Chapter 6
Matrices and linear equations

Essential reading

(For full publication details, see Chapter 1.)

☞ Anthony and Biggs (1996) Chapters 14, 15 and 16.

Further reading

☞ Booth (1998) Chapter 2, Module 5.

☞ Bradley (2008) Section 9.2–9.3.

☞ Dowling (2000) Chapter 10.

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6.1 Introduction

This chapter of the guide deals with matrices. The main use of matrices in this subject is in solving systems of simultaneous linear equations.

6.2 Vectors

An *n-vector*[1] $\mathbf{v}$ is a list of n numbers, written either as a *row-vector*

$$(v_1, v_2, \ldots, v_n),$$

or a *column-vector*

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

(Sometimes the commas are omitted in the notation for a row vector.) The numbers v_1, v_2, and so on are known as the *components*, *entries* or *coordinates* of $\mathbf{v}$. The *zero vector* is the vector with all of its entries equal to 0.

Vectors are useful in geometry, where they represent directions. However, this is not something that will concern us in this course. Vectors are a particularly useful form of

[1] See Anthony and Biggs (1996) Section 14.1.

notation in economic aspects of management. Suppose for simplicity we have a market selling two items, which we shall call 'grommets' and 'widgets'. If a consumer has 5 grommets and 3 widgets, we may say that the consumer has the (commodity) bundle $(5, 3)$. More generally, if there are n goods in a commodity market — goods $X_1, X_2, \ldots, X_n$, say — and a consumer has x_1 units of X_1, x_2 of X_2, and so on, we say that she has the bundle $\mathbf{x} = (x_1 x_2 \ldots x_n)$.

We can define *addition* of two n-vectors by the rule

$$(w_1, w_2, \ldots, w_n) + (v_1, v_2, \ldots, v_n) = (w_1 + v_1, w_2 + v_2, \ldots, w_n + v_n).$$

(The rule is described here for row vectors but the obvious counterpart holds for column vectors.) Also, we can multiply a vector by any single number α (usually called a *scalar* in this context) by the following rule:

$$\alpha(v_1, v_2, \ldots, v_n) = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n).$$

For example,

$$(1, -2, -3) + (4, 5, 7) = (5, 3, 4) \quad \text{and} \quad 4(1, -2, 5) = (4, -8, 20).$$

The operations of addition and multiplication by a scalar may be combined. For example,

$$3(2, 1, 3) + 2(1, 1, 1) = (6, 3, 9) + (2, 2, 2) = (8, 5, 11).$$

The *dot product* of two n-vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ is denoted by $\mathbf{x} \cdot \mathbf{y}$ and is calculated as follows:

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots + x_n y_n.$$

Thus, $\mathbf{x} \cdot \mathbf{y}$ is the number obtained when we multiply together the first entries of $\mathbf{x}, \mathbf{y}$, multiply together the second entries, and so on, and add these n products together.

Example 6.1 Let $\mathbf{x} = (1, 0, 2)$ and $\mathbf{y} = (5, 2, 3)$. Then

$$\mathbf{x} \cdot \mathbf{y} = 1(5) + 0(2) + 2(3) = 11.$$

Warning: The dot product of two vectors is a number, not a vector. This is a mistake many students make and you must avoid it. (A common error is to assume that the dot product of two n-vectors is the n-vector whose entries are obtained by multiplying together the corresponding entries of the two vectors. This is not the case. The dot product is the sum of these products.) As mentioned, we do not have a way of 'multiplying' together two vectors to get a vector. (However, as we see later, two vectors may be multiplied together to give a matrix.)

Let us return to our economic model. If the consumer has 5 grommets and 3 widgets and the price of a grommet is p_1 and the price of a widget is p_2, then this bundle, $(5, 3)$, would have cost the consumer an amount $(5p_1 + 3p_2)$ to buy. More generally, if the consumer has 36 dollars to spend on grommets and widgets, then the bundle (x_1, x_2) can be bought provided its cost, in dollars, which is $p_1 x_1 + p_2 x_2$, is no greater than 36.

