Chapter 2
Basics

Essential reading

(For full publication details, see Chapter 1.)

☞ Anthony and Biggs (1996) Chapters 1, 2, and 7.

Further reading

☞ Binmore and Davies (2001) Chapter 2, Sections 2.1–2.6.

☞ Booth (1998) Modules 1, 3, 4, 6–8, 11–15.

☞ Bradley (2008) Sections 1.1–1.6, 2.1, 3.1.1, 4.1–4.3.

☞ Dowling (2000) Chapters 1 and 2.

☞ Ostaszewski (1993) Chapter 1 (though this is more advanced than is required at this stage of the subject), Chapter 5, Sections 5.5 and 5.12.

2.1 Introduction

This chapter discusses some of the very basic aspects of the subject, aspects on which the rest of the subject builds. It is essential to have a firm understanding of these topics before the more advanced topics can be understood.

Most things in economics and related disciplines — such as demand, sales, price, production levels, costs and so on — are interrelated. Therefore, in order to come to rational decisions on appropriate values for many of these parameters it is of considerable benefit to form mathematical models or functional relationships between them. It should be noted at the outset that, in general, the economic models used are typically only approximations to reality, as indeed are all models. They are, nonetheless, very useful in decision-making. Before we can attempt such modelling, however, we need some mathematical basics.

This chapter contains a lot of material, but much of it will be revision. If you find any of the sections difficult, please refer to the texts indicated for further explanation and examples.

2.2 Basic notations

Although there is a high degree of standardisation of notation within mathematical texts, some differences do occur. The notation given here is indicative of what is used in the rest of this guide and in most of the texts.[1] You should endeavour to familiarise yourself with as many of the common notations as possible. For example, $|a|$ means 'the absolute value of a', which equals a if a is non-negative (that is, if $a \geq 0$), and equals $-a$ otherwise. For instance, $|6| = 6$ and $|-2.5| = 2.5$. (This is sometimes termed 'the modulus of a'. Roughly speaking, the absolute value of a number is obtained just by ignoring any minus sign the number has.) As another example, multiplication is sometimes denoted by a dot, as in $a \cdot b$ rather than $a \times b$. Beware of confusing multiplication and the use of a dot to indicate a decimal point. Even more commonly, one simply uses ab to denote the multiplication of a and b. Also, you should be aware of implied multiplications, as in $2(3) = 6$.

Some other useful notations are those for sums, products and factorials. We denote the sum

$$x_1 + x_2 + \cdots + x_n$$

of the numbers $x_1, x_2, \ldots, x_n$ by

$$\sum_{i=1}^{n} x_i.$$

The 'Σ' indicates that numbers are being summed, and the '$i = 1$' and n below and above the Σ show that it is the numbers x_i, as i runs from 1 to n, that are being summed together. Sometimes we will be interested in adding up only some of the numbers. For example,

$$\sum_{i=2}^{n-1} x_i$$

would denote the sum $x_2 + x_3 + \cdots + x_{n-1}$, which is the sum of all the numbers except the first and last.

We denote their product $x_1 \times x_2 \times \cdots \times x_n$ (the result of multiplying all the numbers together) by

$$\prod_{i=1}^{n} x_i.$$

For a positive whole number, n, $n!$ ('n factorial') is the product of all the numbers from 1 up to n. For example, $4! = 1.2.3.4 = 24$. By convention $0!$ is taken to be 1. The factorial can be expressed using the product notation:

$$n! = \prod_{i=1}^{n} i.$$

Example 2.1 Suppose that $x_1 = 1, x_2 = 3, x_3 = -1, x_4 = 5$. Then

$$\sum_{i=1}^{4} x_i = 1 + 3 + (-1) + 5 = 8 \quad \text{and} \quad \sum_{i=2}^{4} x_i = 3 + (-1) + 5 = 7.$$

[1]You may consult Booth (1998) or a large number of other basic maths texts, for further information on basic notations.

We also have, for example,

$$\prod_{i=1}^{4} x_i = 1(3)(-1)(5) = -15 \quad \text{and} \quad \prod_{i=1}^{3} x_i = 1(3)(-1) = -3.$$

Activity 2.1 Suppose that $x_1 = 3$, $x_2 = 1$, $x_3 = 4$, $x_4 = 6$. Find

$$\sum_{i=1}^{4} x_i \qquad \text{and} \qquad \prod_{i=1}^{4} x_i.$$

2.3 Simple algebra

You should try to become confident and capable in handling simple algebraic expressions and equations. You should be proficient in:

- collecting up terms: e.g.
$$2a + 3b - a + 5b = a + 8b.$$
- multiplication of variables: e.g.
$$\begin{aligned}(-a)(b) + (a)(-b) - 3(a)(b) + (-2a)(-4b) &= -ab - ab - 3ab + 8ab \\ &= 3ab,\end{aligned}$$
- expansion of bracketed terms: e.g.
$$\begin{aligned}(2x - 3y)(x + 4y) &= 2x^2 - 3xy + 8xy - 12y^2 \\ &= 2x^2 + 5xy - 12y^2.\end{aligned}$$

You should also be able to factorise quadratic equations, something discussed later in this chapter.

Activity 2.2 Expand $(x^2 - 1)(x + 2)$.

2.4 Sets

A set may be thought of as a collection of objects.[2] A set is usually described by listing or describing its *members* inside curly brackets. For example, when we write $A = \{1, 2, 3\}$, we mean that the objects belonging to the set A are the numbers $1, 2, 3$ (or, equivalently, the set A consists of the numbers $1, 2$ and 3). Equally (and this is what we mean by 'describing' its members), this set could have been written as

$$A = \{n \mid n \text{ is a whole number and } 1 \leq n \leq 3\}.$$

[2] See Anthony and Biggs (1996) Section 2.1.

Here, the symbol | stands for 'such that'. Often, the symbol ':' is used instead, so that we might write

$$A = \{n : n \text{ is a whole number and } 1 \leq n \leq 3\}.$$

As another example, the set

$$B = \{x \mid x \text{is a reader of this guide}\}$$

has as its members all of you (and nothing else). When x is an object in a set A, we write $x \in A$ and say 'x belongs to A' or 'x is a member of A'.

The set which has no members is called the *empty set* and is denoted by $\varnothing$. The empty set may seem like a strange concept, but it has its uses.

We say that the set S is a *subset* of the set T, and we write $S \subseteq T$, if every member of S is a member of T. For example, $\{1, 2, 5\} \subseteq \{1, 2, 4, 5, 6, 40\}$. (Be aware that some texts use $\subset$ where we use $\subseteq$.)

Given two sets A and B, the *union* $A \cup B$ is the set whose members belong to A or B (or both A and B): that is,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Example 2.2 If $A = \{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 7\}$, then $A \cup B = \{1, 2, 3, 4, 5, 7\}$.

Similarly, we define the *intersection*: $A \cap B$ to be the set whose members belong to both A and B:[3]

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Activity 2.3 Suppose $A = \{1, 2, 3, 5\}$ and $B = \{2, 4, 5, 7\}$. Find $A \cap B$.

2.5 Numbers

There are some standard notations for important sets of numbers.[4] The set $\mathbb{R}$ of *real numbers*, may be thought of as the points on a line. Each such number can be described by a decimal representation.

Given two real numbers a and b, we define the *intervals*

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$
$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$
$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$
$$[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$$
$$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$$

[3] See Anthony and Biggs (1996) for examples of union and intersection.
[4] See Anthony and Biggs (1996) Section 2.1.

$$(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$$
$$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}.$$

The symbol ∞ means 'infinity', but it is not a real number, merely a notational convenience. You should notice that when a square bracket, '[' or ']', is used to denote an interval, the number beside the bracket is included in the interval, whereas if a round bracket, '(' or ')', is used, the adjacent number is not in the interval. For example, $[2, 3]$ contains the number 2, but $(2, 3]$ does not.

The set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ of *integers* is denoted by $\mathbb{Z}$.

The positive integers are also known as *natural numbers* and the set of these, i.e. $\{1, 2, 3, \ldots\}$, is denoted by $\mathbb{N}$.

Having defined $\mathbb{R}$, we can define the set $\mathbb{R}^2$ of *ordered pairs* (x, y) of real numbers. Thus $\mathbb{R}^2$ is the set usually depicted as the set of points in a plane, x and y being the coordinates of a point with respect to a pair of axes. For instance, $(-1, 3/2)$ is an element of $\mathbb{R}^2$ lying to the left of and above $(0, 0)$, which is known as the *origin*.

2.6 Functions

Given two sets A and B, a *function* from A to B is a rule which assigns to each member of A *precisely one* member of B.[5] For example, if A and B are both the set $\mathbb{Z}$, the rule which says 'add 2' is a function. Normally we express this function by a formula: if we call the function f, we can write the rule which defines f as $f(x) = x + 2$. Two very important functions in economics are the supply and demand functions for a good.[6] These are discussed later in this chapter.

It is often helpful to think of a function as a machine which converts an input into an output, as shown in Figure 2.1.

