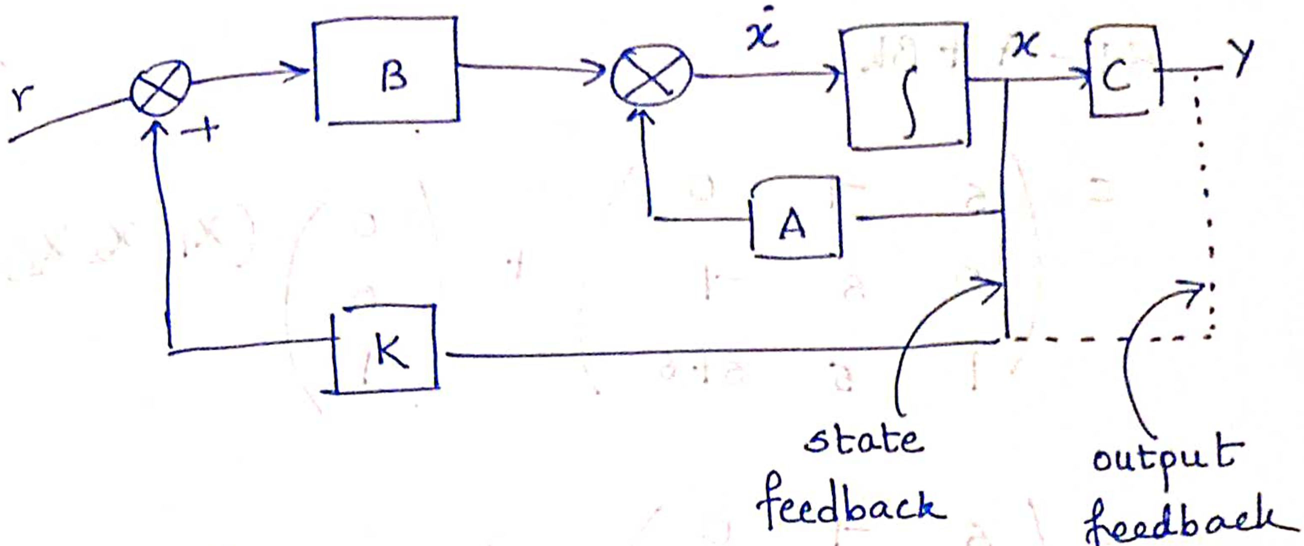
Design in State Space :

State-feedback / Pole placement



$$\dot{x} = Ax + B(r - kx)$$

$$\dot{x} = (A - Bk)\,x + Br$$

$K \rightarrow$ Gain Matrix.

* Check controllability.
* $k = [k_1 \;\; k_2 \; ------ \; k_n]$

$$sI - (A - Bk) = 0$$

$$sI - A + Bk = 0$$

DCF : Direct Substitution Method.

$$(s - u/_1)(s - u/_2) \cdots\cdots (s - u/n) = 0$$

Ques: $\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u.$

Desired eigen values : $-2 \pm j6$, -1

Determine K.

Sol: $\longrightarrow$ it is controllable $\because$ it is given in CCF.

$sI - A + Bk$

$$= \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

$$= \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

$$= \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1+x_1 & 5+x_2 & s+6+x_3 \end{pmatrix}$$

$$|sI - A + Bk| = 0$$

$$s^3 + (6 + \overset{x}{k_3})s^2 + (5 + x_2)s + 1 + \overset{x_1}{k_1} = 0 \quad \text{—(1)}$$

Now,

$\mu_1 = -2 + j6$

$\mu_2 = -2 - j6$

$\mu_3 = -1$

$$(s-(-2+j6))(s-(-2-j6))(s+1)=0$$

$$((s+2)^2+4^2)(s+1)=0$$

$$s^3+5s^2+24s+20=0 \quad \text{—— 2)}$$

Comparing ① & ②,

$$6+x_3=5$$

$$x_3=1$$

$$5+x_2=24$$

$$x_2=19$$

$$1+x_1=20$$

$$x_1=19$$

Type: Ackermman's method:

$$\phi(A)=A^n+\alpha_1 A^{n-1}+\cdots+\alpha_{n-1}A+\alpha_n A$$

$$x=[0 \;\; 0 \;\; \cdots \;\; 1][B \;\; AB \;\cdots\; A^{n-1}B]\,\phi(A)$$

$$(s-\mu_1)(s-\mu_2)\cdots(s-\mu_n)$$

$$=s^n+\alpha_1 s^{n-1}+\cdots\alpha_n.$$

Prev. eq:

$$k=[0 \;\; 0 \;\; 1][B \;\; AB \;\; A^2B]^{-1}\,\phi(A)$$

$$\phi(A)=A^3+5A^2+24A\;20I_3.$$

$$\phi(A)=A^3+5A^2+24A+20I_3.$$

$$k=(0 0 1)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{pmatrix}^{-1}$$

$$K = (1 \quad -6 \quad 31) \begin{pmatrix} -1 & -19 & -1 \\ 1 & 4 & 25 \\ 25 & -124 & -146 \end{pmatrix}$$

Ques: Design a state feedback controller for the system:

$$\dot{X} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 10 \\ 1 \\ 0 \end{pmatrix} u$$

Desired poles $-1 \pm j2, -6$.

* - in exam 2nd order underdamped sys, no poles will be given.
  t_s, t_p and all will be given, so learn all formulae.
  - directly find the poles using formula.

Use Transformation Method & check controllability

$$Q_c = \begin{pmatrix} 10 & -10 & 0 \\ 1 & 8 & -26 \\ 0 & 21 & -75 \end{pmatrix} \qquad |Q_c| \neq 0$$

