonics is

x	y	y cos x	y sin x	y cos 2x	y sin 2x
0°	0	0	0	0	0
30°	9.2	7.9674	4.6000	4.6	7.9674
60°	14.4	7.2	12.4707	-7.2	12.4707
90°	17.8	0	17.8	-17.8	0
120°	17.3	-8.65	14.9822	-8.65	-14.9822
150°	11.7	-10.1326	5.85	5.85	-10.1326
Σ	70.40	-3.6151	55.7029	-23.20	-4.6766

$n = 6$

$\therefore a_0 = \frac{2}{n}\Sigma y = \frac{2}{6}(70.4) = 23.46$

$a_1 = \frac{2}{n}\Sigma y\cos x = \frac{2}{6}(-3.6151) = -1.2050$

$b_1 = \frac{2}{n}\Sigma y\sin x = \frac{2}{6}(55.7029) = 18.5676$

$a_2 = \frac{2}{n}\Sigma y\cos 2x = \frac{2}{6}(-23.20) = -7.7333$

$b_2 = \frac{2}{n}\Sigma y\sin 2x = \frac{2}{6}(-4.6766) = -1.5589$

$\therefore f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x)$

$= \frac{23.46}{2} + (-1.2050\cos x + 18.5676\sin x) + (-7.7333\cos 2x - 1.5589\sin 2x)$

$= 11.73 + (-1.2050\cos x + 18.5676\sin x) + (-7.7333\cos 2x - 1.5589\sin 2x)$

② Express y as a fourier series upto first harmonic for the given data

x	0	30	60	90	120	150	180	210	240	270	300	330
y	1.8	1.1	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

x	y	$y\cos x$	$y\sin x$
0	1.8	1.8	0
30	1.1	0.9526	0.55
60	0.30	0.15	0.2598
90	0.16	0	0.16
120	1.50	-0.75	1.2990
150	1.30	-1.1258	0.65
180	2.16	-2.16	0
210	1.25	-1.0825	-0.625
240	1.30	-0.65	-1.1258
270	1.52	0	-1.52
300	1.76	0.88	-1.5242
330	2.00	1.7320	-1
Σ	16.15	-0.2537	-2.8762

$N = 12$

$\therefore a_0 = \frac{2}{N} \Sigma y = \frac{2}{12}(16.15) = 2.6916$

$a_1 = \frac{2}{N} \Sigma y\cos x = \frac{2}{12}(-0.2537) = -0.0422$

$b_1 = \frac{2}{N} \Sigma y\sin x = \frac{2}{12}(-2.8762) = -0.47936$

$f(x) = \frac{2.6916}{2} + (-0.0422\cos x - 0.47936\sin x)$

$= 1.3458 + (-0.0422\cos x - 0.47936\sin x)$

(5) Obtain the constant term and first sin, cosine terms in the fourier expansion of y from the following table.

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Sol: The given x varies as $0 \le x < 6$

$2l = 6$

$\therefore l = 3$

$\therefore$ The Fourier series expansion upto first harmonic is defined as $f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right) \rightarrow ①$

x	y	$\theta = \frac{\pi x}{3}$	$y\cos\theta$	$y\sin\theta$
0	4	0	4	0
1	8	$60°$	4	6.9282
2	15	$120°$	-7.5	12.9903
3	7	$180°$	-7	0
4	6	$240°$	-3	-5.1961
5	2	$300°$	1	-1.7320
Σ	42	-	-8.5	12.9901

$\therefore a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(42) = 14$

$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{2}{6}(-8.5) = -2.8333$

$b_1 = \frac{2}{N}\Sigma y\sin\theta = \frac{2}{6}(12.9901) = 4.3300$

$\therefore f(x) = 7 - (2.8333)\cos\left(\frac{\pi x}{3}\right) + (4.3301)\sin\left(\frac{\pi x}{3}\right)$

④ Obtain the constant term and the first sin & cosine terms in the Fourier expansion of y from the following table.

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Sol: The given x varies as $0 \le x < 6$

$2l = 6$

$\therefore l = 3$

$\therefore$ The Fourier series expansion upto the first harmonic is defined as

$$f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right) \rightarrow ①$$

x	y	$\theta = \frac{\pi x}{3}$	$y\cos\theta$	$y\sin\theta$
0	9	0	9	0
1	18	60°	9	15.5884
2	24	120°	-12	20.7846
3	28	180°	-28	0
4	26	240°	-13	-22.5166
5	20	300°	10	-17.3205
Σ	125	-	-25	-3.4641

$$\therefore a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(125) = 41.667$$

$$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{2}{6}(-25) = -8.3333$$

$$b_1 = \frac{2}{N}\Sigma y\sin\theta = \frac{2}{6}(-3.4641) = -1.547$$

$$\therefore f(x) = \frac{41.667}{2} - 8.3333\cos\left(\frac{\pi x}{3}\right) - 1.547\sin\left(\frac{\pi x}{3}\right)$$

$$= 20.8335 - 8.3333\cos\left(\frac{\pi x}{3}\right) - 1.547\sin\left(\frac{\pi x}{3}\right)$$

(6) The following table gives the variations of periodic current over a period T. Show by numerical analysis that there is a direct current part 0.75 Amp. The variable current and obtain the amplitude of first harmonic.

t(sec)	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Sol: Given the period to a circuit $2l = T$

$l = T/2$

t	A	$\theta = \frac{\pi t}{\lambda} = \frac{2\pi t}{T}$	$A\cos\theta$	$A\sin\theta$
0	1.98	0°	1.98	0
$T/6$	1.30	60°	0.65	1.1258
$T/3$	1.05	120°	−0.5250	0.9093
$T/2$	1.30	180°	−1.30	0
$\frac{2T}{3}$	−0.88	240°	0.44	0.7621
$\frac{5T}{6}$	−0.25	300°	−0.1250	0.2165
Σ	4.5	−	1.1200	3.0137

$$\therefore a_0 = \frac{2}{N}\Sigma A = \frac{2}{6}(4.5) = 1.5$$

$$a_1 = \frac{2}{N}\Sigma A\cos\theta = \frac{2}{6}(1.12) = 0.3733$$

$$b_1 = \frac{2}{N}\Sigma A\sin\theta = \frac{2}{6}(3.0137) = 1.0045$$

$\therefore$ Direct current $= \frac{a_0}{2} = \frac{1.5}{2} = 0.75$ Amp

$\therefore$ Amplitude $= \sqrt{a_1^2 + b_1^2} = \sqrt{(0.3733)^2 + (1.0045)^2}$

$$= 1.0716$$

Module-3

FOURIER TRANSFORMS

Definition:-

① Let $F(x)$ is a function defined on $[-\infty, \infty]$, then its Fourier transform can be defined as

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx = f(s)$$

where 'F' is called the fourier transform operator and 's' be the parameter either real (or) complex.

The inverse Fourier transform is defined as

$$F^{-1}[F(s)] = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds.$$

② Let $F(x)$ be a function defined on $[0, \infty]$, then the Fourier cosine transform of $F(x)$ is defined as

$$F_c[F(x)] = \int_{0}^{\infty} F(x)\cos(sx)\,dx = f_c(s).$$

and its inverse fourier cosine transform is

$$F^{-1}[f_c(s)] = F(x) = \frac{2}{\pi}\int_{0}^{\infty} f_c(s)\cos(sx)\,ds.$$

Similarly, the Fourier sin transform of $F(x)$ is $F_s[F(x)] = \int_{0}^{\infty} F(x)\sin(sx)\,dx = f_s(s)$ and its inverse sin transform is $F^{-1}[f_s(s)] = F(x) = \frac{2}{\pi}\int_{0}^{\infty} f_s(s)\sin(sx)\,ds.$

① Find the fourier transform of $e^{-a^2x^2}$, $a>0$ and hence show that the fourier transform of $e^{-x^2/2}$ is $\sqrt{2\pi}\, e^{-s^2/2}$.

Sol: Given $F(x) = e^{-a^2x^2}$, $a>0$

∴ WKT

$$\therefore F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{isx} e^{-a^2x^2}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2x^2 + isx}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}\,dx$$

$$= \int_{-\infty}^{\infty} -\left[(ax)^2 - 2(ax)\left(\frac{is}{2a}\right) + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]dx$$

$$= \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 - \frac{i^2s^2}{4a^2}\right]}dx$$

$$= \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]}dx$$

$$= \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{-s^2/4a^2}\,dx$$

$$= e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2}dx \quad \rightarrow ①$$

Let $ax - \frac{is}{2a} = u$

$\Rightarrow a\,dx = du$

$\Rightarrow dx = \frac{1}{a}du$

UL: $x = \infty \Rightarrow u = \infty$

LL: $x = -\infty \Rightarrow u = -\infty$

$\therefore F[e^{-a^2x^2}] = \frac{e^{-s^2/4a^2}}{a}\int_{-\infty}^{\infty} e^{-u^2}du$

$\Rightarrow F[e^{-a^2x^2}] = \frac{e^{-s^2/4a^2}}{a}\sqrt{\pi} \rightarrow (1)$

when $a = \frac{1}{\sqrt{2}}$

$(1) \Rightarrow F[e^{-x^2/2}] = \frac{\sqrt{\pi}}{1/\sqrt{2}} e^{-s^2/4(\frac{1}{2})} = \sqrt{\pi}\, e^{-s^2/2}$

(2) Find the fourier transform of $F(x) = \begin{cases} 1, & |x|<a \\ 0, & |x|>a \end{cases}$ and hence find $\int_0^{\infty}\frac{\sin x}{x}dx$.

Sol:- Given $F(x) = \begin{cases} 1, & |x|<a \\ 0, & |x|>a \end{cases}$

$\Rightarrow F(x) = \begin{cases} 1, & -a<x<a \\ 0, & x>a \end{cases}$

$\therefore$ WKT

$F[F(x)] = \int_{-\infty}^{\infty} e^{isx}F(x)dx$

$= \int_{-\infty}^{-a} e^{isx}F(x)dx + \int_{-a}^{a} e^{isx}F(x)dx + \int_{a}^{\infty} e^{isx}F(x)dx$

$= 0 + \int_{-a}^{a} e^{isx}\cdot dx + 0$

$\therefore F[F(x)] = \int_{-a}^{a} e^{isx}dx$

$= \left[\frac{e^{isx}}{is}\right]_{-a}^{a}$

$= \frac{1}{is}\left[e^{ias} - e^{-ias}\right]$

$= \frac{1}{is}\left[\cancel{\cos(as)} + i\sin(as) - \cancel{\cos(as)} + i\sin(as)\right]$

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$$= 2i\ \frac{\sin(as)}{is}$$

$$\Rightarrow f(s) = \frac{2\sin(as)}{s}$$

WKT the fourier inverse transform is

$$\therefore F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\ \frac{2\sin(as)}{s}\,ds = \begin{cases} 1, & -a<x<a \\ 0, & x>a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} [\cos(sx) - i\sin(sx)]\ \frac{\sin as}{s}\,ds = \pi \begin{cases} 1, & -a<x<a \\ 0, & x>a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos(sx)\sin(as)}{s}\,ds - i\int_{-\infty}^{\infty} \frac{\sin(sx)\sin(as)}{s}\,ds = \pi \begin{cases} 1, & |x|<a \\ 0, & |x|>a \end{cases}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(sx)\sin(as)}{s}\,ds = \pi \begin{cases} 1, & -a<x<a \\ 0, & x>a \end{cases} \quad \rightarrow (1)$$

when $x = 0$

$$\therefore (1) \Rightarrow \int_{-\infty}^{\infty} \frac{\sin(as)}{s}\,ds = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(as)}{s}\,ds = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin(as)}{s}\,ds = \pi$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin(as)}{s}\,ds = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin s}{s}\,ds = \frac{\pi}{2}$$ for $a = 1$.

$$\Rightarrow \int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$ for $s = x$.

③ Find the fourier transform $F(x) = \begin{cases} a^2-x^2, & |x| \le a \\ 0, & |x| > a. \end{cases}$

hence show that (i) $\int_0^\infty \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4}$

(ii) $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = -\frac{3\pi}{16}.$

Sol: Given $F(x) = \begin{cases} a^2-x^2, & -a \le x \le a \\ 0, & x > a \end{cases}$

$\Rightarrow$ WKT $F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx$

$= \int_{-\infty}^{a} e^{isx} F(x)\,dx + \int_{-a}^{a} e^{isx} F(x)\,dx + \int_{a}^{\infty} e^{isx} F(x)\,dx$

$= \int_{-a}^{a} (a^2-x^2) e^{isx}\,dx$

$= (a^2-x^2)\int_{-a}^{a} e^{isx}\,dx - \int_{-a}^{a} \left[-2x \int e^{isx}\,dx\right]dx$

$= \frac{1}{is}(0-0) + \frac{2}{is}\int_{-a}^{a} x\, e^{isx}\,dx$

$= \frac{2}{is}\left\{ x\int_{-a}^{a} e^{isx}\,dx - \int_{-a}^{a}\left[1.\int e^{isx}\,dx\right]dx \right\}$

$= \frac{2}{is}\left\{ \frac{1}{is}\left[x e^{isx}\right]_{-a}^{a} - \frac{1}{i^2s^2}\left[e^{isx}\right]_{-a}^{a} \right\}$

$= \frac{2}{i^2s^2}\left[x\, e^{isx}\right]_{-a}^{a} - \frac{2}{i^3s^3}\left[e^{isx}\right]_{-a}^{a}$

$= -\frac{2}{s^2}\left[a\, e^{ias} + a e^{-ias}\right] + \frac{2}{is^3}\left[e^{ias} - e^{-ias}\right]$

$= -\frac{2a}{s^2}\left[e^{ias} + e^{-ias}\right] + \frac{2}{is^3}\left[e^{ias} - e^{-ias}\right]$

$= -\frac{2a}{s^2}\left[2\cos(as)\right] + \frac{2}{is^3}\left[2i\sin(as)\right]$

$$= \frac{-4a\cos(as)}{s^2} + \frac{4\sin(as)}{s^3}$$

$$= \frac{4\sin(as) - 4as\cos(as)}{s^3}$$

$$\Rightarrow f(s) = \frac{4}{s^3}\left[\sin(as) - as\cos(as)\right]$$

we know that

The fourier inverse transform is

$$F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\cdot\frac{4}{s^3}\left[\sin(as) - as\cos(as)\,ds\right] = \begin{cases} a^2-x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-isx}\frac{\left[\sin(as) - as\cos(as)\right]}{s^3}\,ds = \frac{\pi}{2}\begin{cases} a^2-x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\left[\cos(sx) - i\sin(sx)\right]\frac{\left[\sin(as) - as\cos(as)\right]}{s^3}\,ds = \frac{\pi}{2}\begin{cases} a^2-x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

But $x = 0$

$$\therefore ① \Rightarrow \int_{-\infty}^{\infty}\frac{\sin(as) - as\cos(as)}{s^3}\,ds = \frac{\pi}{2}a^2 \rightarrow ②$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2}, \text{ for } a = 1.$$

$$\Rightarrow 2\int_0^\infty \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2}$$

$$\Rightarrow \int_0^\infty \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4} \quad \text{for } s = x.$$

ii) Let $x = a/2$

$$(1) \Rightarrow \int_{-\infty}^\infty \frac{\sin(as) - s\cos as}{s^3}\cos\left(\frac{as}{2}\right) = \frac{\pi}{2}\left(a^2 - \frac{a^2}{4}\right) = \frac{3\pi a^2}{8}$$

$$\Rightarrow \int_{-\infty}^\infty \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right)ds = \frac{3\pi}{8} \quad \text{for } a = 1$$

$$\Rightarrow 2\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right)ds = \frac{3\pi}{8}$$

$$\Rightarrow \int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right)ds = \frac{3\pi}{16}$$

$$\Rightarrow \int_0^\infty \frac{s\cos s - \sin s}{s^3}\cos\left(\frac{s}{2}\right)ds = \frac{-3\pi}{16}$$

$$\Rightarrow \int_0^\infty \frac{x\sin x - \sin x}{x^3}\cos\left(\frac{x}{2}\right)dx = \frac{-3\pi}{16}$$

(4) Find the fourier transform of $F(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ and hence find $\int_0^\infty \frac{\sin x}{x}\,dx$.