In this way, we obtain the consumer's budget constraint, $p_1x_1 + p_2x_2 \leq 36$. A bundle (x_1, x_2) is affordable if, and only if, it satisfies the budget constraint. The general case is only slightly more complex. Suppose we have n goods $X_1, X_2, \cdots, X_n$ and the cost of one unit of X_i is p_i. Suppose a consumer has an amount M to spend on these goods. Then the budget constraint is 'cost of bundle $\leq M$', which is $p_1x_1 + p_2x_2 + \cdots + p_nx_n \leq M$. But the quantity $p_1x_1 + p_2x_2 + \cdots + p_nx_n$ is the dot product of the price vector $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ and the bundle $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Thus the budget constraint is $\mathbf{p} \cdot \mathbf{x} \leq M$ and bundle $\mathbf{x}$ can be purchased provided it satisfies the budget constraint.

Activity 6.1 Let $\mathbf{x} = (1, 2, 3)$ and $\mathbf{y} = (3, 2, 1)$. Find $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} \cdot \mathbf{y}$.

6.3 Matrices

6.3.1 What is a matrix?

A *matrix*[2] is an array of numbers

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

We denote this array by the single letter A, or by (a_{ij}), and we say that A has m *rows* and n *columns*, or that it is an $m \times n$ matrix. We also say that A is a matrix of *size* $m \times n$. If $m = n$, the matrix is said to be *square*. The number a_{ij} is known as the (i, j)th *entry* of A. The row vector $(a_{i1}, a_{i2}, \cdots, a_{in})$ is *row* i of A, or the ith row of A, and the column vector

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

is *column* j of A, or the jth column of A.

It is useful to think of row and column vectors as matrices. For example, we may think of the row vector $(1, 2, 4)$ as being equal to the 1×3 matrix $(1\ 2\ 4)$. (Indeed, the only visible difference is that the vector has commas and the matrix does not, merely a notational difference.)

6.3.2 Matrix addition and scalar multiplication

Matrices are useful because they provide a compact notation, and because we can 'do algebra' with them[3]. If A and B are two matrices of the same size then we define $A + B$ to be the matrix whose elements are the sums of the corresponding elements in A and

[2] See Anthony and Biggs (1996) Section 15.1.
[3] See Anthony and Biggs (1996) Section 15.1.

B. Formally, the (i,j)th entry of the matrix $A+B$ is $a_{ij}+b_{ij}$ where a_{ij} and b_{ij} are the (i,j)th entries of A and B, respectively. Also, if c is a number, we define cA to be the matrix whose elements are c times those of A; that is, cA has (i,j)th entry ca_{ij}. For example,

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 3 & 3 \end{pmatrix},$$

and

$$3\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 3 \end{pmatrix}.$$

6.3.3 Matrix multiplication

Suppose A and B are matrices such that the number (say n) of columns of A is equal to the number of rows of B. We define the product[4] $C = AB$ to be the matrix whose elements are

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Although this formula looks daunting, it is quite easy to use in practice. What it says is that the element in row i and column j of the product is obtained by taking each entry of row i in turn and multiplying it by the corresponding entry of column j of B, then adding these n products together. In other words, the entry is the dot product of row i of A and column j of B.

Example 6.2 In the following product the element in row 1 and column 2 of the product matrix (indicated in bold type) is found by using, as described above, the row and column printed in bold type:

$$\begin{pmatrix} \mathbf{1} & \mathbf{3} \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mathbf{2} & 1 \\ 2 & \mathbf{3} & 5 \end{pmatrix} = \begin{pmatrix} 7 & \mathbf{11} & 16 \\ 4 & 7 & 7 \end{pmatrix}.$$

The entry is 11 because

$$1 \times 2 + 3 \times 3 = 11.$$

The other elements of the product can be worked out in the same way.

It must be stressed that when A has n columns then B must have n rows if AB is to be defined. In any other case, the product is not defined. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, it follows that AB is an $m \times p$ matrix.

