Sorting Small Groups

An Integer Programming Approach

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Agenda

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Problem Statement

Reacts Young Adults (RYA) Ministry needs to assign *individuals* into groups led by appointed *leaders.*

Stakeholders

Pastor Josh

Manually placing individuals into each group prior to using this proposed method of organizing.

Individuals

Members of the ministry who are interested in participating.

Leaders

Not included in the list of individuals but will lead their perspective group.

Solution Criteria

Solution Criteria

01

02

03

Individuals

1. Input From Everyone
    1.1 Leader Preferences
    1.2 Peer Preferences
    1.3 Equitable and minimize bias

Leaders

2. Even distribution of individuals

Pastor Josh

Timely solution to implement

Proposed Solution

Group Assignment using Integer Programming (IP)

IP consists of Objective Function, Decision Variables, and Constraints

Objective Function

What is the most ideal? Are we looking for a solution that works? Minimize something? Maximize something?

Decision Variables

What do we have to decide on? What are we trying to figure out?

Constraints

What are the factors that limit us in achieving what we want? What is limiting us from giving the stakeholders what they want?

Decision Variables

What do we have to decide on?

A. Who gets placed in which leader's group?

B. Who gets placed with other individuals?

Decision variable: group placement

A. Who gets placed in which leader's group?

$$i = \{1, 2, 3, \ldots, n\}$$
$$l = \{1, 2, \ldots, k\}$$

where n represents the total number of individuals,
and k represents the total number of leaders,
we have the decision variable:

$$x_{il} = \begin{Bmatrix} 1 & \text{if individual } i \text{ is assigned to leader } l \\ 0 & \text{otherwise} \end{Bmatrix}$$

Examples of group placement

A. Who gets placed in which leader's group?

Ex 1. Individual 2 is placed in Leader 4's group?

$$x_{2,4} = 1$$

Ex 2. Individual 2 is NOT placed in Leader 4's group?

$$x_{2,4} = 0$$

Ex 3. Individual 13 is placed in Leader 2's group?

$$x_{13,2} = 1$$

Decision variable: peer placement

B. Who gets placed with other individuals?

$$i = \{1, 2, 3, \ldots, n\}$$
$$j = \{1, 2, 3, \ldots, n\}$$

where n represents the total number of individuals,
and $j \neq i$ because an individual cannot be paired with themselves
we have the second decision variable:

$$y_{ij} = \begin{cases} 1 & \text{if individual } i \text{ is assigned to peer } j \\ 0 & \text{otherwise} \end{cases}$$

Examples of group placement

B. Who gets placed with who in a group?

Ex 1. Individual 2 is placed with Individual 19?

$$y_{2,19} = 1$$

Ex 2. Individual 2 is NOT placed with Individual 19?

$$y_{2,19} = 0$$

Ex 3. Individual 3 is placed with Individual 23?

$$y_{3,23} = 1$$

Visualizing the decision variables

The decision variables will look something like this:

$$X = \begin{bmatrix} x_{11} & \cdot & \cdot & \cdot & x_{1m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{n1} & \cdot & \cdot & \cdot & x_{nm} \end{bmatrix}$$

Objective Function

What do we want?

Consider the preferences in our survey. Using a top-3 ranking system, maximize the group placement of individuals.

Leader Preferences translated into weights

$$i = \{1, 2, 3, \ldots, n\}$$
$$l = \{1, 2, 3, \ldots, k\}$$

where n represents the total number of individuals,
and k represents the total number of leaders,
we retrieve the results from our survey:

$$L_{il} = \begin{cases} 3 & \text{if leader } l \text{ is the first choice for individual } i \\ 2 & \text{if leader } l \text{ is the second choice for individual } i \\ 1 & \text{if leader } l \text{ is the third choice for individual } i \\ 0 & \text{otherwise} \end{cases}$$

Peer Preferences translated into weights

$$i = \{1, 2, 3, \ldots, n\}$$
$$j = \{1, 2, 3, \ldots, n\}$$

where n represents the total number of individuals,
and $j \neq i$ because one cannot prefer oneself,
we retrieve the results from our survey:

$$P_{ij} = \begin{Bmatrix} 3 & \text{if peer } j \text{ is the first choice for individual } i \\ 2 & \text{if peer } j \text{ is the second choice for individual } i \\ 1 & \text{if peer } j \text{ is the third choice for individual } i \\ 0 & \text{otherwise} \end{Bmatrix}$$

Putting ranking and decision variables together into an objective function to maximize

1.3 Equitable and minimize bias

1.1 Leader preferences

1.2 Peer preferences

$$\text{Maximize} \quad \sum_{i \in \text{individuals}} \left(\sum_{l \in \text{leaders}} x_{il} \cdot L_{il} + \sum_{j \in \text{individuals}, j \neq i} y_{ij} \cdot P_{ij} \right)$$

Constraints

What is limiting us?