$$\rho |Q_c| = 3.$$

desired DCE = $(s+1-j2)(s+1+j2)(s+6) = 0$

$$s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0.$$

actual CE

$$K = (\alpha_3 - a_3 \quad \alpha_2 - a_2 \quad \alpha_1 - a_1)\, P_c$$

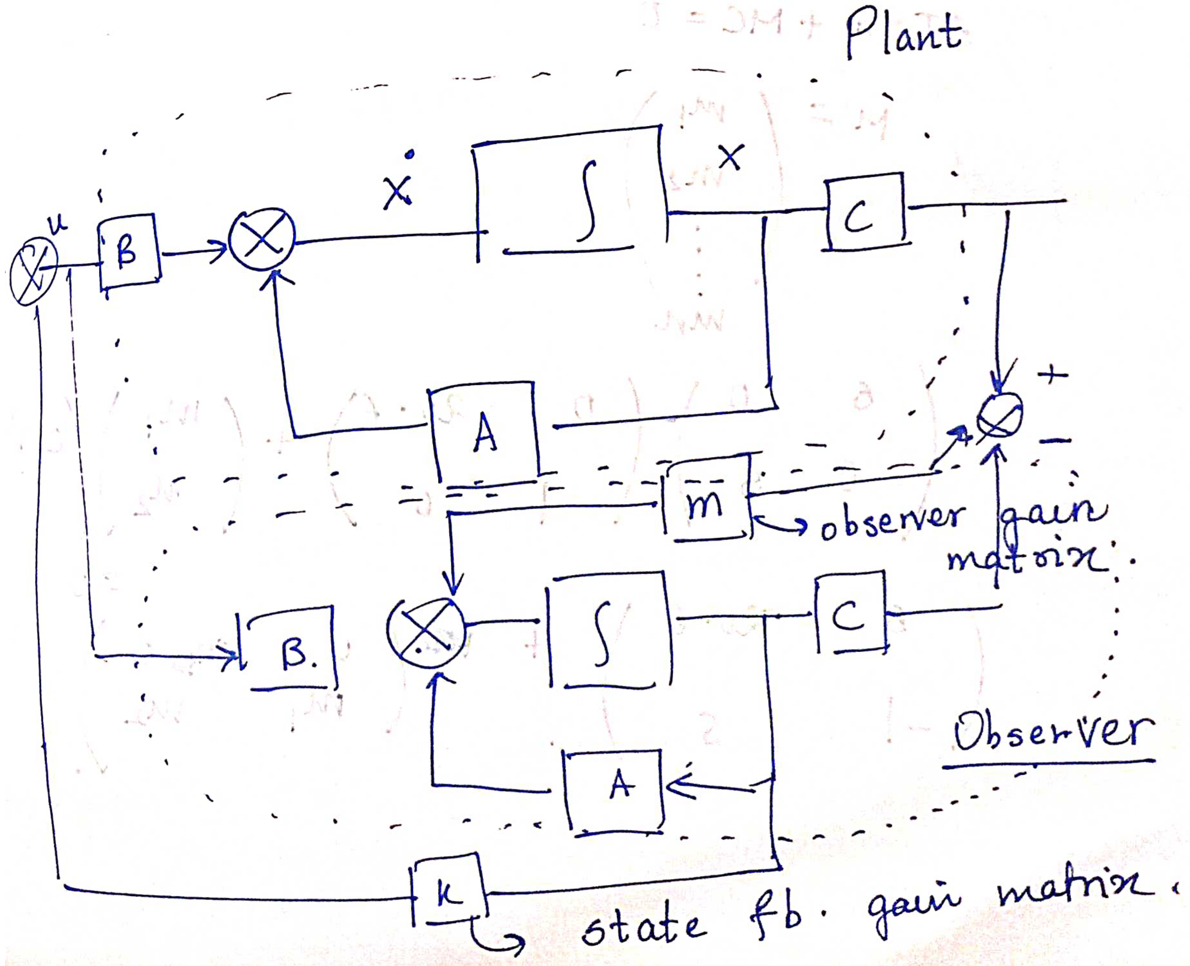
D.C.Q → desired char. eqn.

$$P_c = \begin{bmatrix} P_1 \\ P_1 A \\ P_1 A^2 \end{bmatrix}$$

$$P_1 = [0 \quad 0 \quad 1]\, Q_c^{-1}$$

$0 = |sI - A| \Rightarrow$ ACE → $s^3 + 17s^2 + 11s + 6$.

$$K = (-0.22 \quad 4.22 \quad -2).$$



1. Full order

2. Reduced order.

Error $= e = y - c\hat{x} \longrightarrow e' = \dot{y} - \dot{c}\hat{x}$

$\dot{\hat{x}} = A\hat{x} + Bu + m(y - \hat{x})$

Ques: Design a full order observe for the sys. $\dot{x} = \begin{pmatrix} 0 & 20.6 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$

$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$

Q. Observer poles $= -10, -10$.

1. Observability.

$sI - A + MC = 0$

$M = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix}$

$\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 20.6 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = 0$

$\begin{pmatrix} s & -20.6 \\ -1 & s \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ m_1 & m_2 \end{pmatrix}$

$s^2 + m_2 s + m_1 - 20.6$

$$m = \begin{pmatrix} 120.6 \\ 20 \end{pmatrix}.$$

$$m_1 = 120.6$$

$$m_2 = 20.$$

full order observer system :

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{pmatrix} = \begin{pmatrix} 0 & -100 \\ 1 & 20 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

2nd step.

$$+ \begin{pmatrix} 120.6 \\ 20 \end{pmatrix} y. \cancel{(0\ 1)\ (x_1}$$

1st step

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -100 \\ 1 & 20 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u +$$

$$\begin{pmatrix} -120.6 \\ 20 \end{pmatrix} \left[y - (0\ 1) \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} \right]$$

2) Ackermann's method

$$m = \phi(A) \cdot Q_0^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\phi(A) = A^2 + \alpha_1 A + \alpha_2 I$$

DCE. $(s+10)^2 \Rightarrow s^2 + \underbrace{20}_{\alpha_1}s + \underbrace{100}_{\alpha_2} = 0$

$$A = \begin{pmatrix} 0 & 20.6 \\ 1 & 0 \end{pmatrix}$$

CCF or OCF

↳ trans. matrix is identity matrix.

$$\phi(A) = A^2 + 20A + 100 I_2$$

$$\hookrightarrow \begin{pmatrix} 120.6 & 412 \\ 20 & 120.6 \end{pmatrix}$$

$$m = \begin{pmatrix} 120.6 \\ 20 \end{pmatrix}$$

$$m = \begin{pmatrix} 120.6 & 412 \\ 20 & 120.6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

3) Transformation Method → useful when in ~~tra~~ standard form.

since A in OCF,

$$m = \begin{pmatrix} \alpha_2 & -\alpha_2 \\ \alpha_1 & -\alpha_1 \end{pmatrix}$$

Char eqn $= |sI-A|^{\cancel{e}} = s^2 - 20.6 = 0$

$$m = \begin{pmatrix} 100 + 20.6 \\ 20 \end{pmatrix} = \begin{pmatrix} 120.6 \\ 20 \end{pmatrix}$$

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & 6 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$-2 \pm j4, -1$$

$|sI-A| = 0$

$(s-\mu_1)(s-\mu_2)(s-\mu_3) = 0.$

$$sI - A = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ s+1 & 5 & s+6 \end{vmatrix} = 0.$$

$s^3 + 5s^2 + 24s + 20 = 0.$

$s^3 + 6s^2 + 5s + 1 = 0.$

~~s³+6s²+5s~~

$$k = \begin{pmatrix} 20 & -1 \\ 24 & -5 \\ 5 & -6 \end{pmatrix} = \begin{pmatrix} 19 \\ 19 \\ -1 \end{pmatrix}$$

Reduced Order Observer

$$x(t) = \begin{bmatrix} x_a(t) \\ x_b(t) \end{bmatrix}$$

$x_a(t) \rightarrow$ measurable i/p / known inputs

$x_b(t) \rightarrow$ unmeasurable i/p.

