Chapter 4
Integration

Essential reading

(For full publication details, see Chapter 1.)

☞ Anthony and Biggs (1996) Chapters 25 and 26.

Further reading

☞ Binmore and Davies (2001) Chapter 10, Sections 10.2–10.10.

☞ Booth (1998) Chapter 6.

☞ Bradley (2008) Modules 22–24.

☞ Dowling (2000) Chapters 14 and 15.

4.1 Introduction

The next topic in calculus is integration. This is perhaps one of the most difficult topics in this subject, and I encourage you to practise on lots of examples.

4.2 Integration

Integration is, in essence, the reverse process to differentiation and has a number of applications in economics and related subjects. (It is also essential for solving differential equations, but this topic is not part of this subject.) We start with *indefinite integration.*[1]

Suppose the function f is given, and the function F is such that $F'(x) = f(x)$. Then we say that F is an *anti-derivative* of f. (Sometimes the word *primitive* is used instead of 'anti-derivative'.) For example, $x^4/4$ is an anti-derivative of x^3, and so is $x^4/4 + 5$. Any two anti-derivatives of a given function f differ only by a constant. The general form of the anti-derivative of f is called the *indefinite integral* of $f(x)$, and denoted by

$$\int f(x)\,dx.$$

[1]See Anthony and Biggs (1996) Section 25.3.

Often we call it simply the *integral* of f. It is of the form $F(x) + c$, where F is *any particular* anti-derivative of f and c denotes any constant, known as a *constant of integration*. Thus, for example, we write

$$\int x^3\,dx = \frac{x^4}{4} + c.$$

The process of finding the indefinite integral of f is usually known as *integrating* f, and f is known as the *integrand*.

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Sometimes the variable of integration will be x; other times it will be some other symbol, but this makes no real difference. For instance,

$$\int x^3\,dx = \frac{x^4}{4} + c$$

and

$$\int t^3\,dt = \frac{t^4}{4} + c.$$

Note, however, that if we are to integrate a function of t, then the integral must contain a dt and if we are to integrate a function of x, then the integral must contain a dx.

Just as for differentiation, we shall have a list of *standard integrals*[2] and some rules for combining these. The main standard integrals (which you should memorise) are listed in Table 4.1.

$f(x)$	$\int f(x)\,dx$
$x^n\ (n \neq -1)$	$\dfrac{x^{n+1}}{(n+1)} + c$
$1/x$	$\ln\lvert x\rvert + c$
e^x	$e^x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$

Table 4.1: Standard integtrals.

Note that the integral of $1/x$ is $\ln|x| + c$ rather than $\ln x + c$, because if x is negative, then the derivative of $\ln|x|$ is the derivative of $\ln(-x)$, which is just $1/x$.

Example 4.1 The integral of $\sqrt{x}$, which of course can be written as $x^{1/2}$ is $2x^{3/2}/3 + c$, according to the first rule (taking $n = 1/2$).

Activity 4.1 Integrate the function x^5.

Two important rules of integrals are easily verified:

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx,$$

[2]See Anthony and Biggs (1996) Section 25.5.

for any functions f and g, and

$$\int kf(x)\,dx = k\int f(x)\,dx,$$

for any constant k.

Activity 4.2 If a function f has derivative $x^2 + 2\sin x$, and $f(0) = 1$, what is the function?

4.3 Definite integrals

Let f be a function with an anti-derivative F. The *definite integral*[3] of the function f over the interval $[a, b]$ is *defined to be*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Note that any anti-derivative $G(x)$ of f is of the form

$$G(x) = F(x) + c,$$

for some constant c, so that

$$G(b) - G(a) = (F(b) + c) - (F(a) + c) = F(b) - F(a).$$

Thus, whichever anti-derivative of f is chosen, the quantity on the right-hand side of the definition is the same. In calculations the notation $[F(x)]_a^b$ is often used as a shorthand for $F(b) - F(a)$.

Example 4.2 The definite integral of x^4 over $[0, 1]$ is

$$\int_0^1 x^4\,dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5} - 0 = \frac{1}{5}.$$

Activity 4.3 Calculate $\int_0^2 x^2\,dx$.

Activity 4.4 Determine $\int_{-1}^1 e^t\,dt$.

4.4 Integration by substitution

4.4.1 The method

We now turn our attention to a useful integration technique which may be thought of as doing for integration what the composite function rule does for differentiation. This is the technique of *integration by substitution*.[4] We illustrate with a simple example.

[3] See Anthony and Biggs (1996) Section 25.4.

[4] See Anthony and Biggs (1996) Sections 26.1 and 26.2.