1) An individual is assigned to strictly one leader.
2) An individual is assigned at most {*capacity – 1*} other individual(s).
3) Each leader has at most {*capacity*} individuals assigned in their group.
4) If an individual is paired with another individual, they must be in the same group.

An individual is assigned to strictly one leader.

$$\sum_{l \in leaders} x_{il} = 1, \ \forall \, i = \{1, 2, 3, \ldots, n\}$$

An individual is assigned at most *capacity – 1* other individual(s).

$$\sum_{i \in individuals} y_{ij} \leq \text{Capacity} - 1,\ \forall\, i, j \neq i = \{1, 2, 3, \ldots, n\}$$

Each leader has at most {*capacity*} individuals assigned in their group

2 Even distribution of individuals

$$\sum_{i \in individuals} x_{il} \leq \text{Capacity}, \forall\, l = \{1, 2, \ldots, k\}$$

If an individual is paired with another individual, they must be in the same group.

$$
\begin{aligned}
x_{il} - x_{jl} &\leq M \cdot (1 - y_{ij}) + M \cdot (1 - y_{ji}) \\
x_{jl} - x_{il} &\leq M \cdot (1 - y_{ij}) + M \cdot (1 - y_{ji}) \\
x_{il} - x_{jl} - y_{ij} &\leq 1 \\
x_{il} - x_{jl} - y_{ji} &\leq 1
\end{aligned}
$$

Model Summary

$$\text{Maximize} \sum_{i \in \text{individuals}} \left(\sum_{l \in \text{leaders}} x_{il} \cdot L_{il} + \sum_{j \in \text{individuals}, j \neq i} y_{ij} \cdot P_{ij} \right) \quad \begin{matrix} i = \{1, 2, 3, \ldots, n\} \\ j = \{1, 2, 3, \ldots, n\} \\ l = \{1, 2, 3, \ldots, k\} \end{matrix}$$

$$x_{il} = \begin{cases} 1 & \text{if individual } i \text{ is assigned to leader } l \\ 0 & \text{otherwise} \end{cases}$$

$$y_{ij} = \begin{cases} 1 & \text{if individual } i \text{ is assigned to peer } j \\ 0 & \text{otherwise} \end{cases}$$

$$P_{ij} = \begin{cases} 3 & \text{if peer } j \text{ is the first choice for individual } i \\ 2 & \text{if peer } j \text{ is the second choice for individual } i \\ 1 & \text{if peer } j \text{ is the third choice for individual } i \\ 0 & \text{otherwise} \end{cases}$$

$$L_{il} = \begin{cases} 3 & \text{if leader } l \text{ is the first choice for individual } i \\ 2 & \text{if leader } l \text{ is the second choice for individual } i \\ 1 & \text{if leader } l \text{ is the third choice for individual } i \\ 0 & \text{otherwise} \end{cases}$$

subject to:

$$\sum_{l \in leaders} x_{il} = 1, \ \forall\, i = \{1, 2, 3, \ldots, n\}$$

$$\sum_{i \in individuals} x_{il} \leq \text{Capacity}, \forall\, l = \{1, 2, \ldots, k\}$$

$$\sum_{i \in individuals} y_{ij} \leq \text{Capacity} - 1, \ \forall\, i, j \neq i = \{1, 2, 3, \ldots, n\}$$

$$\begin{aligned} x_{il} - x_{jl} &\leq M \cdot (1 - y_{ij}) + M \cdot (1 - y_{ji}) \\ x_{jl} - x_{il} &\leq M \cdot (1 - y_{ij}) + M \cdot (1 - y_{ji}) \\ x_{il} - x_{jl} - y_{ij} &\leq 1 \\ x_{il} - x_{jl} - y_{ji} &\leq 1 \end{aligned}$$

Python Code Can Solve The Math

Using PuLP library, we can solve the problem without doing the arithmetic. We translate the model formulation in a format in Python so that the computer can solve it.

