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問題 1

$$\alpha(t,T) = v_T(t,T)v(t,T) - m_T(t,T) \quad \sigma(t,T) + v_T(t,T) = 0$$

を証明する

下記の記号を定める

$$\begin{cases} B(t) = (B_1(t), \dots, B_N(t))^\intercal \\ v(t,T) = (v_1(t,T), \dots, v_N(t,T))^\intercal \\ b(t) = (b_1(t), \dots, b_N(t))^\intercal \\ \sigma(t,T) = (\sigma_1(t,T), \dots, \sigma_N(t,T))^\intercal \end{cases}$$

よって，zero-coupon bond の価格

$$P(t,T) = P(0,T) + \int_0^t P(s,T)m(s,T)ds + \int_0^t P(s,T)v(s,T)dB(s)$$

に Ito の公式を適用すると，

$$dP(t,T) = P(s,T)m(s,T)ds + P(s,T)v(s,T)dB(s)$$

一方，$F(P(t,T)) = \log P(t,T)$ に定め，$F(P(t,T))$ に対して Ito の補題を適用する

$$\begin{aligned} dF(P(t,T)) &= \frac{1}{P(t,T)} dP(t,T) \\ &= m(s,T)ds + v(s,T)dB(s) - \frac{1}{2}v(t,T)^\intercal v(t,T)d\langle B^2\rangle(s) \\ &= \Big[m(s,T) - \frac{1}{2}v(t,T)^\intercal v(t,T)\Big]ds + v(s,T)dB(s) \end{aligned}$$

$$\log P(t,T) = \log P(0,T) + \int_0^t \Big[m(s,T) - \frac{1}{2}v(t,T)^\intercal v(t,T)\Big]ds + \int_0^t v(s,T)dB(s)$$

よって，

$$\begin{aligned} f(t,T) &= -\frac{\partial \log P(t,T)}{\partial T} \\ &= -\frac{\partial \log P(0,T)}{\partial T} + \int_0^t \Big[\frac{\partial v(t,T)}{\partial T}v(t,T) - \frac{\partial m(s,T)}{\partial T}\Big]ds - \int_0^t \frac{\partial v(s,T)}{\partial T}dB(s) \\ &= f(0,T) + \int_0^t \alpha(t,T)ds + \int_0^t \sigma(t,T)dB(s) \end{aligned}$$

従って，

$$\alpha(t,T) = v_T(t,T)v(t,T) - m_T(t,T)\ \ \sigma(t,T) + v_T(t,T) = 0$$

問題 2

(4) を証明せよ．

一次元 Ito process$X(t), X(x) = 0$ に対し，$\mathcal{E}(X)(t) := \exp(X(t) - \frac{1}{2}\langle X\rangle(t))$ によって Ito process $\mathcal{E}(X)$ を定義する．
また，$Z(t) := X(t) - \frac{1}{2}(t)$，$f(x) = \exp(x)$ として，Ito formula を適用する：

$$\begin{aligned}\mathcal{E}(X)(t) &= f(Z(t)) \ \text{（伊藤公式より）} \\ &= f(Z(0)) + \int_0^t (\frac{df}{dx})(Z(s))dZ(s) + \frac{1}{2}\int_0^t (\frac{d^2 f}{dx^2})(Z(s))d\langle Z\rangle(s)\end{aligned}$$

一方，$f(Z(0)) = f(x_0) = e^{x_0}$, $\frac{df}{dx} = \exp(x)$ $\frac{d^2 f}{dx^2} = \exp(x)$,

$$\begin{aligned}d\langle Z\rangle(s) &= d\Big\langle X(\cdot) - \frac{1}{2}\langle X\rangle(\cdot)\Big\rangle(s) \\ &= d\Big\langle X(\cdot) - \frac{1}{2}\langle X\rangle(\cdot), X(\cdot) - \frac{1}{2}\langle X\rangle(\cdot)\Big\rangle(s) \\ &= d\Big(\langle X, X\rangle(s) - \frac{1}{2}\langle\langle X\rangle(\cdot), X\rangle(s) - \frac{1}{2}\langle X, \langle X\rangle(\cdot)\rangle(s) + \frac{1}{4}\langle\langle X\rangle(\cdot), \langle X\rangle(\cdot)\rangle(s)\Big) \\ &= d\langle X, X\rangle(s) \\ &= d\langle X\rangle(s)\end{aligned}$$

以上より，

$$\begin{aligned}\mathcal{E}(X)(t) &= e^{x_0} + \int_0^t \exp(Z(s))(dX(s) - \frac{1}{2}d\langle X\rangle(s)) + \frac{1}{2}\int_0^t \exp(Z(s))d\langle X\rangle(s) \\ &= e^{x_0} + \int_0^t \exp(Z(s))dX(s) \\ &= e^{x_0} + \int_0^t \mathcal{E}(s)dX(s)\end{aligned}$$

$x_0 = 0$ ので，$\mathcal{E}(X)(t) = \int_0^t \mathcal{E}(X)(s)dX(s) + 1$
また，

$$\alpha(t) := (\mu_t^S(X(t)) - \mu_t^U(X(t)))^{\intercal} \cdot (\sigma_t(X(t)))^{-1\intercal}$$

$$X(t) := \int_0^t \alpha(s)dB_U(s)$$

と置いてやると，

$$\begin{aligned}\mathcal{E}(X)(t) &= \mathcal{E}(\int_0^t \alpha(s)dB_U(s)) \\ &= \exp(\int_0^t \alpha(s)dB_U(S) - \frac{1}{2}\langle X\rangle(t)) \\ &= \exp(\int_0^t (\mu_t^S(X(t)) - \mu_t^U(X(t)))^{\intercal} \cdot (\sigma_t(X(t)))^{-1\intercal} dB_U(S) - \frac{1}{2}\langle X\rangle(t))\end{aligned}$$

一方，$\langle X\rangle(t)=\langle X,X\rangle(t)=\int_0^t(K^X(s))^2ds=\int_0^t\alpha(s)^2dS$

$$\xi(t)=\mathcal{E}(X)(t)=\xi(0)+\int_0^t\xi(s)dX(s)=\xi(0)+\int_0^t\alpha(s)\xi(s)dB_U(s)$$

問題 3

定理 3.1.2 より，T-claim H の価格は $\pi_t(H)=P(t,T)E^{Q^T}[H|\mathcal{F}_t]$ で与えられる，この事を証明せよ．

ここで，$P(t,T)$ は時刻 t における満期は T の zero-coupon bond の価格である．Numeraire を N とするとき，一般価格公式は

$$\pi_t(H)=N(t)E^{Q^N}[\frac{H}{N(T)}|\mathcal{F}_t]$$

であるので，満期 T の時，$N(t)=P(t,T),N(T)=P(T,T)=1$ を代入すると

$$\pi_t(H)=N(t)E^{Q^T}[\frac{H}{N(T)}|\mathcal{F}_t]=P(t,T)E^{Q^T}[H|\mathcal{F}_t]$$