Example 4.3 Suppose that we are asked to find the indefinite integral

$$\int (3x+5)^{12}\, dx.$$

We note that if we substitute $u = 3x + 5$ the integrand becomes u^{12}, which we know how to integrate. But we have changed the variable of integration. Originally we integrated with respect to x, signified by dx in the integral, now we must integrate with respect to u. We must relate dx to du, for the notation $\int u^{12}\, dx$ has no meaning. (Recall that a function of u must be accompanied by du.) Making the substitution $u = 3x + 5$ is the same as saying that

$$x = \frac{1}{3}u - \frac{5}{3},$$

so $dx/du = 1/3$. Thus (and although this looks strange, it can be justified), $dx = \frac{1}{3}du$. So, when we replace $3x + 5$ by u we should replace dx by $(1/3)du$, giving

$$\int (3x+5)^{12}\, dx = \int u^{12} \left(\frac{1}{3}\right) du = \frac{1}{3}\frac{u^{13}}{13} + c.$$

We need the answer in terms of x, the original variable. Since $u = 3x + 5$ the integral is

$$\frac{1}{39}(3x+5)^{13} + c.$$

The general rule is: when we change the variable by putting $x = x(u)$ in the integral of $f(x)$ with respect to x, we must replace dx by $(dx/du)du$. In other words, to determine

$$\int f(x)\, dx$$

we can work out

$$\int f(x(u))\, x'(u) du,$$

and then substitute back x for $x(u)$.

In practice we overlook the distinction between x as a function of u and the inverse function, in which u is regarded as a function of x, relying on the fact that du/dx is equal to $1/(dx/du)$. This allows us to write 'shorthand' statements like

$$u = 3x + 5, \quad \text{therefore} \quad du = 3\, dx,$$

which determines both du/dx and dx/du.

As we have formally described a change of variable here, it involves expressing the variable of integration, x, in terms of a new variable u. Another way of expressing this is to say that we have made a *substitution*: we have substituted u for x. In practice, in some problems it will be natural to express the new variable u as a function of the original variable x, as in the example above and in some other problems it will be more natural to express x as a function of u. The first approach is probably more common in the type of integrals studied in this subject.

4.4.2 Examples

One key difficulty students have with the substitution method is not knowing which substitution to attempt. This is something you become more proficient at with practice, but it should be borne in mind that there need not be one correct substitution: for a number of problems, more than one substitution might work. In determining what substitution, if any, to try, one approach is to look at the integral and ask yourself what is complicating it. For instance, consider the example given above, where we have to integrate $(3x+5)^{12}$, we note that if we simply had x^{12} rather than $(3x+5)^{12}$, then the problem would be easy. It is for this reason that we seek to transform the integral into a straightforward power, u^{12}, and the way to do this is to set $u = 3x+5$. Fortunately, this works, because the subsequent step of replacing dx by something involving du does not further complicate matters, only introducing a multiplicative factor of $1/3$. Here are a few more examples where the obvious substitution works, and one where it does not quite.

Example 4.4 Consider $\int x(3x+5)^7\,dx$. This is a slightly more complicated integral than the one we worked on above, but it is still the case that what makes the integral difficult is the $(3x+5)^7$. So, we try the substitution $u = 3x+5$. Then $du = 3\,dx$, so $dx = (1/3)du$ and the integral becomes $\int xu^7(1/3)\,du$. But there's something wrong here: the integrand involves both variables x and u, whereas what we want is an integral involving only the new variable u. But, fear not, because, from $u = 3x+5$, we know that $x = (u-5)/3$. So, the integral is

$$\begin{aligned}\frac{1}{3}\int\frac{(u-5)}{3}u^7\,du &= \frac{1}{9}\int\left(u^8-5u^7\right)\,du\\ &= \frac{1}{9}\left(\frac{u^9}{9}-5\frac{u^8}{8}\right)+c\\ &= \frac{u^9}{81}-\frac{5}{72}u^8+c\\ &= \frac{(3x+5)^9}{81}-\frac{5}{72}(3x+5)^8+c,\end{aligned}$$

not an answer you might easily have guessed(!), but which is obtained without too much difficulty using the substitution method.

Example 4.5 Let's think about $\int x(2x^2+7)^8\,dx$. The complicating part of the integral is $2x^2+7$, so we try the substitution $u = 2x^2+7$. With this, $du = 4x\,dx$, so that $dx = (1/(4x))du$ and the integral becomes

$$\int xu^8\left(\frac{1}{4x}\right)\,du = \frac{1}{4}\int u^8\,du.$$

Note that the x cancels with the $1/(4x)$ factor emerging from expressing dx in terms of du. So, here, there is no need to express x in the integrand in terms of u. This integral is now quite straightforward. It evaluates to $u^9/36+c$, so the answer is $(2x^2+7)^9/36+c$.

