Delayed updating and renaming in beta reduction of lambda terms

draft version

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Abstract

In this paper we focus on

1 Name-carrying lambda trees

1.1 Lambda-trees

We use a tree representation for lambda terms in (typed) lambda calculus. See Figure 1. The obtained tree is an undirected rooted tree with labels attached to the *edges* of the tree. We prefer labelling the edges instead of the vertices for reasons to be explained below.

Label L represents a 'lambda' and A stands for an 'application'. Moreover, we use label S for 'subordination'; this extra label enables us to discriminate between main branches and subbranches, which is essential when considering different pathes in a tree.

NOTE: When considering typed lambda calculi containing Π-terms: the tree representation of $\Pi x : M \,.\, N$ is similar to that of $\lambda x : M \,.\, N$, with P_x instead of L_x. In some typed calculi, $*$ occurs as a constant.

Definition 1.1. *tree*: connected acyclic undirected graph, consisting of vertices and edges.

labels: one of the symbols L_x (for each variable x), A, S, or a variable name. (L_x and A represent λ-binding of x, and application, respectively. Label S represents the beginning of a subexpression in λ-calculus.)

Note: In some typed calculi: also P_x is a label (for each variable x). (It represents Π-binding of x.) Also $*$ can be a label.

binders: L_x and P_x are binders. These symbols are intended to bind free variables (e.g., x), in the usual manner. (More on this follows below.)

edge-labelled tree: a tree in which the leaf vertices are omitted and in which each edge has a label.

Meta-variables for trees: $\mathbf{t}$, $\mathbf{t}'$, $\mathbf{t}_1$, ...

Metavariables for labels: ℓ.

So also the *paths* in an edge-labelled tree are edge-labelled. Such a path can be identified with its string of labels. *(From now on, all trees and paths are edge-labelled.)*

Metavariables for paths: p, q, ...

Definition 1.2. A-cell, L-cell, P-cell, var-cell, $*$-cell (binary, binary, binary, nullary, nullary): see Figure 1

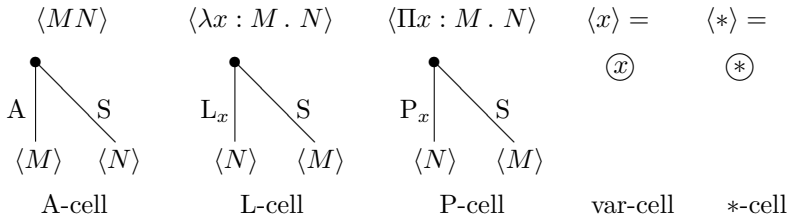

Figure 1: cells of lambda trees

A-, L- and P-cells have *open ends* on the lower sides; they can be downwards connected to other cells at both open ends. Var-cells and $*$-cells can be connected to A-, L- and P-cells, but they cannot be connected downwards to other cells.

Note 1.3. *In* untyped *lambda-calculus,* L*-cells and* P*-cells have only one open end: the edge marked* L_x *or* P_x*, because the types* M *are not given in those calculi.*

Definition 1.4. A λ*-tree* is a rooted (edge-labelled) tree, built by an arbitrary connection of L-cells, A-cells, P-cells, var-cells and $*$-cells, such that there are no open ends. Var-cells and $*$-cells (only) appear at the leaves.

Each λ-tree apparently has a top-cell. The *root* of a λ-tree is the leftmost label of the top-cell of the tree.

Example 1.5. In Example 2 (1) we represent the λ-tree of the following term from typed lambda calculus:

$\lambda\alpha : * \,.\, \lambda\beta : * \,.\, \lambda x : \alpha \,.\, \lambda y : \alpha \to \beta \,.\, y\, x.$

The term marked (1) is written in the *name-carrying* notation, using the *variable names* x, y, α and β. The notation $\alpha \to \beta$ is treated as an abbreviation of $\Pi u : \alpha \,.\, \beta$. The symbol $*$ is a *constant* of typed λ-calculus.

(The term marked (2) will be discussed later.)

Definition 1.6. Let $\mathbf{t}$ be a λ-tree.

We write $p \in \mathbf{t}$ if path p is a (coherent) part of $\mathbf{t}$.

A *root path* p in $\mathbf{t}$ is a path starting in the root. Notation: $p \in^{\wedge} \mathbf{t}$.

A *leaf path* in $\mathbf{t}$ is a path with a leaf as final label. Notation: $p \in_{\vee} \mathbf{t}$.

A *complete path* in $\mathbf{t}$ is a path that is both a root path and a leaf path. Notation: $p \in_{\vee}^{\wedge} \mathbf{t}$.

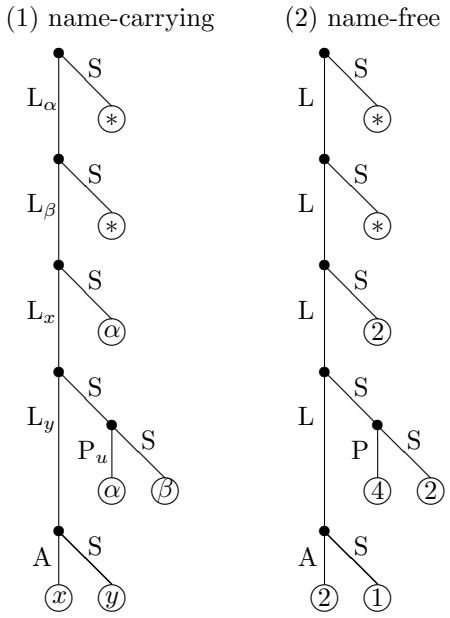

Figure 2: Lambda trees in name-carrying and name-free form

For the sake of convenience, we assume that the binding variables in a λ-tree always differ from each other. This can be expressed as follows.

Convention 1.7. *Let* $\mathbf{t}$ *be a* λ*-tree, let* B_x *and* B'_y *be binders and let the paths* $p\,B_x$ *and* $q\,B'_y \in^\wedge \mathbf{t}$*. Then* $x \equiv y \;\Rightarrow\; (p \equiv q \wedge B \equiv B')$*.*

Hence, in particular: If B_x is a binder in $\mathbf{t}$, then there is no other binder with subscript x in $\mathbf{t}$.

Definition 1.8. Let $p_1, \ldots, p_n$ be all complete paths in a λ-tree $\mathbf{t}$. Then we identify $\mathbf{t}$ with the set $\{p_1, \ldots, p_n\}$.

Remark 1.9. *We consider here the typed version of* λ*-calculus, since that version is widespread in modern applications. Most of the material presented here and below, however, also applies to the* untyped *version of* λ*-calculus, with some adaptations.*

1.2 β-reduction on λ-trees

The relation β-reduction between λ-terms in typed lambda calculi is defined as the compatible relation generated by

$(\lambda x : K\,.\,M)P \to_\beta M[x := P]$.

Here $M[x := P]$ is the result of substituting P for all free x's in M. (We do not give the necessary conditions on the names of variables which prevent undesired bindings in the 'process' of β-reduction.)

We give another description of β-reduction, suited to the *tree format* sketched previously and facilitating the alternative β-reductions described below. We first give some definitions, and a lemma on binding.

Definition 1.10. Let $\mathbf{t}$ be a λ-tree and $p \in^\wedge \mathbf{t}$.

Consider the set S of all complete paths $p\,q\,x \in^\wedge_\vee \mathbf{t}$, so the paths beginning with p, ending in a leaf x and having some path q in between. The set of all these $q\,x$ is itself a tree, called *tree*(p).

We call the set S the *grafted tree* of p in $\mathbf{t}$. We denote this grafted tree by $p\,\mathbf{t}'$, if $\mathbf{t}'$ is *tree*(p).

We use the notation $p\,\mathbf{t}' \in^\wedge_\vee \mathbf{t}$ for this situation.

Metavariable for grafted trees: $\mathbf{g}$.

Definition 1.11. (1) Let $\mathbf{t}$ and $\mathbf{u}$ be λ-trees and $p\,y \in^\wedge_\vee \mathbf{t}$. Then $(p\,y)[x := \mathbf{u}]$ is defined as the grafted tree $p\,\mathbf{u}$ if $x \equiv y$, and as $p\,y$ if $x \not\equiv y$.

(2) Let $\mathbf{t}$ and $\mathbf{u}$ be λ-trees. Then $\mathbf{t}[x := \mathbf{u}]$ is defined as $\{q[x := \mathbf{u}] \mid q \in^\wedge_\vee \mathbf{t}\}$.

Lemma 1.12. *Let* $\mathbf{t}$ *be a* λ*-tree.*

(i) *Assume that* $p\,x \in^\wedge_\vee \mathbf{t}$. *If* x *is a variable in* $\mathbf{t}$ *that is bound in the original* λ*-term by a* λ*, then there is exactly one* $\mathrm{L}_x \in \mathbf{t}$ *binding this* x*, and* $p \equiv p_1\,\mathrm{L}_x\,p_2$ *for some paths* p_1 *and* p_2.

