Counting and Applications

Introduction

- Combinatorial questions arose in the study of gambling games.
- Enumeration, the counting of objects with certain properties, is an important part of combinatorics.
  - Counting is used to determine the complexity of algorithms.
  - Counting is also required to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand.
  - Counting techniques are used extensively when probabilities of events are computed.

The Basics of Counting

- The Product Rule
  - Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Examples

1. There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?
2. How many different bit strings of length seven are there?
3. How many functions are there from a set with m elements to a set with n elements?
4. How many one-to-one functions are there from a set with m elements to one with n elements?

Examples

5. Telephone Numbering Plans
   - Reading assignment from page 387 of the textbook.
6. What is the value of k after the following code has been executed?

```
k:= 0
for i₁ := 1 to n₁
    for i₂ := 1 to n₂
        .
        .
        .
        for iₘ := 1 to nₘ
          k := k + 1
```

7. Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

The Sum Rule

- If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.
  - The sum rule can be extended to more than two tasks. Suppose that a task can be done in one of n_l ways, in one of n_2 ways, ... , or in one of n_m ways, where none of the set of n_j ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with 1<i< j<m. Then the number of ways to do the task is $n_1 + n_2 + . . . + n_m$.

Examples

1. Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Examples

2. A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Examples

- What is the value of k after the following piece of code gets executed?

$$
\begin{array}{l}
k := 0 \\
\textbf{for } i_1 := 1 \textbf{ to } n_1 \\
\qquad k := k + 1 \\
\textbf{for } i_2 := 1 \textbf{ to } n_2 \\
\qquad k := k + 1 \\
\qquad\quad \vdots \\
\textbf{for } i_m := 1 \textbf{ to } n_m \\
\qquad k := k + 1
\end{array}
$$

More Complex Counting Problems

- In some cases, it is not enough to use only the sum rule or the product rule.
- Example: In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC?

Examples

- Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
- Example on IPv4 addressing: Reading Assignment.

The Inclusion-Exclusion Principle

- Adding the number of ways to do tasks in two ways, where some tasks in the first way are also included in the second way, leads to an overcount.
- To correctly count the number of ways to do the two tasks, we add the number of ways to do it in one way and the number of ways to do it in the other way, and then subtract the number of ways to do the task in a way that belongs to both ways.

Examples

- How many bit strings of length eight either start with a 1 bit or end with the two bits 00?
- A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these people majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

THE DIVISION RULE

- There are n/d ways to do a task if it can be done using a procedure that can be carried out in *n* ways, and for every way *w*, exactly *d* of the *n* ways correspond to way *w*.