Example 4.6 Consider the fairly similar-looking integral $\int x^3(2x^2+7)^8\,dx$. Again, we try the substitution $u = 2x^2+7$. We have, as before, $dx = (1/(4x))du$ and so the integral becomes

$$\int x^3u^8\left(\frac{1}{4x}\right)du = \frac{1}{4}\int x^2u^8\,du.$$

Here, the x terms in the integrand do not entirely cancel. But we know that $x^2 = (u-7)/2$, so the integral simplifies as

$$\begin{aligned}\frac{1}{4}\int\frac{(u-7)}{2}u^8\,du &= \frac{1}{8}\int(u^9-7u^8)\,du\\ &= \frac{1}{8}\left(\frac{u^{10}}{10}-\frac{7u^9}{9}\right)+c\\ &= \frac{(2x^2+7)^{10}}{80}-\frac{7(2x^2+7)^9}{72}+c.\end{aligned}$$

Example 4.7 The integral $\displaystyle\int\frac{x+1}{x^2+2x+7}\,dx$ is a different sort of integral, but there is a method that often (though not always) works. We let u, the new variable, be the denominator (that is, the bottom line) of the integrand, $u = x^2+2x+7$. Then, $du = (2x+2)dx$, so $dx = du/(2x+2)$ and the integral is

$$\int\frac{x+1}{u}\frac{du}{2x+2} = \frac{1}{2}\int\frac{du}{u}.$$

Notice how the $x+1$ on the numerator of the integrand and the $2x+2$ factor cancel with each other to give just the constant factor $1/2$. Now the integral is

$$\frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u|+c = \frac{1}{2}\ln|x^2+2x+7|+c.$$

In fact, x^2+2x+7 is positive for all x, so we do not need the absolute value signs, and the answer is simply $(1/2)\ln(x^2+2x+7)+c$.

Example 4.8 The integral $\displaystyle\int\frac{x}{x^2+2x+7}\,dx$ looks similar, but we shall see that the same substitution does not enable us to determine the integral. (Can you anticipate why?) Putting $u = x^2+2x+7$ as before, the integral becomes

$$\int\frac{x}{u}\frac{du}{2x+2} = \int\frac{x}{2x+2}\frac{du}{u}.$$

Here, we do not get the same sort of cancellation as before. To express $x/(1+x)$ in terms of u would be difficult, and would lead to a 'messy' integral which we could not easily determine. The reason that the substitution does not work here is that the numerator is not a multiple of the derivative $(2x+2)$ of the bottom line.

In the context of these last two examples, it is worth mentioning a general rule:

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c.$$

This follows on making the substitution $u = f(x)$. Noting that $du = f'(x)\,dx$, the integral is exactly $\int(1/u)\,du$, which is $\ln|u| + c$, equal to $\ln|f(x)| + c$.

Activity 4.5 Use the substitution $u = x^2$ to determine $\displaystyle\int xe^{x^2}\,dx$.

Activity 4.6 Determine $\displaystyle\int x(2x^2+2)^{1/2}\,dx$ by using substitution.

Activity 4.7 Determine the integral

$$\int x\sqrt{x-1}\,dx,$$

by using the substitution $u = x - 1$. Now determine it using the substitution $u = \sqrt{x-1}$. (You should, of course, get the same answer. The point I'm emphasising here is that there can be more than one appropriate substitution.)

4.4.3 The substitution method for definite integrals

In the case of a definite integral there is no need to revert to the original variable before evaluating the anti-derivative: we simply use the appropriate values of the new variable. If we change from the variable x to the variable u, and the interval of integration for x was $[a, b]$, the interval for u will be $[\alpha, \beta]$, where α and β are the values of u which correspond to $x = a$ and $x = b$ respectively. Formally

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{u=\alpha}^{u=\beta} f(x(u))\,x'(u)du,$$

where $x(\alpha) = a$ and $x(\beta) = b$. This result holds provided that u increases or decreases from α to β as x goes from a to b.

Example 4.9 Making the substitution $u = 3x + 5$,

$$\begin{aligned}\int_0^1 (3x+5)^2\,dx &= \int_5^8 u^2\,\frac{1}{3}du \\ &= \frac{1}{3}\left[\frac{u^3}{3}\right]_5^8 \\ &= \frac{1}{9}(8^3 - 5^3).\end{aligned}$$

Here, we have used the fact that since $u = 3x + 5$, the values of u corresponding to $x = 0$ and $x = 1$ are 5 and 8.