Given $F(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

$$\Rightarrow F(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

$\therefore$ WK T

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx$$

$$= \int_{-\infty}^{-1} e^{isx} F(x)\,dx + \int_{-1}^{1} e^{isx} F(x)\,dx + \int_{1}^{\infty} e^{isx} F(x)\,dx$$

$$= 0 + \int_{-1}^{1} e^{isx} \cdot 1\,dx + 0$$

$$\therefore F[F(x)] = \int_{-1}^{1} e^{isx}\,dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-1}^{1}$$

$$= \frac{1}{is}\left[e^{is} - e^{-is}\right]$$

$$= \frac{1}{is}\left[\cos(s) + i\sin(s) - \cos(s) + i\sin(s)\right]$$

$$= 2i\,\frac{\sin(s)}{is}$$

$$\Rightarrow f(s) = \frac{2\sin(s)}{s}$$

WK T

The fourier inverse transform is

$$\therefore F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\,\frac{2\sin(s)}{s}\,ds = \begin{cases} 1, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} [\cos(sx) - i\sin(sx)]\,\frac{\sin(s)}{s} = \pi \begin{cases} 1, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos(sx)\sin(s)}{s}\,ds - i\int_{-\infty}^{\infty} \frac{\sin(sx)\sin s}{s}\,ds = \begin{cases} \pi\ 1, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(sx)\sin(s)}{s}\,ds = \pi \begin{cases} 1, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

when $x=0$

$$\therefore (1) \Rightarrow \int_{-\infty}^{\infty} \frac{\sin(s)}{s} ds = \pi(1)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(s)}{s} ds = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin(s)}{s} ds = \pi$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin(s)}{s} ds = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \text{ for } s=x$$

(5) Find the fourier transform $F(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x|>1, \end{cases}$

hence show that

(i) $\displaystyle\int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3} dx = \frac{\pi}{4}$

(ii) $\displaystyle\int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = -\frac{3\pi}{16}$

Given $F(x) = \begin{cases} 1-x^2, & -1 \le x \le 1 \\ 0, & x>1 \end{cases}$

⇒ WKT

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx$$

$$= \int_{-\infty}^{-1} e^{isx} F(x)\,dx + \int_{-1}^{1} e^{isx} F(x)\,dx + \int_{1}^{\infty} e^{isx} F(x)\,dx$$

$$= \int_{-1}^{1} (1^2 - x^2) e^{isx}\,dx$$

$$= (1-x^2)\int_{-1}^{1} e^{isx}\,dx - \int_{-1}^{1} [-2x \int e^{isx}\,dx]\,dx$$

$$= \frac{1}{is}[(1-x^2)e^{isx}]_{-1}^{1} + \frac{2}{is}\int_{-1}^{1} x e^{isx}\,dx$$

$$= \frac{1}{is}(0-0) + \frac{2}{is}\int_{-1}^{1} x e^{isx}\,dx$$

$$= \frac{2}{is}\left\{ x\int_{-1}^{1} e^{isx}\,dx - \int_{-1}^{1} [1 \cdot \int e^{isx}\,dx]\,dx \right\}$$

$$= \frac{2}{is}\left\{ \frac{1}{is}[x e^{isx}]_{-1}^{1} - \frac{1}{i^2 s^2}[e^{isx}]_{-1}^{1} \right\}$$

$$= \frac{2}{i^2 s^2}[x e^{isx}]_{-1}^{1} - \frac{2}{i^3 s^3}[e^{isx}]_{-1}^{1}$$

$$= \frac{-2}{s^2}[e^{is} + e^{-is}] + \frac{2}{is^3}[e^{is} - e^{-is}]$$

$$= \frac{-2}{s^2}[2\cos(s)] + \frac{2}{is^3}[2i\sin(s)]$$

$$= \frac{-4\cos(s)}{s^2} + \frac{4\sin(s)}{s^3}$$

$$= \frac{4\sin(s) - 4s\cos(s)}{s^3}$$

$$\Rightarrow f(s) = \frac{4}{s^3}[\sin(s) - s\cos(s)]$$

WKT The fourier inverse transform is

$$F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} \frac{4}{s^3}[\sin s - s\cos s]\,ds = \begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-isx} \left[\frac{\sin s - s\cos s}{s^3}\right] ds = \frac{\pi}{2}\begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} [\cos(sx) - i\sin(sx)]\left[\frac{\sin(s) - s\cos s}{s^3}\right] ds = \frac{\pi}{2}\begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos(sx)\,ds = \frac{\pi}{2}\begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases} \rightarrow (1)$$

Put $x=0$

$$\therefore (1) \Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2} \rightarrow (2)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{4}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4} \quad \text{for } s=x$$

(ii) Let $x=\frac{1}{2}$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right)ds = \frac{\pi}{2}\left(1^2 - \frac{1^2}{4}\right) = \frac{3\pi}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right)ds = \frac{3\pi}{8}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right)ds = \frac{3\pi}{8}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$$

$$\Rightarrow \int_0^{\infty} \frac{s\cos s - \sin s}{s^3} \cos\left(\frac{s}{2}\right) ds = -\frac{3\pi}{16}$$

$$\Rightarrow \int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = -\frac{3\pi}{16} \text{ for } s = x.$$

(6) Find the inverse fourier transform of e^{-s^2}.

Let $f(s) = e^{-s^2}$

$$\therefore F(x) = F^{-1}[f(s)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} f(s) ds$$

$$\Rightarrow F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} e^{-s^2} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(s^2 + isx)} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[s^2 + 2(s)\left(\frac{ix}{2}\right) + \left(\frac{ix}{2}\right)^2 - \left(\frac{ix}{2}\right)^2\right]} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[\left(s + \frac{ix}{2}\right)^2 - \frac{i^2x^2}{4}\right]} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[\left(s + \frac{ix}{2}\right)^2 + \frac{x^2}{4}\right]} ds$$

$$F(x) = \frac{e^{-x^2/4}}{2\pi} \int_{-\infty}^{\infty} e^{-\left(s + \frac{ix}{2}\right)^2} ds$$

Let $s + \frac{ix}{2} = u$

$\Rightarrow ds = du$

$$\therefore (1) \Rightarrow F(x) = \frac{e^{-x^2/4}}{2\pi} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$\Rightarrow F(x) = \frac{e^{-x^2/4}}{2\pi} \sqrt{\pi}$$

$$\Rightarrow F(x) = \frac{1}{2\sqrt{\pi}\sqrt{\pi}} e^{-x^2/4} \sqrt{\pi}$$

$$\Rightarrow F(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}$$

(2) Find the fourier cosine transform of $F(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

Sol: Given $F(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

WKT the fourier cosine transform of $F(x)$ is

$$F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^1 F(x)\cos(sx)\,dx + \int_1^2 F(x)\cos(sx)\,dx + \int_2^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^1 x\cos(sx)\,dx + \int_1^2 (2-x)\cos(sx)\,dx + 0$$

$$\Rightarrow F_c(s) = \int_0^1 x\cos(sx)\,dx + \int_1^2 (2-x)\cos(sx)\,dx$$

$$\therefore \int_0^1 x\cos(sx)\,dx = x\int_0^1 \cos(sx)\,dx - \int_0^1 \left[1\int \cos(sx)\,dx\right]dx$$

$$= \frac{1}{s}\left[x\sin(sx)\right]_0^1 + \frac{1}{s^2}\left[\cos(sx)\right]_0^1$$

$$= \frac{1}{s}\left[\sin s - 0\right] + \frac{1}{s^2}\left[\cos s - \cos 0\right]$$

$$\Rightarrow \int_0^1 x\cos(sx)\,dx = \frac{1}{s}\sin s + \frac{1}{s^2}\cos s - \frac{1}{s^2}$$

$$\int_1^2 (2-x)\cos(sx)\,dx = (2-x)\int_1^2 \cos(sx)\,dx - \int_1^2 \left[(-1)\int \cos(sx)\,dx\right]dx$$

$$= \frac{1}{s}\left[(2-x)\sin(sx)\right]_1^2 - \frac{1}{s^2}\left[\cos(sx)\right]_1^2$$

$$= \frac{1}{s}\left[0 - \sin s\right] - \frac{1}{s^2}\left[\cos 2s - \cos s\right]$$

$$\Rightarrow \int_1^2 (2-x)\cos sx\,dx = \frac{1}{s^2}\cos s - \frac{1}{s^2}\cos 2s - \frac{1}{s}\sin s$$

$$\therefore (1) \Rightarrow f(s) = \frac{1}{s}\sin s + \frac{1}{s^2}\cos s - \frac{1}{s^2} + \frac{1}{s^2}\cos s - \frac{1}{s^2}\cos 2s - \frac{1}{s}\sin s$$

$$\Rightarrow f_c(s) = \frac{2}{s^2}\cos s - \frac{1}{s^2}\cos 2s - \frac{1}{s^2}$$

(8) Find the fourier cosine transform of $F(x) = \begin{cases} 4x, & 0<x<1 \\ 4-x, & 1<x<4 \\ 0, & x>4 \end{cases}$

Sol: Given $F(x) = \begin{cases} 4x, & 0<x<1 \\ 4-x, & 1<x<4 \\ 0, & x>4 \end{cases}$

WKT The fourier cosine transform of $F(x)$ is

$$F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^1 F(x)\cos(sx)\,dx + \int_1^4 F(x)\cos(sx)\,dx + \int_4^\infty F(x)\cos(sx)\,dx$$

$$= 4\int_0^1 x\cos(sx)\,dx + \int_1^4 (2-x)\cos(sx)\,dx + 0$$

$$\Rightarrow f_c(s) = \int_0^1 4x\cos(sx)\,dx + \int_1^4 (2-x)\cos(sx)\,dx$$

$$\therefore \int_0^1 4x\cos(sx)\,dx = 4x\int_0^1 \cos(sx)\,dx - \int_0^1 \left[4.\int \cos(sx)\,dx\right]dx$$

$$= \frac{1}{s}\left[4x\sin(sx)\right]_0^1 + \frac{4}{s^2}\left[\cos(sx)\right]_0^1$$

$$= \frac{4}{s}\left[\sin(s) - 0\right] + \frac{4}{s^2}\left[\cos s - \cos 0\right]$$

$$\Rightarrow \int_0^1 4x\cos(sx)\,dx = \frac{4}{s}\sin s + \frac{4}{s^2}\cos s - \frac{4}{s^2}$$

$$\int_1^4 (4-x)\cos(sx)\,dx = (4-x)\int_1^4 \cos(sx)\,dx - \int_1^4 \left[(-1)\int \cos(sx)\,dx\right]dx$$

$$= \frac{1}{s}\left[(4-x)\sin(sx)\right]_1^4 - \frac{1}{s^2}\left[\cos(sx)\right]_1^4$$

$$= \frac{1}{s}\left[0 - 3\sin s\right] - \frac{1}{s^2}\left[\cos 4s - \cos s\right]$$

$$\Rightarrow \int_1^4 (4-x)\cos(sx)\,dx = \frac{1}{s^2}\cos s - \frac{1}{s^2}\cos 4s - \frac{1}{s}\sin s$$

$$\therefore (1) \Rightarrow f(s) = \frac{4}{s}\sin s + \frac{4}{s^2}\cos s - \frac{4}{s^2} + \frac{1}{s^2}\cos s - \frac{1}{s^2}\cos 4s - \frac{3}{s}\sin s$$

$$\Rightarrow f_c(s) = \frac{1}{s}\sin s + \frac{5}{s^2}\cos s - \frac{1}{s^2}\cos 4s - \frac{4}{s^2}.$$

(9) Find the fourier sin transform of $e^{-|x|}$. Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$ for $m>0$.

$\therefore$ Given $F(x) = e^{-|x|}$

WKT $|x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$

$\therefore F(x) = \begin{cases} e^{-x}, & \text{for } x>0 \\ e^{x}, & \text{for } x<0 \end{cases}$

$\therefore$ WKT the fourier sin transform of $F(x)$ is

$$F_s[F(x)] = f_s(s) = \int_0^\infty F(x)\sin(sx)\,dx$$

$$= \int_0^\infty e^{-x}\sin(sx)\,dx$$

$$\Rightarrow F_s[F(x)] = \left\{\frac{e^{-x}}{(-1)^2+s^2}\left[-1\sin(sx) - s\cos(sx)\right]\right\}_{x=0}^{\infty}$$

$$= \frac{-1}{1+s^2}\left[e^{-x}\left[\sin(sx) + s\cos(sx)\right]\right]_{x=0}^{\infty}$$

$$= \frac{-1}{1+s^2}\left\{0 - 1\left[0 + s(1)\right]\right\}$$

$$= \frac{-1}{1+s^2}(-s)$$

$$\Rightarrow f_s(s) = \frac{s}{s^2+1}$$

By the inverse fourier sin transform WKT

$$\frac{2}{\pi}\int_0^\infty f_s(s)\sin(sx)\,ds = F(x)$$

$$\Rightarrow \int_0^\infty \frac{s}{s^2+1}\sin(sx)\,ds = \frac{\pi}{2}F(x)$$

$$\Rightarrow \int_0^\infty \frac{s\sin(sx)}{s^2+1}\,ds = \frac{\pi}{2}e^{-|x|} \rightarrow ①$$

$$① \Rightarrow \int_0^\infty \frac{s\sin(ms)}{s^2+1}\,ds = \frac{\pi}{2}e^{-m}$$

$$\Rightarrow \int_0^\infty \frac{x\sin(mx)}{x^2+1}\,dx = \frac{\pi}{2}e^{-m} \text{ for } s=x.$$

⑩ Find the fourier sin and cosine transform of $F(x) = \begin{cases} x, & 0<x<2 \\ 0, & x>2 \end{cases}$

$$F(x) = \begin{cases} x, & 0<x<2 \\ 0, & x>2 \end{cases}$$

$\therefore$ WKT the fourier sin transform of $F(x)$ is

$$F_s[F(x)] = f_s(s) = \int_0^\infty s F(x)\sin(sx)\,dx$$

$$= \int_0^2 x\sin(sx)\,dx + \int_2^\infty 0\sin(sx)\,dx$$

$$\Rightarrow F_s[F(x)] = \int_0^2 x\sin(sx)\,dx$$

$$= x\int_0^2 \sin(sx)\,dx - \int_0^2 [1.\int \sin(sx)\,dx]\,dx$$

$$\Rightarrow F_s[F(x)] = \int_0^2 x\sin(sx)\,dx$$

$$= x\int_0^2 \sin(sx)\,dx - \int_0^2 [1.\int \sin(sx)\,dx]\,dx$$

$$= \frac{1}{s}[x\cos(sx)]_0^2 + \frac{1}{s^2}[\sin(sx)]_0^2$$

$$= \frac{1}{s}[x\cos(sx)]_0^2 + \frac{1}{s^2}[\sin(sx)]_0^2$$

$$= \frac{1}{s}[2\cos(sx) - 0] + \frac{1}{s^2}[\sin 2s - 0]$$

$$\Rightarrow F_S[F(x)] = f_S(s) = -\frac{2\cos(sx)}{s} + \frac{1}{s^2}\sin 2s$$

$\therefore$ The fourier cosine transform of $F(x)$ is

$$F_C[F(x)] = f_c(s) = \int_0^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^{2} x\cos(sx)\,dx + \int_2^\infty 0\,dx$$

$$\Rightarrow F_C[F(x)] = \int_0^2 x\cos(sx)\,dx$$

$$= x\int_0^2 \cos(sx)\,dx - \int_0^2 [1.\int\cos(sx)\,dx]\,dx$$

$$= \frac{1}{s}[x\sin(sx)]_0^2 + \frac{1}{s^2}[\cos(sx)]_0^2$$

$$= \frac{1}{s}[2\sin(2s) - 0] + \frac{1}{s^2}[\cos 2s - \cos(0)]$$

$$\Rightarrow F_C[F(x)] : f_c(s) = \left[\frac{2\sin(2s)}{2} + \frac{1}{s^2}\cos 2s - \frac{1}{s^2}\right]$$

Z-transforms And Difference Equations :-

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^4 + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

Definition :-

Suppose $f(n)$ be a function in the variable n, such that the z-transform of $f(n)$ can be defined as

$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z)$, where z is called the z-transform operator.

Some important results:-

(1) $f(n) = 1$

$$\therefore Z[1] = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$= \frac{1}{z^0} + \frac{1}{z^1} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

$$= 1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \left(\frac{z-1}{z}\right)^{-1}$$

$$\Rightarrow \boxed{Z[1] = \frac{z}{z-1}}$$

(2) $f(n) = a^n$

$$\therefore Z[a^n] = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \left(\frac{a}{z}\right)^0 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + \dots$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \dots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

$$= \left[\frac{z-a}{z}\right]^{-1}$$

$$\Rightarrow \boxed{Z[a^n] = \frac{z}{z-a}}$$

for $a = -1$

$$\therefore Z[(-1)^n] = \frac{z}{z+1}$$

③ $f(n) = n$

$$\therefore Z[n] = \sum_{n=0}^{\infty} n \cdot z^{-n}$$

$$= 0 \cdot z^0 + 1 \cdot z^{-1} + 2 \cdot z^{-2} + 3 \cdot z^{-3} + 4 \cdot z^{-4} + \dots$$

$$= 1\left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + 4\left(\frac{1}{z}\right)^4 + \dots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + 4\left(\frac{1}{z}\right)^3 + \dots\right]$$

$$= \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2}$$

$$= \frac{1}{z}\left[\frac{z-1}{z}\right]^{-2}$$

$$= \frac{1}{z} \frac{z^2}{(z-1)^2}$$

$$\Rightarrow \boxed{Z[n] = \frac{z}{(z-1)^2}}$$

④ $f(n) = n^2$

$$\therefore \text{WKT } Z[n^p] = -z \frac{d}{dz} Z[n^{p-1}] \rightarrow ①$$

$$\forall\ p = 2, 3, 4 \dots$$

let $p = 2$

$$\therefore ① \Rightarrow Z[n^2] = -z \frac{d}{dz} Z[n]$$

$$= -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z\,2(z-1)(1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z-1}{(z-1)^3}\right]$$

$$= \frac{z(z+1)}{(z-1)^3}$$

$$\Rightarrow \boxed{Z(n^2) = \frac{z^2+z}{(z-1)^3}}$$

(5) Z-transform of $\sin n\theta$ and $\cos n\theta$

Let $f(n) = e^{in\theta} = \cos n\theta + i\sin n\theta$

$$\therefore Z[e^{in\theta}] = Z[(e^{i\theta})^n]$$

$$= \frac{z}{(z-e^{i\theta})}$$

$$= \frac{z}{z-e^{i\theta}} \times \frac{z-e^{-i\theta}}{z-e^{-i\theta}}$$

$$= \frac{z[z-e^{-i\theta}]}{z^2 - ze^{-i\theta} - ze^{i\theta} + 1}$$

$$\Rightarrow Z[\cos n\theta + i\sin n\theta] = \frac{z^2 - z[\cos\theta - i\sin\theta]}{z^2 - z(2\cos\theta) + 1}$$

$$\Rightarrow Z[\cos n\theta] + iZ[\sin n\theta] = \frac{(z^2 - z\cos\theta) + i(z\sin\theta)}{z^2 - 2z\cos\theta + 1}$$

$$\Rightarrow Z[\cos n\theta] + iZ[\sin n\theta] = \left[\frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}\right] + i\left[\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}\right]$$

$$\therefore Z[\cos n\theta] = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1} \qquad \therefore Z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

(6) WKT $\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$

$$\Rightarrow \cosh(n\theta) = \frac{e^{n\theta} + e^{-n\theta}}{2}$$

$$\Rightarrow z[\cosh(n\theta)] = \frac{1}{2}\, z\left[e^{n\theta} + e^{-n\theta}\right]$$

$$= \frac{1}{2}\left\{ z\left[e^{n\theta}\right] + z\left[e^{-n\theta}\right]\right\}$$

$$= \frac{1}{2}\left\{ z\left[e^{\theta}\right]^{n} + z\left[e^{-\theta}\right]^{n}\right\}$$

$$= \frac{1}{2}\left\{ \frac{z}{z-e^{\theta}} + \frac{z}{z-e^{-\theta}} \right\}$$

$$= \frac{z}{2}\left[\frac{1}{z-e^{\theta}} + \frac{1}{z-e^{-\theta}} \right]$$

$$= \frac{z}{2}\left[\frac{2z-(e^{\theta}+e^{-\theta})}{z^{2}-z(e^{\theta}+e^{-\theta})+1} \right]$$

$$\Rightarrow z[\cosh n\theta] = \frac{z^{2}-z\cosh\theta}{z^{2}-2z\cosh\theta+1}$$

③ WKT $\sinh\theta = \dfrac{e^{\theta}-e^{-\theta}}{2} \Rightarrow \sinh(n\theta) = \dfrac{e^{n\theta}-e^{-n\theta}}{2}$

$$\Rightarrow z[\sinh(n\theta)] = \frac{1}{2}\, z\left[e^{n\theta} - e^{-n\theta}\right]$$

$$= \frac{1}{2}\left\{ z\left[e^{n\theta}\right] - z\left[e^{-n\theta}\right]\right\}$$

$$= \frac{1}{2}\left\{ z\left[e^{\theta}\right]^{n} - z\left[e^{-\theta}\right]^{n}\right\}$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} - \frac{z}{z-e^{-\theta}} \right]$$

$$= \frac{z}{2}\left[\frac{1}{z-e^{\theta}} - \frac{z}{z-e^{-\theta}} \right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-\theta}-z+e^{\theta}}{(z-e^{\theta})(z-e^{-\theta})} \right]$$

$$= \frac{z}{2}\left[\frac{e^{\theta}-e^{-\theta}}{z^{2}-2z\cosh\theta+1} \right]$$

$$= \frac{z}{2}\left[\frac{2\sinh\theta}{z^{2}-2z\cosh\theta+1} \right]$$

$$\Rightarrow z[\sinh n\theta] = \frac{z\sinh\theta}{z^{2}-2z\cosh\theta+1}$$

8. if $Z[f(n)] = F(z)$, then (i) $Z[a^n f(n)] = F(z/a)$

(ii) $Z[a^{-n} f(n)] = F(az)$ it is also called the damping room.