Example Results

Solver

```
Result - Optimal solution found

Objective value:                76.00000000
Enumerated nodes:               0
Total iterations:               2061
Time (CPU seconds):             0.34
Time (Wallclock seconds):       0.37

Option for printingOptions changed from normal to all
Total time (CPU seconds):       0.34   (Wallclock seconds):       0.38
```

Everyone gets 5 points in the test run (n=15, l=3)

Objective value = 76

Total = 76

Individuals = 15

Avg point per person = 76/15 = 5.07

Max point per person = 3+2+1+3+2+1 = 12

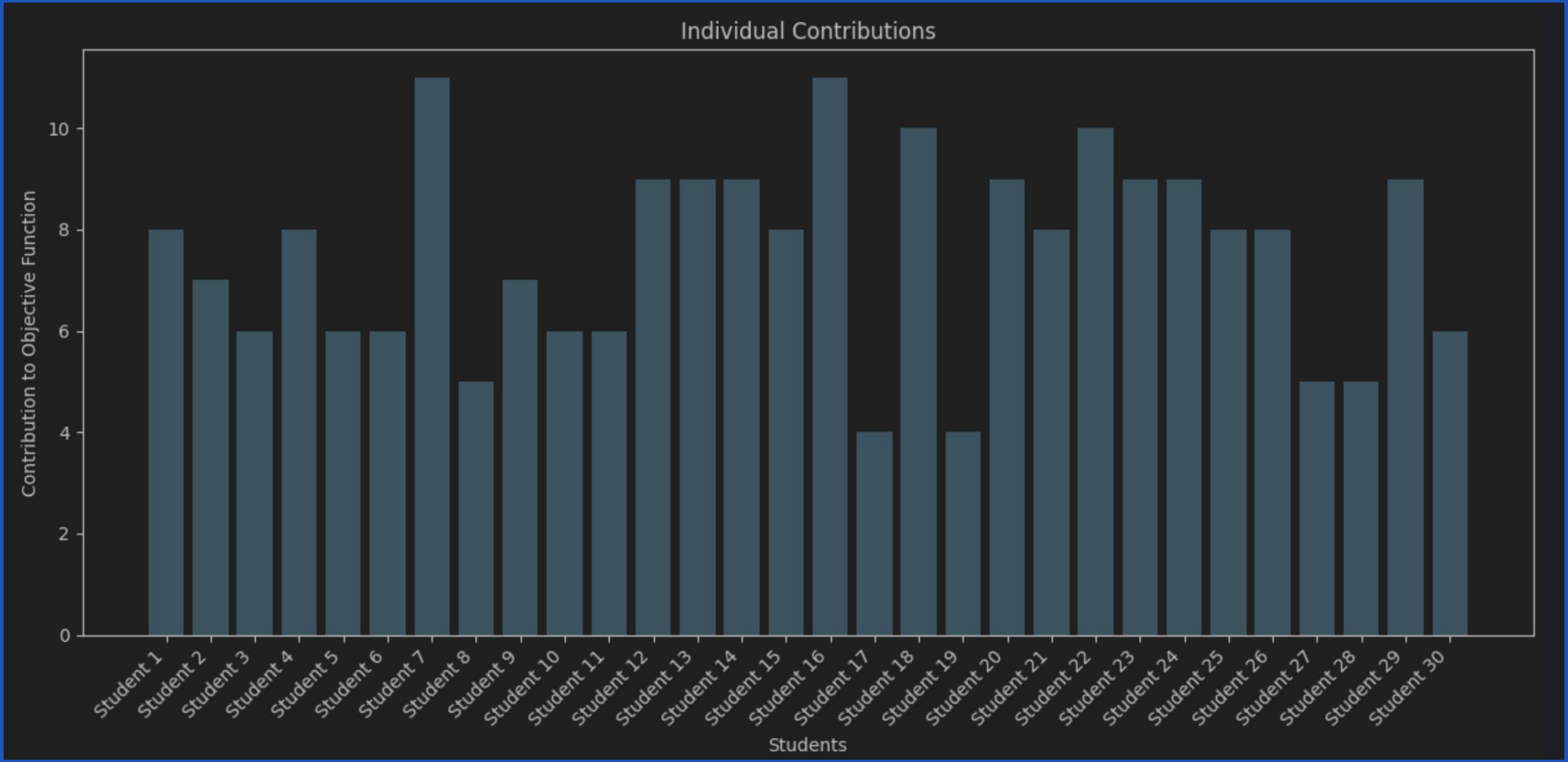
Everyone gets 7 points in the test run (n=30, l=6)