Activity 4.8 Determine $\displaystyle\int_0^1 \frac{x+2}{x^2+4x+5}\,dx.$

4.5 Integration by parts

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The technique of *integration by parts*[5] may be thought of as resulting from the product rule for differentiation, which tells us that the derivative of $u(x)v(x)$ is $u'(x)v(x) + u(x)v'(x)$. Hence the anti-derivative of $u'(x)v(x) + u(x)v'(x)$ is $u(x)v(x)$, or equivalently

$$\int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx = u(x)v(x).$$

Rearranging, we get the rule for *integration by parts*:

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx.$$

Thus, we can express an integral of the form $\int u'(x)v(x)\,dx$ as a known function $(u(x)v(x))$ minus another integral. The second integral may be easier than the first. And this is the point of this rather complicated-looking rule: we might get a simpler integral as a result of replacing one 'part' $u'(x)$ by its integral $u(x)$ and the other 'part' $v(x)$ by its derivative $v'(x)$.

Often, this rule is written in the shorthand form

$$\int f\,dg = fg - \int g\,df.$$

Example 4.10 Consider the integral

$$\int x \ln x\,dx.$$

Taking $u'(x) = x$ and $v(x) = \ln x$ in the integration by parts rule, we have

$$\begin{aligned}
\int u'(x)v(x)\,dx &= u(x)v(x) - \int u(x)v'(x)\,dx \\
&= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \frac{1}{x}\,dx \\
&= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx \\
&= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c.
\end{aligned}$$

[5] See Anthony and Biggs (1996) Section 26.3.

Example 4.11 You might have wondered why, although $\ln x$ is a very important function, it does not feature in our list of standard integrals. The reason is that the integral of $\ln x$ is not particularly easy to remember. There is a rather 'sneaky' way of finding it, using integration by parts. The integral $\int \ln x\, dx$ may be thought of as the integral of $1 \times \ln x$. Taking $u'(x) = 1$ and $v(x) = \ln x$ (so $u(x) = x$ and $v'(x) = 1/x$) and using integration by parts, we have

$$\int \ln x\, dx = x \ln x - \int x \frac{1}{x}\, dx = x \ln x - \int 1\, dx = x \ln x - x + c.$$

Activity 4.9 Use integration by parts to find $\displaystyle\int xe^x\, dx$.

4.6 Partial fractions

This is a way of rewriting integrands of the form $p(x)/q(x)$, where p and q are polynomials, in a simpler form which makes them easier to integrate.[6] Here is an example.

Example 4.12 Consider

$$\int \frac{x}{x^2 - x - 2}\, dx.$$

The integrand is of the form $p(x)/q(x)$, where $p(x) = x$ and $q(x) = x^2 - x - 2$. Further, $q(x)$ factorises as $(x+1)(x-2)$. We claim that we can find constants A_1 and A_2 such that

$$\frac{x}{x^2 - x - 2} = \frac{x}{(x+1)(x-2)} = \frac{A_1}{x+1} + \frac{A_2}{x-2}.$$

Multiplying through by $(x+1)(x-2)$, we obtain

$$A_1(x-2) + A_2(x+1) = x.$$

Taking $x = -1$ gives $-3A_1 + 0A_2 = -1$, so that $A_1 = 1/3$. Taking $x = 2$ gives $3A_2 = 2$ and $A_2 = 2/3$. There is another way of working out the numbers A_1, A_2, using the 'cover-up' rule. To calculate A_1, for example, we put $x = -1$ in the original expression

$$\frac{x}{[(x+1)](x-2)},$$

omitting ('covering up') the term in square brackets: that is,

$$A_1 = \frac{-1}{(-1-2)} = \frac{1}{3}.$$

[6] See Anthony and Biggs (1996) Section 26.4.

Similarly, we can calculate A_2 as

$$A_2 = \frac{2}{(2+1)} = \frac{2}{3}.$$

The identity

$$\frac{x}{x^2 - x - 2} = \frac{1/3}{x+1} + \frac{2/3}{x-2}$$

is called an expansion in *partial fractions*. It can easily be checked by multiplying out.

Now we can determine the integral, as follows.

$$\begin{aligned} \int \frac{x}{x^2 - x - 2}\, dx &= \int \frac{x}{(x+1)(x-2)}\, dx \\ &= \frac{1}{3} \int \left(\frac{1}{x+1} + \frac{2}{x-2} \right) dx \\ &= \frac{1}{3} \ln(x+1) + \frac{2}{3} \ln(x-2) + c. \end{aligned}$$

To be precise, we should use $\ln|x+1|$ and $\ln|x-2|$ since we cannot calculate the logarithm of a negative number.