The definition of matrix multiplication allows us to use some familiar algebraic rules, but care is needed. Among the rules which we can use are:

$$A(BC) = (AB)C \quad \text{and} \quad A(B+C) = AB + AC.$$

On the other hand, it is most important to note that AB and BA are not usually equal. Indeed it is quite possible that one of the products is defined but the other is not. Even if both are defined, they are generally not equal.[5]

[4]See Anthony and Biggs (1996) Section 15.2.
[5]See Anthony and Biggs (1996) Section 15.2.

Activity 6.2 If $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}$, show that $AB = \begin{pmatrix} 5 & 5 \\ 11 & 11 \end{pmatrix}$.

6.3.4 The identity matrix

A useful matrix is the *identity matrix*:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

which has the number 1 in each of the positions on the 'main diagonal', and 0 elsewhere. Note that I is a square matrix. Note that there is an identity matrix of any size $n \times n$.

The identity matrix has the property that, whenever A is an $n \times n$ matrix, we have

$$IA = AI = A.$$

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6.4 Linear equations

A *system of m linear equations in n unknowns* $x_1, x_2, \ldots, x_n$ is a set of m equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

The numbers a_{ij} are usually known as the *coefficients* of the system. We say that $(x_1^*, x_2^*, \cdots, x_n^*)$ is a *solution* of the system if *all* m equations hold true when $x_1 = x_1^*$, $x_2 = x_2^*$ and so on. Sometimes a system of linear equations is known as a set of *simultaneous* equations; such terminology emphasises that a solution is an assignment of values to each of the n unknowns such that *each and every* equation holds with this assignment.

In order to deal with large systems of linear equations we usually write them in matrix form. First we observe that vectors are just special cases of matrices: a row vector or list of n numbers is simply a matrix of size $1 \times n$, and a column vector is a matrix of size $n \times 1$. The rule for multiplying matrices tells us how to calculate the product $A\mathbf{x}$ of an $m \times n$ matrix A and an $n \times 1$ column vector $\mathbf{x}$. According to the rule, $A\mathbf{x}$ is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{n2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}.$$

Note that $A\mathbf{x}$ is a column vector with m rows, these being the left-hand sides of our system of linear equations. If we define another column vector $\mathbf{b}$, whose components are the right-hand sides b_i, the system is equivalent to the matrix equation

$$A\mathbf{x} = \mathbf{b}.$$

We often use the phrase *linear system* to mean 'system of linear equations' and we say that a linear system is *square* if the number of equations is the same as the number of unknowns; that is, if the matrix A is square.

6.5 Elementary row operations

An elementary way of solving the system

$$3x + 2y = 2, \tag{6.1}$$
$$5x + y = 2, \tag{6.2}$$

of linear equations is to 'eliminate' one of the variables, as follows. We can eliminate y by multiplying equation (6.2) by 2 and subtracting equation (6.1) from this new equation. Explicitly, multiplying the second equation by 2 gives the equation

$$10x + 2y = 4,$$

and so, using equation (6.1),

$$(10x + 2y) - (3x + 2y) = 4 - 2.$$

That is, $2 \times (6.2) - (6.1)$ gives

$$7x = 2 \quad \text{so that} \quad x = \frac{2}{7},$$

and substituting this value for x back into either equation yields $y = 4/7$. This technique generalises to larger systems of equations and leads to what is often called the *Gaussian Elimination*, or *Gauss-Jordan*, or *Gaussian*, method for solving systems of equations. This method works even when the number of equations and the number of unknowns are different.

It is a simple observation that the set of solutions of a system of linear equations is unaltered by the following three operations, since the restrictions on the variables $x_1, x_2, \cdots$ given by the new equations imply the restrictions given by the old ones (that is, we can undo the manipulations made on the old system):

- multiply both sides of an equation by a non-zero constant,
- add a multiple of one equation to another,
- interchange two equations.

These observations form the motivation behind a method[6] to solve linear equations.

[6] See Anthony and Biggs (1996) Chapters 16 and 17.