(ii) *Assume that* $p\,\mathrm{L}_x \in^\wedge \mathbf{t}$. *If* $p\,\mathrm{L}_x\,q\,x \in^\wedge_\vee \mathbf{t}$, *then* x *is bound by* L_x. *In a* bound *term, all* x*'s are bound by this* L_x, *so there are no* x*'s 'outside' tree*$(p\,\mathrm{L}_x)$.

Hence, the binder of a variable x in $\mathbf{t}$ can be found on the path leading backwards from x to the root of $\mathbf{t}$. (A similar lemma holds for variables bound by a Π in the original λ-term.)

Moreover, an L_x in $\mathbf{t}$ binds all (free) x's that occur in $\mathbf{t}$.

Definition 1.13. (i) Let $\mathbf{t}$ be a λ-tree and let $p\,\mathrm{L}_x\,q\,x \in^\wedge_\vee \mathbf{t}$, such that L_x binds x. Then the path $\mathrm{L}_x\,q\,x$ is called the L-*block of* x.

(ii) A λ-tree is called *closed* if all leaf variables x are bound by an L_x.

Obviously, in a closed λ-tree, every leaf variable x corresponds to exactly one L-block ending in that x.

We can describe β-reduction of a λ-tree $\mathbf{t}$ as follows.

Definition 1.14. Let $\mathbf{t}$ be a λ-tree and assume that $p\,\mathrm{A}\,\mathrm{L}_x \in^\wedge \mathbf{t}$.

Now $\mathbf{t} \;\to_\beta\; \mathbf{t}[\mathit{tree}(p) := \mathit{tree}(p\,\mathrm{A}\,\mathrm{L}_x)[x := \mathit{tree}(p\,\mathrm{S})]]$.

Again, we disregard some necessary conditions on the variable names. Note: if $p\,\mathrm{A}\,\mathrm{L}_x \in^\wedge \mathbf{t}$, then also $p\,\mathrm{S} \in^\wedge \mathbf{t}$.

We conclude this section with some remarks about the 'natural' order between paths in a tree $\mathbf{t}$.

Definition 1.15. (*i*) The labels are ordered according to the order relation generated by $\mathrm{A} < \mathrm{S}$, $\mathrm{L}_x < \mathrm{S}$ and $\mathrm{P}_x < \mathrm{S}$, for all x.

(*ii*) Let p, $q \in^{\wedge}_{\vee} \mathbf{t}$. Then p and q are ordered by the lexicographic ordering generated by $<$.

(*iii*) The smallest path in a λ-tree $\mathbf{t}$, with respect to the order $<$, is called the *spine* of $\mathbf{t}$.

It will be clear that the spine of a λ-tree does not contain the label S.

Example 1.16. In Figure 2, (1), we have $\mathrm{L}_\alpha\,\mathrm{L}_\beta\,\mathrm{L}_x\,\mathrm{S}\,\mathrm{P}_u\,\alpha \;<\; \mathrm{L}_\alpha\,\mathrm{L}_\beta\,\mathrm{S}\,\alpha$.

The spine of the λ-tree in that example is $\mathrm{L}_\alpha\,\mathrm{L}_\beta\,\mathrm{L}_x\,\mathrm{L}_y\,\mathrm{A}\,x$.

1.3 Some variants of beta-reduction

1.3.1 Balanced beta-reduction

There is a variant of β-reduction that is interesting for certain purposes. We call it *balanced β-reduction*. In the literature, it appears under the name β_1 (Nederpelt, 1973). {To be added: more references plus discussion about the literature}

Firstly, we give the following definition.

Definition 1.17. A path p in a λ-tree $\mathbf{t}$ is called *balanced*, denoted $bal(p)$, if it is constructed by means of the following inductive rules:

(*i*) $bal(\varepsilon)$, i.e., the empty string is balanced;

(*ii*) if $bal(p)$, then $bal(\mathrm{A}\,p\,\mathrm{L}_x)$, for every variable x;

(*iii*) if $bal(p)$ and $bal(q)$, then $bal(p\,q)$.

In case (*ii*), we say that the mentioned A *matches* the L.

Examples of balanced paths: ε, A L, A A L L, A L A L, A A L A A L L L.

Note the close correspondence between nested paths and (consecutive) nested pairs of parentheses. Only A- and L-labels occur in balanced paths, so there is no other label involved, such as S.

We define *balanced* β-reduction, with symbol $\rightarrow_b$, as follows.

Definition 1.18. Let $\mathbf{t}$ be a λ-tree, let $b \in \mathbf{t}$ be a balanced path and assume that $p\,\mathrm{A}\,b\,\mathrm{L}_x \in^{\wedge} \mathbf{t}$.

Then $\mathbf{t} \;\rightarrow_b\; \mathbf{t}[tree(p\,\mathrm{A}\,b\,\mathrm{L}_x) := tree(p\,\mathrm{A}\,b\,\mathrm{L}_x)[x := tree(p\,S)]]$.

Compare this with Definition 1.14.

Consider two λ-trees $\mathbf{t}$ and $\mathbf{t}'$ such that $\mathbf{t} \rightarrow_b \mathbf{t}'$ as described in Definition 1.18, so each x has been replaced by $tree(p\,S)$ in $tree(p\,\mathrm{A}\,b\,\mathrm{L}_x)$. Now we have that $\mathbf{t}$ is a subtree of $\mathbf{t}'$, provided that we omit the var-labels x in $\mathbf{t}$. So, balanced β-reduction has the property that it *extends* the original underlying tree, but for one variable that vanishes.

Definition 1.19. Let $\mathbf{t}$ be a λ-tree, $b \in \mathbf{t}$ a balanced path and assume that $\mathrm{A}\,b\,\mathrm{L}_x \in \mathbf{t}$. Let $\mathrm{A}\,b\,\mathrm{L}_x\,p\,x$ be a path in $\mathbf{t}$.

(*i*) The path $\mathrm{A}\,b\,\mathrm{L}_x\,p\,x \in \mathbf{t}$ (where L_x binds x) is called the A-*block of* x.

(*ii*) An A-block $\mathrm{A}\,b\,\mathrm{L}_x\,p\,x \in \mathbf{t}$ is a *front extension* of the L-block $\mathrm{L}_x\,p\,x$. This A-block and L-block are called *corresponding*.

(*iii*) The path $\mathrm{A}\,b\,\mathrm{L}_x$ is called the *redex block* or **r**-block of x.

Lemma 1.20. (*i*) *The* A*-block of a certain* $x \in \mathbf{t}$*, if it exists, is unique, just as the* L*-block of* x *and the* r*-block of* x*.*

(*ii*) *In a closed term, each var-label* x *corresponds to exactly one* L*-block; but even when the term is closed, not every* L*-block has a corresponding* A*-block.*

(*iii*) *An* A*-block of a certain* $x \in \mathbf{t}$ *is a join of exactly one* r*-block and exactly one* L*-block, overlapping at the* L_x *binding* x*.*

1.3.2 Focused beta-reduction

Sometimes there is a need for another form of β-reduction. One occasion is when β-reduction is invoked to model *definition unfolding*: then a defined notion occurring in M, say x, is replaced by the definiens, say P. Such an action generally occurs for only one instance of the definiendum x. So instead of replacing *all* occurrences of x in M, one aims at *precisely one* occurrence.

When adapting β-reduction to this situation, there are several things to be considered:

(*i*) The substitution $[x := P]$ should only act on *one* free x in M.

(*ii*) Hence, x must occur free in M.

(*iii*) The redex $(\lambda x : K\,.\,M)P$ should remain active after the intended reduction, since there may be other free x's in M which still need a type annotation (i.e., K); moreover, there must remain a possibility to substitute P for one or more of these x's in a later stage of the process.

All this is covered in the following definition of *focused* β-reduction, for which we use the symbol $\to_f$.

Definition 1.21. Let $\mathbf{t}$ be a λ-tree, let $b \in \mathbf{t}$ be a balanced path and assume that $r\,x_0 \in^{\wedge}_{\vee} \mathbf{t}$ be a complete path in $\mathbf{t}$ such that $r \equiv p\,\mathrm{A}\,b\,\mathrm{L}_x\,q\,k$. We call the balanced reduction identified by p and focussed on the path $q\,k$, a (p, q)-*reduction*. It is defined by

$\mathbf{t} \;\to_f\; \mathbf{t}[r\,x_0 := r\;\mathit{tree}(p\,\mathrm{S})]$.

The possibility of having *balanced* β-reduction is necessary to be able to deal with redexes which otherwise would be forbidden by the maintenance of the redex $(\lambda x : K\,.\,M)P$ after β-reduction, as described in requirement (*iii*). See the following example.

Example 1.22. We have, in λ-calculus with normal untyped β-reduction:

$(\lambda x\,.\,((\lambda y\,.\,M)Q))P \to_\beta (\lambda x\,.\,(M[y := Q]))P \to_\beta M[y := Q][x := P]$.

In *focused* β-reduction, this becomes:

$(\lambda x\,.\,((\lambda y\,.\,M)Q))P \to_f (\lambda x\,.\,(\lambda y\,.\,M[y_0 := Q])Q)P \to_f$

$(\lambda x\,.\,((\lambda y\,.\,M[y_0 := Q])Q)[x_0 := P])P$.

Here y_0 and x_0 are selected instances of the free y's and x's in M, respectively.