Generally, the method of partial fractions involves rewriting expressions of the form $p(x)/q(x)$, where p and q are polynomials, as a sum of simpler terms. In particular, if $p(x)$ is linear (that is, of the form $ax + b$) and $q(x)$ is a quadratic with two different roots, then the method of partial fractions applies. Suppose that $q(x) = (x - a_1)(x - a_2)$, where $a_1 \neq a_2$. Then it is possible to write

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x - a_1)(x - a_2)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} \qquad (*)$$

for some numbers A_1 and A_2. Cross-multiplying equation $(*)$, we get

$$p(x) = A_1(x - a_2) + A_2(x - a_1).$$

The numbers A_1 and A_2 may be found by substituting $x = a_1$, $x = a_2$ in turn into this identity. When $p(x)/q(x)$ is expressed in this way, it is possible to evaluate the integral, because we have

$$\begin{aligned} \int \frac{p(x)}{q(x)}\, dx &= \int \left(\frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} \right) dx \\ &= A_1 \ln|x - a_1| + A_2 \ln|x - a_2| + c. \end{aligned}$$

Activity 4.10 Find $\displaystyle\int \frac{dx}{x^2 + 4x + 3}$.

4.7 Applications of integration

We have seen that marginals are derivatives; for instance, the marginal cost MC is the derivative (with respect to quantity) of the total cost function TC. This means that if we are given the marginal cost and we want to find the total cost function then we have to reverse the previous procedure, by integrating. However, we shall also need some additional information, since we know that when we integrate we have a constant of integration which has to be determined. Often this information is provided to us by the fixed cost, which, you will recall, is the cost $TC(0)$ of producing no units. The following example illustrates this.

Example 4.13 Suppose that the marginal cost function is given by $MC(q) = 2e^{0.5q}$ and that the fixed cost is 20. Then the total cost function is the integral of the marginal cost:

$$TC(q) = \int 2e^{0.5q}\,dq = 4e^{0.5q} + c,$$

for some constant c. To determine c, we use the fact that when $q = 0$, the cost $TC(0)$ must equal the fixed cost, 20; thus, $4e^0 + c = 20$, $\ 4 + c = 20$ and so $c = 16$, and $TC(q) = 4e^{0.5q} + 16$. (An extremely common mistake in a problem of this type is to assume that the constant equals the fixed cost, which, as we see in this example, need not be the case. Beware!)

Activity 4.11 Find the total cost function if the marginal cost is $q + 5q^2 + e^q$ and the fixed cost is 10.

Learning outcomes

At the end of this chapter and the relevant reading, you should be able to:

- explain what is meant by an (indefinite) integral and a definite integral
- state and use the standard integrals
- use integration by substitution
- use integration by parts
- integrate using partial fractions
- calculate functions from their marginals

Sample examination/practice questions

Question 4.1

Determine the following integral:

$$\int \frac{x}{x^2 + 5x + 6}\,dx.$$

Question 4.2

Evaluate $\int_1^2 x^2(x-1)^{1/2}\,dx$ using an appropriate substitution.

Question 4.3

Determine $\int \frac{2x+3}{x^2+3x+2}\,dx$.

Question 4.4

Determine $\int_1^e \frac{\ln x\sqrt{\ln x}}{x}\,dx$.

Question 4.5

A company produces only product XYZ. When producing Q units the marginal cost MC is given by

$$MC = 1 - \frac{1}{(Q+1)^2}.$$

If the average cost per unit when producing 4 units is 3.05, what is the total cost of producing 5 units of XYZ?

Question 4.6

A company's marginal cost function is

$$MC = 32 + 18q - 12q^2.$$

Its fixed cost is 43. Determine the firm's total cost function, average cost function, and variable cost.

Question 4.7

The marginal revenue function for a commodity is given by

$$MR = 10 - 2x^2,$$

and the total cost function for the commodity is

$$TC = x^2 + 4x + 2,$$

where x is the number of units produced. Find the revenue function, and determine the maximal profit.

Question 4.8

For a particular company, the marginal cost is a function of output as follows:

$$MC = 10 - q + q^2.$$

Determine the extra cost which is incurred when production is increased from 2 to 4.

Question 4.9

A firm's marginal cost function is

$$\frac{20}{\sqrt{Q}}e^{\sqrt{Q}} + Q^3 + \frac{1}{Q+1}.$$

The firm's fixed costs are 20. Determine the total cost function.

Answers to activities

Feedback to activity 4.1

This is one of the standard derivatives, and the answer is $x^6/6 + c$.

