Chapter 7
Sequences and series

Essential reading

(For full publication details, see Chapter 1.)

☞ Anthony and Biggs (1996) Chapters 3 and 4.

Further reading

☞ Bradley (2008) Chapter 5.

☞ Dowling (2000) Chapter 17.

7.1 Introduction

In this chapter we turn our attention to sequences, series and their applications. This is a rather small topic, but it is sufficiently different from the other topics to merit a separate chapter. A more complete investigation of sequences and series would involve the study of difference equations, but this is not part of this subject. (Difference equations are, however, covered in **05B Mathematics 2**.)

7.2 Sequences

A *sequence*[1] of numbers $y_0, y_1, y_2, \ldots$ is an infinite and ordered list of numbers with one term, y_t, corresponding to each non-negative integer, t. We call y_{t-1} the tth term of the sequence. Notice that, in our notation, the first term is y_0 and y_t is actually the $(t+1)$st term of the sequence. (Be careful not to be confused by this, as some texts differ.) For example, y_t could represent the price of a commodity t years from now, or the balance in a bank account t years from now. Often, a sequence is defined explicitly by a formula. For instance, the formula $y_t = t^2$ generates the sequence

$$y_0 = 0, \quad y_1 = 1, \quad y_2 = 4, \quad y_3 = 9, \quad y_4 = 16, \ldots$$

and the sequence $3, 5, 7, 9, \ldots$ may be described by the formula

$$y_t = 2t + 3 \;\; (t \geq 0).$$

[1]See Anthony and Biggs (1996) Section 3.1.

7.3 Arithmetic progressions

The arithmetic progression with first term a and common difference d has its terms given by the formula $y_t = a + dt$. For example, the arithmetic progression with first term 5 and common difference 3 is $5, 8, 11, 14, \ldots$. Note that y_t is obtained from y_{t-1} by adding the common difference d. In symbols, $y_t = y_{t-1} + d$.

7.4 Geometric progressions

Another very important type of sequence is the geometric progression. The geometric progression with first term a and common ratio x is given by the formula $y_t = ax^t$. Notice that successive terms are related through the relationship $y_t = xy_{t-1}$. For example, the geometric progression with first term 3 and common ratio 1/2 is given by $y_t = 3(1/2)^t$; that is, the sequence is $3, 3/2, 3/4, 3/8, \ldots$.

7.5 Compound interest

7

Perhaps the simplest occurrence of geometric progressions in economics is in the study of compound interest.[2] Suppose that we have a savings account for which the annual percentage interest rate is constant at 8%. What this means is that if we have \$$P$ in the account at the beginning of a year then, at the end of that year, the account balance is increased by 8% of \$$P$. In other words, the balance increases to \$$(P + 0.08P)$. Generally, if the annual percentage rate of interest is $R\%$, then the interest rate is $r = R/100$ and in the course of one year, a balance of \$$P$ becomes \$$(P + rP) =$ \$$(1 + r)P$. One year after that, the balance in dollars becomes \$$(1 + r)[(1 + r)P]$, which is \$$(1 + r)^2P$. Continuing in this way, we can see that if P dollars are deposited in an account where interest is paid annually at rate r, and if no money is taken from or added to the account, then after t years we have a balance of $P(1 + r)^t$ dollars. This process is known as *compounding* (or compound interest), because interest is paid on interest previously added to the account.

> **Activity 7.1** Suppose that 1000 dollars is invested in an account that pays interest at a fixed rate of 7%, paid annually. How much is there in the account after 4 years?

7.6 Compound interest and the exponential function

When we looked at the exponential function in Chapter 2 of the guide, you might well have asked where on earth this strange number e came from. It does seem strange, so let me try to justify it by giving you another definition of the exponential function.[3] In order to do this, we have to have some idea of what is meant by the *limit* of a function.

[2] See Anthony and Biggs (1996) Sections 4.3 and 7.3.

[3] See Anthony and Biggs (1996) Section 7.2 or Ostaszewski (1993) Section 11.1, for a discussion of this approach to the exponential function.