1. Find the z-transform of $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$

Let $f(n) = \cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$

$$= \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow f(n) = \frac{1}{\sqrt{2}}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{\sqrt{2}}\sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow Z[f(n)] = \frac{1}{\sqrt{2}} Z\left[\cos\left(\frac{n\pi}{2}\right)\right] - \frac{1}{\sqrt{2}} Z\left[\sin\left(\frac{n\pi}{2}\right)\right]$$

WKT $Z[\cos n\theta] = \dfrac{z^2 - z\cos\theta}{z^2-2z\cos\theta+1}$

$$\therefore Z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2 - z\cos(\pi/2)}{z^2-2z\cos\left(\frac{\pi}{2}\right)+1}$$

$$\therefore Z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2-0}{z^2-2z(0)+1}$$

$$\Rightarrow Z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2+1}$$

$Z[\sin(n\theta)] = \dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$

$$\Rightarrow Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z\sin(\pi/2)}{z^2-2z\cos\left(\frac{\pi}{2}\right)+1}$$

$$= \frac{z}{z^2-0+1}$$

$$\Rightarrow Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z}{z^2+1}$$

$\therefore$ ① $\Rightarrow F(z) = \frac{1}{\sqrt{2}} \frac{z^2}{z^2+1} - \frac{1}{\sqrt{2}} \frac{z}{z^2+1}$

$\Rightarrow \boxed{F(z) = \frac{z^2-z}{\sqrt{2}\,(z^2+1)}}$

② Find the z-transform of $2n + \sin\left(\frac{n\pi}{4}\right) + 1$.

Let $f(n) = 2n + \sin\left(\frac{n\pi}{4}\right) + 1$

$\Rightarrow z[f(n)] = 2z[n] + z\left[\sin\left(\frac{n\pi}{4}\right)\right] + z[1] \rightarrow$ ①

WKT $z[n] = \frac{z}{(z-1)^2}$

$\Rightarrow z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

$\Rightarrow z\left[\sin\left(\frac{n\pi}{4}\right)\right] = \frac{z\sin(\pi/4)}{z^2 - 2z\cos(\pi/4) + 1}$

$= \frac{z(1/\sqrt{2})}{z^2 - 2z(1/\sqrt{2}) + 1}$

$\Rightarrow z[\sin(n\pi/4)] = \frac{z/\cancel{\sqrt{2}}}{\frac{\sqrt{2}z^2 - 2z + \sqrt{2}}{\cancel{\sqrt{2}}}}$

$\Rightarrow z\left[\sin\left(\frac{n\pi}{4}\right)\right] = \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}}$

$z[1] = \frac{z}{z-1}$

$\therefore$ ① $\Rightarrow F(z) = \frac{2z}{(z-1)^2} + \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}} + \frac{z}{z-1}$

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③ Find the z-transform (i) $\cos\left(\frac{n\pi}{2}+\theta\right)$

(ii) $3^n\left(\cos\left(\frac{n\pi}{4}\right)\right)$

Let $f(n) = \cos\left(\frac{n\pi}{2}+\theta\right)$

$\Rightarrow f(n) = \cos\left(\frac{n\pi}{2}\right)\cos\theta - \sin\left(\frac{n\pi}{2}\right)\sin\theta$

$\Rightarrow z[f(n)] = \cos\theta\, z\left[\cos\left(\frac{n\pi}{2}\right)\right] - \sin\theta\, z\left[\sin\left(\frac{n\pi}{2}\right)\right]$

WKT

$$z[\cos n\theta] = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

$$\Rightarrow z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2 - z\cos(\pi/2)}{z^2 - 2z\cos(\pi/2) + 1}$$

$$\Rightarrow z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2+1}$$

$$\Rightarrow z[\sin(n\theta)] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\Rightarrow z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z\sin(\pi/2)}{z^2 - 2z\cos(\pi/2) + 1}$$

$$= \frac{z}{z^2+1}$$

$\therefore$ ① $\Rightarrow$ $$F(z) = \frac{z^2\cos\theta}{z^2+1} - \frac{z\sin\theta}{z^2+1}$$

$$\Rightarrow F(z) = \frac{z^2\cos\theta - z\sin\theta}{z^2+1}$$

Let $f(n) = \cos\left(\frac{n\pi}{4}\right)$

$\Rightarrow z[f(n)] = z\left[\cos\left(\frac{n\pi}{4}\right)\right]$

$$\Rightarrow F(z) = \frac{z^2 - z\cos(\pi/4)}{z^2 - 2z\cos(\frac{\pi}{4}) + 1}$$

$$\Rightarrow F(z) = \frac{z^2 - z(\frac{1}{\sqrt{2}})}{z^2 - 2z(\frac{1}{\sqrt{2}}) + 1}$$

$$\Rightarrow F(z) = \frac{\sqrt{2}z^2 - z}{\sqrt{2}z^2 - 2z + \sqrt{2}}$$

WKT $z[a^n f(n)] = F(\frac{z}{a})$

$$\Rightarrow z[3^n f(n)] = F(\frac{z}{3})$$

$$\Rightarrow z[3^n \cos(\frac{n\pi}{4})] = \frac{\sqrt{2}(\frac{z}{3})^2 - (\frac{z}{3})}{\sqrt{2}(\frac{z}{3})^2 - 2(\frac{z}{3}) + \sqrt{2}}$$

$$= \frac{\frac{\sqrt{2}z^2}{9} - \frac{z}{3}}{\frac{\sqrt{2}z^2}{9} - \frac{2z}{3} + \sqrt{2}}$$

$$= \frac{\sqrt{2}z^2 - 3z}{\sqrt{2}z^2 - 6z + 9\sqrt{2}}$$

(5) Find the z-transform of

(i) $a^n \sin n\theta$

(ii) $a^{-n} \cos n\theta$

(iii) $\sin(3n + \pi)$

(i) Let $f(n) = \sin n\theta$

$$\Rightarrow z[f(n)] = z[\sin n\theta]$$

$$\Rightarrow F(z) = \frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$$

WKT $Z[a^n f(n)] = F\left(\frac{z}{a}\right)$

$$\Rightarrow Z[a^n \sin(n\theta)] = \frac{\frac{z}{a}\sin\theta}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\theta + 1}$$

$$= \frac{\frac{z}{a}\sin\theta}{\frac{z^2 - 2az\cos\theta + a^2}{a^2}}$$

$$= \frac{az\sin n\theta}{z^2 - 2az\cos\theta + a^2}$$

(ii) Let $f(n) = \cos n\theta$

$$\Rightarrow Z[f(n)] = Z[\cos n\theta]$$

$$\Rightarrow F(z) = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

WKT $Z[a^{-n} f(n)] = F(az)$

$$\Rightarrow Z[a^{-n}\cos(n\theta)] = \frac{a^2z^2 - az\cos\theta}{a^2z^2 - 2az\cos\theta + 1}$$

(iii) Let $f(n) = \sin(3n + 5)$

$$\Rightarrow f(n) = \sin(3n)\cdot\cos 5 + \cos(3n)\cdot\sin 5$$

$$\Rightarrow Z[f(n)] = \cos 5\cdot Z[\sin(3n)] + \sin 5\cdot Z[\cos(3n)]$$

$$\Rightarrow F(z) = \cos 5\cdot\left[\frac{z\sin 3}{z^2 - 2z\cos 3 + 1}\right] + \sin 5\left[\frac{z^2 - z\cos 3}{z^2 - 2z\cos 3 + 1}\right]$$

Inverse Z - transforms :-

Suppose $F(z)$ be a Z-transform of $f(x)$, then the inverse Z-transform of $F(z)$ can be defined as

$$Z^{-1}[F(z)] = f(n).$$

Standard Results :-

1) $Z^{-1}\left[F\left[\frac{z}{a}\right]\right] = a^n f(n)$

2) $Z^{-1}[F(az)] = a^{-n} f(n)$

3) $Z^{-1}\left[\frac{z}{z-1}\right] = 1$

4) $Z^{-1}\left[\frac{z}{z-a}\right] = a^n$

5) $Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$

6) $Z^{-1}\left[\frac{z^2+z}{(z-1)^3}\right] = n^2$

7) $Z^{-1}\left[\frac{az}{(z-a)^2}\right] = a^n . n$

8) $Z^{-1}\left[\frac{z}{z+1}\right] = (-1)^n$

9) $Z^{-1}\left[\frac{z^2}{z^2+1}\right] = \cos\left(\frac{n\pi}{2}\right)$

10) $Z^{-1}\left[\frac{z}{z^2+1}\right] = \sin\left(\frac{n\pi}{2}\right)$

① Find the inverse Z-transform of the following :-

1. $\frac{z}{(z-2)(z-3)}$ 2. $\frac{z}{z^2+7z+10}$ 3. $\frac{3z^2+2z}{(5z-1)(5z+2)}$

4. $\frac{8z^2}{(2z-1)(4z-1)}$ 5. $\frac{2z^2+3z}{(z+2)(z-4)}$ 6. $\frac{z^3-20z}{(z-2)^3(z-4)}$ 7. $\frac{18z^2}{(2z-1)(4z+1)}$

① $\frac{z}{(z-2)(z-3)}$

Let $F(z) = \frac{z}{(z-2)(z-3)}$

$\Rightarrow \frac{F(z)}{z} = \frac{1}{(z-2)(z-3)}$

$\Rightarrow \frac{F(z)}{z} = \frac{1}{z-3} - \frac{1}{z-2}$

$\Rightarrow F(z) = \frac{z}{z-3} - \frac{z}{z-2}$

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$$\Rightarrow z^{-1}[F(z)] = z^{-1}\left[\frac{z}{z-3}\right] - z^{-1}\left[\frac{z}{z-2}\right]$$

$$\Rightarrow f(n) = 3^n - 2^n$$

(2) $\dfrac{z}{z^2+7z+10}$

$$F(z) = \frac{z}{z^2+7z+10}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{1}{z^2+7z+10}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{1}{z^2+2z+5z+10}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{1}{(z+2)(z+5)}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{1}{3}\,\frac{1}{z+2} - \frac{1}{3}\,\frac{1}{z+5}$$

$$\Rightarrow z^{-1}[F(z)] = \frac{1}{3}z^{-1}\left[\frac{z}{z-(-2)}\right] - \frac{1}{3}z^{-1}\left[\frac{z}{z-(-5)}\right]$$

$$\Rightarrow f(n) = \frac{1}{3}(-2)^n - \frac{1}{3}(-5)^n$$

$$\Rightarrow f(n) = \frac{1}{3}\left[(-2)^n - (-5)^n\right]$$

(3) $\dfrac{3z^2+2z}{(5z-1)(5z+2)}$

Let $F(z) = \dfrac{3z^2+2z}{(5z-1)(5z+2)}$

$$\Rightarrow \frac{F(z)}{z} = \frac{3z+2}{(5z-1)(5z+2)} \rightarrow (1)$$

$$\frac{3z+2}{(5z-1)(5z+2)} = \frac{A}{5z-1} + \frac{B}{5z+2}$$

$$\Rightarrow 3z+2 = A(5z+2) + B(5z-1) \rightarrow (2)$$

when $z = \frac{1}{5}$

$③ \Rightarrow \frac{3}{5} + 2 = A(3)$

$\Rightarrow \frac{13}{5} = 3A$

$\Rightarrow A = \frac{13}{15}$

when $z = \frac{-2}{5}$

$\therefore ③ \Rightarrow 3\left(\frac{-2}{5}\right) + 2 = B[-2-1]$

$\Rightarrow \frac{-6}{5} + 2 = -3B$

$\Rightarrow \frac{-6+10}{5} = -3B$

$\Rightarrow B = \frac{-4}{15}$

$\therefore ① \Rightarrow \frac{F(z)}{z} = \frac{13}{5} \cdot \frac{1}{(5z-1)} - \frac{4}{15} \cdot \frac{1}{(5z+2)}$

$\Rightarrow F(z) = \frac{13}{15 \times 5} \cdot \frac{z}{(z - \frac{1}{5})} - \frac{4}{15 \times 5} \frac{z}{(z + \frac{2}{5})}$

$\Rightarrow F(z) = \frac{13}{75}\left[\frac{z}{z - 1/5}\right] - \frac{4}{75}\left[\frac{z}{z - (\frac{-2}{5})}\right]$

$\Rightarrow z^{-1}[F(z)] = \frac{13}{75} z^{-1}\left[\frac{z}{z - 1/5}\right] - \frac{4}{75} z^{-1}\left[\frac{z}{z - (\frac{-2}{5})}\right]$

$\Rightarrow f(n) = \frac{13}{75}\left(\frac{1}{5}\right)^n - \frac{4}{75}\left(\frac{-2}{5}\right)^n$

④ $\frac{18z^2}{(2z-1)(4z+1)}$

Let $F(z) = \frac{18z^2}{(2z-1)(4z+1)}$

$\Rightarrow \frac{F(z)}{z} = \frac{18z}{(2z-1)(4z+1)} \rightarrow ①$

$\frac{18z}{(2z-1)(4z+1)} = \frac{A}{(2z-1)} + \frac{B}{(4z+1)}$

$\Rightarrow 18z = A(4z+1) + B(2z-1) \rightarrow ②$

when $z = 1/2$

$② \Rightarrow 9 = 3A \Rightarrow A = 3$

when $z = -\frac{1}{4}$

② $\Rightarrow 18\left(-\frac{1}{4}\right) = B\left[2\left(-\frac{1}{2}\right)-1\right]$

$\Rightarrow -\frac{9}{2} = -\frac{3}{2}B$

$\Rightarrow B = 3$

$\therefore$ ① $\Rightarrow \frac{F(z)}{z} = \frac{3}{(2z-1)} + \frac{3}{(4z+1)}$

$\Rightarrow F(z) = \frac{3z}{(2z-1)} + \frac{3z}{(4z+1)}$

$\Rightarrow F(z) = \frac{3}{2}\left[\frac{z}{z-1/2}\right] + \frac{3}{4}\left[\frac{z}{z+1/4}\right]$

$\therefore z^{-1}[F(z)] = \frac{3}{2} z^{-1}\left[\frac{z}{z-1/2}\right] + \frac{3}{4} z^{-1}\left[\frac{z}{z-(-1/4)}\right]$

$\Rightarrow f(n) = \frac{3}{2}\left(\frac{1}{2}\right)^n + \frac{3}{4}\left(-\frac{1}{4}\right)^n$

⑤ $\frac{2z^2+3z}{(z+2)(z-4)}$

$\Rightarrow F(z) = \frac{2z^2+3z}{(z+2)(z-4)}$

$\Rightarrow \frac{F(z)}{z} = \frac{2z+3}{(z+2)(z-4)} \rightarrow$ ①

$\frac{2z+3}{(z+2)(z-4)} = \frac{A}{(z+2)} + \frac{B}{(z-4)}$

$2z+3 = A(z-4) + B(z+2) \rightarrow$ ②

when $z = 4$ . . . when $z = -2$

② $\Rightarrow 11 = B(6)$ $\qquad -1 = A(-6)$

$B = \frac{11}{6}$ $\qquad A = 1/6$

$\therefore$ ① $\Rightarrow \frac{F(z)}{z} = \frac{1}{6}\frac{1}{(z+2)} + \frac{11}{6}\frac{1}{(z-4)}$

$$\Rightarrow F(z) = \frac{1}{6}\frac{z}{(z+2)} + \frac{11}{6}\frac{z}{(z-4)}$$

$$\Rightarrow z^{-1}[F(z)] = \frac{1}{6} z^{-1}\left[\frac{z}{z+2}\right] + \frac{11}{6} z^{-1}\left[\frac{z}{z-4}\right]$$

$$\Rightarrow f(n) = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n$$

6) $\dfrac{8z^2}{(2z-1)(4z-1)}$

$$\Rightarrow F(z) = \frac{8z^2}{(2z-1)(4z-1)}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{8z}{(2z-1)(4z-1)} \rightarrow ①$$

$$\frac{8z}{(2z-1)(4z-1)} = \frac{A}{(2z-1)} + \frac{B}{(4z-1)}$$

$$8z = A(4z-1) + B(2z-1) \rightarrow ②$$

when $z = \frac{1}{4}$

$$② \Rightarrow 2 = B\left(2\left(\tfrac{1}{4}\right)-1\right)$$

$$\Rightarrow 2 = B\left(\tfrac{1}{2}-1\right)$$

$$\Rightarrow 2 = B\left(\tfrac{1-2}{2}\right)$$

$$\Rightarrow B = -4$$

$z = \frac{1}{2}$

$$② \Rightarrow 4 = A(2-1)$$

$$\Rightarrow 4 = A$$

$$\Rightarrow A = 4$$

$$\therefore ① \Rightarrow \frac{F(z)}{z} = \frac{4}{(2z-1)} - \frac{4}{(4z-1)}$$

$$\Rightarrow F(z) = 4\frac{z}{(2z-1)} - 4\frac{z}{(4z-1)}$$

$$\Rightarrow z^{-1}[F(z)] = 4z^{-1}\left[\frac{z}{2z-1}\right] - 4z^{-1}\left[\frac{z}{4z-1}\right]$$

$$= \frac{4}{2} z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] - \frac{4}{4} z^{-1}\left[\frac{z}{z-1/4}\right]$$

$$\Rightarrow f(n) = 2\left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{4}\right)^n$$

Difference Equations:

Step ①: Express the given difference equation in the notation of $y_n, y_{n+1}, y_{n+2}, ---$

Step ②: Apply z-transform on both sides and substitute

$$z[y_{n+2}] = z^2[\bar{y}(z) - y_0 - \frac{y_1}{z}]$$

$$z[y_{n+1}] = z[\bar{y}(z) - y_0]$$

$$z[y_n] = \bar{y}(z)$$

Step ③: write $\bar{y}(z)$ has a function of z, hence apply the inverse z-transform and find $y(n)$.