Solver

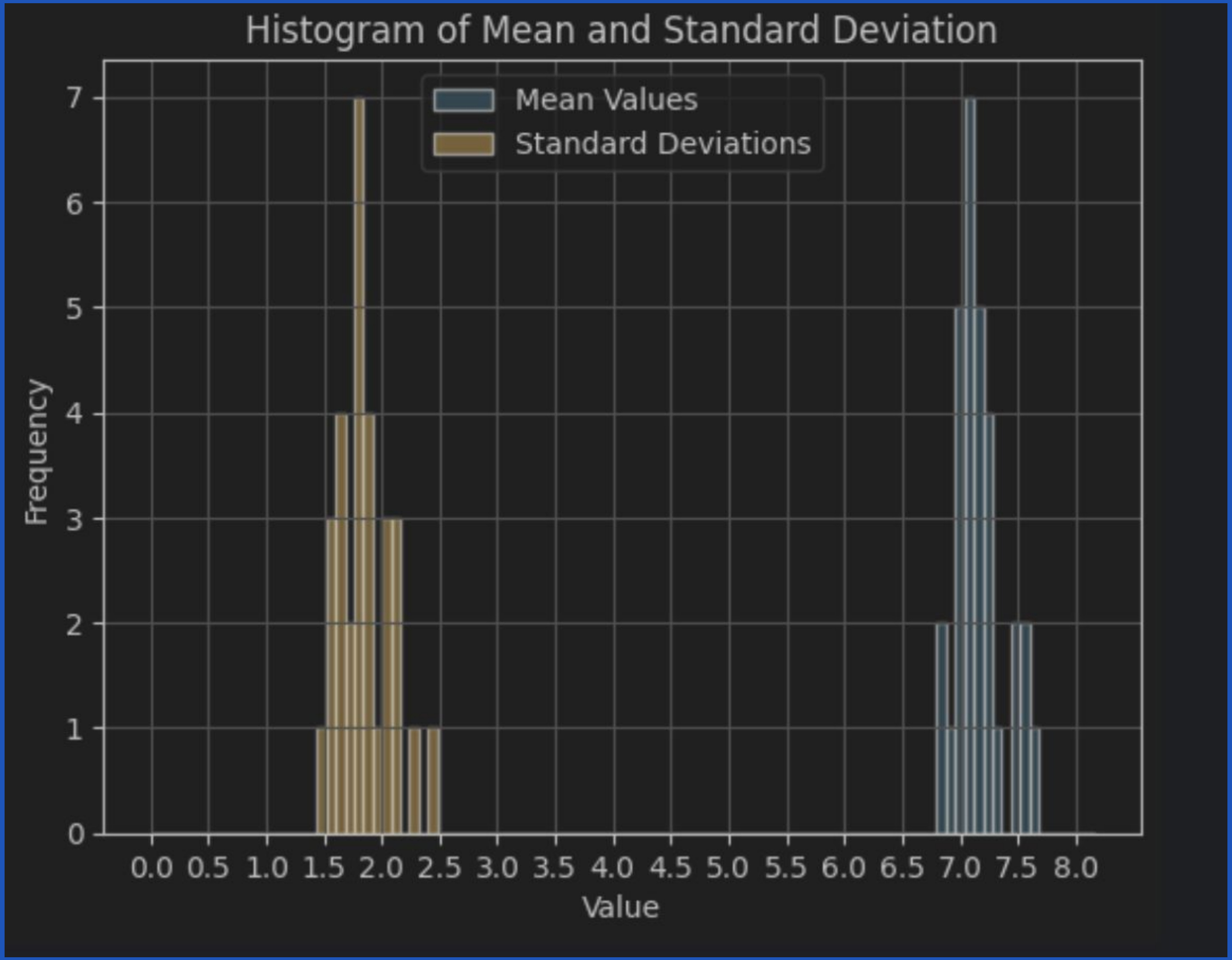
```
Result - Optimal solution found

Objective value:                213.00000000
Enumerated nodes:               12
Total iterations:               30025
Time (CPU seconds):             15.67
Time (Wallclock seconds):       15.81

Option for printingOptions changed from normal to all
Total time (CPU seconds):       15.69   (Wallclock seconds):       15.83
```

```
Mean Contribution: 7.533333333333333
Standard Deviation: 1.960530071414244
```

Everyone Seems To Be Pretty Happy with the Results



Model Performance Analysis through repeated trials=30



Mean of means: Confidence Interval

```
Mean: 7.1644
95% Confidence Interval: (7.0872, 7.2417)
```

Mean of Standard Deviations

```
Mean: 1.8415
```

- We are *95% confident* that the true average of points being assigned to individuals is **7.16**
- Because the most points you can get in one preference group (peer vs. leader) is 3 + 2 + 1 = 6, this means that on average, an individual must receive 1 more point from the other preference group to meet the average of 7.16 > 6 + 1
- Concern: Keep in mind that the input data is generated *randomly*
    - the input data for our model will more than likely not be *independent* due to the nature of human preferences being more cluttered
    - Friends are likely to list each other in their rankings
    - People who aren’t as close are likely to exclude another

Questions?

Overview

Provide a detailed subitem that explains its focus and purpose clearly, offering concise insights.

Item title

Provide a detailed subitem that explains its focus and purpose clearly, offering concise insights.

Competitor Comparison

Item title

Add a brief subitem here.

Item title

Add a brief subitem here.

Item title

Add a brief subitem here.

Item title

Add a brief subitem here.