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du.$$

$$\begin{pmatrix} \dot{x}_a \\ \dot{x}_b \end{pmatrix} = \begin{pmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{pmatrix} \begin{pmatrix} x_a \\ x_b \end{pmatrix} + \begin{pmatrix} B_a \\ B_b \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_a \\ x_b \end{pmatrix}$$

$$\dot{x}_a = A_{aa} \cdot x_a + A_{ab} \cdot x_b + B_a \cdot u.$$

$$\underbrace{\dot{x}_a - A_{aa} x_a - B_a u}_{\text{Known i/p}} = A_{ab} \cdot x_b. \quad \rightarrow 1)$$

$$\dot{x}_b = A_{ba} x_a + A_{bb} x_b + B_b u.$$

$$\dot{x}_b = A_{bb} \cdot x_b + (A_{ba} \cdot x_a + B_b u)$$

$$y = x_a \rightarrow \text{known.}$$

$$\dot{y} = \dot{x}_a \rightarrow 2)$$

Put 2) in 1)

$$\underbrace{\dot{y} - A_{aa}\cancel{x}\cdot y - B_a u}_{y} = \underset{C}{A_{ab}} \cdot \underset{X}{x_b} \quad \text{— o/p. eqn.}$$

$X \rightarrow$ mapping.

$$\dot{\hat{x}} = A\hat{x} + Bu + m(y - C\hat{x})$$

Memorise the eqn.

R.O. Obs. eq.
→ Reduced order

$$\dot{\hat{x}}_b = (A_{bb} - mA_{ab})\hat{x}_b + (A_{ba} - mA_{aa})y$$
$$+ (B_b u - mB_a)u + m\dot{y}$$

derivative → introduces noise

$$x_b - \overset{m}{\hat{y}} = x_b'$$

~~observable~~ observer. eqn

Ques: Design a reduced order observer. eqn for satellite control sys.

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$s = -10$.

Obtain ROO eqn.

$$\dot{x} = \left(\begin{array}{c:c} \overset{A_{aa}}{0} & \overset{A_{ab}}{1} \\ \hdashline \underset{A_{ba}}{0} & \underset{A_{bb}}{0} \end{array}\right) \begin{pmatrix} x_a \\ x_b \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

B_a (top entry), B_b (bottom entry)

$$|sI - A + mC| = 0.$$

$$s - 0 + m_1 \times 1 = 0.$$

$$s + m_1 = 0.$$

$5 + 10 = 0$

$m = 10$

Pole Placement:

$\mu \rightarrow$ closed loop poles

$u(t)$, $y(t) \rightarrow$ scalars.

$r(t) \rightarrow$ reference i/p.

$u(t) = -Kx(t)$.

$$\dot{x} = Ax + Bu(t)$$
$$= Ax(t) - BKx(t)$$
$$= (A - BK)x(t).$$

Characteristic Eqn:

$$|sI - A + BK| = 0$$

Direct Substitution Method:

Step 1: Check controllability.

If not controllable, K is not possible.

Step 2: Assume $K = (x_1 \quad x_2 \quad x_3)$

Step 3: Equate characteristic eqn to the desired poles

$$|sI - A + BK| = (s-\mu_1)(s-\mu_2)(s-\mu_3)\ldots$$

Equate the coefficients on both sides & find x_1, x_2, x_3.

Ackermann's Method :

Step 1: $\phi(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1}A + \alpha_n$.

$(s-\mu_1)(s-\mu_2)\cdots(s-\mu_3) = s^n + \alpha_1 s^{n-1} \cdots \alpha_n$.

→ equating coeff on both sides we can find α

Step 2:

$k = (0 \; 0 \; 1)\,(B \;\; AB \;\; A^2B)^{-1}\,\phi(A)$.

Transformation Method :

$|sI-A| \longrightarrow$ ACE $\longrightarrow$ get coeff $a_1, a_2 \cdots a_n$.

$(s-\mu_1)(s-\mu_2)(s-\mu_3) \longrightarrow$ DCE.

$s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0$.

$k = (\alpha_3 - a_3 \quad \alpha_2 - a_2 \quad \alpha_1 - a_1) \cdot P_c$.

$$P_c = \begin{bmatrix} P_1 \\ P_1 A \\ P_1 A^2 \end{bmatrix}$$

$P_1 = [0 \quad 0 \quad 1]\, Q_c^{-1}$

Module 4

Non-linear control

- System does not obey super-position & homogenity.

1. Friction

$$y = B \cdot \frac{dx}{dt}$$

↳ viscous friction.

Two types of non-linearities:

↙ inherent

↘ intentional

columb friction:- non-linear frictⁿ

2. Spring:

$y = mx. \longrightarrow$ linear

NL: $y = kx + k_1 x^2$

$x_1 \rightarrow y_1$

$x_2 \rightarrow y_2$

$x_1 + x_2 \rightarrow y_3 = y_1 + y_2$

} ~~linear~~ superposition.

$y_1 = kx_1$

$y_2 = kx_2$.

$x_3 = x_1 + x_2$

$y_3 \Rightarrow k(x_1 + x_2) = kx_3.$

} homogeneous.

$y_1 = k x_1 + k_1 x_1^3$

$y_2 = k x_2 + k_2 x_2^3$.

Practice linear differential & non-linear differential.

$\frac{d^2x}{dt^2} \longrightarrow$ linear $\qquad \left(\frac{d^2x}{dt^2}\right)^n \longrightarrow$ non-linear.