To solve a linear system $A\mathbf{x} = \mathbf{b}$ using the new method we first form the augmented matrix $(A\mathbf{b})$, which is A with column $\mathbf{b}$ tagged on. For example, if the system is

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 3 & 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

then the augmented matrix is the 3×4 matrix

$$(A\mathbf{b}) = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 2 \\ 3 & 5 & 4 & 1 \end{pmatrix}.$$

We use this form because of the important fact that elementary operations on the equations of the system correspond to the same operations on the rows of the augmented matrix. For that reason we shall now refer to them as *elementary row operations*. The method now proceeds as follows: we use a sequence of elementary row operations on the augmented matrix until we have changed it into a matrix of the form

$$(C\mathbf{d}) = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{pmatrix},$$

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which is said to be in *echelon form* or *reduced form*. Here, the $*$ symbols merely indicates the presence of some numbers. Note that in an echelon matrix, the first non-zero entry in each row is 1 (we call this the *leading* 1), the position of the leading 1 moves to the right as we go down the rows.

To see why this will be useful, let us carry out the procedure for the linear system given above. In doing so, I shall explain which row operations are being applied at each step, but it is not generally necessary to give such detail when you are answering a question of this type. We start with the augmented matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 2 \\ 3 & 5 & 4 & 1 \end{pmatrix}.$$

We eliminate the second and third entries of the first column by subtracting multiples of the first row. To cancel the 2 in the second row, we subtract twice the first row from the second. We may conveniently denote this as $R_2 \to R_2 - 2R_1$, meaning that row 2 changes to what was row 2, minus twice row 1. We shall also, to eliminate the 3 in the third row and first column, perform the operation $R_3 \to R_3 - 3R_1$ (that is, we subtract 3 times the first row from the third). This gives us the following transformation:

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 2 \\ 3 & 5 & 4 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & -1 & 1 & -2 \end{pmatrix}.$$

Now, clearly we can simplify this by dividing the second row throughout by the number -2. (That is, we perform the operation $R_2 \to R_2/(-2)$.) So the next transformation is

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & -1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -2 \end{pmatrix}.$$

Now we want to delete the -1 in the third row and second column. To do so, we add row 2 to row 3. (Note that we would not want to use row 1 to cancel at this stage, because if we did we would lose the 0 we have worked to obtain in the first column of the third row.) The next step is therefore to perform the row operation $R_3 \to R_3 + R_2$ to get

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -2 \end{pmatrix}.$$

Now, to reduce finally to echelon form, we simply divide the last row by 2, i.e. we perform the row operation $R_2 \to R_2/2$, to get

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

I've been very careful to explain (for your benefit) what the row operations were at each stage, but as I mentioned above, we need not include all this detail. The reduction above can be written simply as

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$$\begin{aligned} \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & 2 \\ 3 & 5 & 4 & 1 \end{pmatrix} &\to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & -1 & 1 & -2 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -2 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -2 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \end{aligned}$$

Now, the initial system of equations has the same set of solutions as the system

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

This system of equations is

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ x_2 + x_3 &= 0, \\ x_3 &= -1. \end{aligned}$$

But it is easy to solve these equations by working backwards from the third equation to the first one. Immediately, we have $x_3 = -1$. The second equation then gives $x_2 = 1$, and then the first gives

$$x_1 = 1 - 2x_2 - x_3 = 0.$$

We give another example.

Example 6.3 Use the method of elementary row operations to solve the following system of equations.

$$\begin{aligned} 3x_1 - 3x_2 + 5x_3 &= 6, \\ x_1 + 7x_2 + 5x_3 &= 4, \\ 5x_1 + 10x_2 + 15x_3 &= 9. \end{aligned}$$

Solution: The augmented matrix corresponding to the system of equations is

$$\begin{pmatrix} 3 & -3 & 5 & 6 \\ 1 & 7 & 5 & 4 \\ 5 & 10 & 15 & 9 \end{pmatrix},$$

which we reduce to echelon form using elementary row operations, as follows.