The second of the two one-step, focused, reductions could not be executed without the possibility to have a balanced λ-term $(\lambda y\,.\,M[y_0 := Q])Q$ between the λx and the P.

1.3.3 Erasing reduction

After having applied balanced or focused β-reduction, one also desires a reduction that gets rid of the 'remains', i.e., the A and the L_x in grafted trees $p\,\mathrm{A}\,b\,\mathrm{L}_x\,\mathbf{t}'$ where $x \notin FV(\mathbf{t}')$. Such pairs $\mathrm{A}\ldots\mathrm{L}_x$ are superfluous, since $\mathbf{L}_x$ has no x that is bound to it. We call the corresponding reduction *erasing* β-reduction and use symbol $\to_e$ for it.

Definition 1.23. Let $\mathbf{t}$ be a λ-tree, let $b \in \mathbf{t}$ be a balanced path and assume that $p\,\mathrm{A}\,b\,\mathrm{L}_x\,\mathbf{t}' \in^{\wedge}_{\vee} \mathbf{t}$. Assume moreover that $x \notin FV(\mathbf{t}')$.

Then $\mathbf{t} \;\to_e\; \mathbf{t}[\mathit{tree}(p) := \mathit{tree}(p\,\mathrm{A})[\mathit{tree}(p\,\mathrm{A}\,b) := \mathit{tree}(p\,\mathrm{A}\,b\,\mathrm{L}_x)]]$.

1.4 Theorems on β-reduction and its variants

E.g.: Postponement of $\to_e$ after $\to_b$ and after $\to_f$;

$\mathbf{t} \to_\beta \mathbf{t}' \Rightarrow \mathbf{t} \to_b \mathbf{t}'$ (trivial);

$\mathbf{t} \to_b \mathbf{t}' \Rightarrow \mathbf{t} \twoheadrightarrow_f \mathbf{t}'$ (trivial).

{*To be done*}

2 Name-free lambda-terms

2.1 The binding of variables

A recurrent nuisance in the formalization of lambda calculus is the *naming* of variables, which plays a dominant role in the establishment of binding. Let's consider some λ-term in which λx binds variable x. Then one may replace both mentioned x's by a y, on the condition that this is done consistently through the term, and that one prevents that the variable renaming does lead to a name-clash. (This relation is called α-reduction.) For example, the mentioned renaming is forbidden in the term $\lambda x\,.\,\lambda y\,.\,x$, for obvious reasons.

Another cause of worry is that *beta*-reduction has a spreading effect on all kinds of variables, so that it is sometimes a precarious matter to ensure that no 'undesired' binding between a λ and a variable arises.

To prevent these matters, N.G. de Bruijn invented a *name-free* version for terms in λ-calculus (de Bruijn, 1972). Instead of using names such as x to record bindings in terms, he employed *natural numbers*. The idea is to see the λ-term as a tree, comparable to the way we introduced λ-trees in the previous chapter. The principle is, that a variable with number n is bound to the λ which can be found by following the root path ending in this n, and choosing the n-th λ along this path as the binder. In a λ-tree, we call such an end label n a *numerical variable* or a *num-label*.

In Example 2 (2), the name-carrying λ-tree of Example 2 (1) has been exhibited, as an example of a name-free tree.

This *name-free* representation of λ-calculus, straightforward as it seems, is not so simple as it appears. A pleasant feature of it is that α-reduction is no longer required. But on the other hand, when applying β-reduction, a lot of updating is necessary. This updating is not very easy and it may require extra calculations that complicate matters.

Example 2.1. Consider the term $t_1 \equiv (\lambda x\,.\,\lambda y\,.\,(\lambda z\,.\,y)x)$ in untyped lambda calculus. This term β-reduces to $t_2 \equiv \lambda x\,.\,\lambda y\,.\,y$.

In name-free notation, $t_1 \equiv \lambda\,\lambda\,(\lambda\,2)\,2$. (Note that the third λ is not on the root path of the free x; so, the x in t_1 becomes not 3, but 2 in the name-free version.)

The name-free version of t_2 is $\lambda\,\lambda\,1$: the first number 2 in t_1 must be updated after the β-reduction: it returns as the number 1 in t_2. (The second 2 in t_1, together with the third λ, vanish by the β-reduction.)

Note 2.2. *There is another version of name-free trees for λ-terms, in which the labels are situated not at the edges – as in our proposal – but at the nodes (see de Bruijn, 1978a). Moreover, the labels* S *that we use, are omitted. This works just as well, but for two details. We show this with the same Example 2 (2).*

Imagine the same tree, but with all labels 'raised' to the closest node. So, for example, the root vertex is now labelled L.

(1) When we consider the trees as not being embedded in the plane, then it is unclear in this case which is the main term and which is the subterm for a given branching. But this can be easily solved by defining trees as non-planar.

(2) A more serious matter is that an extra provision is necessary for determining the binder of a variable. For example, in the tree of Example 2 (2) with all labels raised, the root path of the variable numbered 4 becomes one L *longer, viz.* L L L L P*. Now observe that the fourth* L *is never meant to bind a variable on this path. So one has to neglect that fourth* L *when counting backwards from 4 to 0 in the process of finding its binder. (Simply changing variable number 4 into 5 is not the proper way to solve this problem; for example, this makes Lemma 1.12, (2), untrue, thus undermining the intended binding structure.)*

Definition 2.3. We use the symbol $\mathcal{T}^{car}$ for the set of *name-carrying*, closed λ-trees. The symbol $\mathcal{T}^{fre}$ means the set of *name-free*, closed λ-trees.

The following sections discuss β-reduction in the name-free case. Since reduction itself is independent of typing, we do not distinguish between typed or untyped version of λ-calculus. So when discussing the sets $\mathcal{T}^{car}$ and $\mathcal{T}^{fre}$ we do not bother whether the terms are typed or not.

2.2 Name-free reductions with instant updating

In the present section we consider name-free β-reduction and some of its variants and each time we describe the updating that is required. We consider β-reduction and its variants, all with *instant updating*. That is, each one-step reduction ends in a λ-tree in which the num-labels have immediately been updated. In Section 3, we try to simplify the name-free β-reduction of λ-terms, by postponing the updates (*delayed updating*).

2.2.1 Beta-reduction

We give a description of β-reduction with instant updating in Definition 2.5.

Notation 2.4. (*i*) *We use the notation* $[x := A; y := B]$ *for simultaneous substitution.*

(*ii*) *The* lenght $|p|$ *of a path* p *is the number of labels (including num-labels) in* p*.*

(*iii*) *The* L-length $\|p\|$ *of a path* p *is the number of binders (i.e.,* P *or* L*) occurring in* p*.*

Definition 2.5. Let $\mathbf{u} \in \mathcal{T}^{fre}$ and assume that $p\,\mathrm{A}\,\mathrm{L} \in^{\wedge} \mathbf{u}$. Then the β-reduction based on the redex identified by p, also called *p-reduction*, is

$\mathbf{u} \to_\beta \mathbf{u}\,[tree(p) := \{q\,k \in^{\wedge}_{\vee} tree(p\,\mathrm{A}\,\mathrm{L})\,[upd_1\ ;\ upd_2]\}]$, where

$$\begin{cases} upd_1 \equiv k := k-1 \text{ if } k > \|q\|{+}1, \\ upd_2 \equiv k := \{r\,l \in^{\wedge}_{\vee} tree(p\,\mathrm{S})\,[upd_3]\} \text{ if } k = \|q\|{+}1 \end{cases}$$

$$\text{and } upd_3 \equiv l := l + \|q\| \text{ if } l > \|r\|$$

We now explain the contents of this definition.

Just as in Definition 1.14, we consider two subtrees of the original λ-tree $\mathbf{u}$: $tree(p\,\mathrm{A}\,\mathrm{L})$ and $tree(p\,\mathrm{S})$. Both trees need updating because of the performed β-reduction. In the first tree, we have two simultaneous updatings, upd_1 and upd_2. They act on *all* complete paths $q\,k$ in $tree(p\,\mathrm{A}\,\mathrm{L})$, and the choice depends on the value of the leaf k in the path considered. We discern two cases: $k > \|q\|{+}1$ and $k = \|q\|{+}1$. (It is understood that in the case not mentioned, viz. $k < \|q\|{+}1$, the *identical* update is meant, so $k := k$.) See Figure 3 for a visual explanation.

Note that, in the case $k = \|q\|{+}1$, the k is replaced by a copy of the full $tree(p\,S)$, updated if $l > \|r\|$. (Also here, the intention is that an identical update $l := l$ is applied on $r\,l \in^\wedge_\vee tree(p\,S)$, in the missing case $l \leq \|r\|$.) The 'original' $tree(p\,S)$ vanishes.