① Solve the difference equation using z-transform $y_{n+2} - 4y_n = 0$, subject to the conditions $y_0 = 0$, $y_1 = 2$.

Given $y_{n+2} - 4y_n = 0 \quad y_0 = 0 \quad y_1 = 2$

$$\Rightarrow z[y_{n+2}] - 4z[y_n] = z[0]$$

$$\Rightarrow [z^2\bar{y}(z) - y_0 - \frac{y_1}{z}] - 4\bar{z}z = 0$$

$$\Rightarrow [z^2\bar{y}(z) - 0 - \frac{2}{z}] - 4\bar{y}(z) = 0$$

$$\Rightarrow z^2\bar{y}(z) - 2z - 4\bar{y}(z) = 0$$

$$\Rightarrow (z^2 - 4)\bar{y}(z) = 2z$$

$$\Rightarrow \bar{y}(z) = \frac{2z}{(z^2-4)}$$

$$\Rightarrow \bar{y}(z) = \frac{2z}{(z+2)(z-2)}$$

$$\Rightarrow \frac{\bar{y}(z)}{z} = \frac{2}{(z-2)(z+2)} \rightarrow ①$$

$$\frac{z}{(z-2)(z+2)} = \frac{A}{(z-2)} + \frac{B}{(z+2)}$$

$$\Rightarrow 2 = A(z+2) + B(z-2) \rightarrow ②$$

when $z=2$ when $z=-2$

② $\Rightarrow 2=4A$ ② $\Rightarrow 2=-4B$

$\Rightarrow A=\frac{1}{2}$ $\Rightarrow B=-\frac{1}{2}$

$\therefore$ ① $\Rightarrow \frac{\bar{y}(z)}{z} = \frac{1}{2}\cdot\frac{1}{z-2} - \frac{1}{2}\cdot\frac{1}{z+2}$

$\Rightarrow \bar{y}(z) = \frac{1}{2}\cdot\frac{z}{z-2} - \frac{1}{2}\cdot\frac{z}{z+2}$

$\Rightarrow z^{-1}[\bar{y}(z)] = \frac{1}{2}\, z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{2}\, z^{-1}\left[\frac{z}{z+2}\right]$

$\Rightarrow y(n) = \frac{1}{2}(2)^n - \frac{1}{2}(-2)^n$

② Using z-transform solve the difference equation $y_{n+2}+6y_{n+1}+9y_n = 2^n$, subject to the conditions $y_0=0, y_1=0$.

Sol: Given $y_{n+2}+6y_{n+1}+9y_n = 2^n$ $y_0=0$, $y_1=0$

$\Rightarrow z[y_{n+2}] + 6z[y_{n+1}] + 9z[y_n] = z[2^n]$

$\Rightarrow z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right] + 6z[\bar{y}(z) - y_0] + 9\bar{y}(z) = \frac{z}{z-2}$

$\Rightarrow z^2\bar{y}(z) + 6z\,\bar{y}(z) + 9\bar{y}(z) = \frac{z}{z-2}$

$\Rightarrow (z^2+6z+9)\bar{y}(z) = \frac{z}{z-2}$

$\Rightarrow \bar{y}(z) = \frac{z}{(z-2)(z+3)^2}$

$\Rightarrow \frac{\bar{y}(z)}{z} = \frac{1}{(z-2)(z+3)^2} \rightarrow$ ①

$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$

$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2) \rightarrow$ ②

when $z=2$

② $\Rightarrow 1 = A(2+3)^2$

$\Rightarrow 1 = 25A \Rightarrow A = 1/25$

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when $z=-3$ $\qquad$ $A+B=0$

② $\Rightarrow 1 = C(-5)$ $\qquad$ $\Rightarrow B=-A$

$\Rightarrow C = -\frac{1}{5}$ $\qquad$ $\Rightarrow B = -\frac{1}{25}$

$\therefore$ ① $\Rightarrow \frac{\bar{y}(z)}{z} = \frac{1}{25}\cdot\frac{1}{z-2} - \frac{1}{25}\frac{1}{z+3} - \frac{1}{5}\cdot\frac{1}{(z+3)^2}$

$\Rightarrow \bar{y}(z) = \frac{1}{25}\cdot\frac{z}{z-2} - \frac{1}{25}\cdot\frac{z}{z+3} - \frac{1}{5}\cdot\frac{z}{(z+3)^2}$

$\Rightarrow \bar{y}(z) = \frac{1}{25}\cdot\frac{z}{z-2} - \frac{1}{25}\cdot\frac{z}{z-(-3)} + \frac{1}{5}\cdot\frac{(-3z)}{(z-(-3))^2}$

$\Rightarrow z^{-1}[\bar{y}(z)] = \frac{1}{25}\cdot z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} z^{-1}\left[\frac{z}{z-(-3)}\right] + \frac{1}{15}\cdot\frac{(-3z)}{(z-(-3))^2}$

$\Rightarrow y(n) = \frac{1}{25}\cdot(2^n) - \frac{1}{25}(-3)^n + \frac{1}{15}(-3)^n . n.$

③ Solve $u_{n+2} - 3u_{n+1} + 2u_n = 2^n$. Given $u_0=0$, $u_1=1$ by using z-transform.

Given $u_{n+2} - 3u_{n+1} + 2u_n = 2^n$. $u_0=0$ $u_1=1$

$\Rightarrow z[u_{n+2}] - 3z[u_{n+1}] + 2z[u_n] = z[2^n]$

$\Rightarrow z^2[\bar{u}(z) - u_0 - \frac{u_1}{z}] - 3z[\bar{u}(z) - u_0] + 2\bar{u}(z) = \frac{z}{z-2}$

$\Rightarrow z^2\bar{u}(z) - z - 3z\bar{u}(z) + 2\bar{u}(z) = \frac{z}{z-2}$

$\Rightarrow (z^2-3z+2)\bar{u}(z) = \frac{z}{z-2} + z$

$\Rightarrow (z-1)(z-2)\bar{u}(z) = \frac{z + z(z-2)}{z-2}$

$\Rightarrow (z-1)(z-2)\bar{u}(z) = \frac{z^2-z}{z-2}$

$\Rightarrow \bar{u}(z) = \frac{z^2-z}{(z-1)(z-2)^2}$

$$\Rightarrow \bar{u}(z) = \frac{z\cancel{(z-1)}}{\cancel{(z-1)}(z-2)^2}$$

$$\Rightarrow \bar{u}(z) = \frac{z}{(z-2)^2}$$

$$\Rightarrow \bar{u}(z) = \frac{1}{2} \cdot \frac{2z}{(z-2)^2}$$

$$\Rightarrow z^{-1}[\bar{u}(z)] = \frac{1}{2} z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$\Rightarrow u(n) = \frac{1}{2} n 2^n$$

$$\Rightarrow u(n) = 2^{n-1} n.$$

④ Solve the difference equation $u_{n+2} + 2u_{n+1} + u_n = n$, subject to the conditions $u_0 = 0$, $u_1 = 0$.

Given $u_{n+2} + 2u_{n+1} + u_n = n$, $u_0 = 0$, $u_1 = 0$

$$\Rightarrow z[u_{n+2}] + 2z[u_{n+1}] + z[u_n] = z[n]$$

$$\Rightarrow z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] + 2z[\bar{u}(z)] + \bar{u}(z) = \frac{z}{(z-1)^2}$$

$$\Rightarrow z^2\bar{u}(z) + 2z\bar{u}(z) + \bar{u}(z) = \frac{z}{(z-1)^2}$$

$$\Rightarrow (z^2 + 2z + 1)\bar{u}(z) = \frac{z}{(z-1)^2}$$

$$\Rightarrow (z+1)^2 \bar{u}(z) = \frac{z}{(z-1)^2}$$

$$\Rightarrow \bar{u}(z) = \frac{z}{(z-1)^2(z+1)^2}$$

$$\Rightarrow \frac{\bar{u}(z)}{z} = \frac{1}{(z-1)^2(z+1)^2} \rightarrow ①$$

$$\therefore \frac{1}{(z-1)^2(z+1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{(z+1)} + \frac{D}{(z+1)^2}$$

$$\Rightarrow 1 = A(z-1)(z+1)^2 + B(z+1)^2 + C(z+1)(z-1)^2 + D(z-1)^2 \rightarrow ②$$

when $z=1$

② $\Rightarrow 1 = 4B$

$B = \frac{1}{4}$

when $z=-1$

② $\Rightarrow 1 = 4D$

$D = \frac{1}{4}$

$A+B+C+D=1$

$\Rightarrow A + \frac{1}{4} + C + \frac{1}{4} = 1$

$\Rightarrow -A + C = \frac{1}{2} \rightarrow$ ③

when $A+C=0 \rightarrow$ ④

③ + ④ $\Rightarrow 2C = \frac{1}{2}$

$C = \frac{1}{4}$

$A = -\frac{1}{4}$

$\therefore$ ① $\Rightarrow \frac{\bar{u}(z)}{z} = -\frac{1}{4}\frac{1}{z-1} + \frac{1}{4}\frac{1}{(z-1)^2} + \frac{1}{4}\cdot\frac{1}{(z+1)} + \frac{1}{4}\frac{1}{(z+1)^2}$

$\Rightarrow \bar{u}(z) = -\frac{1}{4}\frac{z}{z-1} + \frac{1}{4}\cdot\frac{z}{(z-1)^2} + \frac{1}{4}\cdot\frac{z}{(z+1)} + \frac{1}{4}\cdot\frac{z}{(z+1)^2}$

$\Rightarrow z^{-1}[\bar{u}(z)] = -\frac{1}{4}z^{-1}\left[\frac{z}{z-1}\right] + \frac{1}{4}z^{-1}\left[\frac{z}{(z-1)^2}\right] + \frac{1}{4}z^{-1}\left[\frac{z}{z-(-1)}\right]$
$-\frac{1}{4}\left[\frac{(-1)z}{(z-(-1))^2}\right]$

$\Rightarrow u(n) = -\frac{1}{4} + \frac{1}{4}n + \frac{1}{4}(-1)^n - \frac{1}{4}(-1)^n.n$ //.

⑤ Solve the difference equation $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$, subject to the conditions $u_0 = 0$, $u_1 = 1$.

Given $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$, $u_0 = 0$, $u_1 = 1$

$\Rightarrow z[u_{n+2}] + 4z[u_{n+1}] + 3z[u_n] = z[3^n]$

$\Rightarrow z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] + 4z\left[\bar{u}(z) - u_0\right] + 3\bar{u}(z) = \frac{z}{z-3}$

$\Rightarrow z^2\bar{u}(z) - z + 4z\,\bar{u}(z) + 3\bar{u}(z) = \frac{z}{z-3}$

$\Rightarrow [z^2 + 4z + 3]\bar{u}(z) = \frac{z}{z-3} + z$

$\Rightarrow (z+3)(z+1)\,\bar{u}(z) = \frac{z + z(z-3)}{z-3}$

$\Rightarrow \bar{u}(z) = \frac{z + z^2 - 3z}{(z+1)(z-3)(z+3)}$

$$\Rightarrow \bar{u}(z) = \frac{z^2-2z}{(z+1)(z+3)(z-3)}$$

$$\Rightarrow \frac{\bar{u}(z)}{z} = \frac{z-2}{(z+1)(z+3)(z-3)} \rightarrow ①$$

$$\frac{z-2}{(z+1)(z+3)(z-3)} = \frac{A}{z+1} + \frac{B}{z+3} + \frac{C}{z-3}$$

$$\Rightarrow z-2 = A(z+3)(z-3) + B(z+1)(z-3) + C(z+1)(z+3) \rightarrow ②$$

when $z=-3$ | $z=3$ | $z=-1$

$② \Rightarrow -5 = B(-2)(-6)$ | $② \Rightarrow 1 = C(4)(6)$ | $② \Rightarrow -3 = A(2)(-4)$

$\Rightarrow -5 = 12B$ | $\Rightarrow 1 = 24C$ | $\Rightarrow -3 = -8A$

$\Rightarrow B = \frac{-5}{12}$ | $\Rightarrow C = \frac{1}{24}$ | $\Rightarrow A = \frac{3}{8}$

$$\therefore ① \Rightarrow \frac{\bar{u}(z)}{z} = \frac{3}{8}\frac{1}{z+1} - \frac{5}{12}\frac{1}{z+3} + \frac{1}{24}\cdot\frac{1}{z-3}$$

$$\Rightarrow \bar{u}(z) = \frac{3}{8}\frac{z}{z+1} - \frac{5}{12}\cdot\frac{z}{z+3} + \frac{1}{24}\cdot\frac{z}{z-3}$$

$$\Rightarrow z^{-1}[\bar{u}(z)] = \frac{3}{8}z^{-1}\left[\frac{z}{z+1}\right] - \frac{5}{12}z^{-1}\left[\frac{z}{z+3}\right] + \frac{1}{24}z^{-1}\left[\frac{z}{z-3}\right]$$

$$\Rightarrow u(n) = \frac{3}{8}(-1)^n - \frac{5}{12}(-3)^n + \frac{1}{24}(3)^n$$

Module-4

Numerical Solution For First Order And First Degree Differential Equation

Taylor's Series Method:-

Step ①:- Write the given differential equation has

$$\frac{dy}{dx} = y' = f(x,y) \text{ to the initial condition } y(x_0) = y_0.$$

Step ②:- Find $y'(x_0), y''(x_0), y'''(x_0), \ldots$

Step ③:- Write the Taylor series expansion has

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \ldots$$

and simplify.

① Employ the taylor series method to find y at $x = 0.1$ correct to 4-decimal places, given $\frac{dy}{dx} = 2y + 3e^x$, $y_{(0)} = 0$.

Given $\frac{dy}{dx} = y' = 2y + 3e^x$, $y_{(0)} = 0$

$\Rightarrow x_0 = 0$, $y_0 = 0$

$y'(x_0) = 2y_0 + 3e^{x_0} = 2(0) + 3e^0 = 3$

$y''(x) = 2y' + 3e^x$

$\Rightarrow y''(x_0) = 2y'_{(x_0)} + 3e^{x_0}$

$= 2(3) + 3e^0$

$= 9$

$y'''(x) = 2y'' + 3e^x$

$\Rightarrow y'''(x_0) = 2y''_{(x_0)} + 3e^{x_0}$

$= 2(9) + 3$

$= 21$

$y^{iv}(x) = 2y''' + 3e^{x}$

$\Rightarrow y^{iv}(x_0) = 2y'''(x_0) + 3e^{x_0}$

$= 2(21) + 3$

$= 45$

∴ WKT

$\Rightarrow y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \dots$

$\Rightarrow y(x) = 0 + \frac{x}{1!}(3) + \frac{x^2}{2!}(9) + \frac{x^3}{3!}(21) + \frac{x^4}{4!}(45) + \dots$

$\Rightarrow y(x) = 3 \cdot x + \frac{9}{2} x^2 + \frac{7}{2} x^3 + \frac{45}{4!} x^4 + \dots$

$\Rightarrow y(0.1) = 3(0.1) + \frac{9}{2}(0.1)^2 + \frac{7}{2}(0.1)^3 + \frac{45}{4!}(0.1)^4 + \dots$

$\Rightarrow y(0.1) \cong 0.3487$

② If $y' + y + 2x = 0$, $y(0) = -1$, then find $y(0.1)$ using Taylor's series method.

Given $y' + y + 2x = 0$

$\Rightarrow y' = -(y + 2x)$, $y(0) = -1$

$\Rightarrow x_0 = 0 \quad y_0 = -1$

$y'(x_0) = -(y_0 + 2x_0) = -(-1 + 0) = 1$

$y''(x) = -(y' + 2)$

$\Rightarrow y''(x_0) = -(y'(x_0) + 2)$

$= -(1 + 2)$

$= -3$

$y'''(x) = -(y'')$

$\Rightarrow y'''(x_0) = -(y''(x_0))$

$= -3$

$y^{iv}(x) = -(y''')$

$\Rightarrow y^{iv}(x_0) = -(y'''(x_0))$

$= 3$

$\therefore$ WKT

$$\Rightarrow y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \dots$$

$$\Rightarrow y(x) = -1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-3) + \frac{x^3}{3!}(3) + \frac{x^4}{4!}(-3) + \dots$$

$$\Rightarrow y(x) = -1 + x - \frac{3}{2}x^2 + \frac{1}{2}x^3 - \frac{1}{8}x^4 + \dots$$

$$\Rightarrow y(0.1) = -1 + (0.1) - \frac{3}{2}(0.1)^2 + \frac{1}{2}(0.1)^3 - \frac{1}{8}(0.1)^4$$

$$\Rightarrow y(0.1) \cong -0.1945$$

③ If $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$, then find $y(0.1)$ using Taylor's series method.