$$\begin{aligned} \begin{pmatrix} 3 & -3 & 5 & 6 \\ 1 & 7 & 5 & 4 \\ 5 & 10 & 15 & 9 \end{pmatrix} &\to \begin{pmatrix} 1 & 7 & 5 & 4 \\ 3 & -3 & 5 & 6 \\ 5 & 10 & 15 & 9 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 7 & 5 & 4 \\ 0 & -24 & -10 & -6 \\ 0 & -25 & -10 & -11 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 7 & 5 & 4 \\ 0 & 1 & 5/12 & 1/4 \\ 0 & -25 & -10 & -11 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 7 & 5 & 4 \\ 0 & 1 & 5/12 & 1/4 \\ 0 & 0 & 5/12 & -19/4 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 7 & 5 & 4 \\ 0 & 1 & 5/12 & 1/4 \\ 0 & 0 & 1 & -57/5 \end{pmatrix}. \end{aligned}$$

This last matrix, in echelon form, represents the system

$$\begin{aligned} x_1 + 7x_2 + 5x_3 &= 4, \\ x_2 + \frac{5}{12}x_3 &= \frac{1}{4}, \\ x_3 &= -\frac{57}{5}. \end{aligned}$$

Its solution (which is the same as that of the original system) may be determined by back-substitution:

$$x_3 = -\frac{57}{5}, \quad x_2 = \frac{1}{4} - \frac{5}{12}x_3 = 5 \quad \text{and} \quad x_1 = 4 - 7x_2 - 5x_3 = 26.$$

Activity 6.3 Use the method of elementary row operations to solve the following system of equations.

$$\begin{aligned} x_1 + x_2 + x_3 &= 6, \\ 2x_1 + 4x_2 + x_3 &= 5, \\ 2x_1 + 3x_2 + x_3 &= 6. \end{aligned}$$

Once we think we have the solution to a system of equations, it is quite straightforward to check whether they are indeed correct: all we have to do is substitute the supposed solutions into the original equations, and check that each equation holds true.

6.6 Applications of matrices and linear equations

Matrices are extremely useful in management and economics. We illustrate with two examples.

Example 6.4 A company manufactures three goods, X, Y and Z, each of which is made from three types of input, A, B and C. Each unit of X requires 1 unit of A, 7 units of B and 3 units of C. Each unit of Y requires 4 units of A, 3 units of B and 1 unit of C. Furthermore, one unit of Z requires 2 units of A, 4 units of B and 2 units of C. In a particular day's production the company uses up 105 units of A, 135 units of B and 55 units of C.

(a) Create a matrix equation to represent the usage of A, B and C in the day's production of x, y and z units of X, Y and Z respectively.

(b) Using matrix algebra, determine the values of x, y and z.

Solution: (a) Consider first the total amount of A used in the production of the goods X, Y, Z. Since each unit of X requires 1 unit of A, the total amount of A used in the production of X is $1 \times x = x$. Similarly, the total amount used in producing Y is $4y$ and the amount used in producing Z is $2z$. Therefore the total amount of A used is $x + 4y + 2z$. On the other hand, we know that this is 105, since this figure is given in the problem. Therefore

$$x + 4y + 2z = 105.$$

Similarly, considering in turn the total amounts of B and C used, we have

$$7x + 3y + 4z = 135 \quad \text{and} \quad 3x + y + 2z = 55.$$

These three equations expressed in matrix form become

$$\begin{pmatrix} 1 & 4 & 2 \\ 7 & 3 & 4 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 105 \\ 135 \\ 55 \end{pmatrix}. \tag{6.3}$$

Note that we have been very careful here and have thought about what the underlying equations are. A hurried approach might be to try to write a matrix equation directly by looking at the numbers given in the questions. But it is tempting to look at the problem and 'read off' the equation

$$\begin{pmatrix} 1 & 7 & 3 \\ 4 & 3 & 1 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 105 \\ 135 \\ 55 \end{pmatrix}.$$

However, this is *wrong*! The matrix just written is the so-called 'transpose' of the correct one (that is, it is obtained from the correct one by interchanging rows and columns). The moral of this digression is: think, and be careful!