** Non-linear ques of diff eqn confirmed in CAT 2.

Non-linear spring

$y = kx + k_1 x^3$

$\hookrightarrow \begin{cases} \text{hard} & k_1 > 0 \\ \text{soft} & k < 0 \\ \text{linear} & k = 0. \end{cases}$

3. Backlash - delay in the output.

4.

1. Harmonics - freq > fundamental freq.
2. Sub-harmonics - freq < fundamental freq.

$\Big\}$ aper-iodic

3. Jump resonance. - discontinuity in response.
4. Stability
5. Periodic Oscillations - limit cycle.

Phase - plane analysiss. — 20 Marks.

Describing func

↳ we write math. TF.

Derivations there.

Isoplane / Delta.

Finite Escape time

At $t \to \infty$ object reaches final state.

for non linear. $x^* \to$ initial state.

For linear time taken is less than ∞.

Isolated Equilibrium

Non-linear systems have many equilibrium points.

Autonomous sys ⇒ no input. → response is due to initial condⁿ.

$$\dot{x} = Ax + \cancel{Bu}^{\,0}.$$

Chaos : Randomness

↳ even though def. i/p is given, o/p is not predictable.

Multiple modes: Eigen values are modes of system.

Linear sys — 1 mode

non linear sys - multiple modes under unforced cond, i.e. autonomous control.

Inherent $\longrightarrow$ friction, spring, backlash.

Intentional $\longrightarrow$ 1) Relay

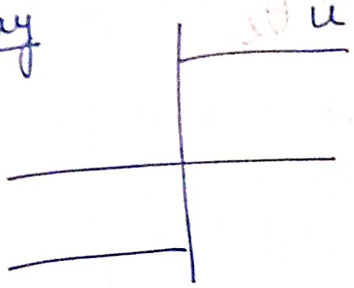
2) Saturation.

3) Deadzone.

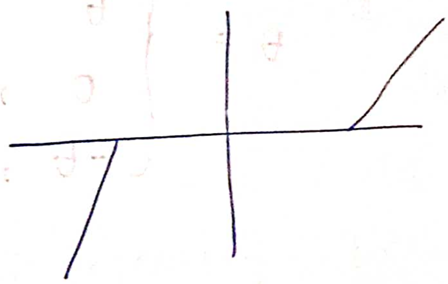
4) Quantization.

5) Hysteris.

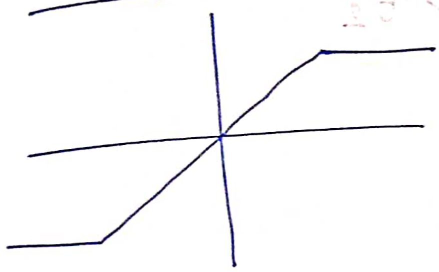
1) Relay



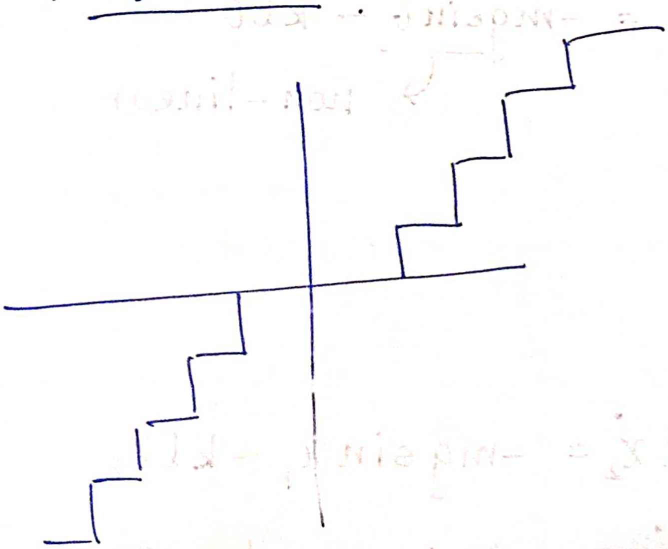
3) Deadzone



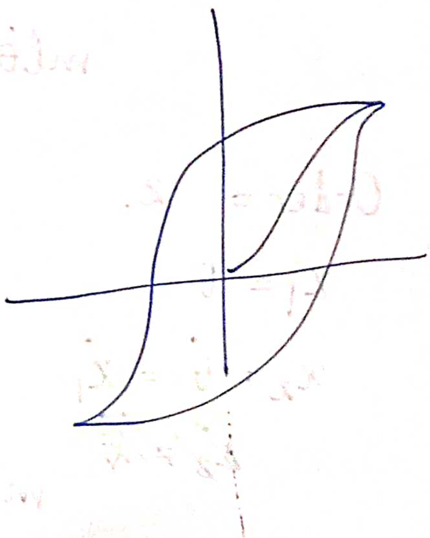
2) Sat^n



4) Quantizat^n



5)



Relay, Sat^n, Deadzone & Hys. can be described using TF.

1) Relay:

$$\theta = \begin{cases} 1 & , u > 0 \\ 0 & , u = 0 \\ -1 & , u < 1 \end{cases}$$

2) Sat^n:

$$y: \theta = \begin{cases} \theta_1 & y > \theta_1 \\ \theta & \theta_1 \leq \theta \leq \theta_2. \\ \theta_2 & y \leq \theta_2. \end{cases}$$

3) Deadzone.

$$\theta = \begin{cases} \theta & , \theta > \theta_1 \\ 0 & , \theta_1 > \theta > \theta_2 \\ -\theta & , \theta < \theta_2. \end{cases}$$

Expression for Simple Pendulum:

$$ml\ddot{\theta} = \underbrace{-mg\sin\theta}_{\text{non-linear}} - kl\dot{\theta}$$

Order = 2.

$x_1 = \theta$

$x_2 = \dot{\theta} = \dot{x}_1$

$\cancel{\dot{x}_2 \neq \dot{\theta}}$

$$ml\dot{x}_2 = -mg\sin x_1 - klx_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2.$$

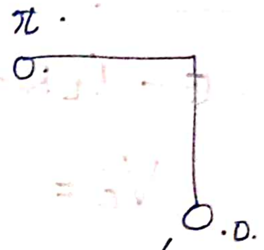
To attain equilibrium point, (singular point).

$\dot{x} = 0 \longrightarrow$ condn.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_3 \end{pmatrix} = 0.$$

$\therefore \dot{x}_1 = \dot{\theta} = 0.$

$$-\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 = 0$$



$x_1 = n\pi \longrightarrow$ general soln.