Feedback to activity 4.2

We know that the function is an anti-derivative of $x^2 + 2\sin x$. Integrating this, by the standard integrals and the rules just seen, we have

$$\int (x^2 + 2\sin x)\, dx = \frac{x^3}{3} - 2\cos x + c,$$

where c is a constant of integration. But we know more about f: we know that $f(0) = 1$, so we know that

$$f(x) = \frac{x^3}{3} - 2\cos x + c,$$

where the constant c is such that $f(0) = 1$. Now, substituting $x = 0$ into the expression for f, we have

$$f(0) = 0 - 2\cos 0 + c = -2 + c,$$

and for this to equal 1, c must be 3. Therefore

$$f(x) = \frac{x^3}{3} - 2\cos x + 3.$$

Feedback to activity 4.3

The integral $\int x^2\, dx$ is $x^3/3 + c$, so

$$\int_0^2 x^2\, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3} - 0 = \frac{8}{3}.$$

Feedback to activity 4.4

The fact that this involves variable t rather than x should not confuse us. We have

$$\int_{-1}^1 e^t\, dt = \left[e^t\right]_{-1}^1 = e^1 - e^{-1} = e - \frac{1}{e}.$$

Feedback to activity 4.5

With $u = x^2$ we have $du = 2x\,dx$ and so $dx = du/(2x)$. Therefore

$$\begin{aligned}\int xe^{x^2}\,dx &= \int xe^u \frac{du}{2x} \\ &= \frac{1}{2}\int e^u\,du \\ &= \frac{1}{2}e^u + c \\ &= \frac{1}{2}e^{x^2} + c.\end{aligned}$$

A slightly quicker approach to making this substitution is to note that since $du = 2x\,dx$ and the integral already has $x\,dx$, we have

$$\int xe^{x^2}\,dx = \frac{1}{2}\int e^u\,du.$$

It amounts to the same thing.

Feedback to activity 4.6

We make the substitution $u = (2x^2 + 2)$. We have $du = 4x\,dx$, so the integral reduces to

$$\int \frac{1}{4}u^{1/2}\,du = \frac{1}{4}\frac{2u^{3/2}}{3} + c = \frac{1}{6}(2x^2+2)^{3/2} + c.$$

Feedback to activity 4.7

With $u = x - 1$, we have $du = dx$ and so, on noting that $x = u + 1$, the integral becomes

$$\begin{aligned}\int x\sqrt{u}\,dx &= \int (u+1)\sqrt{u}\,du \\ &= \int (u+1)u^{1/2}\,du \\ &= \int (u^{3/2} + u^{1/2})du \\ &= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + c \\ &= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + c.\end{aligned}$$

Now we try the second suggested substitution. Setting $u = \sqrt{x-1}$ we have

$$\frac{du}{dx} = \frac{1}{2}\frac{1}{\sqrt{x-1}} = \frac{1}{2u},$$

and so $dx = 2u\,du$. The integral becomes $\int xu(2u)du$ and we need to replace x by its expression in terms of u. We have $u = \sqrt{x-1}$ and so $u^2 = x - 1$, from which we obtain

$x = u^2 + 1$. So the integral is

$$\begin{aligned}\int (u^2+1)u(2u)\,du &= 2\int (u^4+u^2)\,du \\ &= 2\left(\frac{u^5}{5}+\frac{u^3}{3}\right)+c \\ &= \frac{2}{5}\left(\sqrt{x-1}\right)^5 + \frac{2}{3}\left(\sqrt{x-1}\right)^3 + c.\end{aligned}$$

This is (of course!) the same as the answer obtained using the first substitution. But which is easier? Well, the details of the substitution are easier for the first, but the actual integration of the transformed integral is easier for the second (because it does not involve fractional powers). On balance, probably the first substitution is easier. But both are correct! Do not think there's necessarily only one way to solve a problem.

Feedback to activity 4.8

We use the substitution $u = x^2 + 4x + 5$. Then $du = (2x+4)dx$ and so the integral is

$$\int_5^{10} \frac{(1/2)du}{u} = \frac{1}{2}\Big[\ln|u|\Big]_5^{10} = \frac{1}{2}\Big(\ln(10) - \ln(5)\Big) = \frac{1}{2}\ln(2).$$

Feedback to activity 4.9

Recall that the integration by parts rule is

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx.$$

We need to integrate xe^x. If we were to take $u' = x$ and $v = e^x$ then uv' would be $(1/2)x^2e^x$, so the integral on the right of the integration by parts equation would be even more difficult than the one we started with. There is, though, another possibility: we can take $u' = e^x$ and $v = x$, in which case $u = e^x$ and $v' = 1$. Then we have

$$\int xe^x\,dx = xe^x - \int 1.e^x\,dx = xe^x - e^x + c.$$

Feedback to activity 4.10

The integrand is

$$\frac{1}{x^2+4x+3} = \frac{1}{(x+1)(x+3)},$$

and the partial fractions rule says that

$$\frac{1}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3},$$

for some numbers A and B. Multiplying both sides by $(x+1)(x+3)$, we obtain

$$1 = A(x+3) + B(x+1).$$

Taking $x = -3$ gives $-2B = 1$, so $B = -1/2$. Taking $x = -1$ gives $2A = 1$, so $A = 1/2$. The integral is therefore

$$\frac{1}{2}\int\left(\frac{1}{x+1} - \frac{1}{x+3}\right)dx = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+3| + c.$$