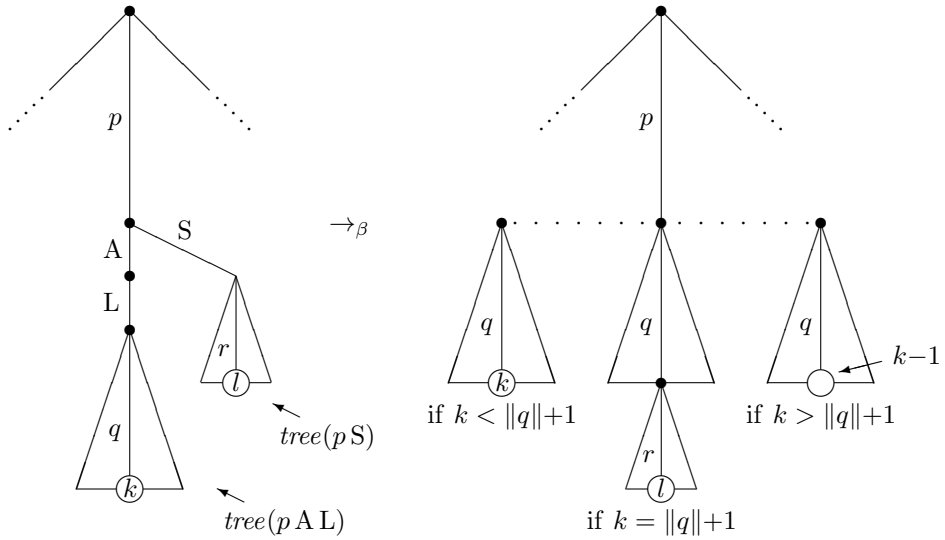

Figure 3: A picture of name-free β-reduction with updating

2.2.2 Balanced beta-reduction

For balanced β-reduction, the updates are somewhat different, due to the non-vanishing of the A-L-couple involved, and the remaining of the original 'argument' $tree(p\,S)$.

Definition 2.6. Let $\mathbf{u} \in \mathcal{T}^{fre}$, let $b \in \mathbf{u}$ be a balanced path and assume that $p\,\mathrm{A}\,b\,\mathrm{L} \in^\wedge \mathbf{u}$. Then the balanced reduction based on the redex identified by p (again called p-reduction) is

$$\mathbf{u} \to_b \mathbf{u}\,[tree(p\,\mathrm{A}\,b\,\mathrm{L}) := \{q\,k \in^\wedge_\vee tree(p\,\mathrm{A}\,b\,\mathrm{L})\,[upd_1]\}], \text{ where}$$
$$upd_1 \equiv k := \{r\,l \in^\wedge_\vee tree(p\,\mathrm{S})[upd_2]\} \textit{ if } k = \|q\|{+}1$$
$$\text{and } upd_2 \equiv l := l + \|q\|{+}1 + \|b\| \textit{ if } l > \|r\|$$

As in Section 2.2.1, an identical substitution (i.e., nothing changes) applies in the missing cases for k and l.

2.2.3 Focused beta-reduction

In the case of focused β-reduction, a simple adaptation of Section 2.2.2 is required. This leads to the following.

Definition 2.7. Let $\mathbf{u} \in \mathcal{T}^{fre}$, let $b \in \mathbf{u}$ be a balanced path, assume that $p\,\mathrm{A}\,b\,\mathrm{L} \in^{\wedge} \mathbf{u}$ and that $q\,k$ is a fixed, complete path in $tree(p\,\mathrm{A}\,b\,\mathrm{L})$.

Consider a β-reduction identified by p. When focusing on q as the *focus path*, we speak about a (p,q)-*reduction* and define this reduction by

$\mathbf{u} \rightarrow_f \mathbf{u}\,[tree(p\,\mathrm{A}\,b\,\mathrm{L}) := tree(p\,\mathrm{A}\,b\,\mathrm{L})[upd_1]]$, where

$$upd_1 \equiv q\,k := q\,\{r\,l \in^{\wedge}_{\vee} tree(p\,\mathrm{S})[upd_2]\} \;\; if \;\; k = \|q\|{+}1$$
$$\text{and } upd_2 \equiv l := l + \|q\|{+}1 + \|b\| \;\; if \;\; l > \|r\|$$

2.2.4 Erasing reduction

For erasing reduction, the following definition applies.

Definition 2.8. Let $\mathbf{u} \in \mathcal{T}^{fre}$, let $b \in \mathbf{u}$ be a balanced path, assume that $p\,\mathrm{A}\,b\,\mathrm{L} \in^{\wedge} \mathbf{u}$ and that no numerical variable in $tree(p\,\mathrm{A}\,b\,\mathrm{L})$ is bound by the mentioned L. (Otherwise said: for no $q\,k \in^{\wedge}_{\vee} tree(p\,\mathrm{A}\,b\,\mathrm{L})$ we have that $k = \|q\|{+}1$.)

Then $\mathbf{u} \rightarrow_e \mathbf{u}[tree(p) := tree(p\,\mathrm{A})[tree(p\,\mathrm{A}\,b) := \{q\,k \in^{\wedge}_{\vee} tree(p\,\mathrm{A}\,b\,\mathrm{L})\}[upd]]$, where

$$upd \equiv k := k - 1 \;\; if \;\; k > \|q\|$$

3 Delayed updating

Another possibility is to make reduction easy, by not considering the updates of the numerical variables, until required. In this case we *delay all updates*. In the text below, we restrict this case to the *focused* balanced β-reduction described above (Section 2.2.3), but it can easily be extended to the more general balanced version of β-reduction discussed in Section 2.2.2.

3.1 Focused beta-reduction with delayed updating

We us the symbol '$\rightarrow_{df}$' for the delayed, focused β-reduction.

Definition 3.1. Let $\mathbf{u} \in \mathcal{T}^{fre}$, let $b \in \mathbf{u}$ be a balanced path, assume that $p\,\mathrm{A}\,b\,\mathrm{L} \in^{\wedge} \mathbf{u}$ and that $q\,k$ is a fixed, complete path in $tree(p\,\mathrm{A}\,b\,\mathrm{L})$, where k is bound by L.

Then $\mathbf{u} \rightarrow_{df} \mathbf{u}\,[tree(p\,\mathrm{A}\,b\,\mathrm{L}) := tree(p\,\mathrm{A}\,b\,\mathrm{L})[q\,k := q\,k\,tree(p\,\mathrm{S})]]$.

Note that the leaf k stays where it is, and $tree(p\,\mathrm{S})$ is attached to it in $\rightarrow_{df}$-reduction, whereas k is *replaced* by an updated $tree(p\,\mathrm{S})$ in the original focused reduction. The remaining presence of the number k is to enable updating at a later stage. For a pictorial representation, see Figure 4. (Note: if $tree(p\,\mathrm{S})$ consists of a single num-variable only, then this variable is added to the variable k, *inside* the circular vertex of k; see Section 3.4 for an example.)

The above definition implies that we have to revise our definition of λ-trees, since *var-cells now need not be end-labels*; var-cells may have connections to other cells at their bottom end. Such internal variables not being an end-label, we call *inner num-labels*. To distinguish them from the original end-labels, we call the latter *outer* num-labels (or leaves).

Consequently, other definitions should be extended. For example, the definition of a *balanced path* (Definition 1.17) must be adapted such that it allows inner num-labels *inside* the string of A's and L's.

In the remainder of this paper, we assume that these definition revisions have been made.

Note 3.2. *In Definition 3.1, the copy of* $tree(p\,\mathrm{S})$ *attached to* k*, is not updated. Moreover, the complete tree* $\mathbf{u}$ *(including the* k*-node) remains intact as integral part of* $\mathbf{u}'$*. Hence,* $\mathbf{u} \rightarrow_{df} \mathbf{u}'$ *implies that* $\mathbf{u} \subseteq \mathbf{u}'$*.*

The latter fact is clearly an advantage. But it comes with a price: when we wish to establish the relation between a leaf-variable of the copied $tree(p\,\mathrm{S})$ and its binder, we have to do more work. We discuss this in the following subsection.

Definition 3.3. The symbol $\mathcal{T}^{fre}$ concerns the set of name-free, closed trees *without* inner variables. The set of these trees where also inner variables are permitted, we denote by $\mathcal{T}^{fre+}$.

3.2 Tracing the binder in reduction with delayed updating

Let $\mathbf{u} \in \mathcal{T}^{fre}$ and $\mathbf{u} \twoheadrightarrow_{df} \mathbf{u}'$, so $\mathbf{u}'$ is the result of a series of df-reductions. These reductions may introduce inner variables, so it is not immediately clear what the binders are for (inner or outer) variables. In this section we investigate how one can determine the binder of a variable in $\mathbf{u}'$.

Let $p\,n \in^{\wedge} \mathbf{u}'$, so $p\,n$ is a root path. Here n can be an inner or an outer num-label. We describe a pushdown automaton $\mathcal{P}_{fre}$ to find:

– either the label $\mathrm{L} \in p$ that 'binds' n (this label always exists, since $\mathcal{T}^{fre}$ only contains closed terms),

– or the label $\mathrm{A} \in p$ that matches the $\mathrm{L} \in p$ 'binding' n, if such an A exists (this needs not to be the case).