Consider the function $f(y) = 1/y$. As y gets larger and larger, $f(y)$ gets closer and closer to 0. This idea of 'getting closer and closer' to a given number is the essence of what we mean by a limit. We say that $f(y)$ *tends to 0 as y tends to infinity*, or that 0 is the limit of $f(y)$ as y tends to infinity. The notation used for this is

$$f(y) \to 0 \text{ as } y \to \infty,$$

or

$$\lim_{y \to \infty} f(y) = 0,$$

where the symbol ∞ stands for 'infinity'. Do not think that ∞ is a number; it is merely a convenient notation. A rigorous, formal, approach to the exponential function is to *define* e to be the limit

$$e = \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^y.$$

Then, for any x, we define e^x to be the limit

$$e^x = \lim_{y \to \infty} \left(1 + \frac{x}{y}\right)^y.$$

This way of thinking about e is useful when we consider compound interest. What happens if interest is added more frequently than once a year? Suppose, for example, that instead of 8% interest paid at the end of the year, we have 4% interest added twice-yearly, once at the middle of the year and once at the end. If \$100 is invested, the amount after one year will be

$$100(1 + 0.04)^2 = 108.16,$$

dollars which is slightly more than the \$108 which results from the single annual addition. If the interest is added quarterly (so that 2% is added four times a year), the amount after one year will be

$$100(1 + 0.02)^4 = 108.24,$$

dollars (approximately). In general, when the year is divided into m equal periods, the rate is r/m over each period, and the balance after one year is

$$P\left(1 + \frac{r}{m}\right)^m,$$

where P is the initial deposit. Taking m larger and larger — formally, letting m tend to infinity — we find ourselves in the situation of *continuous compounding*. Now, from above,

$$\lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m = e^r,$$

so the balance after one year is Pe^r. If invested for a further year, we would have $Pe^r e^r = P(e^r)^2 = Pe^{2r}$. After t years continuous compounding, the balance of the account would be Pe^{rt}.

7.7 Series

Let us continue with the story of our investor. It is natural to investigate how the balance varies if the investor adds a certain amount to the account each year. Suppose that she adds $\$P$ to the account at the beginning of each year, so that at the beginning of the first year the balance is $\$P$. At the beginning of the second year the balance, in dollars, will be $P(1+r)+P$; this represents the money from the first year with interest added, and the new, further, deposit of $\$P$. Convince yourself that, continuing in this way, the balance at the beginning of year t is, in dollars,

$$P + P(1+r) + \cdots + P(1+r)^{t-2} + P(1+r)^{t-1}.$$

How can we calculate this expression? Note that it is the sum of the first t terms (that is, term 0 to term $t-1$) of the geometric progression with first term P and common ratio $(1+r)$. Before coming back to this, we shall discuss such things in a more general setting.

Given a sequence $y_0, y_1, y_2, y_3, \ldots$, a *finite series* is a sum of the form

$$y_0 + y_2 + \cdots + y_{t-1},$$

the first t terms added together, for some number t. There are two important results about series, concerning the cases where the corresponding sequence is an *arithmetic progression* (in which case the series is called an arithmetic series) and where it is a geometric progression (in which case the series is called a *geometric series*).

7.7.1 Arithmetic series

The main result here is that if $y_t = a + dt$ describes an arithmetic progression and S_t is the sequence

$$S_t = y_0 + y_1 + y_2 + \cdots + y_{t-1},$$

then

$$S_t = \frac{t(2a + (t-1)d)}{2}.$$

There is a useful way of remembering this result. Notice that S_t may be rewritten as

$$S_t = t\frac{(a + (a + (t-1)d))}{2} = t\frac{(y_0 + y_{t-1})}{2},$$

so that we have the following easily remembered result: an arithmetic series has a sum equal to the number of terms, t, times the value of the average of the first and last terms $(y_0 + y_{t-1})/2$. Equivalently, the average value S_t/t of the t terms is the average, $(y_0 + y_{t-1})/2$ of the first and last terms.

Activity 7.2 Find the sum of the first n terms of an arithmetic series whose first term is 1 and whose common difference is 5.