Feedback to activity 4.11

We have
$$TC(q) = \int MC\,dq = \int (q + 5q^2 + e^q)dq = \frac{q^2}{2} + \frac{5q^3}{3} + e^q + c.$$
We know that $TC(0) = FC = 10$, so $0 + 0 + e^0 + c = 10$; in other words, $1 + c = 10$ and $c = 9$. Therefore the total cost function is
$$TC = \frac{q^2}{2} + \frac{5q^3}{3} + e^q + 9.$$

Answers to Sample examination/practice questions

Answer to question 4.1

Noting that
$$\frac{x}{x^2 + 5x + 6} = \frac{x}{(x+2)(x+3)},$$
we use partial fractions. We have
$$\frac{x}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3},$$
for some numbers A and B. Multiplying both sides by both factors in the usual way, we have
$$A(x+3) + B(x+2) = x.$$
Taking $x = -3$ gives $-B = -3$, so $B = 3$. Taking $x = -2$ we get $A = -2$. Hence
$$\begin{aligned}\int \frac{x}{x^2 + 5x + 6}\,dx &= \int \left(\frac{-2}{x+2} + \frac{3}{x+3}\right)dx \\ &= -2\ln|x+2| + 3\ln|x+3| + c.\end{aligned}$$

Answer to question 4.2

Let us try $u = x - 1$. We have $du = dx$. When $x = 1$, $u = 0$ and when $x = 2$, $u = 1$. Furthermore, since $x = u + 1$, we may write x^2 as $(u+1)^2$. The integral therefore becomes
$$\begin{aligned}\int_0^1 (u+1)^2 u^{1/2}\,du &= \int_0^1 (u^2 + 2u + 1)u^{1/2}\,du \\ &= \int_0^1 (u^{5/2} + 2u^{3/2} + u^{1/2})\,du \\ &= \left[\frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_0^1 \\ &= \frac{2}{7} + \frac{4}{5} + \frac{2}{3} \\ &= \frac{184}{105}.\end{aligned}$$

Answer to question 4.3

For this integral, the substitution $u = x^2 + 3x + 2$ gives

$$\int \frac{du}{u} = \ln|u| + c = \ln|x^2 + 3x + 2| + c.$$

An alternative approach, however, is to use partial fractions, because the denominator factorises as $(x+1)(x+2)$. Partial fractions tells us that for some numbers A and B,

$$\frac{2x+3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

In the usual way, we have

$$A(x+2) + B(x+1) = 2x+3,$$

for all x. Taking $x = -2$ reveals that $-B = -1$, so $B = 1$; and taking $x = -1$, we obtain $A = 1$. Therefore the integral equals

$$\int \left(\frac{1}{x+1} + \frac{1}{x+2} \right) dx = \ln|x+1| + \ln|x+2| + c.$$

This is the same answer as we obtained using substitution, because

$$\ln|x+1| + \ln|x+2| = \ln|(x+1)(x+2)| = \ln|x^2 + 3x + 2|.$$

Note that here (again), there is more than one way to solve the problem.

Answer to question 4.4

With $u = \ln x$, we have $du = (1/x)dx$. Also, the values $x = 1$ and $x = e$ correspond to $u = 0$ and $u = 1$. So

$$\int_1^e \frac{\ln x\sqrt{\ln x}}{x}\, dx = \int_0^1 u\sqrt{u}\, du = \int_0^1 u^{3/2}\, du = \left[\frac{2}{5} u^{5/2} \right]_0^1 = \frac{2}{5}.$$

Answer to question 4.5

We know that

$$\begin{aligned} TC &= \int MC\, dQ \\ &= \int \left(1 - \frac{1}{(Q+1)^2} \right) dQ \\ &= \int \left(1 - (Q+1)^{-2} \right) dQ \\ &= Q + (Q+1)^{-1} + c. \end{aligned}$$

So, for some constant c,

$$TC = Q + \frac{1}{Q+1} + c.$$

Now, we know that the average cost when $Q = 4$ is 3.05, so the total cost when $Q = 4$ is $4(3.05) = 12.2$. But

$$TC(4) = 4 + (1/5) + c = 4.2 + c,$$

so we must have $c = 8$. So $TC = Q + 1/(Q + 1) + 8$. When $Q = 5$ the total cost is therefore $5 + 1/6 + 8 = 79/6$.