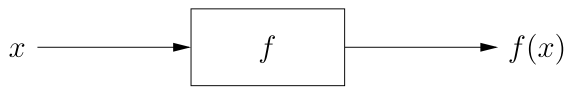

Figure 2.1: A function as a machine

2.7 Inverse functions

As the one-way arrows in Figure 2.1 indicate, a function is a one-way relationship: the function f takes a number x as input and it returns another number, $f(x)$. Suppose you were told that the output, $f(x)$, was a number y, and you wanted to know what the input was. In some cases, this is easy. For example, if $f(x) = x + 2$ and the output $f(x)$ is the number y, then we must have $y = f(x) = x + 2$. Solving for x in terms of y, we find that $x = y - 2$. In other words, there is only one possible input x which could have produced output y for this function, namely $x = y - 2$. In a situation such as this, where for *each and every* y there is *exactly one* x such that $f(x) = y$, we say that the

[5]See Anthony and Biggs (1996) Section 2.2.
[6]See Anthony and Biggs (1996) Section 1.2.

function f has an inverse function.[7] The inverse function is denoted by f^{-1}, and it is the rule for reversing f. Formally, $f^{-1}(x)$ is defined by

$$x = f^{-1}(y) \iff f(x) = y.$$

(The symbol $\iff$ means 'if and only if' or 'is equivalent to'). When $f(x) = x + 2$, we have seen that

$$y = f(x) \iff x = y - 2,$$

so the inverse function (which takes as input a number y and returns the number x such that $f(x) = y$) is given by

$$f^{-1}(y) = y - 2.$$

(This could also be written as $f^{-1}(x) = x - 2$ or $f^{-1}(z) = z - 2$; there is nothing special about the symbol used to denote the *variable*, i.e. the input to the function.[8])

It should be emphasised that not every function has an inverse. For instance, the function $f(x) = x^2$, from $\mathbb{R}$ to $\mathbb{R}$, has no inverse. To see this, we can simply observe that there is not exactly one number x such that $f(x) = y$, where $y = 1$; for, both when $x = 1$ and $x = -1$, $f(x) = x^2 = 1$. (Of course, this observation is true for any positive number y.) So, in this case, we cannot definitively answer the question 'If $f(x) = 1$, what is x?'.

Activity 2.4 If $f(x) = 3x + 2$, find a formula for $f^{-1}(x)$.

2.8 Composition of functions

If we are given two functions f and g, then we can apply them consecutively to obtain what is known as the *composite* function h, given by the rule

$$h(x) = f(g(x)).$$

The composite function h is denoted $h = fg$ and is often described in words as 'g followed by f' or as 'f after g'.[9] It is also sometimes denoted by $f \circ g$.

Example 2.3 Suppose that $f(x) = x + 1$ and $g(x) = x^4$. Then the composite function $h = fg$ is given by

$$fg(x) = f\left(g(x)\right) = f(x^4) = x^4 + 1.$$

On the other hand, the function $k = gf$ is given by

$$gf(x) = g(f(x)) = g(x + 1) = (x + 1)^4.$$

Note, then, that in general, the compositions fg and gf are different.

[7]See Anthony and Biggs (1996) Section 2.2.
[8]See Anthony and Biggs (1996) Section 2.2, for discussion of 'dummy variables'.
[9]See Anthony and Biggs (1996) Section 2.3.

Activity 2.5 If $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, find a formula for the composition fg.

2.9 Powers

When n is a positive integer, the nth *power*[10] of the number a, a^n, is simply the product of n copies of a, that is,

$$a^n = \underbrace{a \times a \times a \times \cdots \times a}_{n \text{ times}}.$$

The number n is called the *power*, *exponent* or *index*. We have the *power rules* (or *rules of exponents*):

$$a^x a^y = a^{x+y}, \quad (a^x)^y = a^{xy},$$

whenever x and y are positive integers. The power a^0 is defined to be 1. When n is a positive integer, a^{-n} means $1/a^n$. For example, 3^{-2} is $1/3^2 = 1/9$. The power rules hold when x and y are any integers, positive, negative or zero. When n is a positive integer, $a^{1/n}$ is the 'positive nth root of a'; this is the number x such that $x^n = a$. Formally, suppose n is a positive integer and let S be the set of all non-negative real numbers. Then the function $f(x) = x^n$ from S to S has an inverse function f^{-1}. We can think of f^{-1} as the definition of raising a number to the power of $1/n$: explicitly, $f^{-1}(y) = y^{1/n}$. Of course, $a^{1/2}$ is usually denoted by $\sqrt{a}$, and is the *square root* of a. When m and n are integers and n is positive, $a^{m/n}$ is $\left(a^{1/n}\right)^m$. So, the power rules still apply.

2.10 Graphs

In this section, we consider the graphs of functions. The graphing of functions is very important in its own right, and familiarity with graphs of common functions and the ability to produce graphs systematically is a necessary and important aspect of the subject.

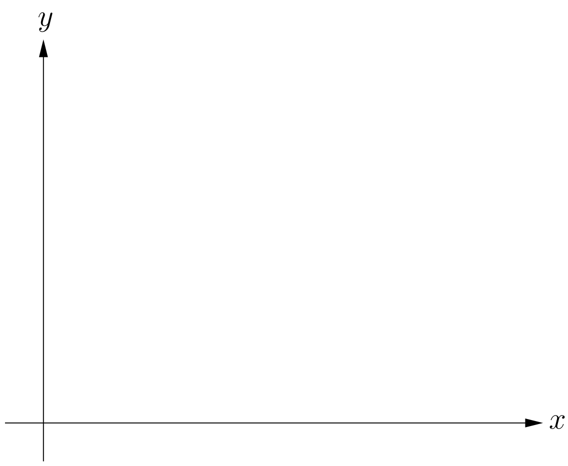

Figure 2.2: The x and y-axes

[10] See Anthony and Biggs (1996) Section 7.1.

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The *graph*[11] of a function $f(x)$ is the set of all points in the plane of the form $(x, f(x))$. Sketches of graphs can be very useful. To sketch a graph, we start with the x-axis and y-axis, as in Figure 2.2. (This figure only shows the region in which x and y are both non-negative, but the x-axis extends to the left and the y-axis extends downwards.)

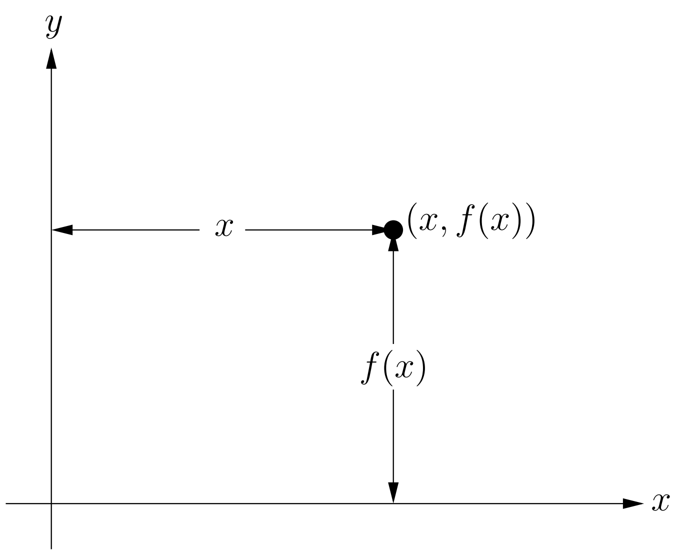


Figure 2.3: Plotting the a point on a graph

We then plot all points of the form $(x, f(x))$. Thus, at x units from the origin (the point where the axes cross), we plot a point whose height above the x-axis (that is, whose y-coordinate) is $f(x)$. This is shown in Figure 2.3. The graph is sometimes described as the graph $y = f(x)$ to signify that the y-coordinate represents the function value $f(x)$.

Joining together all points of the form $(x, f(x))$ results in a *curve*, called the *graph* of $f(x)$. This is often described as the *curve with equation* $y = f(x)$. Figure 2.4 gives an example of what this curve might look like.

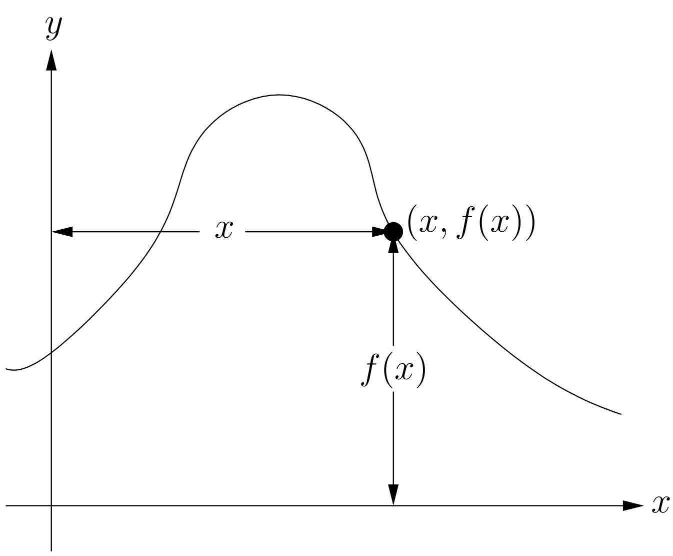


Figure 2.4: The curve $y = f(x)$

[11]See Anthony and Biggs (1996) Section 2.4.

These figures indicate what is meant by the graph of a function, but you should not imagine that the correct way to sketch a graph is to plot a few points of the form $(x, f(x))$ and join them up; this approach rarely works well and more sophisticated techniques are needed. (Many of these will be discussed later.)

We shall discuss the graphs of some standard important functions as we progress. We start with the easiest of all: the graph of a *linear function*. In the next section we look at the graphs of quadratic functions. The linear functions are those of the form $f(x) = mx + c$ and their graphs are straight lines, with *gradient*, or *slope*, m, which cross the y-axis at the point $(0, c)$. Figure 2.5 illustrates the graph of the function $f(x) = 2x + 3$ and Figure 2.6 the graph of the function $f(x) = -x + 2$.