In algorithm $\mathcal{P}_{fre}$, we employ *states* that are pairs of natural numbers: (k, l). We start with the insertion of such a pair in the tail of the string $p\,n$, *between* p and n. The automaton moves the pair *to the left* through p, one step at a time, successively passing the labels in p and meanwhile adapting the numbers in the

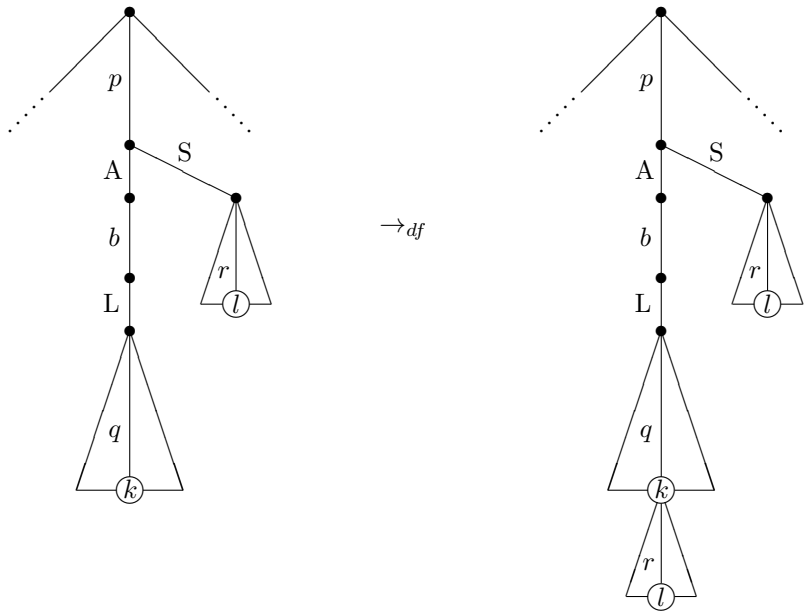

Figure 4: A picture of name-free β-reduction with delayed updating

pair. The automaton stops when the desired label (either the binding L or the A matching that L) has been found.

The automaton has an outside *stack* that will contain certain states that are *pushed* on the top of the stack; a state on top of the stack can also be *popped back*, i.e., inserted into the path p, again.

The *transitions* are described in Definition 3.4. A possible one-step transition is denoted by the symbol $\rightarrow$ (the reflexive, transitive closure of this relation is denoted $\twoheadrightarrow$). The procedure may be complicated by several recursive calls (see Lemma 3.6).

The *action* of $\mathcal{P}_{fre}$ is described in Definition 3.5 and in Lemma 3.6.

Definition 3.4. The algorithm $\mathcal{P}_{fre}$ is specified by the following rules:

1. *first step: stack* $= \emptyset$
2. p L (m,k) $q \ \rightarrow \ p$ $(m$–$1,k)$ L q, if $m > 0$
3. p A (m,k) $q \ \rightarrow \ p$ (m,k) A q, if $m > 0$
4. p S (m,k) $q \ \rightarrow \ p$ (m,k) S q, if $m > 0$
5. p j (m,k) $q \ \rightarrow \ p$ $(j,1)$ j q, if $m > 0$; *push* (m,k)
6. p L $(0,l)$ $q \ \rightarrow \ p$ $(0,l$+$1)$ L q, if $l > 0$
7. p A $(0,l)$ $q \ \rightarrow \ p$ $(0,l$–$1)$ A q, if $l > 0$
8. p j $(0,l)$ $q \ \rightarrow \ p$ $(0,l)$ j q, if $l > 0$

9*a*. p $(0,0)$ $q \ \rightarrow \ p$ *pop* q, if *stack* $\neq \emptyset$

9*b*. p $(0,0)$ $q \ \rightarrow$ *stop*, if *stack* $= \emptyset$.

The **use of the algorithm** is as follows. Let $\mathbf{u} \in \mathcal{T}^{fre+}$ and $p\,n \in^{\wedge} \mathbf{u}$. Assume that we desire to apply algorithm $\mathcal{P}_{fre}$. Then transform $p\,n$ into $p\,(n,i)\,n$ for i either 0 or 1, and *start* the algorithm. The i decides whether it delivers an L-block or an A-block.

Definition 3.5. Let $\mathbf{u} \in \mathcal{T}^{fre+}$ and $p\,n \in^{\wedge} \mathbf{u}$.

(*i*) If $\mathcal{P}_{fre}$ is applied to $p\,(n,0)\,n$ and stops in $p'\,(0,0)\,q\,n$, then $q\,n$ is called the L-*block* of n.

(*ii*) If $\mathcal{P}_{fre}$ is applied to $p\,(n,1)\,n$ and stops in $p'\,(0,0)\,q\,n$, then $q\,n$ is called the A-*block* of n.

After the start step, the procedure $\mathcal{P}_{fre}$ has two, possibly recursive, rounds with different actions.

Lemma 3.6. (*i*)*: Round (1) (steps (2) to (4)), gives a count-down of the* L*-labels in the path leftwards from an (inner or outer) num-label n; in this round,* A*- labels and* S*-labels have no effect. When the count-down results in $m = 0$, then we just passed the* L *binding the original n. So we found the* L*-block of n. If the stack is empty, so $\mathcal{P}_{fre}$ is not in a recursion, then the* L*-block of the original variable has been found (step (9b)).*

(*ii*)*: When an inner variable j is met in Round (1) (step (5)), then $\mathcal{P}_{fre}$ starts a recursive round, in which the* A*-block of that j is determined. The count-down of Round (1) resumes immediately left of that* A*-block (step (9a)).*

(*iii*) *The* A*-block of an inner variable j is found by executing Round (1), giving the* L*-block of j, immediately followed by Round (2) (steps (6) to (8)) determining the* r*-block connected with j. The latter is done by counting upwards for* L*'s and downwards for* A*'s. The* L*-block and the* r*-block combine in the desired* A*-block of j.*

(*iv*) *Inner variables met in the search of an* r*-block, are skipped (step (8)). So no recursion is started inside an* r*-block.*

(*v*) *When determining an* r*-block, no label S may be encountered, as can be seen in steps (6) to (8), where S is missing. (This corresponds with Definition 1.17.)*

Lemma 3.7. (*i*)*: If $q\,n$ is the* L*-block of n, then $q\,n \equiv \mathrm{L}\,q'\,n$ and the* L *binds the n.*

(*ii*)*: If $q\,n$ is the* A*-block of n, then $q\,n \equiv \mathrm{A}\,q'\,n$.*

3.3 An example of the action of algorithm $\mathcal{P}_{fre}$

We give an example of the execution of algorithm $\mathcal{P}_{fre}$.

Example 3.8. Firstly, we look for the L-binder of the final num-label – i.e., 3 – in the path $\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L\,S\,3}$.

So we take $i = 0$, and start $\mathcal{P}_{fre}$:

$\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L\,S}\,(\mathbf{3},\mathbf{0})\,3 \;\; (stack = \emptyset) \rightarrow$

$\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L}\,(\mathbf{3},\mathbf{0})\,\mathrm{S}\,3 \rightarrow$

$$
\begin{array}{l}
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,(\mathbf{2},\mathbf{0})\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,(\mathbf{1},\mathbf{0})\,L\,L\,L\,S\,3} \;\; (\mathit{push}\ (\mathbf{1},\mathbf{0})) \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,(\mathbf{4},\mathbf{1})\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,(\mathbf{3},\mathbf{1})\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,(\mathbf{3},\mathbf{1})\,S\,L\,4\,L\,L\,S\,3} \;\; (\mathit{push}\ (\mathbf{3},\mathbf{1})) \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,(\mathbf{2},\mathbf{1})\,2\,S\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,(\mathbf{2},\mathbf{1})\,A\,2\,S\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,(\mathbf{1},\mathbf{1})\,L\,A\,2\,S\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,A\,L\,(\mathbf{0},\mathbf{1})\,\underbrace{\mathrm{L\,L\,A\,2}}_{\text{L-block of 2}}\,\mathrm{S\,L\,4\,L\,L\,S\,3}} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,(\mathbf{0},\mathbf{2})\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,L\,(\mathbf{0},\mathbf{1})\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,A\,(\mathbf{0},\mathbf{2})\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,A\,(\mathbf{0},\mathbf{1})\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,L\,(\mathbf{0},\mathbf{0})\,\underbrace{\mathrm{A\,A\,L\,A\,L\,L\,L\,A\,2}}_{\text{A-block of 2}}\,\mathrm{S\,L\,4\,L\,L\,S\,3}} \;\; (\mathit{pop}\ (\mathbf{3},\mathbf{1})) \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,(\mathbf{3},\mathbf{1})\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,L\,(\mathbf{2},\mathbf{1})\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,L\,(\mathbf{1},\mathbf{1})\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,A\,(\mathbf{0},\mathbf{1})\,\underbrace{\mathrm{L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4}}_{\text{L-block of 4}}\,\mathrm{L\,L\,S\,3}} \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,(\mathbf{0},\mathbf{0})\,\underbrace{\mathrm{A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4}}_{\text{A-block of 4}}\,\mathrm{L\,L\,S\,3}} \;\; (\mathit{pop}\ (\mathbf{1},\mathbf{0})) \rightarrow \\
\mathrm{A\,A\,L\,L\,A\,(\mathbf{1},\mathbf{0})\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,L\,(\mathbf{1},\mathbf{0})\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L\,S\,3} \rightarrow \\
\mathrm{A\,A\,L\,(\mathbf{0},\mathbf{0})\,\underbrace{\mathrm{L\,A\,A\,L\,L\,L\,A\,A\,L\,A\,L\,L\,L\,A\,2\,S\,L\,4\,L\,L\,S\,3}}_{\text{L-block of 3}}} \;\; (\mathit{stack} = \emptyset, \text{ hence } \mathit{stop})
\end{array}
$$

Example 3.9. (i) We leave it to the reader to derive the A-block of the final num-label 3 in the path of Example 3.8, by starting with $i = 1$. The derivation is very similar to the one given, slightly extending it.