7.7.2 Geometric series

We now look at geometric series. It is easily checked (by multiplying out the expression) that, for any x,

$$(1-x)(1+x+x^2+\cdots+x^{t-1}) = 1-x^t.$$

So, if $x \neq 1$ and $y_t = ax^t$, then the geometric series

$$S_t = y_0 + y_2 + \cdots + y_{t-1} = a + ax + ax^2 + \cdots + ax^{t-1},$$

is therefore given by

$$S_t = \frac{a(1-x^t)}{1-x}.$$

Example 7.1 In our earlier discussion on savings accounts, we came across the expression

$$P + P(1+r) + \cdots + P(1+r)^{t-2} + P(1+r)^{t-1}.$$

We now see that this is a geometric series with t terms, first term P and common ratio $1+r$. Therefore it equals

$$P\frac{1-(1+r)^t}{1-(1+r)} = \frac{P}{r}\left((1+r)^t - 1\right).$$

Activity 7.3 Find an expression for

$$2 + 2(3) + 2(3^2) + 2(3^3) + \cdots + 2(3)^n.$$

7.8 Finding a formula for a sequence

Often we can use results on series to determine an exact formula for the members of a sequence of numbers. The following example illustrates this.

Example 7.2 Suppose a sequence of numbers is constructed as follows. The first number, y_0, is 1, and each other number in the sequence is obtained from the previous number by multiplying by 2 and adding 1 (so that $y_t = 2y_{t-1} + 1$, for $t \geq 1$). What is the general expression for y_t in terms of t?

We can see that

$$\begin{aligned}
y_1 &= 2y_0 + 1 = 2(1) + 1 = 2 + 1,\\
y_2 &= 2y_1 + 1 = 2(2+1) + 1 = 2^2 + 2 + 1,\\
y_3 &= 2y_2 + 1 = 2(2^2+2) + 1 = 2^3 + 2^2 + 2 + 1,\\
y_4 &= 2y_3 + 1 = 2(2^3 + 2^2 + 2 + 1) + 1 = 2^4 + 2^3 + 2^2 + 2 + 1.
\end{aligned}$$

In general, it would appear that

$$y_t = 2^t + 2^{t-1} + \cdots + 2^2 + 2 + 1.$$

But this is just a geometric series: perhaps this is clearer if we write it as

$$y_t = 1 + 2 + 2^2 + \cdots + 2^{t-1} + 2^t,$$

from which it is clear that this is the sum of the first $t+1$ terms of the geometric progression with first term 1 and common ratio 2. Thus, using the formula for the sum of a geometric series, we have

$$y_t = \frac{1 - 2^{t+1}}{1 - 2} = 2^{t+1} - 1.$$

7.9 Limiting behaviour

When x is greater than 1, as t increases, x^t will eventually become greater than any given number, and we say that x^t *tends to infinity as* t *tends to infinity.*[4] We write this in symbols as

$$x^t \to \infty \text{ as } t \to \infty \qquad \text{or} \qquad \lim_{t\to\infty} x^t = \infty.$$

On the other hand, when $x < 1$ and $x > -1$, we have

$$x^t \to 0 \text{ as } t \to \infty \qquad \text{or} \qquad \lim_{t\to\infty} x^t = 0.$$

We notice that, while x^t gets closer and closer to 0 for all values of x in the range $-1 < x < 1$, its behaviour depends to some extent on whether x is positive or negative. When x is negative, the terms are alternately positive and negative, and we say that the approach to zero is *oscillatory.* For example, when $x = -0.2$, the sequence x^t is

$$1, -0.2, 0.04, -0.008, 0.0016, -0.00032, 0.000064, -0.0000128, \ldots$$

for $t \geq 0$. When x is less than -1, the sequence is again oscillatory, but it does not approach any limit, the terms being alternately large-positive and large-negative. In this case, we say that x^t *oscillates increasingly.*

As an application of this, let us consider again the geometric series

$$S_t = a + ax + ax^2 + \cdots + ax^{t-1}.$$

We have, using the formula for a geometric series, that

$$S_t = \frac{a(1 - x^t)}{1 - x}.$$

If $-1 < x < 1$ then $x^t \to 0$ as $t \to \infty$. This means that S_t approaches the number

$$a\frac{1 - 0}{1 - x} = \frac{a}{1 - x},$$

[4]See Anthony and Biggs (1996) Section 3.3.

as t increases. In other words,

$$S_t \to \frac{a}{1-x} \quad \text{as} \quad t \to \infty.$$

We call this limit the *sum to infinity* of the sequence given by $y_t = ax^t$. Note that a geometric sequence has a sum to infinity which is finite only if the common ratio is strictly between -1 and 1.