- How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

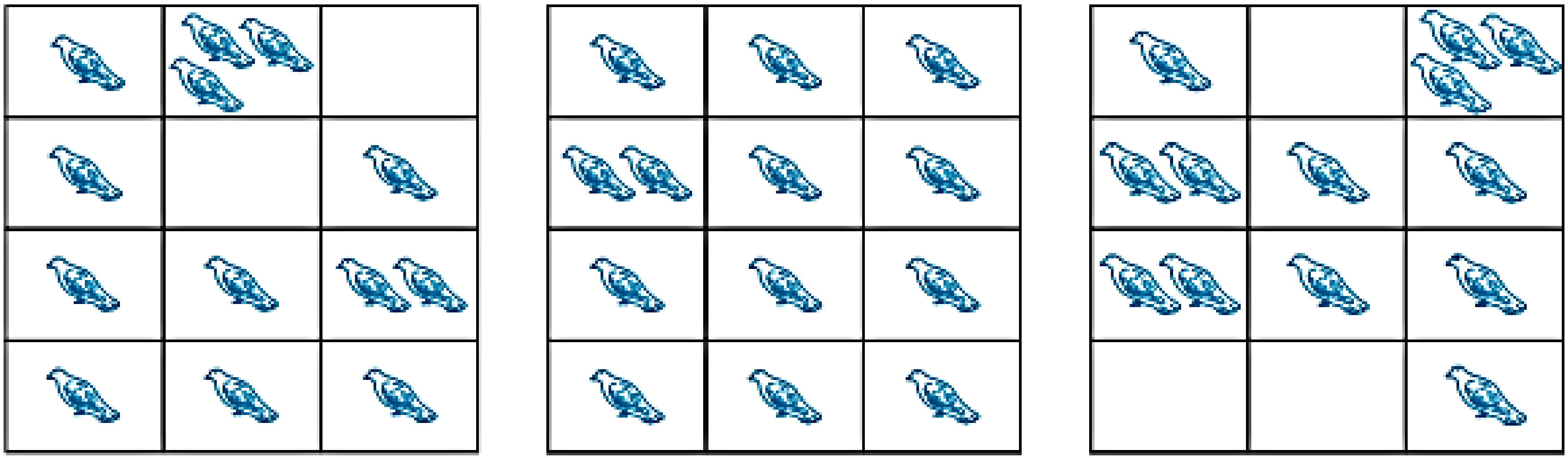
Tree Diagrams

- A branch in a rooted tree will correspond to a choice emanating from that root.
- Example: How many bit strings of length four do not have two consecutive 1s?
- A playoff between two teams consists of at most five games. The first team that wins three games wins the playoff. In how many different ways can the playoff occur?

The Pigeonhole Principle

- Theorem: If k is a positive integer and k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

- Corollary: A function f from a set with k + 1 or more elements to a set with k elements is not one-to-one.



Examples

- Example 1: How many people do you need to ensure that two of them have the same birthday (day and month)
- Example 2: How many English words do you need to ensure that two of them begin with the same letter.
- Example 3: How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

The Generalized Pigeonhole Principle

- Theorem If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Examples

- Example 1: What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

- 26

Examples

- Example 2: How many cards must be selected from a standard deck of 52 cards to guarantee that at least
  - three cards of the same suit are chosen?
    - 9
  - three hearts are selected?
    - 42

Examples

- Example 3: What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers?
    - Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.
    - 4

Permutations and Combinations

- Introduction
  - Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the **order of these elements matters**.
  - Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the **order of the elements selected does not matter**.
    - In how many ways can we select three students from a group of five students to stand in line for a picture?
    - 60
    - How many different committees of three students can be formed from a group of four students?
    - 4

Permutations

- A ***permutation*** of a set of distinct objects is an ordered arrangement of these objects.
- An ordered arrangement of r elements of a set is called an ***r-permutation***.
- Let S = {1, 2, 3}.
  - An ordered permutation of S is ……
  - A 2-permutation of S is ……
- The number of r-permutations of a set with n elements is denoted by P(n, r)
  - Which we can compute using the product rule!

P(n,r)

- If n is a positive integer and r is an integer with $1 \leq r \leq n$, then, there are

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

  - Note that P(n,0) = 1.

- Corollary: If n and r are integers with $0 \leq r \leq n$ then

$$P(n,r) = \frac{n!}{(n-r)!}$$

- P(n,n) = n!

Examples

- Example 1: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?
  - P(100,3)
- Example 2: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?
  - 7!
- Example 3: How many permutations of the letters ABCDEFGH contain the string ABC?
  - 6!

Combinations

- An ***r-combination*** of elements of a set is an unordered selection of r elements from the set.
  - an r-combination is simply a subset of the set with r elements.
- The number of r-combinations of a set with n distinct elements is denoted by C(n, r).
- Note that C(n, r) is also denoted by $\binom{n}{r}$ and is called a binomial coefficient.

Combinatorial Proofs

- **Definition 1**: A *combinatorial proof* of an identity is a proof that uses one of the following methods.
  - A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
  - A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.

Combinatorial Proofs

- Here are two combinatorial proofs that

$$C(n, r) = C(n, n - r)$$

when r and n are nonnegative integers with $r < n$:

  - *Bijective Proof*: Suppose that S is a set with n elements. The function that maps a subset A of S to $\bar{A}$ is a bijection between the subsets of S with r elements and the subsets with $n - r$ elements. Since there is a bijection between the two sets, they must have the same number of elements. ◄
  - *Double Counting Proof*: By definition the number of subsets of S with r elements is $C(n, r)$. Each subset A of S can also be described by specifying which elements are not in A, i.e., those which are in $\bar{A}$. Since the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements. ◄

Important Results on Combinations

- Theorem: The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

- Corollary: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Examples

- Example 1: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?
  - C(10,5)
- Example 2: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?
  - C(30,6)

Examples

- Example 3: How many bit strings of length n contain exactly r 1s?
- Example 4: Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?
  - $C(9,3) \cdot C(11,4)$

Binomial Coefficients

- The Binomial Theorem: Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Examples

1. What is the expansion of $(x + y)^4$?

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$
$$= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

2. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.$$

Examples

3. Find $\sum_{k=0}^{n} \binom{n}{k}$

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$$

4. Find $\sum_{k=1}^{n} (-1)^k \binom{n}{k}$

$$0 = 0^n = ((-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

Pascal’s Identity and Triangle

- Pascal's Identity: Let n and k be positive integers with $n \geq k$. Then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

Vandermonde's Identity

VANDERMONDE'S IDENTITY Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

COROLLARY 4 If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

THEOREM 4 Let n and r be nonnegative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Generalized Permutations and Combinations

- How to solve counting problems where elements may be used more than once?
- How to solve counting problems in which some elements are not distinguishable?
- How to solve problems involving counting the ways we to place distinguishable elements in distinguishable boxes?

Permutations with Repetition

- The number of *r*-permutations of a set of *n* objects with **repetition** allowed is n^r

- Example: How many strings of length *n* can be formed from uppercase English alphabets?

  26^r

Combinations with Repetition

- How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

Solution: To solve this problem we list all the ways possible to select the fruit. There are 15 ways:

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

The solution is the number of 4-combinations with repetition allowed from a three-element set, $\{apple, orange, pear\}$. ◀

Combinations with Repetition

Example: How many ways are there to select 5 bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills? Assume that the order in which bills are chosen does not matter and there are at least 5 bills of each type.

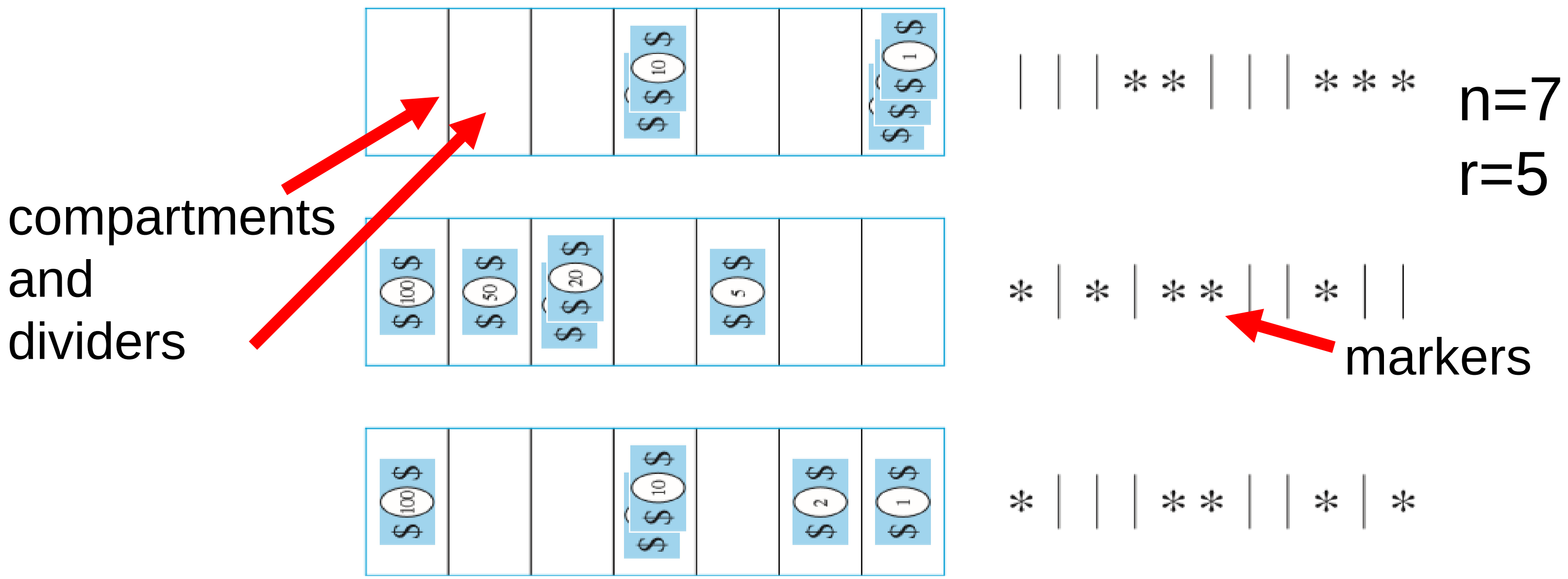
Combinations with Repetition

<u>*Approach:*</u> Place five markers in the compartments

i.e., # ways to arrange five stars and six bars ...