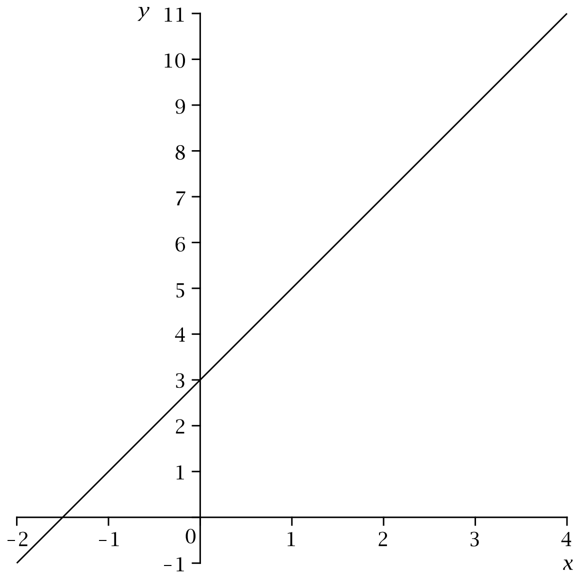


Figure 2.5: The graph of the line $y = 2x + 3$

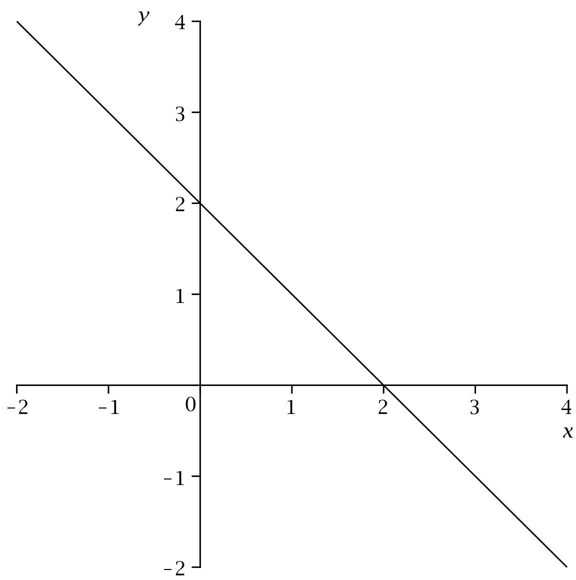


Figure 2.6: The graph of the line $y = -x + 2$

Activity 2.6 Sketch the curves $y = x + 3$ and $y = -3x - 2$.

2.11 Quadratic equations and curves

A common problem is to find the set of solutions of a *quadratic* equation[12]

$$ax^2 + bx + c = 0,$$

where we may as well assume that $a \neq 0$, because if $a = 0$ the equation reduces to a linear one. (Note that, by a solution, we mean a value of x for which the equation is true.) In some cases the quadratic expression can be factorised, which means that it can be written as the product of two linear terms (of the form $x - a$ for some a). For example $x^2 - 6x + 5 = (x - 1)(x - 5)$, so the equation $x^2 - 6x + 5$ becomes $(x - 1)(x - 5) = 0$. Now the only way that two numbers can multiply to give 0 is if at least one of the numbers is 0, so we can conclude that $x - 1 = 0$ or $x - 5 = 0$; that is, the equation has two solutions, 1 and 5. Although factorisation may be difficult, there is a general technique for determining the solutions to a quadratic equation, as follows.[13] Suppose we have the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$. Then:

- if $b^2 - 4ac < 0$, the equation has *no* real solutions;
- if $b^2 - 4ac = 0$, the equation has *exactly one* solution, $x = -\dfrac{b}{2a}$;
- if $b^2 - 4ac > 0$, the equation has *two* solutions,

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

For example, consider the quadratic equation $x^2 - 2x + 3 = 0$; here we have $a = 1$, $b = -2$, $c = 3$. The quantity $b^2 - 4ac$ (called the *discriminant*) is $(-2)^2 - 4(1)(3) = -8$, which is negative, so this equation has no solution. (Technically, it has no *real* solutions. It does have solutions in 'complex numbers', but this is outside the scope of this subject.) This is less mysterious than it may seem. We can write the equation as $(x - 1)^2 + 2 = 0$. Rewriting the left-hand side of the equation in this form is known as 'completing the square'. Now, the square of a number is always greater than or equal to 0, so the quantity on the left of this equation is always at least 2 and is therefore never equal to 0. The above formulae for the solutions to a quadratic equation are obtained using the technique of completing the square.[14]

It is instructive to look at the graphs of quadratic functions and to understand the connection between these and the solutions to quadratic equations. First, let's look at the graph of a typical quadratic function $y = ax^2 + bx + c$. Figure 2.7 shows the curves one obtains for two typical quadratics $ax^2 + bx + c$. For the first, a is positive and for the second a is negative. We have omitted the x and y axes in these figures; it is the *shape* of the graph that we want to emphasise first. Note that the first graph has a 'U'-shape and that the second is the same sort of shape, upside-down. To be more formal, the curves are *parabolae*.

[12] See Anthony and Biggs (1996) Section 2.4.
[13] See Anthony and Biggs (1996) Section 2.4.
[14] See Anthony and Biggs (1996) Section 2.4, if you haven't already.

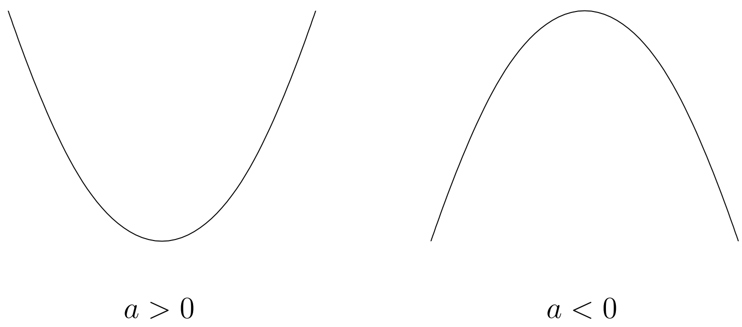


Figure 2.7: Typical quadratic curves $y = ax^2 + bx + c$

Figure 2.8 is the graph of the quadratic function $f(x) = x^2 - 6x + 5$.

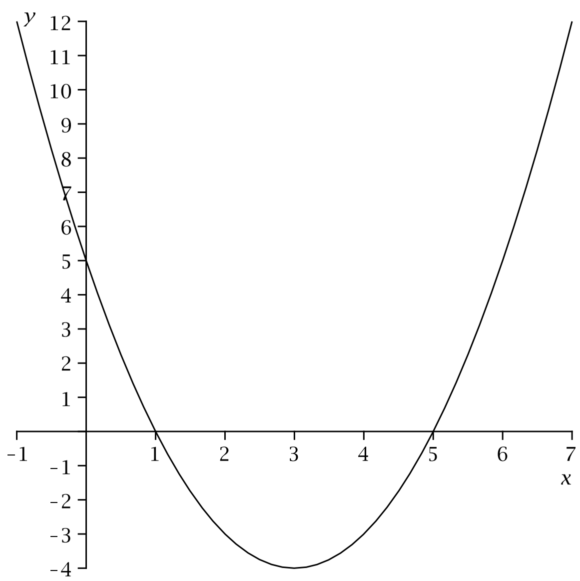


Figure 2.8: The graph of the quadratic $y = x^2 - 6x + 5$

Note that, since the number in front of the x^2 term (what we called a above) is positive, the curve is of the first type displayed in Figure 2.7. What we want to emphasise with this specific example is the positioning of the curve with respect to the axes. There is a fairly straightforward way to determine where the curve crosses the y-axis. Since the y-axis has equation $x = 0$, to find the y-coordinate of this crossing (or intercept), all we have to do is substitute $x = 0$ into the function. Since $f(0) = 0^2 - 6(0) + 5 = 5$, the point where the curve crosses the y-axis is $(0, 5)$. (Generally, the point where the graph of a function $f(x)$ crosses the y-axis is $(0, f(0))$.) The other important points on the diagram are the points where the curve crosses the x-axis. Now, the curve has equation $y = f(x)$, and the x-axis has equation $y = 0$, so the curve crosses (or meets) the x-axis when $y = f(x) = 0$. (This argument, so far, is completely general: to find where the graph of $f(x)$ crosses the x-axis, we solve the equation $f(x) = 0$. In general, this may have no solution, one solution or a number of solutions, depending on the function.) Thus, we have to solve the equation $x^2 - 5x + 6 = 0$. We did this earlier, and the solutions are $x = 1$ and $x = 5$. It follows that the curve crosses the x-axis at $(1, 0)$ and $(5, 0)$.

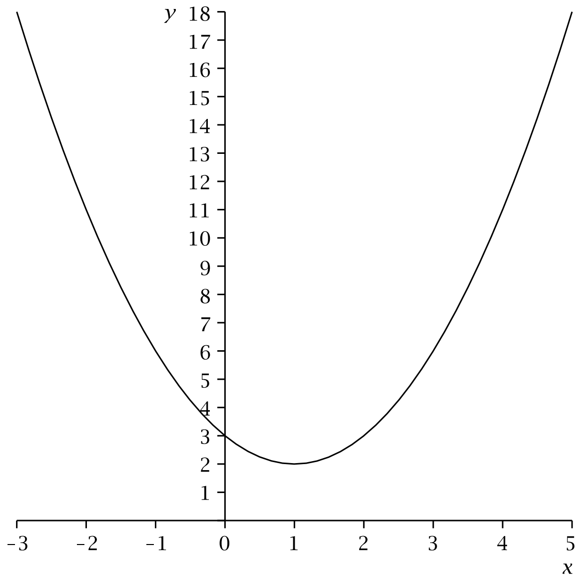


Figure 2.9: The graph of the quadratic $y = x^2 - 2x + 3$

Figure 2.9 shows the graph of another quadratic, $f(x) = x^2 - 2x + 3$.

Notice that this one does *not* cross the x-axis. This is because the quadratic equation $x^2 - 2x + 3 = 0$ (which we met earlier) has *no* solutions. You might ask what the coordinates of the lowest point of the 'U' are. Later, we shall encounter a general technique for answering such questions. For the moment, we can determine the point by using the observation, made earlier, that the function is $(x-1)^2 + 2$. Now, $(x-1)^2 \geq 0$ and is equal to 0 only when $x = 1$, so the lowest value of the function is 2, which occurs when $x = 1$; that is, the lowest point of the 'U' is the point $(1, 2)$. You can obtain quite a lot of information about quadratic curves without using very sophisticated techniques.