$n = 0, 1.$

$x_2 = 0.$

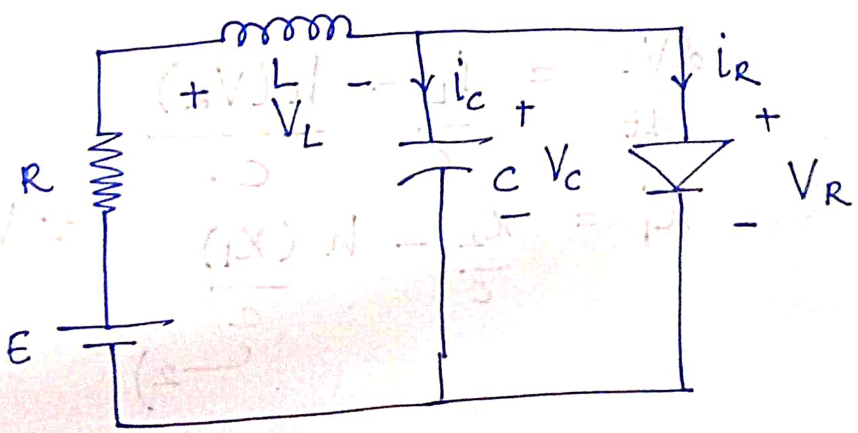
$\therefore (0, 0), (0, \pi)$

most stable eq. point.

Non-linearities $\longrightarrow$ with memory. (backlash) $\downarrow$ depends on the history.

$\quad\quad\quad\quad \hookrightarrow$ without memory.

Tunnel Diode



2 state variables — i_L, V_c.

$$i_R = C \cdot \frac{dV_c}{dt}$$

$$V_L = L \cdot \frac{di_L}{dt}$$

$V_R = h(V_R) \longrightarrow$ non-linear

remaining are linear.

KVL

$E - i_L R - V_L - V_c = 0.$

$V_c = E - i_L R - V_L.$

$$E - i_L R - L \cdot \frac{di_L}{dt} \cdot - V_c = 0.$$

$$\frac{di_L}{dt} = -\frac{V_c}{L} + \frac{E}{L} - i_L \cdot \frac{R}{L}.$$

$$\dot{x_2} = -\frac{x_1}{L} + \frac{E}{L} - x_2 \cdot \frac{R}{L} \quad \text{—1)}$$

$\{ V_c = V_R.$

same for current & voltage.

KCL

$i_C + i_R = i_L$

$$C \cdot \frac{dV_c}{dt} + h(V_R) = i_L.$$

$$\frac{dV_c}{dt} = \frac{i_L}{C} - \frac{h(V_R)}{C}.$$

$$\dot{x_1} = \frac{x_2}{C} - \frac{h(x_1)}{C} \quad \text{—2)}$$

$\because V_R = V_c.$

$\dot{x}_1 = 0$ for eq. point.

$\dot{x}_2 = 0$.

$\therefore \cancel{x_2 = 0,\ h(x_1) = 0}$.

$\cancel{x_1 = 0,\ E = 0,\ x_2 = 0}$.

$$0 = \frac{x_2}{c} - \frac{h(x_1)}{c}$$

$$x_2 = h(x_1).$$

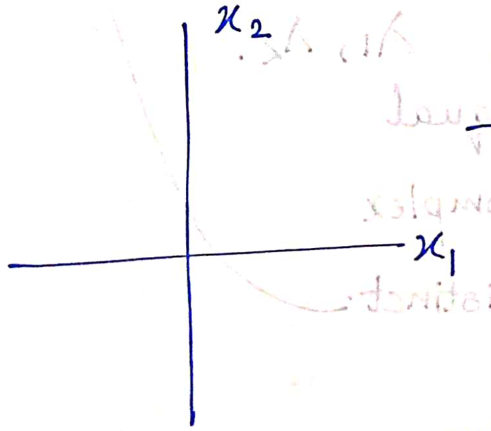
$$-\frac{x_1}{L} + \frac{E}{L} - x_2\frac{R}{L} = 0.$$

$$x_1 = E - x_2 R.$$

$$x_1 = E - h(x_1) \cdot R.$$

$$\boxed{h(x_1) = \frac{E - x_1}{R}}$$

Any non-linear sys. we convert into linear. second order & we find equilibrium points.



Phase-plane.

System moves towards equilibrium point & we analyse it using phase variable.

Trajectory - based on traj. we analyse the system.

Linearization technique - Taylor series

Module 5

$\dot{x} = f(x, u)$ } → generalized eqn.
$\dot{x} = f(x)$

Characteristics

1. Nodal point — stable, unstable, saddle.

Different types of equilibrium points.

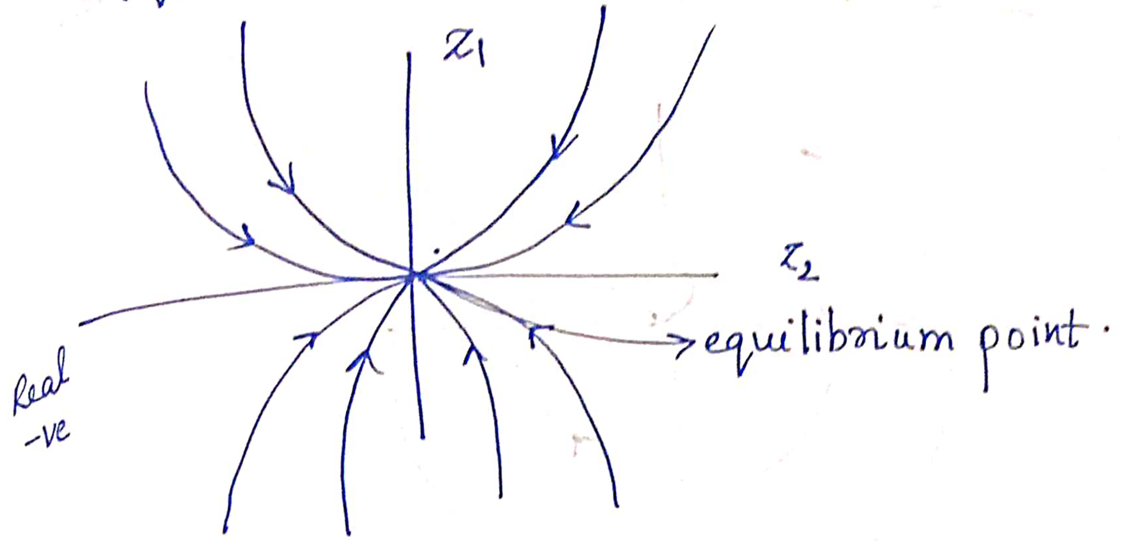
2. Focus point — stable, unstable, centre / vertex.