It's useful, perhaps, to point out how *not* to answer this question. A naive approach might be to argue as follows: the total cost at $Q = 4$ is 12.2, and the marginal cost when $Q = 4$ is $1 - (1/4^2) = 15/16$. Since the marginal cost is the cost of producing one additional item, the cost of producing 5 is therefore $12.2 + 15/16 = 13.1375$. This is incorrect. Why? The reason is that the marginal cost gives, approximately, the cost of producing one more item, but that for this approximation to be good, increasing production by one item must be a relatively small increase. But increasing from 4 to 5 is a big relative change in production. If we were increasing from 400 to 401 (say), the approximation would be better. (Recall that the formal mathematical definition of marginal cost is that it is the derivative of the total cost, and that this is *approximately* the cost of producing one additional item.)

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Answer to question 4.6

We have

$$TC = \int MC\,dq = \int (32 + 18q - 12q^2)dq = 32q + 9q^2 - 4q^3 + c,$$

and we know that the fixed cost, which is $TC(0)$, is 43, so

$$43 = 0 + 0 + 0 + c,$$

and hence

$$TC = 32q + 9q^2 - 4q^3 + 43.$$

Then, the average cost is

$$AC = \frac{TC}{q} = 32 + 9q - 4q^2 + \frac{43}{q},$$

and

$$VC = TC - FC = (32q + 9q^2 - 4q^3 + 43) - 43 = 32q + 9q^2 - 4q^3.$$

Answer to question 4.7

The total revenue is given by

$$TR = \int MR\,dx = \int (10 - 2x^2)dx = 10x - \frac{2}{3}x^3 + c.$$

But what should c be? Well, think about it: what's the revenue from selling 0 items? It's 0, of course, so $TR(0) = 0$ and hence $c = 0$. Therefore

$$TR = 10x - \frac{2}{3}x^3.$$

The profit function is

$$\Pi = TR - TC = \left(10x - \frac{2}{3}x^3\right) - (x^2 + 4x + 2) = 6x - \frac{2}{3}x^3 - x^2 - 2.$$

Setting $\Pi'(x) = 0$ we obtain

$$6 - 2x^2 - 2x = 0,$$

or, equivalently,

$$x^2 + x - 3 = 0.$$

Solving this, we obtain $x = (-1 \pm \sqrt{13})/2$ and clearly, for it to have economic significance, it is the positive solution $(-1 + \sqrt{13})/2$ that is relevant. The second derivative $\Pi''(x)$ is $-4x - 2$, which is negative here, so this gives a maximum. The maximum value of the profit is obtained by substituting this value into the profit function. This turns out to be 2.64536.

Answer to question 4.8

What we need here is $TC(4) - TC(2)$. Since

$$TC = \int MC\, dq,$$

we have

$$\begin{aligned}
TC(4) - TC(2) &= \int_2^4 MC\, dq \\
&= \int_2^4 (10 - q + q^2)\, dq \\
&= \left[10q - \frac{q^2}{2} + \frac{q^3}{3}\right]_2^4 \\
&= \left(10(4) - \frac{4^2}{2} + \frac{4^3}{3}\right) - \left(10(2) - \frac{2^2}{2} + \frac{2^3}{3}\right) \\
&= \frac{98}{3}.
\end{aligned}$$

(Alternatively, you could determine TC by indefinite integration to start with, and then calculate $TC(4) - TC(2)$. Of course, the constant won't be known since we aren't told the fixed costs. But this does not matter since, in working out the difference $TC(4) - TC(2)$, the constant will cancel.)

Answer to question 4.9

We have

$$TC = \int MC\, dQ = \int \left(\frac{20}{\sqrt{Q}} e^{\sqrt{Q}} + Q^3 + \frac{1}{Q+1}\right) dQ.$$

Now, to determine the integral of $e^{\sqrt{Q}}/\sqrt{Q}$, we use the substitution $u = \sqrt{Q}$. We have $du = (1/(2\sqrt{Q}))dQ$ and

$$\int \frac{e^{\sqrt{Q}}}{\sqrt{Q}}\, dQ = \int 2e^u\, du = 2e^u + c = 2e^{\sqrt{Q}} + c.$$

So,

$$TC = 40e^{\sqrt{Q}} + \frac{Q^4}{4} + \ln|Q+1| + c.$$

Now,

$$20 = FC = TC(0) = 40e^0 + 0 + \ln(1) + c = 40 + c,$$

so $c = -20$ and

$$TC = 40e^{\sqrt{Q}} + \frac{Q^4}{4} + \ln|Q+1| - 20 = 40e^{\sqrt{Q}} + \frac{Q^4}{4} + \ln(Q+1) - 20.$$

(We've used the fact that Q, as a quantity, is non-negative to observe that $|Q+1| = Q+1$.)