Given $y' = x^2y - 1 \rightarrow ①$

$\Rightarrow y' = x^2y - 1$, $y_0 = 1$
$x_0 = 0$

$\therefore y'(x_0) = x_0^2 y_0 - 1 = 0(1) - 1 = -1$

$\therefore ① \Rightarrow y''(x) = 2xy + x^2y' - 0$

$= 2xy + x^2y'$

$y''(x_0) = 2x_0y_0 + x_0^2y_0'$

$= 2(0)(1) + 0(-1)$

$= 0$

$\Rightarrow y'''(x) = 2(1.y + xy') + (2xy' + x^2y'')$

$= 2y + 2xy' + 2xy' + x^2y''$

$= 2y + 4xy' + x^2y''$

$y'''(x_0) = 2y_0 + 4x_0y_0' + x_0^2y_0''$

$= 2(1) = 2$

$\therefore$ WKT

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \cdots$$

$$\Rightarrow y(x) = 1 + \frac{x}{1!}(-1) + \frac{x^3}{3!}(2) + \cdots$$

$$\Rightarrow y(x) = 1 - \frac{x}{1!} + \frac{x^3}{3} + \cdots$$

$$\therefore y(0.1) = 1 - \frac{0.1}{1!} + \frac{(0.1)^3}{3} + \cdots$$

$\Rightarrow y(0.1) = 1 - 0.1 + 0.0003$

$\Rightarrow y(0.1) \cong 0.9003$

④ Employ the Taylor series method to find 'y' at $x = 0.1$ correct to 4 decimal places, given $\frac{dy}{dx} = 3x + y^2$, $y(0) = 1$.

Given $\frac{dy}{dx} = y' = 3x + y^2 \rightarrow ①$ $\quad y(0) = 1$

$x_0 = 0 \quad y_0 = 1$

$y'(x_0) = 3x_0 + y_0^2$

$= 3(0) + 1^2$

$= 1$

$y''(x) = 3 + 2yy'$

$\Rightarrow y''(x_0) = 3 + 2y_0y_0'$

$= 3 + 2(1)(1)$

$= 5$

$y'''(x) = 0 + 2[yy'' + (y')^2]$

$= 2[yy'' + (y')^2]$

$\Rightarrow y'''(x_0) = 2[y_0 y_0'' + (y_0')^2]$

$= 2[(1)(5) + 1^2]$

$= 12$

$\therefore$ WKT

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y(x_0) + \frac{(x-x_0)^2}{2!} y^2(x_0) + \cdots$$

$$\Rightarrow y(x) = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(5) + \frac{x^3}{3!}(12) + \cdots$$

$$\Rightarrow y(x) = 1 + x + \frac{5}{2}x^2 + 2x^3 + \cdots$$

$$\Rightarrow y(0.1) = 1 + (0.1) + \frac{5}{2}(0.1)^2 + 2(0.1)^3 + \cdots$$

$$\Rightarrow y(0.1) = 1 + 0.1 + 0.025 + 0.002$$

$y(0.1) \simeq 1.127$

(6) Solve $\frac{dy}{dx} = e^x - y$, $y(0) = 1$ using taylor series method considering upto the 4^{th} degree terms and find $y(0.1)$.

Given $\frac{dy}{dx} = y' = e^x - y$ $\quad y(0) = 1$

$x_0 = 0 \quad y_0 = 1$

$\Rightarrow y'(x_0) = e^{x_0} - y_0$

$= e^0 - 1$

$= 1 - 1 = 0$

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$\Rightarrow y''(x_0) = e^{x_0} - y_0'$

$= 1 - 0$

$= 1$

$\Rightarrow y'''(x) = e^{x} - y''$

$y'''(x_0) = e^{x_0} - y_0''$

$= 1 - 1 = 0$

$\Rightarrow y^{iv}(x) = e^{x} - y'''$

$y^{iv}(x_0) = e^{x_0} - y_0'''$

$= 1 - 0$

$= 1$

$\therefore$ WKT

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \ldots$$

$$\Rightarrow y(x) = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \ldots$$

$$\Rightarrow y(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\Rightarrow y(0.1) = 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{24} + \ldots$$

$$\Rightarrow y(0.1) \cong 1.0050$$

6) Solve $\frac{dy}{dx} = x^3 + y$ $y(1) = 1$ using taylor series method considering upto the 4^{th} degree terms and find $y(1.1)$

Given $\frac{dy}{dx} = y' = x^3 + y$

$y(1) = 1 \Rightarrow x_0 = 1, \ y_0 = 1$

$$\Rightarrow y'(x_0) = x_0^3 + y_0$$
$$= 1 + 1 = 2$$
$$y''(x) = 3x^2 + y'$$
$$\Rightarrow y''(x_0) = 3x_0^2 + y_0'$$
$$= 3(1) + 2$$
$$= 5$$
$$y'''(x) = 6x + y''$$
$$\Rightarrow y'''(x_0) = 6x_0 + y_0''$$
$$= 6(1) + 5$$
$$= 11$$
$$y^{IV}(x) = 6 + y'''$$
$$\Rightarrow y^{IV}(x_0) = 6 + y_0'''$$
$$= 6 + 11$$
$$= 17$$

$\therefore$ WKT

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots$$

$$\Rightarrow y(x) = 1 + \frac{(x-x_0)}{1!}(2) + \frac{(x-1)^2}{2!}(11) + \frac{(x-1)^4}{4!}(17) + \cdots$$

$$\Rightarrow y(1.1) = 1 + \frac{2(0.1)}{1!} + \frac{5}{2}(0.1)^2 + \frac{11}{6}(0.1)^3 + \frac{17}{24}(0.1)^4 + \cdots$$

$$\Rightarrow y(1.1) \cong 1.2269$$

Modified Euler's method :-

Step ① :- consider $\frac{dy}{dx} = f(x,y)$ to the initial condition

$y(x_0) = y_0$ having the step size 'h'.

Step ② :-

To get $y(x_1) = y_1$

I-1: $y(x_1) = y_1^{(1)} = y_0 + h f(x_0, y_0)$

I-2: $y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

I-3: $y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

I-4: $y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})]$

Step ③ :-

To find $y(x_2) = y_2$

Consider $y(x_1) = y_1$ has the initial condition

I-1: $y(x_2) = y_2^{(1)} = y_1 + h f[x_1, y_1]$

I-2: $y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$

I-3: $y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$

① Use modified Euler's method to compute $y_{(0.2)}$. Given $\frac{dy}{dx} - xy^2 = 0$, under the Initial condition $y_{(0)} = 2$. Perform three iterations at each step, taking $h = 0.1$.

Sol:- Given $\frac{dy}{dx} - xy^2 = 0$

$$\Rightarrow \frac{dy}{dx} = xy^2 = f(x,y) \rightarrow ①$$

and $y_{(0)} = 2 \Rightarrow x_0 = 0,\ y_0 = 2$

to find $y_{(x_1)} = y_1$

$\Rightarrow y_{(0.1)} = ?$

$$\Rightarrow y(x_0+h) = y_{(x_1)} = y_1^{(1)} = y_0 + hf(x_0, y_0)$$

$$\Rightarrow y_1^{(1)} = 2 + (0.1) f(0,2)$$

$$\Rightarrow y_1^{(1)} = 2 + 0 = 2$$

$$\Rightarrow y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(1)})]$$

$$y_1^{(2)} = 2 + \frac{0.1}{2}[f(0,2) + f(0.1,2)]$$

$$y_1^{(2)} = 2 + 0.05[0 + 0.4]$$

$$y_1^{(2)} = 2.02$$

$$\Rightarrow y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(2)})]$$

$$y_1^{(3)} = 2 + \frac{0.1}{2}[f(0,2) + f(0.1, 2.02)]$$

$$= 2 + 0.05[0 + 0.4080]$$

$$y_1^{(3)} = 2.0204$$

$$\therefore y_{(0.1)} = 2.0204$$

$\Rightarrow x_1 = 0.1 \quad y_1 = 2.0204$

To find $y(x_2) = y_2$

$\Rightarrow y_{(0.2)} = ?$

$\Rightarrow y(x_1+h) = y_{(x_2)} = y_2^{(1)} = y_1 + h f(x_1, y_1)$

$y_2^{(1)} = 2.0204 + (0.1) f(0.1, 2.0204)$

$= 2.0204 + (0.1)(0.4082)$

$y_2^{(1)} = 2.0612$

$\Rightarrow y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= 2.0204 + \frac{0.1}{2}[f(0.1, 2.0204) + f(0.2, 2.0612)]$

$= 2.0204 + 0.05[0.4082 + 0.8497]$

$y_2^{(1)} = 2.0833$

$\Rightarrow y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$= 2.0204 + \frac{0.1}{2}[f(0.1, 2.0204) + f(0.2, 2.0833)]$

$= 2.0204 + 0.05[0.4082 + 0.8680]$

$y_2^{(3)} = 2.0842$

$\therefore y(0.2) \cong 2.0842$

② Use modified Euler's method to compute $y_{(0.1)}$. Given $\frac{dy}{dx} = -xy^2$ under the initial condition $y_{(0)} = 2$. Perform three iterations at each step, taking $h = 0.05$.

Sol: Given $\frac{dy}{dx} = -xy^2 = f(x,y) \rightarrow (1)$

and $y_{(0)} = 2$, $x_0 = 0$, $y_0 = 2$

To find $y(x_1) = y_1 \Rightarrow y_{(0.05)} = ?$

$\Rightarrow y_{(x_0+h)} = y_{(x_1)} = y_1^{(1)} = y_0 + hf(x_0, y_0)$

$= 2 + (0.05) f(0,2)$

$= 2 - 0$

$y_1^{(1)} = 2$

$\Rightarrow y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(1)})]$

$= 2 + \frac{0.05}{2}[f(0,2) + f(0.05, 2)]$

$= 2 + 0.025 [0 - 0.2]$

$\Rightarrow y_1^{(2)} = 1.995$

$\Rightarrow y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(2)})]$

$= 2 + \frac{0.05}{2}[f(0,2) + f(0.05, 1.995)]$

$= 2 + 0.025 [0 - 0.1990]$

$\Rightarrow y_1^{(3)} = 1.9950$

$\therefore y_{(0.05)} = 1.9950$

$\Rightarrow x_1 = 0.05$, $y_1 = 1.9950$

To find $y(x_2) = y_2 \Rightarrow y_{(0.1)} = ?$

$\Rightarrow y(x_1+h) = y(x_2) = y_2^{(1)} = y_1 + hf(x_1, y_1)$

$= 1.9950 + (0.05) f(0.05, 1.9950)$

$\Rightarrow y_2^{(1)} = 1.9850$

$\Rightarrow y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= 1.9950 + 0.025[f(0.05, 1.9950) + f(0.1, 1.9850)]$

$= 1.9950 + 0.025[-0.19900 - 0.3940]$

$\Rightarrow y_2^{(2)} = 1.9802$

$\Rightarrow y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$= 1.9950 + 0.025[f(0.05, 1.9950) + f(0.1, 1.9802)]$

$= 1.9950 + 0.025[-0.19900 - 0.3921]$

$\Rightarrow y_2^{(3)} = 1.9802$

$\therefore y_{(0.1)} \cong 1.9802$

(5) Use modified Euler's method to compute $y(0.1)$. Given $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$. Perform three iterations by taking $h = 0.1$.

Sol: Given $\frac{dy}{dx} = 3x + \frac{y}{2} = f(x,y)$

$y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1, h = 0.1$

$\therefore y(x_0+h) = y(x_1) = y_1^{(1)} = y_0 + h\, f(x_0, y_0)$

$\Rightarrow y_1^{(1)} = 1 + (0.1) f(0,1)$

$= 1 + (0.1)(0.5)$

$= 1.05$

$\Rightarrow y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$

$= 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1, 1.05)\right]$

$= 1.0662$

$\Rightarrow y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(2)})\right]$

$= 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1, 1.0662)\right]$

$= 1 + 0.05\left[0.5 + 0.8331\right]$

$= 1.0666$

$\therefore y(0.1) \simeq 1.0665$

(6) Using modified Euler's method to find $y(0.1)$ Given $\frac{dy}{dx} = x^2 - y = f(x,y)$, $y(0) = 1$, $x_0 = 0$, $y_0 = 1$ perform three iterations by taking $h = 0.05$.

Sol: Given $\frac{dy}{dx} = x^2 - y = f(x,y)$

$y(0) = 1 \Rightarrow x_0 = 0 \quad y_0 = 1. \quad h = 0.05$

To find $y(x_1) = y_1$

$\Rightarrow y(x_0+h) = y(x_1) = y_1^{(1)} = y_0 + hf(x_0, y_0)$

$= 1 + (0.05) f(0,1)$

$= 1 - 0.05$

$= 0.95$

$\Rightarrow y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$= 1 + \frac{0.05}{2}[f(0,1) + f(0.05, 0.95)]$

$= 1 + \frac{0.05}{2}[-1 - 0.9475]$

$= 0.9513$

$\Rightarrow y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

$= 1 + \frac{0.05}{2}[f(0,1) + f(0.05, 0.9513)]$

$= 1 + \frac{0.05}{2}[-1 - 0.9488]$

$y_1^{(3)} = 0.9513$

$\therefore y(0.05) = 0.9513$

$x_1 = 0.05 \quad y_1 = 0.9513$

To find $y(x_2) = y_2$

$y(x_1+h) = y_2^{(1)} = y_1 + h(f(x_1, y_1)$

$= 0.9513 + (0.05) f(0.05, 0.9513)$

$= 0.9513 + (0.05)(-0.9488)$

$= 0.90386$

$\Rightarrow y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= 0.9513 + \frac{0.05}{2}[f(0.05, 0.9513) + f(0.1, 0.90386)]$

$= 0.9513 + \frac{0.05}{2}[-0.9488 - 0.8938]$

$= 0.9052$

$\Rightarrow y_2^{(3)} = 0.95 + \frac{0.05}{2}[f(0.05, 0.9513) + f(0.1, 0.9052)]$

$= 0.95 + \frac{0.05}{2}[-0.9488 - 0.8932]$

$= 0.9052$

$\therefore y(0.1) \cong 0.9052$

(5) Solve the differential equation $\frac{dy}{dx} = x\sqrt{y}$ under the initial condition $y(1)=1$ by using modified Euler's method find y at $x = 1.4$. Perform three iterations at each step, taking $h=0.2$.

Given: $\frac{dy}{dx} = x\sqrt{y} = f(x,y)$

$y(1)=1 \Rightarrow x_0=1,\ y_0=1,\ h=0.2$

To find $y(x_1)=y_1$

$y(x_0+h) = y(x_1) = y_0 + h f(x_0, y_0)$

$= 1 + (0.2) f(1,1)$

$= 1 + 0.2 \times 1$

$= 1.2$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$= 1 + \frac{0.2}{2}[f(1,1) + f(1.2, 1.2)]$

$= 1 + 0.1 [1 + 1.3145]$

$= 1.2314$

$y_1^{(3)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})]$

$= 1 + \frac{0.2}{2} [f(1, 1) + f(1.2, 1.2314)]$

$= 1 + (0.1) [1 + 1.3316]$

$= 1.2331$

$\therefore y_{(1.2)} \cong 1.2331$

$\Rightarrow x_1 = 1.2 \quad y_1 = 1.2331$

To find $y(x_2) = y_2$

$y(x_1 + h) = y_2^{(1)} = y_1 + h f(x_1, y_1)$

$= 1.2331 + (0.2) f(1.2, 1.2331)$

$= 1.2331 + (0.2)(1.3325)$

$= 1.4996$

$y_2^{(2)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= 1.2331 + \frac{0.2}{2} [f(1.2, 1.2331) + f(1.4, 1.4996)]$

$= 1.2331 + 0.1 [(1.3325) + (1.7144)]$

$= 1.5377$

$y_2^{(3)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$= 1.2331 + \frac{0.2}{2} [f(1.2, 1.2331) + f(1.4, 1.5377)]$

$= 1.2331 + 0.1 [1.3325 + 1.7360]$

$= 1.5399$

$\therefore y(1.4) \cong 1.5399$

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Runge - Kutta method of 4^{th} order :-

Step ①:- write the given differential equation $\frac{dy}{dx} = f(x,y)$ to the initial condition $y(x_0) = y_0$ having the step size 'h'.

Step ②:- Find $y(x_0+h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

where $K_1 = hf(x_0, y_0)$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$K_4 = hf(x_0+h, y_0 + K_3)$

① Use 4^{th}-order Runge - Kutta method to solve $(x+y)\frac{dy}{dx} = 1$. $y(0.4) = 1$, to find $y_{(0.5)}$ taking $h = 0.1$.

Sol: Given $(x+y)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{(x+y)} = f(x,y)$

and $y(0.4) = 1 \Rightarrow x_0 = 0.4$, $y_0 = 1$, $h = 0.1$.

$\Rightarrow K_1 = hf(x_0, y_0)$

$= (0.1) f(0.4, 1)$

$= (0.1)(0.7147)$

$K_1 = 0.07142$

$\Rightarrow K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= (0.1) f(0.45, 1.0357)$

$K_2 = 0.0673$

$\Rightarrow K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= (0.1)\, f(0.45, 1.0336)$

$K_3 = 0.0674$

$\Rightarrow K_4 = hf(x_0+h, y_0+k_3)$

$= (0.1)\, f(0.5, 1.0674)$

$K_4 = 0.06379$

$\therefore y(x_1) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$\Rightarrow y_{(0.5)} = 1 + \frac{1}{6}[0.07142 + 2(0.0673) + 2(0.0674) + 0.06379]$

$\Rightarrow y_{(0.5)} = 1 + \frac{1}{6}[0.40461]$

$\Rightarrow y_{(0.5)} \cong 1.0674$

② Use R-K method of 4^{th} order to solve $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y_{(0)} = 1$ to find $y_{(0.2)}$, take $h = 0.2$.