Activity 2.7 Sketch the curve $y = x^2 + 4x + 3$. Where does it cross the x-axis?

2.12 Polynomial functions

Linear and quadratic functions are examples of a more general type of function: the *polynomial functions*. A polynomial function is one of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

The right-hand side is known simply as a *polynomial*. The number a_i is known as *the coefficient of* x^i. If $a_n \neq 0$ then n, the largest power of x in the polynomial, is known as the *degree* of the polynomial. Thus, the linear functions are precisely the polynomials of degree 1 and the quadratics are the polynomials of degree 2. Polynomials of degree 3 are known as cubics. The simplest is the function $f(x) = x^3$, the graph of which is shown in Figure 2.10. Notice that the graph of the function $f(x) = x^3$ only crosses the x-axis at the origin, $(0, 0)$. That is, the equation $f(x) = 0$ has just one solution. (We say that the function has just one *zero*.) In general, a polynomial function of degree n has at most n zeroes. For example, since

$$x^3 - 7x + 6 = (x-1)(x-2)(x+3),$$

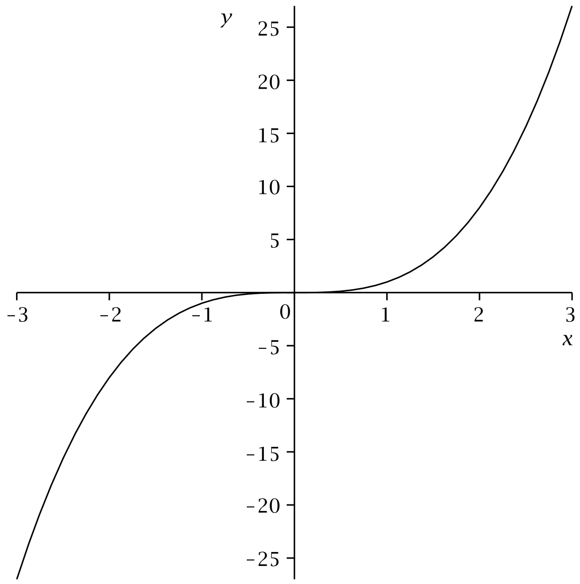


Figure 2.10: The graph of the curve $y = x^3$

the function $f(x) = x^3 - 7x + 6$ has three zeroes; namely, $1, 2, -3$. Unfortunately, there is no general straightforward formula (as there is for quadratics) for the solutions to $f(x) = 0$ for polynomials f of degree larger than 2.

Activity 2.8 Factorise $f(x) = x^3 + 4x^2 + 3x$.

2.13 Simultaneous equations

An important type of problem arises when we have several equations which we have to solve 'simultaneously'.[15] This means that we must find the intersection of the solution sets of the individual equations. We have already met an example of this: when we want to find the points (if any) where the curve $y = f(x)$ meets the x-axis, we are essentially solving two equations simultaneously. The first is $y = f(x)$ and the second (the equation of the x-axis) is $y = 0$. This is one way of thinking about how the equation $f(x) = 0$ arises. We shall spend a lot of time later looking at problems in which the aim is to solve simultaneously more than two equations. For the moment, we shall just illustrate with a simple example.

Any two lines which are not parallel cross each other exactly once, but how do we find the crossing point? Let's consider the lines with equations $y = 2x - 1$ and $y = x + 2$. These are not parallel, since the gradient of the first is 2, whereas the gradient of the second is 1. Figure 2.11 shows the two lines. Our aim is to determine the coordinates of the crossing point C.

To find C, let us suppose that $C = (X, Y)$. Then, since C lies on the line with equation $y = 2x - 1$, we must have $Y = 2X - 1$. But C also lies on the line $y = x + 2$, so $Y = X + 2$. Therefore the coordinates X and Y of C satisfy the following two

[15]See Anthony and Biggs (1996) Sections 1.3 and 2.4.

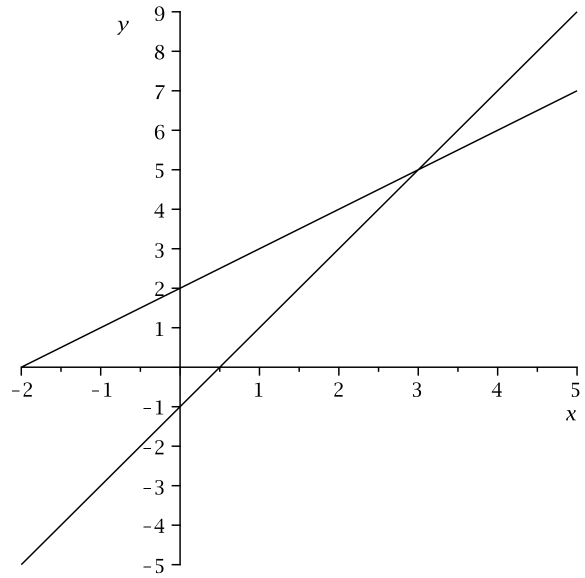


Figure 2.11: The lines $y = 2x - 1$ and $y = x + 2$

equations, *simultaneously*:

$$Y = 2X - 1$$
$$Y = X + 2.$$

It follows that

$$Y = 2X - 1 = X + 2.$$

From $2X - 1 = X + 2$ we obtain $X = 3$. Then, to obtain Y, we use either the fact that $Y = 2X - 1$, obtaining $Y = 5$, or we can use the equation $Y = X + 2$, obtaining (of course) the same answer. It follows that $C = (3, 5)$.

> **Activity 2.9** Find the point of intersection of the lines with equations $y = 2x - 3$ and $y = 2 - \frac{1}{2}x$.

2.14 Supply and demand functions

Supply and demand functions[16] describe the relationship between the price of a good, the quantity supplied to the market by the manufacturer, and the amount the consumers wish to buy. The *demand function* q^D of the price p describes the demand quantity: $q^D(p)$ is the quantity which would be sold if the price were p. Similarly, the *supply function* q^S is such that $q^S(p)$ is the amount supplied when the market price is p.

In the simplest models of the market, it is assumed that the supply and demand functions are linear — in other words, their graphs are straight lines. For example, it could be that $q^D(p) = 4 - p$ and $q^S(p) = 2 + p$. Note that the graph of the demand function is a downward-sloping straight line, whereas the graph of the supply function is upward-sloping. This is to be expected, since, for example, as the price of a good

[16] See Anthony and Biggs (1996) Section 1.2.

increases, the consumers are prepared to buy less of the good, and so the demand function decreases as price increases.

Sometimes, the supply and demand relationships are expressed through equations. For instance, in the example just given we could equally well have described the relationship between demand quantity and price by saying that the *demand equation* is $q + p = 4$. The graphs of the demand function and supply function are known, respectively, as the *demand curve* and the *supply curve*.

There is another way to view the relationship between price and quantity demanded, where we ask how much the consumers (as a group; that is, on aggregate) would be willing to pay for each unit of a good, given that a quantity q is available. From this viewpoint we are expressing p in terms of q, instead of the other way round. We write $p^D(q)$ for the value of p corresponding to a given q, and we call p^D the *inverse demand function*. It is, as the name suggests, the inverse function to the demand function. For example, with $q^D(p) = 4 - p$, we have $q = 4 - p$ and so $p = 4 - q$; thus, $p^D(q) = 4 - q$. In a similar way, when we solve for the price in terms of the supply quantity, we obtain the *inverse supply function* $p^S(q)$.

The market is in *equilibrium*[17] when the consumers have as much of the commodity as they want and the suppliers sell as much as they want. This occurs when the quantity supplied matches the quantity demanded, or, *supply equals demand*. To find the equilibrium price p^*, we solve $q^D(p) = q^S(p)$ and then to determine the equilibrium quantity q^* we compute $q^* = q^D(p^*)$ (or $q^* = q^S(p^*)$). (Generally there might be more than one equilibrium, but not when the supply and demand are linear.) Geometrically, the equilibrium point(s) occur where the demand curve and supply curve intersect.

Activity 2.10 Suppose the demand function is $q^D(p) = 20 - 2p$ and that the supply function is $q^S(p) = \frac{2}{3}p - 4$. Find the equilibrium price p^* and equilibrium quantity q^*.

Not all supply and demand equations are linear. Consider the following example.

Example 2.4 Suppose that we have demand curve,

$$q = 250 - 4p - p^2,$$

and supply curve

$$q = 2p^2 - 3p - 40.$$

Let us find the equilibrium price and quantity, and sketch the curves for $1 \leq p \leq 10$.