3. Center / Vertex. point.

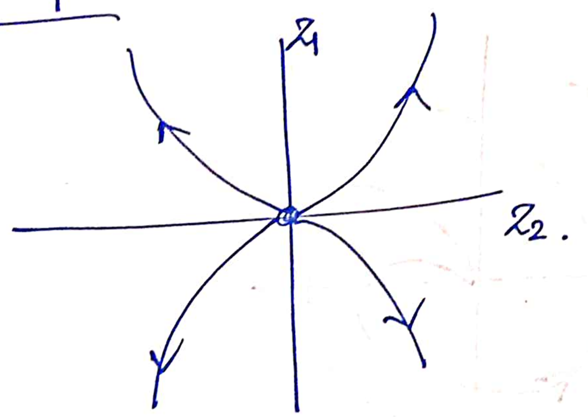
if. $\dot{X} = AX = 0$. λ_1, λ_2.

1) Real & equal
2) complex complex
3) Real & distinct.
4) imaginary.
5) one eigen val = 0, Real.
6) multiple EV are zero.

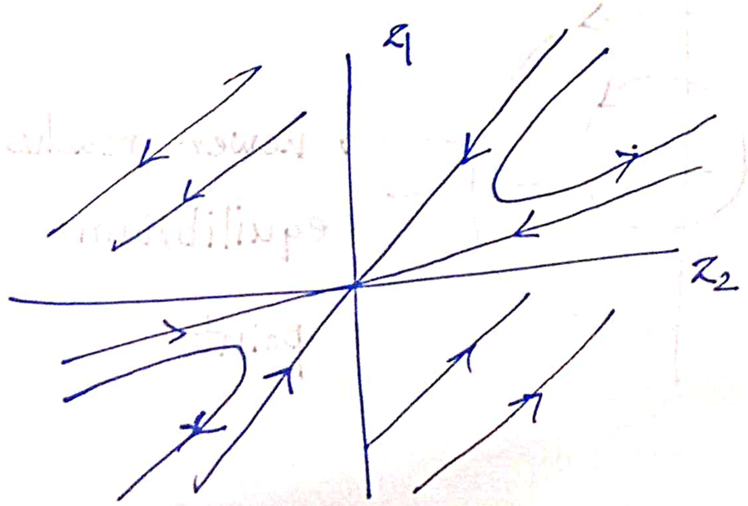
If eigen values are real & both are -ve we get a stable nodal point



If EV are real & both are +ve → unstable nodal point.

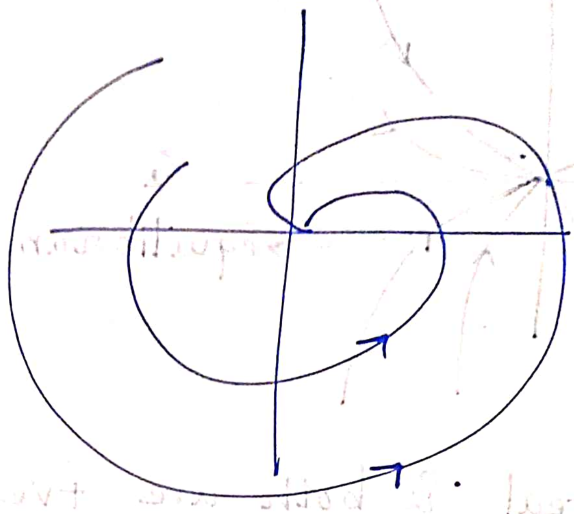


If EV are real & one +ve & one -ve → saddle nodal point

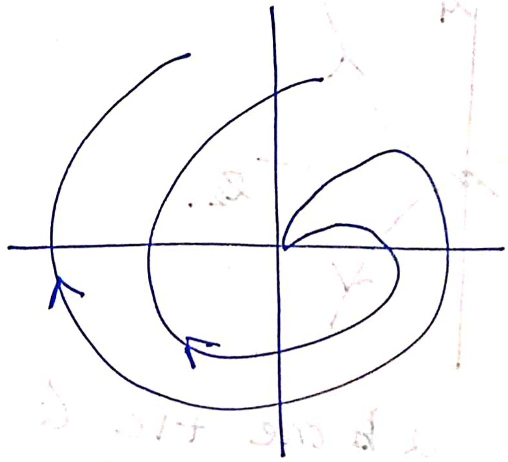


Complex – Focus point.

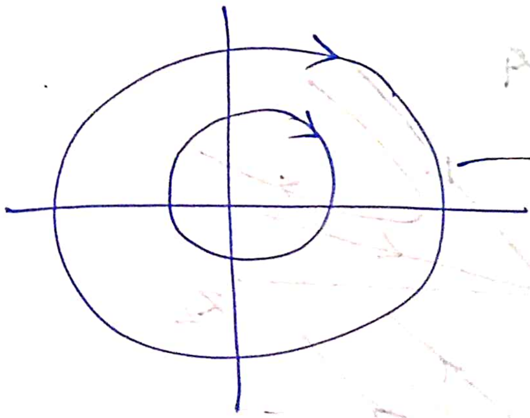
Stable focus pt → Real part –ve.



Unstable focus pt → Real part +ve

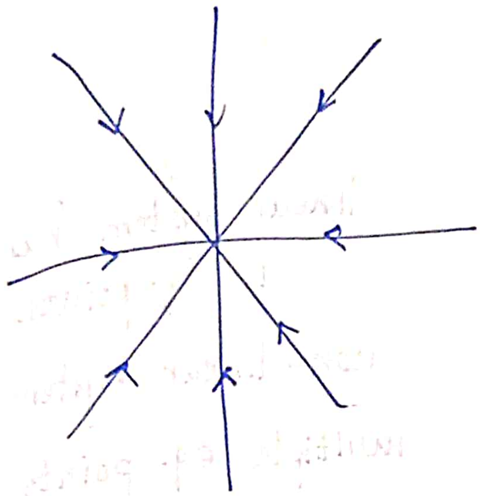


Center pt → Real part = 0. imaginary

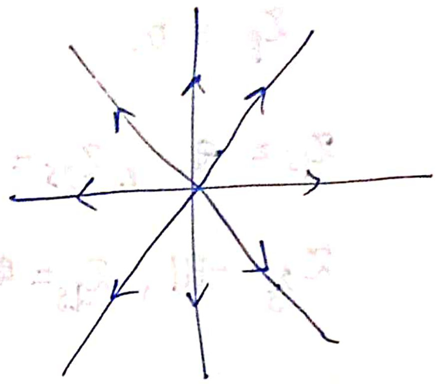


→ newer reaches equilibrium point.