(b) To answer this part of the problem, we need to solve the matrix equation (6.3) to determine x, y, z. We use the standard technique. First, we write down the augmented matrix,

$$\begin{pmatrix} 1 & 4 & 2 & 105 \\ 7 & 3 & 4 & 135 \\ 3 & 1 & 2 & 55 \end{pmatrix},$$

which we then reduce to echelon form using elementary row operations, as follows.

$$\begin{pmatrix} 1 & 4 & 2 & 105 \\ 7 & 3 & 4 & 135 \\ 3 & 1 & 2 & 55 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 2 & 105 \\ 0 & -25 & -10 & -600 \\ 0 & -11 & -4 & -260 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 4 & 2 & 105 \\ 0 & 1 & 2/5 & 24 \\ 0 & -11 & -4 & -260 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 4 & 2 & 105 \\ 0 & 1 & 2/5 & 24 \\ 0 & 0 & 2/5 & 4 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 4 & 2 & 105 \\ 0 & 1 & 2/5 & 24 \\ 0 & 0 & 1 & 10 \end{pmatrix}.$$

Therefore,

$$x + 4y + 2z = 105, \quad y + \frac{2}{5}z = 24 \quad \text{and} \quad z = 10,$$

so we have $z = 10$, $y = 20$ and $x = 5$.

Example 6.5 The supply function for a commodity takes the form

$$q^S(p) = ap^2 + bp + c,$$

for some constants a, b, c. When $p = 1$, the quantity supplied is 5; when $p = 2$, the quantity supplied is 12; when $p = 3$, the quantity supplied is 23. Find the constants a, b, c.

Solution: The given information means that

$$q^S(1) = 5, \quad q^S(2) = 12 \quad \text{and} \quad q^S(3) = 23,$$

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that is,

$$\begin{aligned} a(1^2) + b(1) + c &= 5, \\ a(2^2) + b(2) + c &= 12, \\ a(3^2) + b(3) + c &= 23. \end{aligned}$$

So we have the following system of linear equations for a, b, c:

$$\begin{aligned} a + b + c &= 5, \\ 4a + 2b + c &= 12, \\ 9a + 3b + c &= 23. \end{aligned}$$

We solve this in the usual way, by reducing the augmented matrix to echelon form.

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 & 5 \\ 4 & 2 & 1 & 12 \\ 9 & 3 & 1 & 23 \end{pmatrix} &\to \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & -2 & -3 & -8 \\ 0 & -6 & -8 & -22 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & -2 & -3 & -8 \\ 0 & 0 & -1 & -2 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & 3/2 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} a + b + c &= 5, \\ b + \frac{3}{2}c &= 4, \\ c &= 2, \end{aligned}$$

so that $c = 2$, $b = 1$ and $a = 2$. The supply function is therefore given explicitly by $q^S(p) = 2p^2 + p + 2$.

Learning outcomes

At the end of this chapter and the relevant reading, you should be able to:

- explain what is meant by a vector, either a row or column vector
- add vectors, multiply a vector by a number (or scalar), and compute the dot product of two vectors
- explain what is meant by a matrix
- add matrices and multiply a matrix by a number
- multiply matrices
- explain what is meant by the $n \times n$ identity matrix

- solve linear systems using row operations
- apply matrices and linear equations to problems in management

You do not need to know about determinants and the methods for linear equations based on them, such as Cramer's rule. You need not know about matrix inverses and their calculation, inconsistent systems, or systems with infinitely many solutions.

Sample examination/practice questions

Question 6.1

Express the following set of equations in matrix form and hence solve them **using a matrix method**:

$$\begin{aligned} x + y + z &= 1, \\ 2x - y + z &= -1, \\ x + 3y - z &= 7. \end{aligned}$$

Question 6.2

A high class dressmaker makes three types of dresses. She makes cheap 'everyday' dresses, medium-priced 'cocktail' dresses and expensive 'ballroom' dresses. The making of the dresses involves the 'inputs' of fabric, labour, fastenings and machine time. Table 6.1shows the units of input required per dress for each dress type.