To find the equilibrium, the simplest approach is to set the demand quantity equal to the supply quantity, giving $250 - 4p - p^2 = 2p^2 - 3p - 40$. To solve this, we convert it into a quadratic equation in the standard form (that is, one of the form $ax^2 + bx + c = 0$, though clearly here we shall use p rather than x). We have $3p^2 + p - 290 = 0$. Using the formula for the solutions of a quadratic equation, we have solutions

$$p = \frac{-1 \pm \sqrt{1 - 4(3)(-290)}}{2(3)} = \frac{-1 \pm \sqrt{3481}}{6} = \frac{-1 \pm 59}{6},$$

[17] See Anthony and Biggs (1996) Section 1.3.

which is $p = -10$ and $p = 29/3$. (A calculator has been used here, so, given that calculators are not permitted in the exam, this precise example would not appear in an exam. This type of example, with easier arithmetic, could, however, do so: see the Sample exam questions at the end of this chapter.) You could also have solved this equation using factorisation. It is not so easy in this case, but we might have been able to spot the factorisation

$$3p^2 + p - 290 = (3p - 29)(p + 10),$$

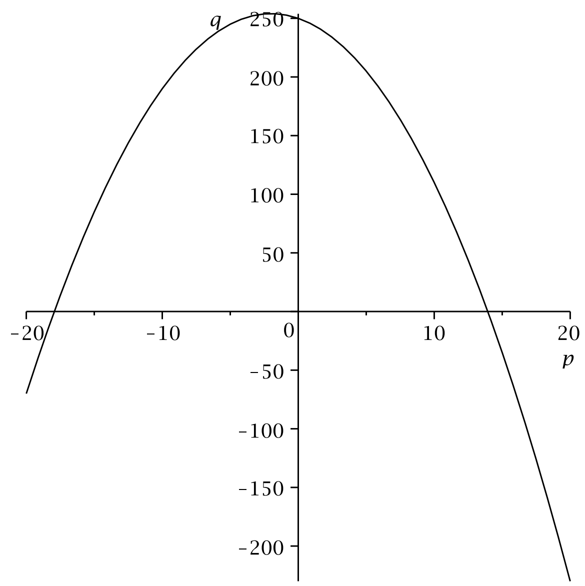
which leads to the same answers. Clearly only the second of these two solutions is economically meaningful. So the equilibrium price is $p = 29/3$. To find the equilibrium quantity, we can use either the supply or demand equations, and we obtain

$$q = 250 - 4\left(\frac{29}{3}\right) - \left(\frac{29}{3}\right)^2 = \frac{1061}{9}.$$

We now turn our attention to sketching the curves. The demand curve $q = 250 - 4p - p^2$ is a quadratic with a negative squared term, and hence has an up-turned 'U' shape. It crosses the q-axis at $(0, 250)$. It crosses the p-axis where $250 - 4p - p^2 = 0$. In standard form, this quadratic equation is $-p^2 - 4p + 250 = 0$ and it has solutions

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(-1)(250)}}{2(-1)} = \frac{4 \pm \sqrt{1016}}{-2} = 13.937, -17.937.$$

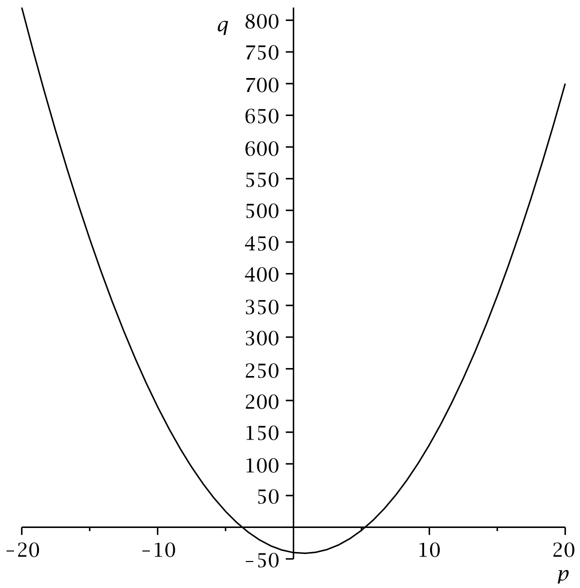
(Again, we use a calculator here, but in the exam such difficult computations would not be required.) With this information, we now know that the curve is as in the following sketch.



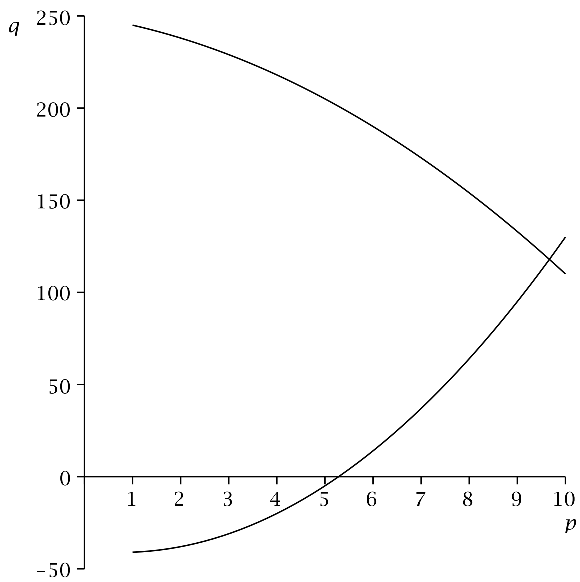
For the supply curve, we have $q = 2p^2 - 3p - 40$, which is a 'U'-shaped parabola. This curve crosses the q-axis at $(0, -40)$. It crosses the P-axis when $2p^2 - 3p - 40 = 0$. This equation has solutions

$$\frac{3 \pm \sqrt{9 - 4(2)(-40)}}{4} = \frac{3 \pm \sqrt{329}}{4} = 5.285, -3.785$$

and the curve therefore looks like the following.



The question asks us to sketch the curves for the range $1 \leq p \leq 10$. Sketching both on the same diagram we obtain:



Note that the equilibrium point $(29/3, 1061/9)$ is where the two curves intersect.

Activity 2.11 Suppose the market demand function is given by $p = 4 - q - q^2$ and that the market supply function is $p = 1 + 4q + q^2$. Sketch both these functions on the same graph.

2.15 Exponentials

An *exponential-type* function is one of the form $f(x) = a^x$ for some number a. (Do not confuse it with the function which raises a number to the power a. An exponential-type function has the form $f(x) = a^x$, whereas the 'ath power function' has the form $f(x) = x^a$.)

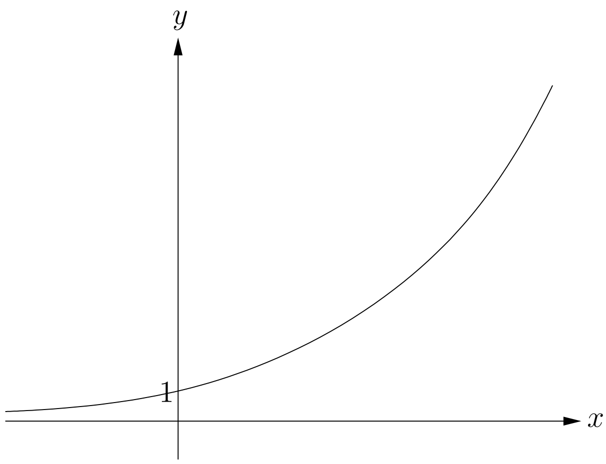


Figure 2.12: The graph of the function $f(x) = a^x$, when $a > 1$

There are some important points to notice about $f(x) = a^x$ and its graph, for $a > 0$. First of all, a^x is always positive, for every x. Furthermore, if $a > 1$ then a^x becomes larger and larger, without bound, as x increases. We say that a^x *tends to infinity* as x tends to infinity. Also, for such an a, as x becomes more and more negative, the function a^x gets closer and closer to 0. In other words, a^x tends to 0 as x tends to 'minus infinity'. This behaviour can be seen in Figure 2.12 for the case in which a is a number larger than 1. If $a < 1$ the behaviour is quite different; the resulting graph is of the form shown in Figure 2.13. (You can perhaps see why it has this shape by noting that $a^x = (1/a)^{-x}$.)

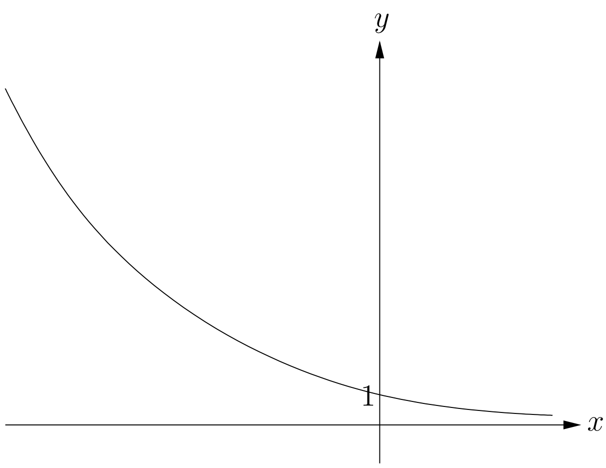


Figure 2.13: The graph of the function $f(x) = a^x$, when $0 < a < 1$

Some very important properties of exponential-type functions, exactly like the power laws, hold. In particular,

$$a^{r+s} = a^r a^s, \quad (a^r)^s = a^{rs}.$$

Another property is that, regardless of a, a^0 is equal to 1, and the point $(0, 1)$ is the only place where the graph of a^x crosses the y-axis.

We now define the *exponential function*. This is the most important exponential-type function. It is defined to be $f(x) = e^x$, where e is the special number $2.71828\ldots$ (The function e^x is also sometimes written as $\exp(x)$.) The most important facts about e^x to remember from this section are the shape of its graph, and its properties. The graph is shown in Figure 2.14. We shall see in the next chapter one reason why the number e is so special.

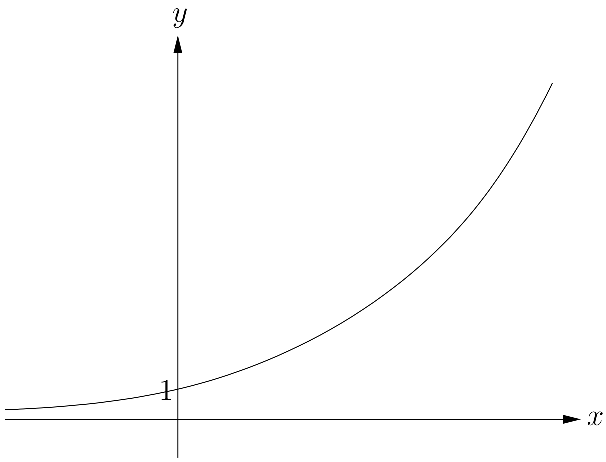


Figure 2.14: The graph of the function $f(x) = e^x$

2.16 The natural logarithm

Formally, the *natural logarithm*[18] of a positive number x, denoted $\ln x$ (or, sometimes, $\log x$), is the number y such that $e^y = x$. In other words, the natural logarithm function is the *inverse* of the exponential function e^x (regarded as a function from the set of all real numbers to the set of positive real numbers). Sometimes $\ln x$ is called the *logarithm to base e*. The reason for this is that we can, more generally, consider the inverse of the exponential-type function a^x. This inverse function is called the *logarithm to base a* and we use the notation $\log_a x$. Thus, $\log_a x$ is the answer to the question 'What is the number y such that $a^y = x$?'.