Given $\frac{dy}{dx} = 3x + \frac{y}{2} = f(x, y)$

$\Rightarrow K_1 = hf(x_0, y_0)$

$= 0.2\, f(0, 1)$

$K_1 = 0.1$

$\Rightarrow K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= (0.2)\, f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\right)$

$= (0.2)\, f(0.1, 1.05)$

$K_2 = 0.165$

$\Rightarrow K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

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$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.165}{2}\right)$

$= (0.2) f(0.1, 1.0825)$

$K_3 = 0.16825$

$\Rightarrow K_4 = hf(x_0 + h, y_0 + K_3)$

$= (0.2) f(0 + 0.2, 1 + 0.16825)$

$= (0.2) f(0.2, 1.16825)$

$K_4 = 0.23682$

$\Rightarrow y_{(0.2)} = 1 + \frac{1}{6}[0.1 + 2 \times 0.165 + 2 \times 0.16825 + 0.23682]$

$y_{(0.2)} \cong 1.16722$

(3) Use R-K method to find $y_{(0.2)}$, given $\frac{dy}{dx} = \sqrt{x+y}$, taking $h = 0.2$ inition condition $y_{(0)} = 1$.

Sol: Given $\frac{dy}{dx} = \sqrt{x+y} = f(x, y)$

$h = 0.2$

$y_{(0)} = 1 \Rightarrow x_0 = 0, \; y_0 = 1$

$\Rightarrow K_1 = hf(x_0, y_0)$

$= 0.2 f(0, 1)$

$= 0.2$

$\Rightarrow K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$

$= (0.2) f(0.1, 1.1)$

$= 0.2190$

$\Rightarrow K_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$

$= (0.2) f\left(0+\frac{0.2}{2}, 1+\frac{0.2190}{2}\right)$

$= (0.2) f(0.1, 1.1095)$

$= 0.2199$

$\Rightarrow K_4 = hf(x_0+h, y_0+k_3)$

$= (0.2) f(0+0.2, 1+0.2199)$

$= (0.2) f(0.2, 1.2199)$

$= 0.2383$

$\Rightarrow y(x_0+h) = y_0+\frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$= 1+\frac{1}{6}[0.2+2\times 0.2190+2\times 0.2199+0.2383]$

$y_{(0.2)} = 1.21935$

(4) Using R-K method of 4th order find $y_{(0.2)}$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$ $y_{(0)}=1$ taking $h=0.2$.

Sol:- Given $\frac{dy}{dx} = \frac{y-x}{y+x} = f(x,y)$

$x_0=0, y_0=1$

$\Rightarrow K_1 = hf(x_0,y_0)$

$= 0.2 f(0,1)$

$K_1 = 0.2$

$\Rightarrow K_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$

$= (0.2) f\left(0+\frac{0.2}{2}, 1+\frac{0.2}{2}\right)$

$= (0.2) f(0.1, 1.1)$

$K_2 = 0.1667$

$\Rightarrow K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= (0.2)\, f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1667}{2}\right)$

$= 0.2\, f(0.1, 1.0833)$

$K_3 = 0.1662$

$\Rightarrow K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.2\, f(0.2, 1 + 0.1662)$

$= 0.1414$

$\Rightarrow y(x_0+h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 1 + \frac{1}{6}[0.2 + 0.3334 + 0.3324 + 0.1414]$

$= 1.1679$

(5) Using R-K method of 4^{th} order find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$ $y(0) = 1$ taking $h = 0.1$.

Given $\frac{dy}{dx} = \frac{y-x}{y+x} = f(x, y)$

$y(0) = 1, \; x_0 = 0, \; y_0 = 1$ $\quad h = 0.1$

Stage 1:- $\Rightarrow K_1 = hf(x_0, y_0)$

$= (0.1)\, f(0, 1)$

$= 0.1$

$\Rightarrow K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= (0.1)\, f(0.05, 1.05)$

$= 0.091$

$\Rightarrow K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= (0.1)\, f(0.05, 1.0455)$

$= 0.0909$

$\Rightarrow K_4 = hf(x_0 + h, y_0 + K_3)$

$= (0.1)\, f(0.1, 1.0909)$

$= 0.0832$

$\therefore y(x_0 + h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= 1 + \frac{1}{6}(0.1 + 0.182 + 0.1818 + 0.0832)$

$y_{(0.1)} = 1.091167 \cong 1.0912$

Stage-2 :-

$f(x, y) = \frac{y - x}{y + x}, \; x_0 = 0.1, \; y_0 = 1.0912, \; h = 0.1$

$\Rightarrow K_1 = hf(x_0, y_0) = (0.1)\, f(0.1, 1.0912)$

$= 0.0832$

$\Rightarrow K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= (0.1)\, f(0.15, 1.1328)$

$= 0.0766$

$\Rightarrow K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= (0.1)\, f(0.15, 1.1295)$

$= 0.07655$

$\Rightarrow K_4 = hf(x_0 + h, y_0 + K_3)$

$= (0.1)\, f(0.2, 1.16775)$

$= 0.07075$

$\therefore y(x_0 + h) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$y_{(0.1+0.1)} = 1.0912 + \frac{1}{6}(0.0832 + 0.1532 + 0.1531 + 0.07076)$

$\Rightarrow y_{(0.2)} = 1.167908 \cong 1.1679$

(6) Solve: $(y^2-x^2)dx = (y^2+x^2)dy$ for $x=0.2$ and 0.4 given that $y=1$ at $x=0$ initially. by applying Runge-kutta method of order 4. Compute $y(0.2)$ by taking $h=0.2$.

Sol: Given $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$, $x_0=0$, $y_0=1$, $h=0.2$

Stage ①:- $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$

$\Rightarrow K_1 = hf(x_0, y_0)$

$= (0.2)f(0,1)$

$= 0.2$

$\Rightarrow K_2 = hf(x_0+\frac{h}{2}, y_0+\frac{K_1}{2})$

$= (0.2)f(0.1, 1.1)$

$= 0.1967$

$\Rightarrow K_3 = hf(x_0+\frac{h}{2}, y_0+\frac{K_2}{2})$

$= (0.2)f(0.1, 1.0984)$

$= 0.1967$

$\Rightarrow K_4 = hf(x_0+h, y_0+K_3)$

$= (0.2)f(0.2, 1.1967)$

$= 0.1891$

$\therefore y_{(x_0+h)} = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 1 + \frac{1}{6}[0.2 + 2\times 0.1967 + 2\times 0.1967 + 0.1891]$

$y_{(0.2)} = 1.19598 \cong 1.196$

Stage ②:- $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$, $x_0=0.2$, $y_0=1.196$, $h=0.2$

$\Rightarrow k_1 = hf(x_0,y_0) = (0.2)f(0.2, 1.196)$

$= 0.1891$

$\Rightarrow k_2 = hf(x_0+\frac{h}{2}, y_0+\frac{k_1}{2})$

$= (0.2)f(0.3, 1.29055)$

$= 0.1795$

$\Rightarrow k_3 = hf(x_0+\frac{h}{2}, y_0+\frac{k_2}{2})$

$= (0.2)f(0.3, 1.28575)$

$= 0.1793$

$\Rightarrow k_4 = hf(x_0+h, y_0+k_3)$

$= (0.2)f(0.4, 1.3753)$

$= 0.1688$

$\therefore y(x_0+h) = y_0 + \frac{1}{6}[k_1+2k_2+2k_3+k_4]$

$y(0.4) = 1.196 + \frac{1}{6}[0.1891 + 2\times 0.1795 + 2\times 0.1793 + 0.1688]$

$= 1.37525$

$y_{(0.4)} \cong 1.3753$.

⑦ Use fourth order Runge-Kutta method to compute $y(1.1)$ given that $\frac{dy}{dx} = xy^{1/3}$, $y(1)=1$.

Sol:- Given $f(x,y) = xy^{1/3}$, $x_0=1$, $y_0=1$

Given, $f(x,y) = xy^{1/3}$

$\Rightarrow k_1 = hf(x_0, y_0)$

$= (0.1) f((1)(1)^{1/3})$

$= 0.1$

$\Rightarrow k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= (0.1) f(1.05, 1.05)$

$= 0.1067$

$\Rightarrow k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= (0.1) f(1.05, 1.05335)$

$= 0.1068$

$\Rightarrow k_4 = hf(x_0 + h, y_0 + k_3)$

$= (0.1) f(1.1, 1.1068)$

$= 0.1138$

$\therefore y(x_0 + h) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \frac{1}{6}[0.1 + 2 \times 0.1067 + 2 \times 0.1068 + 0.1138]$

$y(1.1) = 1.1068$

Milne's, Adams - Bashforth Predictor and Corrector Method :-

Step ① :- Consider the given differential equation as $\frac{dy}{dx} = f(x,y)$, to the initial conditions, $y(x_0) = y_0$, $y(x_1) = y_1$ $y(x_2) = y_2$, $y(x_3) = y_3$.

Step ② :- Find $f_0 = f(x_0, y_0)$, $f_1 = f(x_1, y_1)$, $f_2 = f(x_2, y_2)$, $f_3 = f(x_3, y_3)$ and find $y(x_4) = y_4$ using

1. Milne's method :-

$$y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}] \qquad f_4^{(P)} = f(x_4, y_4^{(P)})$$

2. Adams method :-

$$y_4^{(P)} = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$y_4^{(C)} = y_3 + \frac{h}{24}[9f_4^{(P)} + 19f_3 - 5f_2 + f_1] \qquad f_4^{(P)} = f(x_4, y_4^{(P)})$$

(1) Given $\frac{dy}{dx} = x^2(1+y)$, $y_{(1)} = 1$, $y_{(1.1)} = 1.233$, $y_{(1.2)} = 1.548$, $y_{(1.3)} = 1.979$. Evaluate $y_{(1.4)}$ by

1) Milne's Predictor - Corrector method.
2) Adams - Bashforth Predictor - Corrector method.

sol:- Given $\frac{dy}{dx} = x^2(1+y) = f(x,y)$

$y_{(1)} = 1 \Rightarrow x_0 = 1,\ y_0 = 1$

$y_{(1.1)} = 1.233 \Rightarrow x_1 = 1.1,\ y_1 = 1.233$

$y_{(1.2)} = 1.548 \Rightarrow x_2 = 1.2,\ y_2 = 1.548$

$y_{(1.3)} = 1.979 \Rightarrow x_3 = 1.3,\ y_3 = 1.979$ & $h = 0.1$.

$f_0 = f(x_0, y_0) = f(1,1) = 2$

$f_1 = f(x_1, y_1) = f(1.1, 1.233) = 2.7019$

$f_2 = f(x_2, y_2) = f(1.2, 1.548) = 3.6691$

$f_3 = f(x_3, y_3) = f(1.3, 1.979) = 5.0345$

1) Milne's method:-

$$\Rightarrow y_4^{(p)} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(2.7019) - 3.6691 + 2(5.0345)\right]$$

$$\Rightarrow y_4^{(p)} = 2.5738$$

$$\therefore f_4^{(p)} = f(1.4, 2.5738) = 7.0046$$

$$\therefore y_4^{(c)} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_4^{(p)}\right]$$

$$= 1.548 + \frac{0.1}{3}\left[3.6691 + 4(5.0345) + 7.0046\right]$$

$$\Rightarrow y_4^{(c)} = 2.5750$$

$$\therefore y(x_4) = y(1.4)$$

$$= 2.5750$$

2) Adams method:-

$$y_4^{(p)} = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$$= 1.979 + \frac{0.1}{24}\left[276.8975 - 216.4769 + 99.9703 - 18\right]$$

$$\Rightarrow y_4^{(p)} = 2.5722$$

$$\therefore f_4^{(p)} = f(1.4, 2.5722) = 7.0015$$

$$\therefore y_4^{(c)} = y_3 + \frac{h}{24}\left[9f_4^{(p)} + 19f_3 - 5f_2 + f_1\right]$$

$$= 1.979 + \frac{0.1}{24}\left[63.0135 + 95.6555 - 18.3455 + 2.7019\right]$$

$$\Rightarrow y_4^{(c)} = 2.5750$$

② Given $\frac{dy}{dx} = \frac{1}{x+y}$ $y_{(0)} = 2$, $y_{(0.2)} = 2.0933$, $y_{(0.4)} = 2.1755$, $y_{(0.6)} = 2.2493$. Compute y at $x = 0.8$ by using

(i) Milne's - Predictor Corrector method.

(ii) Adams - Bashforth Predictor Corrector method.

Sol: Given. $\frac{dy}{dx} = \frac{1}{x+y} = f(x,y)$

$y_{(0)} = 2 \Rightarrow x_0 = 0$, $y_0 = 2$

$y_{(0.2)} = 2.0933 \Rightarrow x_1 = 0.2$, $y_1 = 2.0933$

$y_{(0.4)} = 2.1755 \Rightarrow x_2 = 0.4$, $y_2 = 2.1755$

$y_{(0.6)} = 2.2493 \Rightarrow x_3 = 0.6$, $y_3 = 2.2493$ and $h = 0.2$

$f_0(x_0, y_0) = f(0, 2) = 0.5$

$f_1(x_1, y_1) = f(0.2, 2.0933) = 0.4360$

$f_2(x_2, y_2) = f(0.4, 2.1755) = 0.3882$

$f_3(x_3, y_3) = f(0.6, 2.2493) = 0.3509$

i) Milne's method:-

$\Rightarrow y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$

$= 2 + \frac{4(0.2)}{3}(2 \times 0.4360 - 0.3882 + 2 \times 0.3509)$

$\Rightarrow y_4^{(P)} = 2.3162$ $\therefore f_4^{(P)} = f(0.8, 2.3162) = 0.32091$

$\therefore y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$

$= 2.1755 + \frac{0.2}{3}[0.3882 + 4 \times 0.3509 + 0.32091]$

$y_4^{(C)} = 2.3163$.

$$y(x_4) = y(0.8)$$
$$= 2.3163$$

(ii) Adams Method :-

$$y_4^{(P)} = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$= 2.2493 + \frac{0.2}{24}[55 \times 0.3509 - 59 \times 0.3882 + 37 \times 0.4360 - 9 \times 0.5]$$

$$y_4^{(P)} = 2.3162$$

$$\therefore f_4^{(P)} = f(0.8, 2.3162) = 0.32090$$

$$\Rightarrow y_4^{(C)} = y_3 + \frac{h}{24}[9f_4^{(P)} + 19f_3 - 5f_2 + f_1]$$

$$= 2.2493 + \frac{0.2}{24}[9 \times 0.32090 + 19 \times 0.3509 - 5 \times 0.3882 + 0.4360]$$

$$y_4^{(C)} = 2.3164$$

$$\therefore y(x_4) = y(0.8)$$
$$= 2.3164$$

③ Apply Milne's and Adam's method to compute $y_{(0.4)}$ given $\frac{dy}{dx} = x + y^2$ with condition

x	y
0.0	1.0000
0.1	1.1000
0.2	1.2310
0.3	1.4020

Sol: Given $\frac{dy}{dx} = x + y^2 = f(x,y)$

$$y_{(0.0)} = 1.0000 \Rightarrow x_0 = 0.0,\ y_0 = 1.0000$$
$$y_{(0.1)} = 1.1000 \Rightarrow x_1 = 0.1,\ y_1 = 1.1000$$
$$y_{(0.2)} = 1.2310 \Rightarrow x_2 = 0.2,\ y_2 = 1.2310$$
$$y_{(0.3)} = 1.4020 \Rightarrow x_3 = 0.3,\ y_3 = 1.4020$$
$$h = 0.1$$

$f_0 = f(x_0, y_0) = f(0.0, 1.0000) = 1$

$f_1 = f(x_1, y_1) = f(0.1, 1.1000) = 1.31$

$f_2 = f(x_2, y_2) = f(0.2, 1.2310) = 1.7154$

$f_3 = f(x_3, y_3) = f(0.3, 1.4020) = 2.2656$

(i) Milne's method:-

$$\Rightarrow y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 1.0000 + \frac{4\times 0.1}{3}[2\times 1.31 - 1.7154 + 2\times 2.2656]$$

$$\Rightarrow y_4^{(P)} = 1.72477$$

$$\therefore f_4^{(P)} = f(0.4, 1.72477) = 3.37483$$

$$\Rightarrow y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$$

$$= 1.2310 + \frac{0.1}{3}[1.7154 + 4\times 2.2656 + 3.37483]$$

$$\Rightarrow y_4^{(C)} = 1.7027$$

(ii) Adams method:-

$$\Rightarrow y_4^{(P)} = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$= 1.4020 + \frac{0.1}{24}[55\times 2.2656 - 59\times 1.7154 + 37\times 1.31 - 9\times 1]$$

$$\Rightarrow y_4^{(P)} = 1.66395$$

$$\therefore f_4^{(P)} = f(x_4, y_4^{(P)}) = f(0.4, 1.66395) = 3.1687$$

$$y_4^{(C)} = y_3 + \frac{h}{24}[9f_4^{(P)} + 19f_3 - 5f_2 + f_1]$$

$$= 1.4020 + \frac{0.1}{24}[9\times 3.1687 + 19\times 2.2656 - 5\times 1.7154 + 1.31]$$

$$\Rightarrow y_4^{(C)} = 1.6699$$

(4) Given $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$, $y_{(1)} = 1$, $y_{(1.1)} = 0.9960$, $y_{(1.2)} = 0.9860$, $y_{(1.3)} = 0.9720$ find $y_{(1.4)}$ using Adam's Bashforth Predictor Corrector method.