(ii) Notice that several L-blocks and A-blocks arise during the execution of the procedure, and that these can, on their own, be derived by means of a sub-calculation of the one above.

3.4 An example of beta-reduction with delayed updating

We consider the following untyped λ-term and three of its delayed, focused β-reducts. The underlining is meant to mark the redex parts, the dot above the variable (e.g., $\dot{y}$) marks the focus of the reduction. Transitions $(i) \rightarrow_f (ii)$ and $(ii) \rightarrow_f (iii)$ are one step reductions, $(iii) \twoheadrightarrow_f (iv)$ is two-step.

(i) $((\lambda x(\underline{\lambda y}\,.\ \lambda z\,.\ \dot{y}\,z)\underline{\lambda u\,.\ x})x)\lambda v\,.\ v \;\rightarrow_f$

(ii) $((\lambda x\,.\ (\lambda y\,.\ \underline{\lambda z}\,.\ (\lambda u\,.\ x)\dot{z})\underline{\lambda u\,.\ x})\underline{x})\lambda v\,.\ v \;\rightarrow_f$

(iii) $((\underline{\lambda x}\,.\ (\lambda y\,.\ \lambda z\,.\ (\lambda u\,.\ x)\dot{x})\lambda u\,.\ \dot{x})x)\underline{\lambda v\,.\ v} \;\twoheadrightarrow_f$

(iv) $((\lambda x\,.\ (\lambda y\,.\ \lambda z\,.\ (\lambda u\,.\ x)\lambda v\,.\ v)\lambda u\,.\ \lambda v\,.\ v)x)\lambda v\,.\ v.$

In Figure 3.4 we represent these four λ-terms as trees. We mark the A-L-couples and the arguments in boxes. In the picture, we use a short arrow to mark the focus(ses).

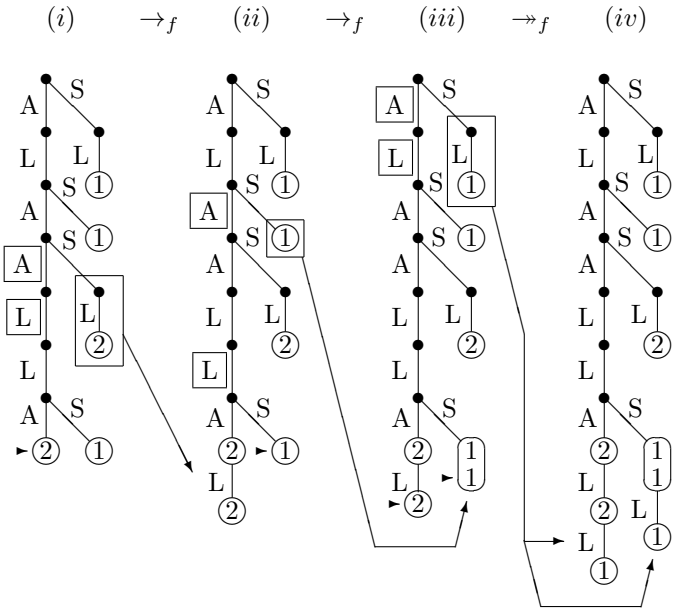

Figure 5: Examples of delayed, focused β-reduction

Figure 5, (iii), shows that focused reduction may cause that a vertex becomes *multiply* labelled. See the lower, right-most vertex, which contains two labels: an inner num-variable (the top-1) and an outer one (the bottom-1). In Figure 5, (iv), both mentioned labels have become inner ones.

Multiple labels are a natural consequence of algorithm $\mathcal{P}_{fre}$. More than two labels for one vertex are possible, as well. We emphasize that we consider the mentioned cases each time as *one* vertex, with labels obeying a *linear order* ('top-to-bottom').

It can be easily checked that the top-1 in Figure 5, (iii), is the final variable of the A-block A A L L S 1. This implies that the bottom-1 is the final variable in the L-block L A A L L S 1 1, so the bottom-1 is bound by the leftmost, uppermost L. (In this example case, there is also an A-block for the bottom-1.)

3.5 From name-carrying to name-free lambda-terms, and vice versa

In this section we compare $\mathcal{T}^{car}$ with $\mathcal{T}^{fre+}$ and define mappings between them. It turns out that there is a natural relation between L-blocks in $\mathcal{T}^{car}$ and L-blocks in $\mathcal{T}^{fre+}$, and between A-blocks in $\mathcal{T}^{car}$ and in $\mathcal{T}^{fre+}$ (cf. Definitions 1.13 (i), 1.19 and 3.5).

We only consider *closed* λ-trees, i.e., λ-trees in which all variables have been bound. We only consider λ-abstraction, so there are no P_x's or P's in the sets of terms studied below. (Hence, we limit ourselves to the set $\lambda_{\underline{\omega}}$ in Barendregt's cube; cf. Barendregt, 1992.) But Π-abstraction behaves similarly.

Definition 3.10. (i) Let $\mathbf{t} \in \mathcal{T}^{car}$. The *structure* of $\mathbf{t}$, or $str(\mathbf{t})$, is $\mathbf{t}$ in which all variables have been omitted. This concerns the variables at the leaves, but also the subscripts of labels L_x, for any x.

(ii) Let $\mathbf{u} \in \mathcal{T}^{fre+}$. The *structure* of $\mathbf{u}$, or $str(\mathbf{u})$, is (again) $\mathbf{u}$ in which all variables have been omitted. This concerns both the outer and the inner variables. In the trees representing a structure, the vertices are all rendered as identical dots. This also applies to the vertices with multiple num-labels.

Definition 3.11. We define a mapping from $\mathcal{T}^{car}$ to $\mathcal{T}^{fre}$. Let $\mathbf{t} \in \mathcal{T}^{car}$. Then $[\mathbf{t}] \in \mathcal{T}^{fre}$ is the tree with the same structure as $\mathbf{t}$ and with numbers n instead of the variable names at the leaves of $\mathbf{t}$. These numbers are chosen such that they *respect* the bindings. That is, if $p\,\mathrm{L}_x\,q\,x$ is a path in $\mathbf{t}$, then the corresponding path $p\,\mathrm{L}\,q\,n$ in $[\mathbf{t}]$ has the property that $n = \|q\|+1$.

The reverse mapping is slightly more complicated. We include λ-trees with inner variables, so we describe a mapping from $\mathcal{T}^{fre+}$ to $\mathcal{T}^{car}$. This mapping consists of three stages.

Let $\mathbf{u}$ be a λ-tree in $\mathcal{T}^{fre+}$.

(i) *Add names* to L-labels: provide all L-labels in $\mathbf{u}$ with a name, such that $p\,\mathrm{L}_x,\ q\,\mathrm{L}_y \in^{\wedge} \mathbf{u} \Rightarrow (x \equiv y \Rightarrow p \equiv q)$.

By this process, the various L-labels in $\mathbf{u}$ get names that are *different* from each other. We call the paths thus obtained *enriched paths*.

(ii) *Replace* the outer num-labels by appropriate names: consider all enriched complete paths $p\,\mathrm{L}_x\,q\,n$ where $\mathrm{L}\,q\,n$ is an L-block in the original tree $\mathbf{u}$. In every such path, replace n by x.

(iii) *Erase* all inner num-labels and their edges.

Definition 3.12. Let $\mathbf{u} \in \mathcal{T}^{fre+}$. The combined procedure (i), followed by (ii), followed by (iii), applied to $\mathbf{u}$, gives a mapping from $\mathcal{T}^{fre+}$ to $\mathcal{T}^{car}$. We denote the resulting tree as $\langle\mathbf{u}\rangle$.

Lemma 3.13. *Let* $\mathbf{t} \in \mathcal{T}^{car}$. *Then* $\langle[\mathbf{t}]\rangle \equiv_{\alpha} \mathbf{t}$.

Note 3.14. *A possibility for speeding up the procedure in stage* (ii)*, is to use the method of* backtracking*, applied to the whole tree, instead of doing the job path by path.*

3.6 The behaviour of name-free lambda-trees under beta-reduction

In this section, we compare balanced β-reduction on λ-terms in $\mathcal{T}^{fre+}$ with focused, balanced β-reduction for $\mathcal{T}^{car}$. We choose to study *focused* β-reduction

only, since the behaviour of the usual β-reduction and of balanced β-reduction is more or less similar. We discuss erasing reduction later in this chapter.

For that purpose, it is necessary that we can refer to redexes in both $\mathcal{T}^{car}$ and $\mathcal{T}^{fre+}$.