The two most common logarithms, other than the natural logarithm, are logarithms to base 2 and 10. For example, since $2^3 = 8$, we have $\log_2 8 = 3$. It may seem awkward to have to think of a logarithm as the inverse of an exponential-type function, but it is really not that strange. Confronted with the question 'What is $\log_a x$?', we simply turn it around so that it becomes, as above, 'What is the number y such that $a^y = x$?'.

There is often some confusion caused by the notations used for logarithms. Some texts use log to mean natural logarithm, whereas others use it to mean $\log_{10}$. In this guide,

[18] See Anthony and Biggs (1996) Section 7.4.

we shall use ln to mean natural logarithm and we shall avoid altogether the use of 'log' without a subscript indicating its base.

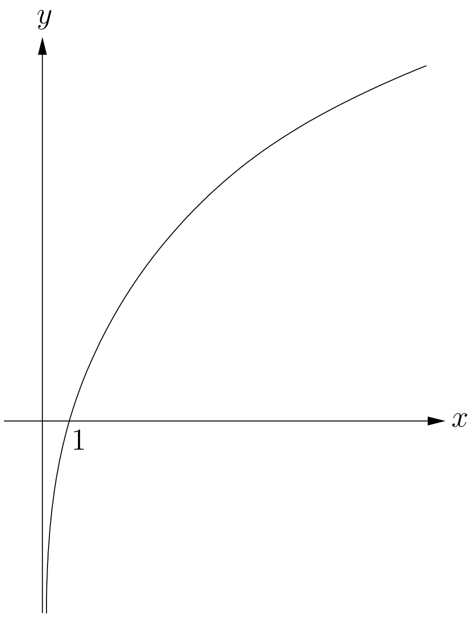


Figure 2.15: The graph of the natural logarithm function, $\ln x$

Figure 2.15 shows the graph of the natural logarithm. Note that it only makes sense to define $\ln x$ for positive x. All the important properties of the natural logarithm follow from those of the exponential function. For example, $\ln 1 = 0$. Why? Because $\ln 1$ is, by its definition, the number y such that $e^y = 1$. The only such y is $y = 0$.

The other very important properties of $\ln x$ (which follow from properties of the exponential function[19]) are:

$$\ln(ab) = \ln a + \ln b, \quad \ln\left(\frac{a}{b}\right) = \ln a - \ln b, \quad \ln(a^b) = b \ln a.$$

These relationships are fairly simple and you will get used to them as you practise.

2.17 Trigonometrical functions

The trigonometrical functions, $\sin x, \cos x, \tan x$ (the *sine function*, *cosine function* and *tangent function*) are very important in mathematics and they will occur later in this subject. We shall not give the definition of these functions here. If you are unfamiliar with them, consult the texts.[20]

It is important to realise that, throughout this subject, angles are measured in *radians* rather than *degrees*. The conversion is as follows: 180 degrees equals π radians, where π is the number $3.141\ldots$ It is good practice *not* to expand π or multiples of π as decimals, but to leave them in terms of the symbol π. For example, since 60 degrees is one third of 180 degrees, it follows that, in radians, 60 degrees is $\pi/3$.

[19]See Anthony and Biggs (1996) Section 7.4.

[20]See Dowling (2000) Section 20.5, for example or Booth (1998) Module 13.

The graphs of the sine function, $\sin x$, and the cosine function, $\cos x$, are shown in Figures 2.16 and 2.17. Note that these functions are periodic: they repeat themselves every 2π steps. (For example, the graph of the sine function between 2π and 4π has exactly the same shape as the graph of the function between 0 and 2π.)

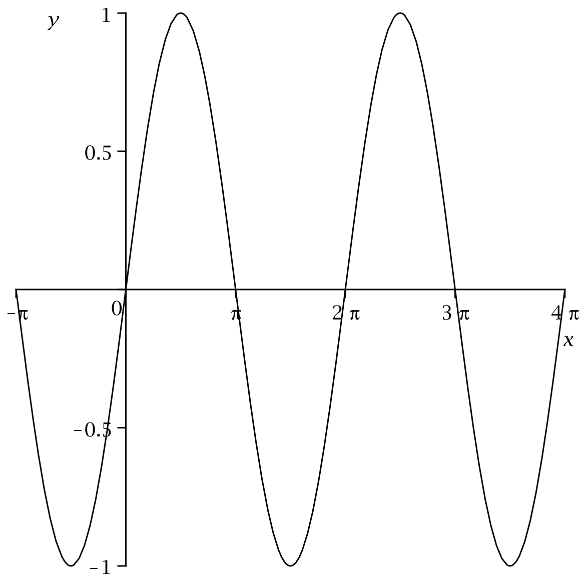


Figure 2.16: The graph of the function $\sin x$

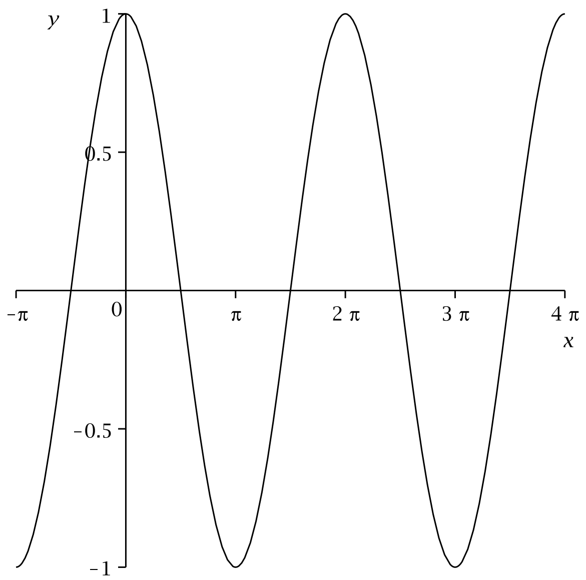


Figure 2.17: The graph of the function $\cos x$

Note also that the graph of $\cos x$ is a 'shift' of the graph of $\sin x$, obtained by shifting the $\sin x$ graph by $\pi/2$ to the left. Mathematically, this is equivalent to the fact that $\cos x = \sin(x + \frac{\pi}{2})$.

The tangent function, $\tan x$, is defined in terms of the sine and cosine functions, as follows:

$$\tan x = \frac{\sin x}{\cos x}.$$

Note that the sine and cosine functions always take a value between 1 and -1. Table 2.1 gives some important values of the trigonometrical functions.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	undefined

Table 2.1: Important values for the trigonometrical functions.

Technically, the tangent function is not defined at $\pi/2$. This means that no meaning can be given to $\tan(\pi/2)$. To see why, note that $\tan x = \sin x/\cos x$, but $\cos(\pi/2) = 0$, and we cannot divide by 0. You might wonder what happens to $\tan x$ around $x = \pi/2$. The graph of $\tan x$ can be found in the textbooks.[21]

There are some useful results about the trigonometrical functions, with which you should familiarise yourself. First, for all x,

$$(\cos x)^2 + (\sin x)^2 = 1.$$

(We use $\sin^2 x$ to mean $(\sin x)^2$, and similarly for $\cos^2 x$.) Then there are the **double-angle formulae**, which state that:

$$\sin(2x) = 2\sin x\cos x, \quad \cos(2x) = \cos^2 x - \sin^2 x.$$

Note that, since $\cos^2 x + \sin^2 x = 1$, the double angle formula for $\cos(2x)$ may be written in another two, useful, ways:

$$\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$$

The double-angle formulae arise from two more general results. It is the case that for any angles θ and ϕ, we have

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

and

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

The double-angle formulae follow from these when we take θ and ϕ to be equal to each other.

Let S be the interval $[-\pi/2, \pi/2]$. Then, regarded as a function from S to the interval $[-1, 1]$, $\sin x$ has an inverse function, which we denote by $\sin^{-1}$; thus, for $-1 \leq y \leq 1$, $\sin^{-1}(y)$ is the angle x (in radians) such that $-\pi/2 \leq x \leq \pi/2$ and $\sin x = y$. In a similar manner, the function $\cos x$ from the interval $[0, \pi]$ to $[-1, 1]$ has an inverse, which we denote by $\cos^{-1}$: so, for $-1 \leq y \leq 1$, $\cos^{-1} y$ is the angle x (in radians) such that $0 \leq x \leq \pi$ and $\cos x = y$. Similarly, the function $\tan x$, regarded as a function from the interval $(-\pi/2, \pi/2)$ to $\mathbb{R}$ has an inverse, denoted by $\tan^{-1}$. Some texts use the notation arcsin for $\sin^{-1}$, arccos for $\cos^{-1}$, and arctan for $\tan^{-1}$.

[21] See, for example, Binmore and Davies (2001) p. 57.

2.18 Further applications of functions

We have already seen that supply and demand can usefully be modelled using very simple functional relationships, such as linear functions. We now discuss a few more applications.

Suppose that the demand equation for a good is of the form $p = ax + b$ where x is the quantity produced. Then, at equilibrium, the quantity x is the amount supplied and sold, and hence the total revenue TR at equilibrium is price times quantity, which is

$$TR = (ax + b)x = ax^2 + bx,$$

a quadratic function which may be maximised either by completing the square, or by using the techniques of calculus (discussed later).