EV=0 i.e. $\lambda_1 = 0$, $\lambda_2 =$ real −ve.



$\lambda_1 = 0$, $\lambda_2 =$ real +ve



Takee one cond of EV, & make phase plane trajectory.

Solve any design problem – Reduced observer.

Comment on research, what happening, why happening.

– Limit Cycle Problems for DA 2.

① $\dot{z_1} = z_2(z_1+1)$

$\dot{z_1} = z_1(z_2+3)$

$\dot{z}_s = 0.$

$z_{1s} = \cancel{4}$ & $z_{2s} = 0$

$z_{1s} = -\cancel{3}1$, $z_{2s} = -3.$

linear system has 1 eq. points.
non-linear system has multiple eq. points.

After linearization,

$$\dot{z} = \begin{pmatrix} z_{2s} & z_{1s}+1 \\ z_{2s}+1 & z_{1s} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$$

$\lambda I - A = 0.$

$\lambda = \pm\sqrt{3}$ → real & distinct
→ one +ve & one −ve.
→ trajectory is saddle point.

Take $(-1, -3)$ as eq. point.

$$\dot{z} = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \text{stable node}$$

Ques Examine the behaviour of the linear ~~framing~~ system ~~&~~ at equilibrium point.

$\dot{x_1} = -x_1$

$\dot{x_2} = -4x_2$

$\dot{x} = 0.$

$\left.\begin{array}{l} x_1 = 0. \\ x_2 = 0 \end{array}\right\} (0,0).$

* Use these problems for DA-2.

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$$

$$\lambda I - A = 0.$$

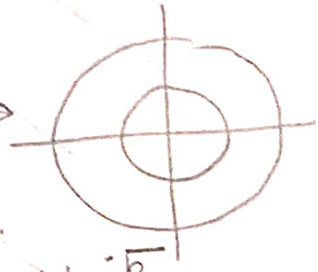
Stable node $\longrightarrow$ solution will always converge to $(0,0)$ for initial cond.

if: $\left.\begin{array}{l} \dot{x}_1 = -x_1 \\ \dot{x}_2 = +4x_2 \end{array}\right\}$ saddle point.

Ques: (1) $\left.\begin{array}{l} \dot{x}_1 = x_1 + 2x_2 \\ \dot{x}_2 = -x_1 + x_2. \end{array}\right\} \to$ stable system.

Determine the behaviour of the system.

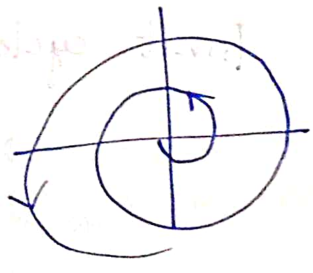
(2) $\left.\begin{array}{l} \dot{x}_1 = -x_1 \mp x_2 \\ \dot{x}_2 = 4x_1 + x_2. \end{array}\right\} \to$ center $\longrightarrow$



$\to$ eigen values $\pm j3$.

$\to$ eigen values : $1 \pm j\sqrt{2}$

(1) $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$



$\dot{x} = 0$, $\quad x_2 = -x_1/2.$

$x_1 = x_2.$

$\lambda I - A$

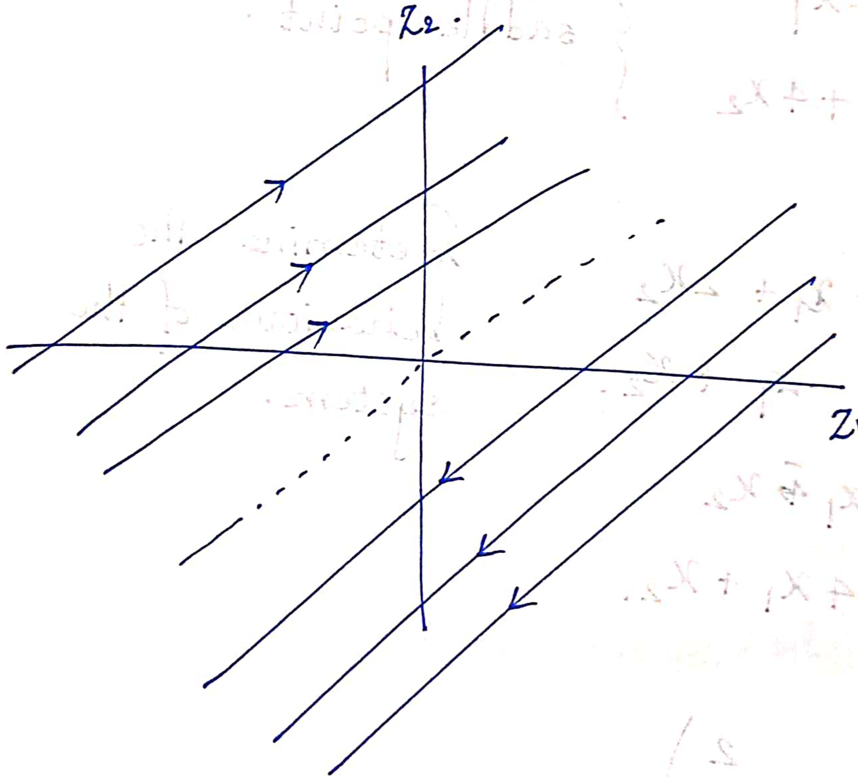
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

Ques: Consider a non-linear function: ← solve for

$$h(x) = 17.76 - 207.58\,x_1 + 688.86\,x_1 - 905.24\,x_1^3 + 41.86\,x^4 .$$

$$A = \begin{pmatrix} 0.5\,h(x_1) & 0.5 \\ -0.2 & -0.3 \end{pmatrix}$$

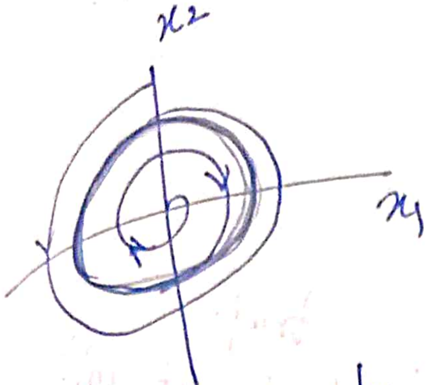
<u>if both EV are zero</u>



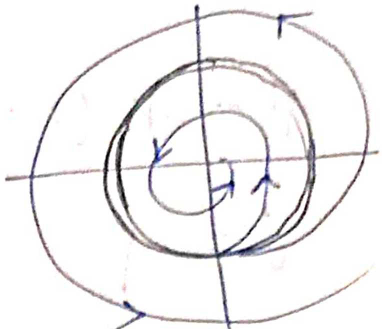
limit cycle is oscillations.