Definition 3.15. Let $r \equiv p\,\mathrm{A}\,b\,\mathrm{L}_x\,q\,x \in^{\wedge}_{\vee} \mathbf{t} \in \mathcal{T}^{car}$, with balanced path b; then p identifies a redex and the pair $\rho \equiv (p, q)$ identifies a *focused* reduction. (Cf. Definition 2.7.)

We write $\mathbf{t} \to^{\rho}_{f} \mathbf{t}'$ for the β-reduction generated by ρ.

Now look at $\mathcal{T}^{fre+}$. Assume that $\mathbf{u} \in \mathcal{T}^{fre+}$ has the same structure as $\mathbf{t}$. There exists a path $r' \equiv p'\,\mathrm{A}\,b'\,\mathrm{L}\,q'\,k \in^{\wedge}_{\vee} \mathbf{u}$ with the same structure as r. That is, striking out all inner variables in r', results in r. Then the pair $\rho' \equiv (p', q')$ identifies a focused reduction in $\mathcal{T}^{fre+}$.

We say that the pairs ρ and ρ' *correspond* to each other and we write $[\rho]$ for ρ'.

Lemma 3.16. *Let $\mathbf{t} \in \mathcal{T}^{car}$ and assume that ρ identifies a focused β-reduction such that $\mathbf{t} \to^{\rho}_{f} \mathbf{t}'$.*

Further, let $\mathbf{u} \in \mathcal{T}^{fre+}$ and $str(\mathbf{u}) \equiv str(\mathbf{t})$. Then $\mathbf{u} \to^{[\rho]}_{df} \mathbf{u}'$ and $str(\mathbf{u}') \equiv str(\mathbf{t}')$.

In the following lemma, we compare a sequence of focused β-reductions in $\mathcal{T}^{car}$ with a corresponding sequence of delayed, focused β-reductions in $\mathcal{T}^{fre+}$.

Lemma 3.17. *Let $\mathbf{t}_1 \in \mathcal{T}^{car}$ and assume that $\mathbf{t}_1 \to^{\rho_1}_{f} \mathbf{t}_2 \to^{\rho_2}_{f} \ldots \to^{\rho_{n-1}}_{f} \mathbf{t}_n$, where $\rho_1 \ldots \rho_{n-1}$ identify the chosen one-step β-reductions.*

Let $\mathbf{u}_1 \equiv [\mathbf{t}_1] \in \mathcal{T}^{fre}$. Then there is a focused, delayed β-reduction:
$\mathbf{u}_1 \to^{[\rho_1]}_{df} \mathbf{u}_2 \to^{[\rho_2]}_{df} \ldots \to^{[\rho_{n-1}]}_{df} \mathbf{u}_n$, where $\mathbf{u}_i \in \mathcal{T}^{fre+} (2 \leq i \leq n)$.
Moreover, $str(\mathbf{t}_i) \equiv str(\mathbf{u}_i)$ for all $1 \leq i \leq n$.

Proof Induction. $\mathbf{t}_1$ and $\mathbf{u}_1$ have the same structure, by Definition 3.11. Use Lemma 3.16. □

Note that, although $\mathbf{u}_i \in \mathcal{T}^{fre+}$, it will in general *not* be the case that $\mathbf{u}_i \equiv [\mathbf{t}_i]$ for $2 \leq i \leq n$.

Lemma 3.18. *Let $\mathbf{u} \in \mathcal{T}^{fre+}$ and $p\,S\,q\,n \in^{\wedge}_{\vee} \mathbf{u}$. Then the leaf n is bound by a certain* L *in some position in either p or q.*

Let a be an A*-block in $\mathbf{u}$. Then leaf n is bound by an* L *in $p\,a\,q\,n$ at the same position in p or in q.*

Proof If $\mathrm{L} \in q$, then trivial.

Assume $\mathrm{L} \in p$, say $p \equiv p_1\,\mathrm{L}\,p_2$. Apply algorithm $\mathcal{P}_{fre}$:

(i) $p\,\mathrm{S}\,q\,(n,0)\,n \twoheadrightarrow^{(1)} p\,\mathrm{S}\,(m,0)\,q\,n \twoheadrightarrow p\,(m,0)\,\mathrm{S}\,q\,n \equiv p_1\,\mathrm{L}\,p_2\,(m,0)\,\mathrm{S}\,q\,n \twoheadrightarrow^{(2)} p_1\,(0,0)\,\mathrm{L}\,p_2\,\mathrm{S}\,q\,n$.

(ii) Let $a \equiv a'\,k$. Then: $p\,a\,q\,(n,0)\,n \twoheadrightarrow^{\text{see }(1)} p\,a'\,k\,(m,0)\,q\,n \twoheadrightarrow^{\text{push }(m,0)} p\,a'\,(k,1)\,k\,q\,n \twoheadrightarrow^{\text{Def. 3.5,}(ii)} p\,(0,0)\,a'\,k\,q\,n \twoheadrightarrow^{\text{pop }(m,0)} p\,(m,0)\,a\,q\,n \equiv p_1\,\mathrm{L}\,p_2\,(m,0)\,a\,q\,n \twoheadrightarrow^{\text{see }(2)} p_1\,(0,0)\,\mathrm{L}\,p_2\,a\,q\,n$.

So $\mathrm{L}\,p_2\,\mathrm{S}\,q\,n$ and $\mathrm{L}\,p_2\,a\,q\,n$ are L-blocks by Definition 3.5,(i), and n is bound by the same L in both cases. □

Note 3.19. *At transition (1), the reached state must be* $(m, 0)$ *for some* m*, and not* (m, l) *for some* $l > 0$*, since the latter case only occurs when the displayed* S *is part of an* r*-block. This would contradict Lemma 3.6,* (iv).

Theorem 3.20. *Let* $\mathbf{t}_1 \in \mathcal{T}^{car}$ *and assume that* $\mathbf{t}_1 \to_f^{\rho_1} \ldots \to_f^{\rho_{n-1}} \mathbf{t}_n$.
Let $\mathbf{u}_1 \equiv [\mathbf{t}_1] \in \mathcal{T}^{fre}$ *and* $\mathbf{u}_1 \to_{df}^{[\rho_1]} \ldots \to_{df}^{[\rho_{n-1}]} \mathbf{u}_n$ *(cf. Lemma 3.17). Then for all* $1 \leq i \leq n$: $\langle \mathbf{u}_i \rangle \equiv_\alpha \mathbf{t}_i$.

Proof Induction.
(i) Let $i = 1$. Then $\langle \mathbf{u}_1 \rangle \equiv \langle [\mathbf{t}_1] \rangle \equiv_\alpha \mathbf{t}_1$ by Lemma 3.13.
(ii) Assume that $\langle \mathbf{u}_i \rangle \equiv_\alpha \mathbf{t}_i$. Now $str(\mathbf{u}_i) \equiv str(\mathbf{t}_i)$ by Lemma 3.17.
Since $\mathbf{t}_i \to_f^{\rho_i} \mathbf{t}_{i+1}$, also $\mathbf{u}_i \to_{df}^{[\rho_i]} \mathbf{u}_{i+1}$ and $str(\mathbf{u}_{i+1}) \equiv str(\mathbf{t}_{i+1})$ (Lemma 3.16).
Let ρ_i be an (p, q)-reduction, so there is a path $p\,\mathrm{A}\,b\,\mathrm{L}_x\,q\,x \in \mathbf{t}_i$, with b a balanced path. The corresponding path in $\mathbf{u}_i$ is $p'\,\mathrm{A}\,b'\,\mathrm{L}\,q'\,n$, with $str(p') \equiv str(p)$, and similarly for b' and q'. By the reduction $[\rho_i]$, this path is transformed into the grafted tree $p'\,\mathrm{A}\,b'\,\mathrm{L}\,q'\,n\,tree(p\,\mathrm{S})$ in $\mathbf{u}_{i+1}$.

Note that the paths in the mentioned grafted tree are the only paths that we have to consider, since all other paths remain unchanged, both in the transition from $\mathbf{t}_i$ to $\mathbf{t}_{i+1}$ as from $\mathbf{u}_i$ to $\mathbf{u}_{i+1}$.

So let $p'\,\mathrm{A}\,b'\,\mathrm{L}\,q'\,n\,r'\,m$ be a complete path in $p'\,\mathrm{A}\,b'\,\mathrm{L}\,q'\,n\,tree(p\,\mathrm{S})$. We have to check whether m is bound and if so, to locate the binding L and compare this with the situation in $\mathbf{t}_{i+1}$.

Since the mentioned $tree(p\,\mathrm{S})$ is a copy of the $tree(p\,\mathrm{S})$ that follows $p\,\mathrm{S}$ in $\mathbf{u}_{i+1}$, we also find a path $r'\,m$ in the latter tree, occurring in both $\mathbf{u}_{i+1}$ and $\mathbf{u}_i$. Since by induction $\langle \mathbf{u}_i \rangle \equiv_\alpha \mathbf{t}_i$, we have that there is a path $p\,\mathrm{S}\,r\,y \in \mathbf{t}_i$ for some r with $str(r) \equiv str(r')$.