Sol: Given $\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1-xy}{x^2} = f(x,y)$

$f_0 = f(x_0, y_0) = f(1,1) = 0$

$f_1 = f(x_1, y_1) = f(1.1, 0.9960) = -0.0790$

$f_2 = f(x_2, y_2) = f(1.2, 0.9860) = -0.1272$

$f_3 = f(x_3, y_3) = f(1.3, 0.9720) = -0.15597$

$y_{(1)} = 1 \Rightarrow x_0 = 1,\ y_0 = 1$

$y_{(1.1)} = 0.9960 \Rightarrow x_1 = 1.1,\ y_1 = 0.9960$

$y_{(1.2)} = 0.9860 \Rightarrow x_2 = 1.2,\ y_2 = 0.9860$

$y_{(1.3)} = 0.9720 \Rightarrow x_3 = 1.3,\ y_3 = 0.9720$ & $h = 0.1$

$\Rightarrow y_4^{(P)} = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$

$= 0.9720 + \frac{h}{24}[55 \times (-0.15597) + 59 \times 0.1272 - 37 \times 0.0790]$

$\Rightarrow y_4^{(P)} = 0.95535$

$\therefore f_4^{(P)} = f(x_4, y_4^{(P)}) = f(1.4, 0.95535) = -0.17219$ -0.172189

$\Rightarrow y_4^{(C)} = y_3 + \frac{h}{24}[9f_4^{(P)} + 19f_3 - 5f_2 + f_1]$

$= 0.9720 + \frac{0.1}{24}[9 \times (-0.17219) + 19 \times (-0.15597) - 5 \times (-0.1272) + (-0.0790)]$

$y_4^{(C)} = 0.95552$

(5) Apply Milne's predictor-corrector method to compute $y_{(2.0)}$, given $\frac{dy}{dx} = \frac{1}{2}(x+y)$ and

x	y
0.0	2.0000
0.5	2.6360
1.0	3.5950
1.5	4.9680

Sol:- Given: $\frac{dy}{dx} = \frac{x+y}{2} = f(x,y)$

$y_{(0.0)} = 2.0000,\ x_0 = 0.0,\ y_0 = 2.0000$

$y_{(0.5)} = 2.6360,\ x_1 = 0.5,\ y_1 = 2.6360$

$h = 0.5$

$y_{(1.0)} = 3.5950,\ x_2 = 1.0,\ y_2 = 3.5950$

$y_{(1.5)} = 4.9680,\ x_3 = 1.5,\ y_3 = 4.9680$

$f_0 = f(x_0, y_0) = f(0.0, 2.0000)$

$= 1$

$f_1 = f(x_1, y_1) = f(0.5, 2.6360) = 1.568$

$f_2 = f(x_2, y_2) = f(1.0, 3.5950) = 2.2975$

$f_3 = f(x_3, y_3) = f(1.5, 4.9680) = 3.234$

$\Rightarrow y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$

$= 2.0000 + \frac{4 \times 0.5}{3}[2 \times 1.568 - 2.2975 + 2 \times 3.234]$

$\Rightarrow y_4^{(P)} = 6.871$

$\therefore f_4^{(P)} = f(2.0, 6.871) = 4.4355$

$\Rightarrow y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$

$= 3.5950 + \frac{0.5}{3}[2.2975 + 4 \times 3.234 + 4.4355]$

$y_4^{(C)} = 6.8732$

$\therefore y(x_4) = y(2.0) = 6.8732$

NUMERICAL SOLUTION FOR SECOND ORDER DIFFERENTIAL EQUATION

Consider the differential equation of second order

$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = Q \rightarrow (1)$, where a_0, a_1, a_2 are the functions of 'x'.

Let $\frac{dy}{dx} = y' = z = f(x,y,z)$, then

$(1) \Rightarrow \frac{dz}{dx} = g(x,y,z)$ to the initial conditions $y(x_0) = y_0$,

$y'(x_0) = y'_0$

$\Rightarrow z(x_0) = z_0$.

R-K METHOD:-

$$y(x_1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$z(x_1) = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4], \text{ where}$$

$k_1 = hf(x_0, y_0, z_0)$, $l_1 = hg(x_0, y_0, z_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$, $l_2 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$, $l_3 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$, $l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$

1. Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, for $x = 0.1$, correct to 4 decimals using initial conditions $y_{(0)} = 1$, $y'_{(0)} = 0$, by using R-K method of 4^{th} order.

Sol:- Given: $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1 \rightarrow (1)$

Let $\frac{dy}{dx} = y' = z = f(x,y,z)$

$\therefore (1) \Rightarrow \frac{dz}{dx} - x^2 z - 2xy = 1$

$\Rightarrow \frac{dz}{dx} = 1 + x^2 z + 2xy = g(x,y,z)$

and given

$y(0) = 1$, $y'(0) = 0$

$\Rightarrow x_0 = 0, y_0 = 1$, $y'_0 = z_0 = 0$, $h = 0.1$

$K_1 = hf(x_0, y_0, z_0)$

$= (0.1) f(0,1,0)$

$= (0.1)(0)$

$K_1 = 0$

$\therefore l_1 = hg(x_0, y_0, z_0)$

$= (0.1)\, g(0,1,0)$

$= (0.1)[1 + (0)(0) + 2(0)(1)]$

$l_1 = 0.1$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.1) f(0.05, 1, 0.05)$

$= (0.1) f(0.05)$

$K_2 = 0.005$

$\therefore l_2 = (0.1)\, g(0.05, 1, 0.05)$

$= (0.1)(1.100125)$

$l_2 = 0.11$

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(2)

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= (0.1) f(0.05, 1.0025, 0.055)$

$= (0.1)(0.055)$

$k_3 = 0.0055$

$\therefore l_3 = (0.1) g(0.05, 1.0025, 0.055)$

$= (0.1) g(1.1003875)$

$l_3 = 0.11$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= (0.1) f(0+0.1, 1+0.0055, 0+0.11)$

$= (0.1) f(0.1, 1.0055, 0.11)$

$= (0.1)(0.11)$

$k_4 = 0.011$

$\therefore y(x_1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \frac{1}{6}[0 + 2\times 0.005 + 2\times 0.0055 + 0.011]$

$y(0.1) \simeq 1.00533$

② Using RK method, solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$, for $x = 0.2$. Correct to 4 decimals using initial conditions $y_{(0)} = 1$, $y'_{(0)} = 0$.

Sol: Given: $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2 \rightarrow ①$

Let $\frac{dy}{dx} = y' = z = f(x, y, z)$

$\therefore ① \Rightarrow \frac{dz}{dx} = xz^2 - y^2 = g(x, y, z)$

and $y'_{(0)} = 0$, $y_{(0)} = 1$

$\Rightarrow x_0 = 0$, $y_0 = 1$ $\quad y'_0 = z_0 = 0$, $h = 0.2$

$K_1 = hf(x_0, y_0, z_0)$

$= (0.2)(0)$

$K_1 = 0$

$\therefore l_1 = hg(x_0, y_0, z_0)$

$= (0.2)\, g(0,1,0)$

$= (0.2)(-1)$

$l_1 = -0.2$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.2)\, f(0.1, 1, -0.1)$

$= (0.2)(-0.1)$

$K_2 = -0.02$

$\therefore l_2 = (0.2)\, g(0.1, 1, -0.1)$

$= (0.2)(-0.999)$

$l_2 = -0.2$

$K_3 = (0.2)\, f(0.1, 0.99, -0.1)$

$= (0.2)(-0.1)$

$K_3 = -0.02$

$\therefore l_3 = (0.2)\, g(0.1, 0.99, -0.1)$

$= (0.2)(-0.989)$

$l_3 = -0.1978$

$K_4 = hf(x_0 + h, y_0 + K_3, z_0 + l_3)$

$= (0.2)\, f(0.2, 0.98, -0.1978)$

$= (0.2)(-0.1978)$

$K_4 = -0.03956$

$\therefore y(x_1) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$y(0.2) = 1 + \frac{1}{6}[0 - 0.04 - 0.04 - 0.03956]$

$\Rightarrow y(0.2) \cong 0.98$

③ Find $y(0.1)$ using R-K method given that $y'' = xy' - y$, $y_{(0)} = 3$, $y'_{(0)} = 0$.

Sol: Given: $y'' = xy' - y \rightarrow ①$

Let $\frac{dy}{dx} = y' = z = f(x, y, z)$

$\therefore ① \Rightarrow \frac{dz}{dx} = xz - y = g(x, y, z)$

and $y_{(0)} = 3$, $y'_{(0)} = 0$

$\Rightarrow x_0 = 0$, $y_0 = 3$ $\quad y'_0 = z_0 = 0$ $\quad h = 0.1$

$k_1 = hf(x_0, y_0, z_0)$

$= (0.1)\, f(0, 3, 0)$

$= (0.1)(0)$

$k_1 = 0$

$\therefore l_1 = hg(x_0, y_0, z_0)$

$= (0.1)\, g(0, 3, 0)$

$= (0.1)(-3)$

$l_1 = -0.3$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.1)\, f\left(0 + \frac{0.1}{2}, 3 + \frac{0}{2}, 0 - \frac{0.3}{2}\right)$

$= (0.1)(-0.15)$

$k_2 = -0.015$

$\therefore l_2 = hg(0.05, 3, -0.15)$

$= (0.1)\, g(-3.015)$

$l_2 = -0.30075$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= (0.1)\, f\left(0.05, 3 - \frac{0.015}{2}, -\frac{0.3015}{2}\right)$

$= (0.1)\, f(0.05, 2.9925, -0.15075) = -0.015075$

$\therefore l_3 = (0.1)g(0.05, 2.9925, -0.15075)$

$l_3 = -0.30000$

$K_4 = hf(x_0+h, y_0+K_3, z_0+l_3)$

$= (0.1) f(0.1, 2.9849, -0.30000)$

$K_4 = -0.03$

$\therefore y_{(0.1)} = 3 + \frac{1}{6}[0 - 0.03 - 0.03015 - 0.03]$

$\Rightarrow y_{(0.1)} \cong 2.9849$

(4) Using R-K method to solve $\frac{d^2y}{dx^2} = x^3\left(y + \frac{dy}{dx}\right)$, $y_{(0)} = 1$, $y'_{(0)} = 0.5$, find y at $x = 0.1$.

Sol: Given: $\frac{d^2y}{dx^2} = x^3\left(y + \frac{dy}{dx}\right) \rightarrow (1)$

Let $\frac{dy}{dx} = y' = z = f(x,y,z)$

$\therefore (1) \Rightarrow \frac{dz}{dx} = x^3(y+z) = g(x,y,z)$

and $y_{(0)} = 1$, $y'_{(0)} = 0.5$

$\Rightarrow x_0 = 0, y_0 = 1 \quad y'_0 = z_0 = 0.5$

$\Rightarrow K_1 = hf(x_0, y_0, z_0) \qquad \Rightarrow l_1 = hg(x_0, y_0, z_0)$

$= 0.1\, f(0,1,0.5) \qquad = 0.1\, g(0,1,0.5)$

$K_1 = 0.05 \qquad \Rightarrow l_1 = 0$

$\Rightarrow K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.1) f(0.05, 1.025, 0.5)$

$\Rightarrow K_2 = 0.05$

$\Rightarrow k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$

$= (0.1)\, f(0.05, 1.025, 0.5)$

$\Rightarrow k_3 = 0.05$

$\Rightarrow l_3 = (0.1)\, g(0.05, 1.025, 0.5)$

$l_3 = 0$

$\Rightarrow k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= (0.1)\, f(0.1, 1.05, 0.5)$

$k_4 = 0.05$

$y_{(0.1)} = 1 + \frac{1}{6}[0.05 + 0.1 + 0.1 + 0.05]$

$y_{(0.1)} \cong 1.05$

① Apply milne's method to compute $y_{(0.8)}$ given that $y'' = 1 - 2yy'$

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
$y' = z$	0	0.1996	0.3972	0.5689

Sol:- Given: $y'' = 1 - 2yy' \rightarrow ①$

Let $y' = z = f(x, y, z)$

$\therefore ① \Rightarrow \frac{dz}{dx} = 1 - 2yz = g(x, y, z)$

and given

$x_0 = 0$ $\quad$ $y_0 = 0$ $\quad$ $y'_0 = z_0 = 0$

$x_1 = 0.2$ $\quad$ $y_1 = 0.02$ $\quad$ $y'_1 = z_1 = 0.1996$

$x_2 = 0.4$ $\quad$ $y_2 = 0.0795$ $\quad$ $y'_2 = z_2 = 0.3972$

$x_3 = 0.6$ $\quad$ $y_3 = 0.1762$ $\quad$ $y'_3 = z_3 = 0.5689$

$\therefore f_0 = f(x_0, y_0, z_0) = 0$

$f_1 = f(x_1, y_1, z_1) = 0.1996$

$f_2 = f(x_2, y_2, z_2) = 0.3972$

$f_3 = f(x_3, y_3, z_3) = 0.5689$

$g_0 = 1 - 2y_0 z_0 = 1$

$g_1 = 1 - 2y_1 z_1 = 0.9920$

$g_2 = 1 - 2y_2 z_2 = 0.93684$

$g_3 = 1 - 2y_3 z_3 = 0.7995$

$\therefore y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$

$= \frac{4 \times 0.2}{3}[2 \times 0.1996 - 0.3972 + 2 \times 0.5689]$

$y_4^{(P)} = 0.3039$

$\Rightarrow z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$

$= \frac{4 \times (0.2)}{3}[1.984 - 0.93684 + 1.599]$

$z_4^{(P)} = 0.7056$

$\therefore f_4^{(P)} = 0.7056$

$y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$

$= 0.0795 + \frac{0.2}{3}[0.3972 + 4 \times 0.5689 + 0.7056]$

$y_4^{(C)} = 0.3047$

$y_{(0.8)} = 0.3047$

② Apply milne's predictor - corrector method to compute $y(0.4)$ given the differential equation $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ and the following table of initial values.

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.3990
z	1	1.2103	1.4427	1.6990

Sol: Given: $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx} \rightarrow ①$

Let $y' = \frac{dy}{dx} = z = f(x,y,z)$

$\therefore ① \Rightarrow \frac{dz}{dx} = 1 + z = g(x,y,z)$

and given

$x_0 = 0 \qquad y_0 = 1 \qquad y_0' = z_0 = 1$

$x_1 = 0.1 \qquad y_1 = 1.1103 \qquad y_1' = z_1 = 1.2103$

$x_2 = 0.2 \qquad y_2 = 1.2427 \qquad y_2' = z_2 = 1.4427$

$x_3 = 0.3 \qquad y_3 = 1.3990 \qquad y_3' = z_3 = 1.6990$

$\therefore f_0 = f(x_0, y_0, z_0) = 1 \qquad g_0 = 1 + z_0 = 2$

$f_1 = f(x_1, y_1, z_1) = 1.2103 \qquad g_1 = 1 + z_1 = 2.2103$

$f_2 = f(x_2, y_2, z_2) = 1.4427 \qquad g_2 = 1 + z_2 = 2.4427$

$f_3 = f(x_3, y_3, z_3) = 1.6990 \qquad g_3 = 1 + z_3 = 2.6990$

$\therefore y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$

$= 1 + \frac{4 \times 0.1}{3}[2 \times 1.2103 - 1.4427 + 2 \times 1.6990]$

$y_4^{(P)} = 1.5835$

$\Rightarrow z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$

$= 1 + \frac{4 \times 0.1}{3}[2 \times 2.2103 - 2.4427 + 2 \times 2.6990]$

$\Rightarrow z_4^{(P)} = 1.9835$

$\therefore f_4^{(P)} = 1.9835$

$\Rightarrow y_4^{(c)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$

$= 1.2427 + \frac{0.1}{3}[1.4427 + 4 \times 1.6990 + 1.9835]$

$y_4^{(c)} = 1.5835$

$y_{(0.4)} = 1.5835$

③ Apply Milne's method to find $y_{(0.4)}$ for the given D.E $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$.

x	0	0.1	0.2	0.3
y	1	1.03995	1.138036	1.29865
z	0.1	0.6995	1.2580	1.8730

Sol: Given: $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0 \rightarrow ①$

Let $y' = \frac{dy}{dx} = z = f(x, y, z)$

$\therefore ① \Rightarrow \frac{dz}{dx} = -3xz + 6y$

$\frac{dz}{dx} = 6y - 3xz = g(x, y, z)$

and

$x_0 = 0$, $y_0 = 1$, $y_0' = z_0 = 0.1$

$x_1 = 0.1$, $y_1 = 1.03995$, $y_1' = z_1 = 0.6995$

$x_2 = 0.2$, $y_2 = 1.138036$, $y_2' = z_2 = 1.2580$

$x_3 = 0.3$, $y_3 = 1.29865$, $y_3' = z_3 = 1.8730$

$\therefore f_0 = f(x_0, y_0, z_0) = 1 \qquad g_0 = 6y_0 - 3x_0z_0 = 6$

$f_1 = f(x_1, y_1, z_1) = 0.6995 \qquad g_1 = 6y_1 - 3x_1z_1 = 6.02985$

$f_2 = f(x_2, y_2, z_2) = 1.2580 \qquad g_2 = 6y_2 - 3x_2z_2 = 6.073416$

$f_3 = f(x_3, y_3, z_3) = 1.8730 \qquad g_3 = 6y_3 - 3x_3z_3 = 6.1062$

$$\therefore y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 1 + \frac{4\times 0.1}{3}[2\times 0.6995 - 1.2580 + 2\times 1.8730]$$

$$\Rightarrow y_4^{(P)} = 1.5183$$

$$\therefore z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$$

$$= 0.1 + \frac{4\times 0.1}{3}[2\times 6.02985 - 6.02985 + 2\times 6.073416]$$

$$z_4^{(P)} = 2.5236$$

$$\therefore f_4^{(P)} = 2.5236$$

$$\therefore y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$$

$$= 1.138036 + \frac{0.1}{3}[1.2580 + 4\times 1.8730 + 2.5236]$$

$$y_4^{(C)} = 1.5138$$

(4) Use Milne's method, obtain an approximate solution at the point $x=0.8$ of the problem $y''=2yy'$, given $y(0)=0$, $y'(0)=1$, $y(0.2)=0.2027$, $y'(0.2)=1.041$, $y(0.4)=0.4228$, $y'(0.4)=1.179$, $y(0.6)=0.6841$, $y'(0.6)=1.468$.