	'Everyday'	'Cocktail'	'Ballroom'
Fabric	5	6	8
Labour	20	25	30
Fastenings	15	20	22
Machine time	7	9	12

Table 6.1: Units of input required per dress for each dress type.

The dressmaker makes a combination of the three dress types which uses exactly 270 units of fabric, 1050 units of labour and 790 units of fastenings. How many of each type of dress does she make? What is the corresponding machine time used?

Question 6.3

The function $f(x)$ is given by

$$f(x) = \frac{a}{1 + x^2} + bx + c,$$

for some constants a, b, c. Given that $f(0) = 8$, $f(1) = 3$ and $f(2) = -8/5$, find a system of linear equations for a, b, c. Solve this system **using a matrix method**.

Answers to activities

Feedback to activity 6.1

$\mathbf{x}+\mathbf{y}=(4,4,4)$ and $\mathbf{x}\cdot\mathbf{y}=10$.

Feedback to activity 6.2

This is quite straightforward. We know that, since A is a 2×3 matrix and B is a 3×2 matrix, that the product AB can be formed and will be a 2×2 matrix. There are therefore four entries to compute. The top left entry is computed by forming the dot product of

$$\text{the first row, } (1\ 2\ 3)\text{, of } A \text{ and the first column, } \begin{pmatrix}2\\0\\1\end{pmatrix}\text{, of } B,$$

and is therefore

$$(1\times 2)+(2\times 0)+(3\times 1)=5.$$

The other entries are computed in a similar way. For example, the bottom right entry of AB (indicated in bold in the following equation) is the dot product of the row and column indicated in bold:

$$AB=\begin{pmatrix}1 & 2 & 3\\ \mathbf{3} & \mathbf{2} & \mathbf{5}\end{pmatrix}\begin{pmatrix}2 & \mathbf{1}\\ 0 & -\mathbf{1}\\ 1 & \mathbf{2}\end{pmatrix}=\begin{pmatrix}5 & 5\\ 11 & \mathbf{11}\end{pmatrix}.$$

Feedback to activity 6.3

We reduce the augmented matrix as follows:

$$\begin{aligned}\begin{pmatrix}1 & 1 & 1 & 6\\ 2 & 4 & 1 & 5\\ 2 & 3 & 1 & 6\end{pmatrix} &\to \begin{pmatrix}1 & 1 & 1 & 6\\ 0 & 2 & -1 & -7\\ 2 & 3 & 1 & 6\end{pmatrix}\\ &\to \begin{pmatrix}1 & 1 & 1 & 6\\ 0 & 1 & -1/2 & -7/2\\ 0 & 1 & -1 & -6\end{pmatrix}\\ &\to \begin{pmatrix}1 & 1 & 1 & 6\\ 0 & 1 & -1/2 & -7/2\\ 0 & 0 & -1/2 & -5/2\end{pmatrix}\\ &\to \begin{pmatrix}1 & 1 & 1 & 6\\ 0 & 1 & -1/2 & -7/2\\ 0 & 0 & 1 & 5\end{pmatrix}.\end{aligned}$$

You should check which operations have been used at each step. For example, going from the first to the second matrix, we have subtracted twice the first row from the second.

The final augmented matrix represents the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 6, \\ x_2 - \frac{1}{2}x_3 &= -\frac{7}{2}, \\ x_3 &= 5. \end{aligned}$$

As we have seen, the solution of this system can be found by 'working backwards':

$$x_3 = 5, \quad x_2 = \frac{1}{2}x_3 - \frac{7}{2} = -1 \quad \text{and} \quad x_1 = -x_2 - x_3 + 6 = 2.$$

Answers to Sample examination/practice questions

Answer to question 6.1

In matrix form, the system of equations is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}.$$

The augmented matrix corresponding to the system of equations is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & -1 \\ 1 & 3 & -1 & 7 \end{pmatrix}.$$

We reduce this with row operations as follows:

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & -1 \\ 1 & 3 & -1 & 7 \end{pmatrix} &\to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & -3 \\ 0 & 2 & -2 & 6 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -2 & 6 \\ 0 & 3 & 1 & 3 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 3 & 1 & 3 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 4 & -6 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -3/2 \end{pmatrix}. \end{aligned}$$