<u>*Solution:*</u> Select the positions of the 5 stars from 11 possible positions !

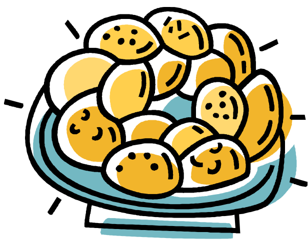
$$C(n+r-1,5)= C(7+5-1,5)=C(11,5)$$



Combinations with Repetition

- The number of r-combinations from a set with n elements when **repetition** of elements is allowed are $C(n+r-1,r)$

Combinations with Repetition



Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements.

$$C(9,6) = C(9,3) = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.

Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with non-distinguishable objects

- Example: How many different strings can be made by reordering the letters of the word

SUCCESS

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

Solution: There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in $C(7,3)$ different ways, leaving four positions free.
- The two Cs can be placed in $C(4,2)$ different ways, leaving two positions free.
- The U can be placed in $C(2,1)$ different ways, leaving one position free.
- The E can be placed in $C(1,1)$ way.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

Permutations with Indistinguishable Objects

- The number of different permutations of *n* objects, where there are n_1 non-distinguishable objects of <u>type 1</u>, n_2 non-distinguishable objects of <u>type 2</u>, …, and n_k non-distinguishable objects of <u>type *k*</u>, is

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

i.e., $C(n,n_1)C(n-n_1,n_2)\ldots C(n-n_1-n_2-\ldots-n_{k-1},n_k)$

$$n_1+n_2+\ldots+n_k=n$$

Distributing Objects into Boxes

- Many counting problems can be solved by counting the ways objects can be placed in boxes.
  - The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).
  - The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).

INDISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES

- Example: How many ways are there to place 10 non-distinguishable balls into 8 distinguishable bins?

$$C(8+10-1,10) = C(17,10) = \frac{17!}{10!7!} = 19{,}448.$$

- INDISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES = Combinations with Repetition

Distributing Distinguishable Objects into Distinguishable Boxes

- The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i=1,2,\ldots,k$, equals

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Distributing Distinguishable Objects into Distinguishable Boxes

- <u>Example:</u> How many ways are there to distribute hands of 5 cards to each of 4 players from the standard deck of 52 cards?

- There are $52!/(5!5!5!5!32!)$ ways to distribute hands of 5 cards each to four players.

INDISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES

- Example: How many ways are there to place 10 non-distinguishable balls into 8 distinguishable bins?

$$C(8+10-1, 10) = C(17, 10) = \frac{17!}{10!7!} = 19{,}448.$$

- INDISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES = Combinations with Repetition

Distributing Objects into Boxes

- ***Distinguishable objects*** **and** ***distinguishable boxes*****.**
  - There are $n!/(n_1!n_2!\cdots n_k!)$ ways to distribute *n* distinguishable objects into *k* distinguishable boxes.

  - Example: There are $52!/(5!5!5!5!32!)$ ways to distribute hands of 5 cards each to four players.

- ***Indistinguishable objects*** **and** ***distinguishable boxes*****.**
  - There are $C(n + r - 1, n - 1)$ ways to place *r* indistinguishable objects into *n* distinguishable boxes.
  - Proof based on one-to-one correspondence between *n*-combinations from a set with *k*-elements when repetition is allowed and the ways to place *n* indistinguishable objects into *k* distinguishable boxes.
  - Example: There are $C(8 + 10 - 1, 10) = C(17,10) = 19{,}448$ ways to place 10 indistinguishable objects into 8 distinguishable boxes.

Distributing Objects into Boxes

- ***Distinguishable objects*** **and** ***indistinguishable boxes*****.**
  - Example: There are 14 ways to put four employees into three indistinguishable offices (*see Example* 10).
  - There is no simple closed formula for the number of ways to distribute *n* distinguishable objects into *j* indistinguishable boxes.

- ***Indistinguishable objects*** **and** ***indistinguishable boxes*****.**
  - Example: There are 9 ways to pack six copies of the same book into four identical boxes (*see Example* 11).
  - The number of ways of distributing *n* indistinguishable objects into *k* indistinguishable boxes equals $p_k(n)$, the number of ways to write *n* as the sum of at most *k* positive integers in increasing order.
  - No simple closed formula exists for this number.