Another very important function in applications is the total cost function of a firm. In the simplest model of a cost function, a firm has a *fixed cost*, that remain fixed independent of production or sales, and it has *variable costs* which, for the sake of simplicity, we will assume for the moment vary proportionally with production. That is, the variable cost is of the form Vx for some constant V, where x represents the production level. The total cost TC is then the sum of these two: $TC = F + Vx$. For a limited range of x this very simplistic relationship often holds well but more complicated models (for instance, involving quadratic and exponential functions) often occur.

Combining the total cost and revenue functions on one graph enables us to perform break-even analysis. The break-even output is that for which total cost equals total revenue. In simplified, linear, models the break-even point (should it exist) is unique. When non-linear relationships are used, a number of break-even points are possible.

Example 2.5 Let us find the break-even points in the case where the total cost function is $TC = 7 + 2x + x^2$ and the total revenue function is $TR = 10x$. To find the break-even points, we need to solve $TC = TR$; that is, $7 + 2x + x^2 = 10x$ or $x^2 - 8x + 7 = 0$. Splitting this into factors, $(x - 7)(x - 1) = 0$, so $x = 7$ or $x = 1$. (Alternatively, the formula for the solutions of a quadratic equation could be used.) Note that there are two break-even points.

Activity 2.12 Find the break-even points in the case where the total cost function is $TC = 2 + 5x + x^2$ and the total revenue function is $TR = 12 + 8x$.

Learning outcomes

At the end of this chapter and the relevant reading, you should be able to:

- determine inverse functions and composite functions
- sketch graphs of simple functions
- sketch quadratic curves and solve quadratic equations
- solve basic simultaneous equations
- find equilibria from supply and demand functions, and sketch these

- find break-even points
- explain what is meant by exponential-type functions and be able to sketch their graphs
- use properties such as $a^{x+y} = a^x a^y$ and $(a^x)^y = a^{xy}$
- explain what is meant by the exponential function e^x
- describe the natural logarithm ($\ln x$), logarithms to base a ($\log_a x$) and their properties
- describe the functions $\sin x, \cos x, \tan x$ and their properties, key values, and graphs
- explain what is meant by inverse trigonometrical functions

You do not need to know about complex (or imaginary) numbers (as you might see discussed in some texts when, in a quadratic equation, $b^2 - 4ac < 0$).

Sample examination/practice questions

The material in this chapter of the guide is essential to what follows. Many exam questions involve this material, but additionally involve other topics, such as calculus. We give just three examples of exam-type questions which make use only of the material in this chapter.

Question 2.1

Suppose the market demand function is given by

$$p = 4 - q - q^2$$

and that the market supply function is

$$p = 1 + 4q + q^2.$$

Determine the equilibrium price and quantity.

Question 2.2

Suppose that the demand relationship for a product is $p = 6/(q+1)$ and that the supply relationship is $p = q + 2$. Determine the equilibrium price and quantity.

Question 2.3

Suppose that the demand equation for a good is $q = 8 - p^2 - 2p$ and that the supply equation is $q = p^2 + 2p - 3$. Sketch the supply and demand curves on the same diagram, and determine the equilibrium price.

Answers to activities

Feedback to activity 2.1

$\sum_{i=1}^{4} x_i$ is $3 + 1 + 4 + 6 = 14$. The product $\prod_{i=1}^{4} x_i$ equals $3 \times 1 \times 4 \times 6 = 72$.

Feedback to activity 2.2

$x^3 + 2x^2 - x - 2.$

Feedback to activity 2.3

$A \cap B$ is the set of objects in both sets, and so it is $\{2, 5\}$.

Feedback to activity 2.4

If $y = f(x) = 3x + 2$ then we may solve this for x by noting that $x = (y - 2)/3$. It follows that $f^{-1}(y) = (y - 2)/3$ or, equivalently, $f^{-1}(x) = (x - 2)/3$.

Feedback to activity 2.5

$(fg)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x^2 + 1}$.

Feedback to activity 2.6

The curve $y = x + 3$ is a straight line with gradient 1, passing through the y-axis at the point $(0, 3)$. Therefore, sketching it we obtain Figure 2.18.

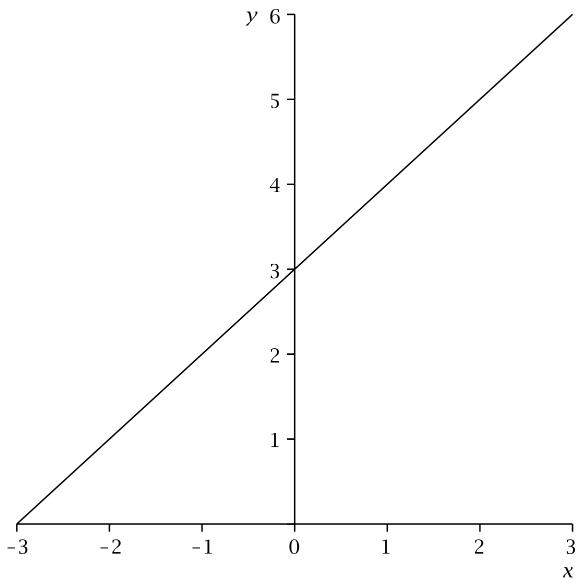


Figure 2.18: Graph of the straight line $y = x + 3$.

The curve $y = -3x - 2$ is a straight line with gradient -3 (and hence sloping downwards), passing through the y-axis at $(0, -2)$. The graph of this curve is shown in Figure 2.19.

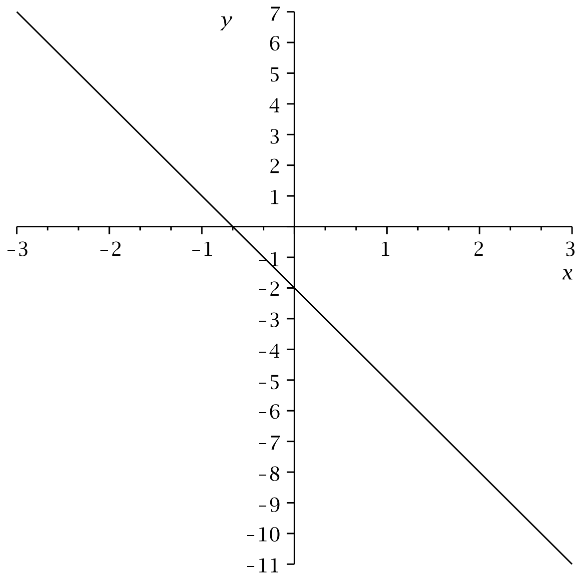


Figure 2.19: Graph of the straight line $y = -3x - 2$.

Feedback to activity 2.7

The graph $y = x^2 + 4x + 3$ is a quadratic, with a positive x^2 term, and hence it has the parabolic 'U'-shape. To locate its position, we find where it crosses the axes. It crosses the y-axis when $x = 0$, and hence at $(0, 3)$. To find where it crosses the x-axis (if at all), we need to solve $y = 0$; that is, $x^2 + 4x + 3 = 0$. There are two ways we can do this. We could spot that this factorises as $(x + 3)(x + 1) = 0$, so that the solutions are $x = -3$ and $x = -1$. Alternatively, we can use the formula for the solutions of a quadratic equation, with $a = 1, b = 4, c = 3$. This gives

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = (-4 \pm \sqrt{4})/2 = -3, -1.$$

With this information, we can sketch the curve (see Figure 2.20).

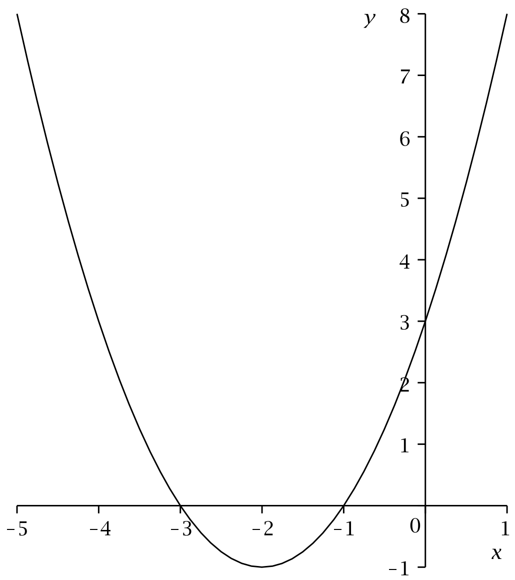


Figure 2.20: Graph of the curve $y = x^2 + 4x + 3$.

Feedback to activity 2.8

We note first that x is a factor, and so we have $x^3 + 4x^2 + 3x = x(x^2 + 4x + 3)$. To factorise the quadratic, we can simply spot the factorisation $x^2 + 4x + 3 = (x + 1)(x + 3)$ or, alternatively, we can solve the quadratic equation $x^2 + 4x + 3 = 0$, which has solutions $-1, -3$, meaning that $x^2 + 4x + 3 = (x - (-1))(x - (-3)) = (x + 1)(x + 3)$. It follows that the factorisation we require is $x(x + 1)(x + 3)$.