This last matrix, in echelon form, represents the system

$$\begin{aligned} x + y + z &= 1, \\ y - z &= 3, \\ z &= -\frac{3}{2}. \end{aligned}$$

Its solution (which is the same as that of the original system) may be determined by back-substitution:

$$z = -\frac{3}{2}, \quad y = 3 + z = \frac{3}{2} \quad \text{and} \quad x = 1 - y - z = 1.$$

Answer to question 6.2

Let x be the number of Everyday dresses, y the number of Cocktail dresses and z the number of Ballroom dresses made. The fact that 270 units of fabric are used means (from the information given in the table), that

$$5x + 6y + 8y = 270.$$

Considering, in turn, labour and fastenings, we obtain the additional two equations

$$20x + 25y + 30z = 1050 \quad \text{and} \quad 15x + 20y + 22y = 790.$$

(We are not given any constraint on the machine time, so there is no fourth equation corresponding to this.) We therefore have to solve the system

$$\begin{aligned} 5x + 6y + 8z &= 270, \\ 20x + 25y + 30z &= 1050, \\ 15x + 20y + 22z &= 790. \end{aligned}$$

We reduce the augmented matrix to echelon form as follows:

$$\begin{aligned} \begin{pmatrix} 5 & 6 & 8 & 270 \\ 20 & 25 & 30 & 1050 \\ 15 & 20 & 22 & 790 \end{pmatrix} &\to \begin{pmatrix} 5 & 6 & 8 & 270 \\ 0 & 1 & -2 & -30 \\ 0 & 2 & -2 & -20 \end{pmatrix} \\ &\to \begin{pmatrix} 5 & 6 & 8 & 270 \\ 0 & 1 & -2 & -30 \\ 0 & 0 & 2 & 40 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 6/5 & 8/5 & 54 \\ 0 & 1 & -2 & -30 \\ 0 & 0 & 1 & 20 \end{pmatrix}. \end{aligned}$$

So we have

$$x + \frac{6}{5}y + \frac{8}{5}z = 54, \quad y - 2z = -30 \quad \text{and} \quad z = 20,$$

from which it follows that

$$z = 20, \quad y = 10 \quad \text{and} \quad x = 10.$$

The dressmaker must therefore have made 10 Everyday dresses, 10 Cocktail dresses and 20 Ballroom dresses.

The corresponding machine time used is

$$7(10) + 9(10) + 12(20) = 400.$$

Answer to question 6.3

$$f(x) = \frac{a}{1+x^2} + bx + c,$$

and $f(0) = 8$, $f(1) = 3$ and $f(2) = -8/5$. So, substituting $x = 0, 1, 2$ in turn, we obtain the equations

$$\begin{aligned} a + c &= 8, \\ \frac{a}{2} + b + c &= 3, \\ \frac{a}{5} + 2b + c &= -\frac{8}{5}. \end{aligned}$$

Multiplying the second equation by 2 and the third by 5 (just to make it easier to deal with), we obtain the system

$$\begin{aligned} a + c &= 8, \\ a + 2b + 2c &= 6, \\ a + 10b + 5c &= -8. \end{aligned}$$

Reducing the augmented matrix to echelon form, we have

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 1 & 8 \\ 1 & 2 & 2 & 6 \\ 1 & 10 & 5 & -8 \end{pmatrix} &\to \begin{pmatrix} 1 & 0 & 1 & 8 \\ 0 & 2 & 1 & -2 \\ 0 & 10 & 4 & -16 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 0 & 1 & 8 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & -1 & -6 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & 1/2 & -1 \\ 0 & 0 & 1 & 6 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} a + c &= 8, \\ b + \frac{c}{2} &= -1, \\ c &= 6. \end{aligned}$$

It follows that

$$c = 6, \qquad b = -4 \qquad \text{and} \qquad a = 2.$$

The unknown function is therefore

$$f(x) = \frac{2}{1+x^2} - 4x + 6.$$