The final y in this path $p\,\mathrm{S}\,r\,y$ is bound by an L_y in either p or r.

(i) Assume y is bound in p. Then $p \equiv p_1\,\mathrm{L}_y\,p_2$. Also, $p' \equiv p'_1\,\mathrm{L}\,p'_2$ and the final m in $p'_1\,\mathrm{L}\,p'_2\,\mathrm{S}\,r'\,m$ is bound by the mentioned L.

Now note that the path $a \equiv \mathrm{A}\,b'\,\mathrm{L}\,q'\,n$ is an A-block in $\mathbf{u}_i$, so also in $\mathbf{u}_{i+1}$. It follows from Lemma 3.18 that the final m in $p'_1\,\mathrm{L}\,p'_2\,\mathrm{A}\,b'\,\mathrm{L}\,q'\,n\,r'\,m$ is bound by the L following p'_1.

Correspondingly, in $\mathbf{t}_{i+1}$ we have that the final variable in $p_1\mathrm{L}_y\,p_2\,\mathrm{A}\,b\,\mathrm{L}_x\,q\,r\,y$ must be y. So the two last-mentioned paths are matching.

(ii) Assume y is bound in r. Then $r \equiv r_1\,\mathrm{L}_y\,r_2$. Also, $r' \equiv r'_1\,\mathrm{L}\,r'_2$ and the final m in $p'\,\mathrm{S}\,r'_1\,\mathrm{L}\,r'_2\,m$ is bound by the mentioned L.

So also the final m in $p'\,\mathrm{A}\,b'\,\mathrm{L}\,q'\,n\,r'_1\,\mathrm{L}\,r'_2\,m \in \mathbf{u}_{i+1}$ is bound by the L following r'_1.

In $\mathbf{t}_{i+1}$ we have that the corresponding path is $p\,\mathrm{A}\,b\,\mathrm{L}_x\,q\,r_1\,\mathrm{L}_z\,r_2\,z$ for some z. (Note that all binding variables in $\mathbf{t}_{i+1}$ must be different, so α-reduction should change $\ldots\mathrm{L}_y\ldots y$ into something like $\ldots\mathrm{L}_z\ldots z$ in the copy of $tree(p\,\mathrm{S})$ that pops up in $\mathbf{t}_{i+1}$ by the $\to_f$-reduction.) Clearly, the bindings of m and z are matching.

It follows that all bindings in $\mathbf{u}_{i+1}$ correspond one-to-one to the bindings in $\mathbf{t}_{i+1}$. This is enough to conclude that $\langle \mathbf{u}_{i+1} \rangle \equiv_\alpha \mathbf{t}_{i+1}$. □

3.7 Theorems on delayed updating

Definition 3.21. If $\mathbf{u}' \in \mathcal{T}^{fre+}$ such that there is an $\mathbf{u} \in \mathcal{T}^{fre}$ with $\mathbf{u} \to_{df} \mathbf{u}'$, then $\mathbf{u}'$ is called *legal*. Such an $\mathbf{u}$ is unique and is called the *origin* of $\mathbf{u}'$.

Theorem 3.22. *Let* $\mathbf{u}' \in \mathcal{T}^{fre+}$ *be legal. Let* $\mathbf{u}$ *be defined as the set of all paths* $p\,k \in^\wedge \mathbf{u}$ *such that* p *has no inner variables. Then* $\mathbf{u}$ *is the origin of* $\mathbf{u}'$.

Definition 3.23. Let $\mathbf{u}' \in \mathcal{T}^{fre+}$ be legal. We define a linear order $<$ between all *inner* variables of $\mathbf{u}'$ as follows.

Assume that k, l are inner variables in $\mathbf{u}'$, where $k \in^\wedge p\,k$ and $l \in^\wedge q\,l$.

(i) If $p \subseteq q$, then $k < l$.

(ii) If $q \subseteq p$, then $l < k$.

(iii) Otherwise, ik $p < q$, then $k < l$.

{ To be added: theorems on legality}

4 Delayed renaming

In the previous Section 3 we considered beta-reduction in the *namefree* notation and discussed the possibility to delay the updating of the num-variables that is needed because the count to find the binder of a num-variable has been disturbed in some instances by the beta-reduction. We introduced a procedure to restore the bond between variable and binder.

The present section deals with similar observations for the *name-carrying* notation. Also in that case, the renaming of variables – often a cumbersome task accompanying the beta-reduction – may be postponed. And there is a similar procedure to determine the binding between a variable and its binder. We describe these matters below.

As before, we concentrate on *focused*, *balanced* beta-reduction.

4.1 Focused beta-reduction with delayed renaming

As a start, we rephrase Definition 3.1 for this case. We use symbol $\to_{dc}$ for delayed, focused β-reduction in $\mathcal{T}^{car}$. See the definition below.

Definition 4.1. Let $\mathbf{t} \in \mathcal{T}^{car}$, let $b \in \mathbf{t}$ be a balanced path, assume that $p\,\mathrm{A}\,b\,\mathrm{L}_x \in^\wedge \mathbf{t}$ and that $q\,x$ is a fixed, complete path in $tree(p\,\mathrm{A}\,b\,\mathrm{L}_x)$, where x is bound by the final L_x in $p\,\mathrm{A}\,b\,\mathrm{L}_x$.

Then $\mathbf{t} \to_{dc} \mathbf{t}\,[tree(p\,\mathrm{A}\,b\,\mathrm{L}_x) := tree(p\,\mathrm{A}\,b\,\mathrm{L}_x)[q\,x := q\,x\,tree(p\,\mathrm{S})]]$.

Hence, we leave the variable x in the tree and copy $tree(p\,\mathrm{S})$ right behind it.

Similarly to Section 3.1, we distinguish *inner* and *outer* variables in trees of $\mathcal{T}^{car}$. (The x in $q\,x$ is clearly an outer variable, the x in $q\,x\,tree(p\,\mathrm{S})$ an inner one.)

The following definition reflects Definition 3.3.

Definition 4.2. The symbol $\mathcal{T}^{car}$ concerns the set of name-carrying, closed trees *without* inner variables. The set of these trees where also inner variables are permitted, we denote by $\mathcal{T}^{car+}$.

Also for name-carrying trees and $\rightarrow_{dc}$-reduction, there is a procedure $\mathcal{P}_{car}$ to locate a binder. This procedure is similar to the one for name-free trees (cf. Definition 3.4).

Definition 4.3. The algorithm $\mathcal{P}_{car}$ is specified by the following rules:

1. *first step: stack* $= \emptyset$
2*a*. $p\ \mathrm{L}_y\ (x,k)\ q\ \rightarrow\ p\ (x,k)\ \mathrm{L}_y\ q$, if $x \not\equiv y$
2*b*. $p\ \mathrm{L}_x\ (x,k)\ q\ \rightarrow\ p\ (0,k)\ \mathrm{L}_x\ q$
3. $p\ \mathrm{A}\ (x,k)\ q\ \rightarrow\ p\ (x,k)\ \mathrm{A}\ q$
4. $p\ \mathrm{S}\ (x,k)\ q\ \rightarrow\ p\ (x,k)\ \mathrm{S}\ q$
5. $p\ y\ (x,k)\ q\ \rightarrow\ p\ (y,1)\ y\ q$; *push* (x,k)
6. $p\ \mathrm{L}_y\ (0,l)\ q\ \rightarrow\ p\ (0,l{+}1)\ \mathrm{L}_y\ q$, if $l > 0$
7. $p\ \mathrm{A}\ (0,l)\ q\ \rightarrow\ p\ (0,l{-}1)\ \mathrm{A}\ q$, if $l > 0$
8. $p\ y\ (0,l)\ q\ \rightarrow\ p\ (0,l)\ y\ q$, if $l > 0$
9*a*. $p\ (0,0)\ q\ \rightarrow\ p\ pop\ q$, if *stack* $\neq \emptyset$
9*b*. $p\ (0,0)\ q\ \rightarrow$ *stop*, if *stack* $= \emptyset$.

The **use of the algorithm** is as follows. Let $\mathbf{u} \in \mathcal{T}^{car}$ and $p\,x \in^{\wedge} \mathbf{u}$. Assume that we desire to apply algorithm $\mathcal{P}_{car}$. Then transform $p\,x$ into $p\,(x,i)\,x$ for i either 0 or 1, and *start* the algorithm. The i decides whether it delivers an L-block or an A-block.

Definition 4.4. Let $\mathbf{u} \in \mathcal{T}^{car}$ and $p\,x \in^{\wedge} \mathbf{u}$.

(*i*) If $\mathcal{P}_{car}$ is applied to $p\,(x,0)\,x$ and stops in $p'\,(0,0)\,q\,x$, then $q\,x$ is called the L-*block* of x.

(*ii*) If $\mathcal{P}_{car}$ is applied to $p\,(x,1)\,x$ and stops in $p'\,(0,0)\,q\,x$, then $q\,x$ is called the A-*block* of x.

This algorithm for the name-carrying version has a similar behaviour as that for the name-free case (cf. Lemma's 3.17 and 3.18 and Theorem 3.20). There are also theorems similar to the ones in Section 3.7 that hold in the name-carrying situation.

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