↳ whether : stable
unstable
semistable.

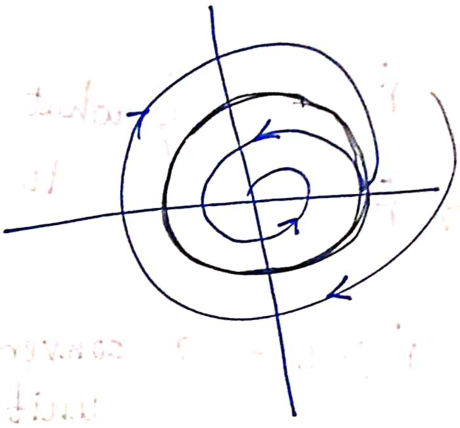
Limit cycles are periodic oscillations.
↳ isolated closed orbit.



if IC converges to limit cycle → stable



IC converges away from limit cycle
— unstable.



One converges & 1 diverges
— semi stable.

(Prepare for CAT)

Ques: Examine the stability of limit cycle.

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)$$

$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1).$$

$x_1 = r\cos\theta$

$x_2 = r\sin\theta$

~~$r\sin\theta(1-r^2+1)$~~

~~$r\sin\theta(2-r^2)$~~

$$\dot{x}_1 = r\cos\theta - r\sin\theta(r^2-1)$$

$$\dot{x}_2 = -r\cos\theta - r\sin\theta(r^2-1)$$

$\cancel{x_1 - x_2 = 2r\cos\theta}$

$\cancel{x_1 - x_2(\,) + x_1 + x_2(\,) = 2r\cos\theta}$.

$\dot{r} = -r\,(r^2-1)$

$\dot{\theta} = -1$

$x_1^2(0) + x_2^2(0) = r^2 = 1$. ↳ traj. starts at unit circle.

we need to analyze the trajectory.

radius of cicle so can't be <0

$\dot{r} = 0$.

$r<1 \Rightarrow \dot{r}$

$r>1 \Rightarrow \dot{r}$ } what happens to traj.

(*)(*) choose value & see if it is converging inside / outside the unit circle.

if $r<1$, $\dot{r}>0 \longrightarrow$ converging towards unit circle from insi

$r>1 \Rightarrow \dot{r}<0 \longrightarrow$ converges from outsic

Ans - stable.

Ques: $\dot{x_1} = x_2 - x_1\,(x_1^2 + x_2^2 - 1)$

$\dot{x_2} = -x_1 - x_2\,(x_1^2 + x_2^2 - 1)$.

$\dot{r} = r\,(r^2-1)$

$\dot{\theta} = -1$.

Diverge within circle. $\dot{r}>$

$\dot{r}<0$ Dive outside circle.

Ques: $\dot{x}_1 = x_2 - x_1 (x_1^2 + x_2^2 - 1)^2$

$\dot{x}_2 = -x_1 - x_2 (x_1^2 + x_2^2 - 1).$

$x_1 = r\cos\theta$

$x_2 = r\sin\theta.$

$\dot{r} = -r(r^2-1)^2$

$\dot{\theta} = -1$

$\left.\begin{array}{l} \dot{r} < 0 \rightarrow \text{diverge.} \\ \dot{r} > 0 \ \text{converge.} \end{array}\right\}$ semistable.

Isodine Method

↳ same slope

$$m = \frac{dx_2}{dx_1} = \frac{f_2(x_2, x_1)}{f_1(x_2, x_1)} = \alpha$$

$$f_2(x_2, x_1) = \alpha f_1(x_2, x_1)$$

Ques: A linear 2nd order system is described by : $\ddot{c} + 2\xi \omega_n \dot{c} + \omega_n^2 c = 0$

$\xi = 0.15$ $\omega_n = 1$ rad/s

↳ singular point.

~~Determine si~~

Construct phase trajectory using isodine method for IC

$c(0) = 1.5$ $\dot{c}(0) = 0$.

Step I: Obtain state space model.

$$\cancel{s \cdot c(0) \quad s^2 C(s) + 2\zeta\omega_n C(s) + \omega_n^2 \, \frac{c(s)}{s} = 0}$$

$$s^2 + 0.3s + 1 = 0.$$

$$-0.15 \pm j\,0.988. \rightarrow \text{stable}$$

$$m = \frac{dx_2/dt}{dx_1/dt}$$

$$c = x_1$$

$$\dot{c} = x_2 = \dot{x}_1$$

$$\dot{x}_2 = -(0.3x_2 + x_1)$$

$$m = \frac{-x_1 - 0.3x_2}{x_2}.$$

$$x_2 \cdot m = -x_1 - 0.3x_2.$$

$$mx_2 + 0.3x_2 + x_1 = 0$$

$$x_2 = \frac{-x_1}{m + 0.3}.$$

$$\boxed{m = \tan\alpha}$$

$m(x)$	-2		-1		-0.5		0		0.5		1		$2.$	
$\alpha.$	-63		-45		-26.56		0		26.56		45		$63.$	
	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2	x_1	x_2
	1	~~0.59~~ 0.6	1	~~1.43~~ 1.4	0.25	1.3	0.25	−0.8	1	−1.3	1	−0.8	1	−0.4
	2	~~1.18~~ 1.2	1.5	~~2.14~~ 2.1	0.5	−2.5	0.75	−2.5	2	−2.52		−1.52		−0.[illegible]

IC of the system (x_1, x_2)

$\hookrightarrow (c, \dot{c})$

$= (1.5, 0)$.

$$m = \frac{-1.5 - 0}{0} = \infty$$

$\therefore \angle = 90^\circ$.