Sol: Given: $y''=2yy' \rightarrow (1)$

Let $y'=z=f(x,y,z)$

$\therefore (1) \Rightarrow \frac{dz}{dx} = 2yz = g(x,y,z)$

and

$x_0 = 0$	$y_0 = 0$	$y_0' = z_0 = 1$
$x_1 = 0.2$	$y_1 = 0.2027$	$y_1' = z_1 = 1.041$
$x_2 = 0.4$	$y_2 = 0.4228$	$y_2' = z_2 = 1.179$
$x_3 = 0.6$	$y_3 = 0.6841$	$y_3' = z_3 = 1.468$

$\therefore f_0 = f(x_0, y_0, z_0) = 1$ $\quad g_0 = 2y_0 z_0 = 0$

$f_1 = f(x_1, y_1, z_1) = 1.041$ $\quad g_1 = 2y_1 z_1 = 0.4220$

$f_2 = f(x_2, y_2, z_2) = 1.179$ $\quad g_2 = 2y_2 z_2 = 0.9969$

$f_3 = f(x_3, y_3, z_3) = 1.468$ $\quad g_3 = 2y_3 z_3 = 2.0085$

$$\therefore y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= \frac{4\times 0.2}{3}[2\times 1.041 - 1.179 + 2\times 1.468]$$

$y_4^{(P)} = 1.0237$

$$\therefore z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$$

$$= 1 + \frac{4\times 0.2}{3}[2\times 0.4220 - 0.9969 + 2\times 2.0085]$$

$\Rightarrow z_4^{(P)} = 2.0304$

$\therefore f_4^{(P)} = 2.0304$

$$\therefore y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$$

$$= 0.4228 + \frac{0.2}{3}[1.179 + 4\times 1.468 + 2.0304]$$

$y_4^{(C)} = 1.02823$

$y_{(0.8)} = 1.02823$

⑤ Obtain the solution of the equation $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$, by computing the value of the dependent variable corresponding to the value 1.4 of the independent variable by applying Milne's method using the following data.

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
z	2	2.3178	2.6725	2.0657

Sol: Given: $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx} \rightarrow ①$

Let $y' = \frac{dy}{dx} = z = f(x,y,z)$

$\therefore ① \Rightarrow 2\frac{dz}{dx} = 4x + z$

$\frac{dz}{dx} = 2x + \frac{z}{2} = g(x,y,z) \rightarrow ②$

and

$x_0 = 1$ $\quad y_0 = 2$ $\quad y'_0 = z_0 = 2$

$x_1 = 1.1$ $\quad y_1 = 2.2156$ $\quad y'_1 = z_1 = 2.3178$

$x_2 = 1.2$ $\quad y_2 = 2.4649$ $\quad y'_2 = z_2 = 2.6725$

$x_3 = 1.3$ $\quad y_3 = 2.7514$ $\quad y'_3 = z_3 = 2.0657$

$\therefore f_0 = f(x_0, y_0, z_0) = 2$ $\qquad g_0 = 2x_0 + \frac{z_0}{2} = 3$

$f_1 = f(x_1, y_1, z_1) = 2.3178$ $\qquad g_1 = 2x_1 + \frac{z_1}{2} = 3.3589$

$f_2 = f(x_2, y_2, z_2) = 2.6725$ $\qquad g_2 = 2x_2 + \frac{z_2}{2} = 3.73625$

$f_3 = f(x_3, y_3, z_3) = 2.0657$ $\qquad g_3 = 2x_3 + \frac{z_3}{2} = 3.63285$

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$$\therefore y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 2 + \frac{4\times0.1}{3}[2\times2.3178 - 2.6725 + 2\times2.0657]$$

$$\Rightarrow y_4^{(P)} = 2.8126$$

$$\therefore z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$$

$$= 2 + \frac{4\times0.1}{3}[2\times3 - 3.73665 + 2\times3.63285]$$

$$z_4^{(P)} = 3.2706$$

$$\therefore f_4^{(P)} = 3.2706$$

$$\Rightarrow y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$$

$$= 2.4649 + \frac{0.1}{3}[2.6725 + 4\times3.0657 + 3.2706]$$

$$\Rightarrow y_4^{(C)} = 3.07176$$

$$y(1.4) = 3.07176$$

Calculus of Variations:-

Euler's Theorem:-

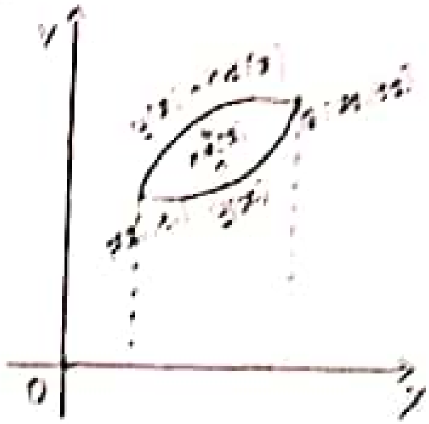
A necessary condition for the integral $I = \int_{x_1}^{x_2} f(x, y, y')dx$ where $y(x_1) = y_1$, $y(x_2) = y_2$, to be as extremum that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

Proof: Let the curve $y = y(x)$ passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ and make I as extremal. Also let $y = y(x) + h\alpha(x)$ be the neighbouring curve passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$ be an extremal.

the curves both are coincident at P and Q.

$\Rightarrow \alpha(x_1)=0\ ,\ \alpha(x_2)=0$



Given : $I = \int_{x_1}^{x_2} f(x, y, y')\,dx \quad \rightarrow (1)$

$$\Rightarrow I = \int_{x_1}^{x_2} f[x,\ y(x)+h\alpha(x),\ y'(x)+h\alpha'(x)]\,dx$$

$$\Rightarrow \frac{dI}{dh} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial h}\right]dx$$

$$= \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial x}(0) + \frac{\partial f}{\partial y}(0) + \frac{\partial f}{\partial y'}(\alpha'(x))\right]dx$$

$$\therefore \frac{dI}{dh} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}(\alpha(x)) + \frac{\partial f}{\partial y'}(\alpha'(x))\right]dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2}\alpha(x)\frac{\partial f}{\partial y}\,dx + \int_{x_1}^{x_2}\alpha'(x)\frac{\partial f}{\partial y'}\,dx$$

$$\Rightarrow \frac{dI}{dh} = \int_{x_1}^{x_2}\alpha(x)\frac{\partial f}{\partial y}\,dx + \frac{\partial f}{\partial y'}\int_{x_1}^{x_2}\alpha'(x)\,dx - \int_{x_1}^{x_2}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\int\alpha(x)\,dx\right]dx$$

$$= \int_{x_1}^{x_2}\alpha(x)\frac{\partial f}{\partial y}\,dx + \frac{\partial f}{\partial y'}\left[\alpha(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\alpha(x)\,dx$$

$$= \int_{x_1}^{x_2}\alpha(x)\frac{\partial f}{\partial y}\,dx + \frac{\partial f}{\partial y'}\left[\alpha(x_2)-\alpha(x_1)\right] - \int_{x_1}^{x_2}\left[\alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx\right]$$

$$= \int_{x_1}^{x_2}\alpha(x)\frac{\partial f}{\partial y}\,dx - \int_{x_1}^{x_2}\alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\alpha(x)\,dx$$

for the extremum of I, then $\frac{dI}{dh}=0$

① Find the extremal for the function $\int_0^{\pi/2}(y^2-y'^2-2y\sin x)\,dx$,

$y(0)=0,\ y(\frac{\pi}{2})=1.$

Sol: Given: $I=\int_{x_1}^{x_2} f(x,y,y')\,dx$

$$=\int_0^{\pi/2}[y^2-y'^2-2y\sin x]\,dx$$

$\therefore f(x,y,y')=y^2-y'^2-2y\sin x$

$\therefore$ WKT for the extremum of I

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

$$\Rightarrow (2y-2\sin x)-\frac{d}{dx}(-2y')=0$$

$$\Rightarrow (y-\sin x)-\frac{d}{dx}\left(-\frac{dy}{dx}\right)=0$$

$$\Rightarrow y-\sin x+\frac{d^2y}{dx^2}=0$$

$$\Rightarrow \frac{d^2y}{dx^2}+y=\sin x$$

$$\Rightarrow (D^2+1)y=\sin x$$

$\therefore$ Auxiliary equation is

$$m^2+1=0$$

$$\Rightarrow m^2=-1$$

$$\Rightarrow m=0\pm i$$

$\therefore y_c=c_1\cos x+c_2\sin x$

$$\therefore y_p=\frac{\sin x}{D^2+1}$$

$$= \frac{-x}{2(1)} \cos x$$

$$y_p = -\frac{x}{2} \cos x$$

$\therefore y = y_c + y_p$

$\Rightarrow y = c_1 \cos x + c_2 \sin x - \frac{x}{2} \cos x \rightarrow ①$

where $x = 0 \Rightarrow y = 0$

$① \Rightarrow 0 = c_1(1)$

$c_1 = 0$

where $x = \frac{\pi}{2} \Rightarrow y = 1$

$① \Rightarrow 1 = c_1(0) + c_2(1) - 0$

$c_2 = 1$

$① \Rightarrow y = \sin x - \frac{x}{2} \cos x$

② On what curves can the functional $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dx$, $y(0) = 0$, $y(\frac{\pi}{2}) = 0$ be extremised.

Sol: Given: $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dx = \int_{x_1}^{x_2} f(x, y, y')\,dx$

$\therefore f(x, y, y') = y'^2 - y^2 + 2xy$

$\therefore \frac{\partial f}{\partial y} = 0 - 2y + 2x = 2x - 2y$

$\frac{\partial f}{\partial y'} = 2y' - 0 + 0 = 2y'$

$\therefore$ wkt $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$\Rightarrow (2x - 2y) - \frac{d}{dx}(2y') = 0$

$\Rightarrow x - y - \frac{d}{dx}(y') = 0$

$\Rightarrow x - y - y'' = 0$

$\Rightarrow y'' + y = x$

$\Rightarrow (D^2+1)y = x \qquad D = \frac{d}{dx}$

$\therefore$ The Auxillary equation is $m^2+1=0$

$m = 0 \pm i$

$\therefore y_c = c_1 \cos x + c_2 \sin x$

$\Rightarrow y_p = \frac{x}{D^2+1}$

$y_p = (1+D^2)^{-1} x$

$y_p = (1 - D^2 + D^4 - D^6 + \dots) x$

$y_p = x$

$\therefore y = y_c + y_p$

$\Rightarrow y = c_1 \cos x + c_2 \sin x + x \rightarrow (1)$

when $x = 0 \Rightarrow y = 0$

(1) $\Rightarrow 0 = c_1(1) + 0 + 0$

$c_1 = 0$

when $x = \frac{\pi}{2} \Rightarrow y = 0$

$\therefore$ (1) $\Rightarrow 0 = c_1(0) + c_2(1) + \frac{\pi}{2}$

$\Rightarrow c_2 = \frac{-\pi}{2}$

$$\boxed{y = \frac{-\pi}{2} \sin x + x}$$

(3) Find the extremal of the functional $\int_0^{\pi/2} (y'^2 - y^2 + 4y \cos x)\, dx$ $y(0) = 0 = y(\pi/2)$.

Given: Let $I = \int_{x_1}^{x_2} f(x, y, y')\, dx = \int_0^{\pi/2} (y'^2 - y^2 + 4y \cos x)\, dx$

$\therefore f(x,y,y') = y'^2 - y^2 + 4y\cos x$

$\therefore \frac{\partial f}{\partial y} = 0 - 2y + 4\cos x = -2y + 4\cos x$

$\frac{\partial f}{\partial y'} = 2y'$

$\therefore$ WKT $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$\Rightarrow -2y + 4\cos x - \frac{d}{dx}(2y') = 0$

$\Rightarrow -y + 2\cos x - y'' = 0$

$\Rightarrow y'' + y = 2\cos x$

$\Rightarrow (D^2+1)y = 2\cos x$

$\therefore$ The Auxillary equation is

$m^2+1 = 0$

$m^2 = -1$

$m = 0 \pm i$

$\therefore y_c = c_1\cos x + c_2\sin x$

$\therefore y_p = \frac{2\cos x}{D^2+1}$ $\qquad \frac{\cos ax}{D^2+a^2} = \frac{x}{2a}\sin ax$

$= \frac{2x}{2(1)}\sin x$ $\qquad \frac{\sin ax}{D^2+a^2} = \frac{-x}{2a}\cos ax$

$y_p = x\sin x$

$\therefore y = y_c + y_p$

$y = c_1\cos x + c_2\sin x + x\sin x \rightarrow (1)$

where $x=0 \Rightarrow y=0$

$\therefore (1) \Rightarrow 0 = c_1(1) + 0 + 0$

$c_1 = 0$

when $x = \pi/2 \Rightarrow y = 0$

① ⇒ $0 = c_1(0) + c_2(1) + \frac{\pi}{2}(1)$

$c_2 = -\frac{\pi}{2}$

$\therefore y = -\frac{\pi}{2}\sin x + x\sin x$

④ Find the extremal of the function $\int_0^1 [y'^2 - y^2 - ye^{2x}]dx$ that passes through the point $(0,0)$ $(1, \frac{1}{e})$.

Sol: Let $I = \int_{x_1}^{x_2} f(x,y,y')dx = \int_0^1 [y'^2 - y^2 - ye^{2x}]dx$

$\therefore f(x,y,y') = y'^2 - y^2 - ye^{2x}$

$\Rightarrow \frac{\partial f}{\partial y} = -2y - e^{2x}$

$\Rightarrow \frac{\partial f}{\partial y'} = 2y'$

$\therefore$ WKT

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$\Rightarrow -2y - e^{2x} - \frac{d}{dx}(2y') = 0$

$\Rightarrow -y - \frac{1}{2}e^{2x} - y'' = 0$

$\Rightarrow y'' + y = -\frac{1}{2}e^{2x}$

$\Rightarrow (D^2+1)y = -\frac{1}{2}e^{2x}$

$\therefore$ The Auxiliary equation is

$m^2 + 1 = 0$

$\Rightarrow m = 0 \pm i$

$\therefore y_c = c_1\cos x + c_2\sin x$

$\therefore y_p = -\frac{1}{2}\frac{e^{2x}}{D^2+1}$

$$= -\frac{1}{2}\frac{e^{2x}}{4+1}$$

$$y_p = -\frac{e^{2x}}{10}$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = c_1 \cos x + c_2 \sin x - \frac{e^{2x}}{10} \rightarrow ①$$

when $x = 0 \Rightarrow y = 0$

$$\therefore ① \Rightarrow 0 = c_1(1) - \frac{1}{10}$$

$$c_1 = 0.1$$

when $x = 1 \Rightarrow y = \frac{1}{e}$

$$\Rightarrow \frac{1}{e} = c_1 \cos(1) + c_2 \sin(1) - \frac{e^2}{10}$$

$$\Rightarrow 0.3679 = (0.1)(0.5403) + c_2(0.8457) - 0.7389$$

$$c_2 = 1.2511$$

$$y = (0.1)\cos x + (1.2511)\sin x - \frac{e^{2x}}{10}$$

⑤ Find the extremal of the function $\int_{x_1}^{x_2} (y'^2 - y^2 + 2y\sec(x))dx$

Sol: Let $I = \int_{x_1}^{x_2} f(x, y, y')dx = \int_{x_1}^{x_2} (y'^2 - y^2 + 2y\sec x)dx$

$$\therefore f(x, y, y') = y'^2 - y^2 + 2y\sec x$$

$$\frac{\partial f}{\partial y'} = 2y'$$

$$\therefore \text{wkt } \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\Rightarrow 2y + 2\sec x - \frac{d}{dx}(2y') = 0$$

$\Rightarrow -y + \sec x - \frac{d}{dx}(y') = 0$

$\Rightarrow -y + \sec x - y'' = 0$

$\Rightarrow y'' + y = \sec x$

$\Rightarrow (D^2+1)y = \sec x$

$\therefore$ The Auxiliary equation is

$m^2+1=0$

$m = 0 \pm i$

$\therefore y_c = c_1 \cos x + c_2 \sin x$

$\therefore$ Let $y = Ay_1 + By_2$

when $y_1 = \cos x$, $y_2 = \sin x$

$y_1' = -\sin x$, $y_2' = \cos x$

$w = y_1 y_2' - y_2 y_1'$

$= (\cos x)(\cos x) - (\sin x)(-\sin x)$

$w = 1$

$\therefore A = -\int \frac{y_2 Q(x)}{w} dx + K_1$

$= -\int \frac{\sin x \cdot \sec x}{1} dx + K_1$

$= -\int \frac{\sin x}{\cos x} dx + K_1$

$= \int -\frac{\sin x}{\cos x} dx + K_1$

$A = \log(\cos x) + K_1$

$\therefore B = \int \frac{y_1 Q(x)}{w} dx + K_2$

$= \int \frac{\cos x \cdot \sec x}{1} dx + K_2$

$= \int 1 \, dx + K_2$

$B = x + K_2$

$\therefore\ y = \left[\log(\cos x) + K_1 \int \log x + (x + K_2)\sin x\right]$

(6) Prove that geodesic of a plane surface are straight lines.

Sol:

Let $$S = \int_{x_1}^{x_2} \frac{ds}{dx}\,dx$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\,dx$$

$$\therefore f(x, y, y') = \sqrt{1 + y'^2}$$

$$\therefore \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{1+y'^2}}\,2y'$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$\therefore$ WKT
$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\Rightarrow 0 - \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0$$

$$\Rightarrow \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0$$

$$\Rightarrow \frac{\left(\sqrt{1+y'^2}\right)y'' - y'\cdot\frac{1}{2\sqrt{1+y'^2}}\cdot 2y'y''}{\left(\sqrt{1+y'^2}\right)^2} = 0$$

$$\Rightarrow y''\sqrt{1+y'^2} - \frac{y'^2\cdot y''}{\sqrt{1+y'^2}} = 0$$

$\Rightarrow y''(1+y'^2) - y'^2 \cdot y'' = 0$

$\Rightarrow y'' + y'^2 y'' - y'^2 y'' = 0$

$y'' = 0$

$D^2 y = 0 \quad , \quad D = \frac{d}{dx}$

$\therefore$ A.E. is $m^2 = 0$

$m = 0, 0$

$\therefore \boxed{y = c_1 + c_2 x} \rightarrow$ Straight line equation.