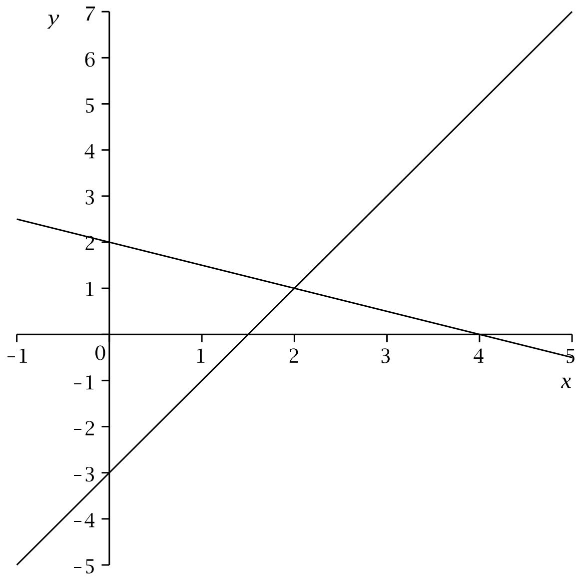


Figure 2.21: Graph of the curves $y = 2x - 3$ and $y = 2 - (1/2)x$.

Feedback to activity 2.9

To find the intersection of the two lines we solve the equations $y = 2x - 3, y = 2 - (1/2)x$ simultaneously. There is more than one way to do so, but perhaps the easiest is to write $2x - 3 = 2 - (1/2)x$, from which we obtain $(5/2)x = 5$ and hence $x = 2$. The y-coordinate of the intersection can then be found from either one of the initial equations: for example, $y = 2x - 3 = 2(2) - 3 = 1$. It follows that the intersection point is $(2, 1)$. Figure 2.21 shows the two curves. (If this were an exam question, it would not be essential to include the sketch as part of your answer. I'm doing so just to help you understand what's going on.)

Feedback to activity 2.10

To find the equilibrium, we solve simultaneously the demand and supply equations; that is, we set supply q^D equal to demand q^S. Since $q^D = 20 - 2p$ and $q^S = (2/3)p - 4$, we set $20 - 2p = (2/3)p - 4$ and hence $(8/3)p = 24$, and $p = 9$. To find the equilibrium quantity, we can use either the supply or the demand equation. Using the demand equation gives $q = 20 - 2(9) = 2$ (and of course we will get the same answer using the supply equation). Figure 2.22 shows the demand and supply curves (which are, of course, straight lines in this case).

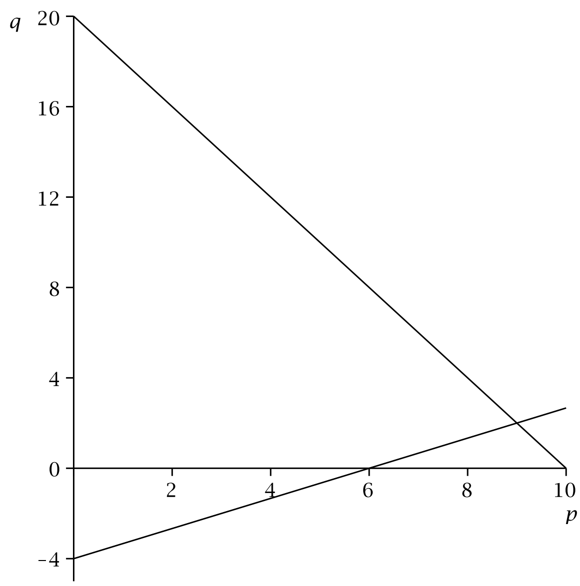


Figure 2.22: Graph of the curves $q^D = 20 - 2p$ and $q^S = (2/3)p - 4$.

Feedback to activity 2.11

Note that, here, p is given in terms of q, whereas in the example preceding this activity, the relationships were given the other way round, by which I mean that the quantities were expressed as functions of the prices. This is not something you should get confused about. We can think about price as a function of quantity or quantity as a function of price. In this problem, q is treated as the independent variable and p as the dependent variable, so we will sketch p against q, with the vertical axis being the P axis and the horizontal axis the q-axis. (This is in contrast to the previous question, where q was the vertical coordinate and p the horizontal.)

Consider the demand curve, with equation $p = 4 - q - q^2$. This is an up-turned 'U'-shape. It crosses the p-axis (when $q = 0$) at $(0, 4)$ and it crosses the q-axis when $4 - q - q^2 = 0$. Solving this quadratic in the usual way, we obtain

$$q = \frac{1 \pm \sqrt{1 - 4(-1)(4)}}{-2} = \frac{1 \pm \sqrt{17}}{-2} = -2.562, 1.562.$$

The supply curve, with equation $p = 1 + 4q + q^2$, crosses the p-axis at $(0, 1)$. It crosses the q-axis when $1 + 4q + q^2 = 0$, which is when

$$q = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2} = \frac{-4 \pm \sqrt{12}}{2} = -0.268, -3.732.$$

Sketching both curves on the same diagram, we obtain Figure 2.23.

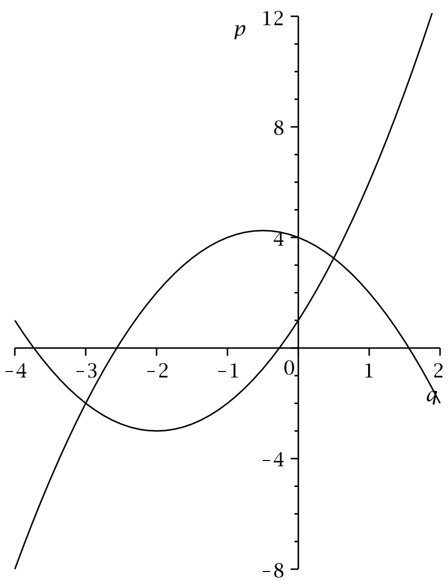


Figure 2.23: Graphs of the curves $p = 4 - q - q^2$ and $p = 1 + 4q + q^2$.

Feedback to activity 2.12

We solve $TC = TR$, which is $2 + 5x + x^2 = 12 + 8x$. Writing this equation in the standard way, it becomes $x^2 - 3x - 10 = 0$. We can factorise this as $(x - 5)(x + 2) = 0$, showing that the solutions are $5, -2$. Or, we can use the formula for the solutions of a quadratic, with $a = 1, b = -3, c = -10$. Either way we see that there are two possible break-even points, $x = 5$ or $x = -2$. But the second of these has no economic significance, since it represents a negative quantity. We therefore deduce that the break-even point is $x = 5$.

Answers to Sample examination/practice questions

Answer to question 2.1

To find the equilibrium quantity, we solve

$$4 - q - q^2 = 1 + 4q + q^2,$$

which is

$$2q^2 + 5q - 3 = 0.$$

Using the formula for the solutions of a quadratic, we have

$$q = \frac{-5 \pm \sqrt{5^2 - 4(2)(-3)}}{4} = \frac{-5 \pm \sqrt{49}}{4} = \frac{-5 \pm 7}{4} = -3, \frac{1}{2}.$$

So the equilibrium quantity is the economically meaningful solution, namely $q = 1/2$. The corresponding equilibrium price is

$$p = 4 - (1/2) - (1/2)^2 = 4 - 1/2 - 1/4 = \frac{13}{4}.$$

2

Answer to question 2.2

We solve

$$\frac{6}{q+1} = q+2.$$

Multiplying both sides by $q+1$, we obtain

$$6 = (q+1)(q+2) = q^2 + 3q + 2,$$

so

$$q^2 + 3q - 4 = 0.$$

This factorises as $(q-1)(q+4) = 0$ and so has solutions 1 and -4. Thus the equilibrium quantity is 1, the positive solution. The equilibrium price, which can be obtained from either one of the equations, is $p = 6/(1+1) = 3$. (Here, I have used the demand equation.)

Answer to question 2.3

Consider first the demand curve. This is a negative quadratic and so has an upturned 'U' shape. It crosses the p-axis when $8 - p^2 - 2p = 0$ or, equivalently, when $p^2 + 2p - 8 = 0$. This factorises as $(p+4)(p-2) = 0$, so the solutions are $p = 2, -4$. Alternatively, we can use the formula for the solutions to a quadratic:

$$p = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{2} = \frac{-2 \pm \sqrt{36}}{2} = \frac{-2 \pm 6}{2} = -4, 2.$$

(No calculator needed!) The supply curve is $q = p^2 + 2p - 3 = (p+3)(p-1)$, which crosses the p-axis at -3 and 1, and has a 'U' shape. We notice also that the demand curve crosses the q-axis at $q = 8$ and the supply curve crosses the q-axis at $q = -3$. The curves therefore look as in Figure 2.24.

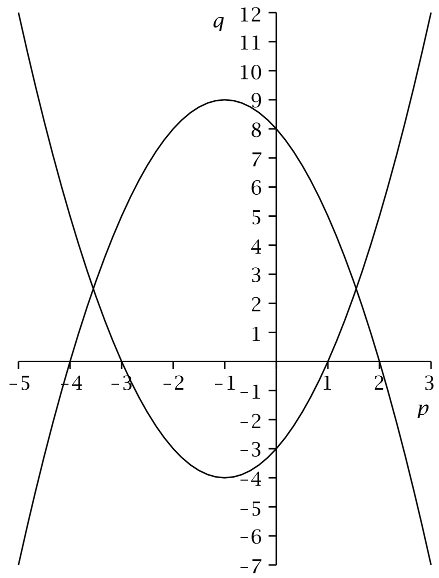


Figure 2.24: Graph of the curves $q = 8 - p^2 - 2p$ and $q = p^2 + 2p - 3$.

The equilibrium price is given by

$$8 - p^2 - 2p = p^2 + 2p - 3,$$

or $2p^2 + 4p - 11 = 0$. This has solutions

$$p = \frac{-4 \pm \sqrt{4^2 - 4(2)(-11)}}{4} = \frac{-4 \pm \sqrt{104}}{4} = -1 \pm \frac{1}{2}\sqrt{26}.$$

We know that $\sqrt{26} > 2$, so the solution $-1 + \sqrt{26}/2$ is positive and is therefore the equilibrium price. (The other solution is obviously negative.) (Note: in an exam, you should leave the answer like this, since you will not have a calculator to work the answer out as a decimal expansion.)