Example 7.3 Consider the sequence with $y_t = 1/2^t$ for $t \geq 0$. The sum of the first t terms of this sequence would be given by

$$S_t = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{t-1}}.$$

Using the formula for the sum of a geometric series, we then have

$$S_t = 2\left[1 - \left(\frac{1}{2}\right)^t\right],$$

and that $S_t \to 2$ as $t \to \infty$.

Activity 7.4 Find an expression for

$$S_t = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots + \left(\frac{2}{3}\right)^t,$$

and determine the limit of S_t as t tends to infinity.

7.10 Financial applications

A number of problems in financial mathematics can be solved using arithmetic and geometric series. Here is an example.

Example 7.4 John has opened a savings account with a bank, and they pay a fixed interest rate of 5% per annum, with the interest paid once a year, at the end of the year. He opened the savings account with a payment of \$100 on 1 January 2003, and will be making deposits of \$200 yearly, on the same date. What will his savings be after he has made N of these additional deposits? (Your answer will be an expression involving N.)

If y_N is the required amount, then we have

$$y_1 = (1.05)100 + 200,$$

and then

$$y_2 = (1.05)y_1 + 200 = 100(1.05)^2 + 200(1.05) + 200,$$

7

so that, in general, we can spot the pattern and observe that

$$
\begin{aligned}
y_N &= 100(1.05)^N + 200(1.05)^{N-1} + 200(1.05)^{N-2} + \cdots + 200(1.05) + 200 \\
&= 100(1.05)^N + 200\left[1 + (1.05) + (1.05)^2 + \cdots + (1.05)^{N-2} + (1.05)^{N-1}\right] \\
&= 100(1.05)^N + 200\frac{1-(1.05)^N}{1-(1.05)} \\
&= 100(1.05)^N + 4000\left[(1.05)^N - 1\right],
\end{aligned}
$$

where we have used the formula for the sum of a geometric series.

Learning outcomes

At the end of this chapter and the relevant reading, you should be able to:

- explain what is meant by arithmetic and geometric progressions, and calculate the sum of finite arithmetic and geometric series
- explain compound interest and calculate balances under compound interest
- apply sequences and series in management and finance
- analyse the long-term behaviour of series and sequences

Sample examination/practice questions

Question 7.1

A geometric progression has a sum to infinity of 3 and has second term, y_1, equal to 2/3. Show that there are two possible values of the common ratio x and find the corresponding values of the first term a.

Question 7.2

Suppose we have an initial amount, A_0, to invest and we add an additional investment F at the end of each subsequent year. All investments earn interest at a rate of $i\%$ per annum, paid at the end of each year.

(a) Use the formula for the sum of a geometric series to derive a formula for the value of the investment, A_n, after n years.

(b) An investor puts \$10000 into an investment account that yields interest of 10% per annum. The investor adds an additional \$5000 at the end of each year. How much will there be in the account at the end of five years? Show that if the investor has to wait N years until the balance is at least 80000, then

$$N \geq \frac{\ln(13/6)}{\ln(1.1)}.$$

Question 7.3

An amount of \$1000 is invested and attracts interest at a rate equivalent to 10% per annum. Find expressions for the total after one year if the interest is compounded:

(a) annually,

(b) quarterly,

(c) monthly,

(d) daily. (Assume the year is not a leap year.)

What would be the total after one year if the interest is 10% compounded continuously?

Question 7.4

Suppose $y_t = 1/2^{2t}$. Find the limit, as $t \to \infty$, of

$$S_t = y_0 + y_2 + \cdots + y_{t-1}.$$

7

Answers to activities

Feedback to activity 7.1

The required amount is $1000(1 + 0.07)^4 = 1310.80$ dollars.

Feedback to activity 7.2

We have

$$S_n = \frac{1}{2}n\,(2(1) + (n-1)5) = \frac{n}{2}(5n-3) = \frac{5}{2}n^2 - \frac{3}{2}n.$$

Feedback to activity 7.3

Noting that there are $n+1$ terms in the series, and that it is the sum of a geometric progression with first term 2 and common ratio 3, the expression is

$$\frac{2(1-3^{n+1})}{1-3} = 3^{n+1} - 1.$$

Feedback to activity 7.4

S_t is the sum of the first t terms of a geometric progression with first term 2/3 and common ratio 2/3, so

$$S_t = \left(\frac{2}{3}\right)\frac{1-(2/3)^t}{1-(2/3)} = 2\left[1 - \left(\frac{2}{3}\right)^t\right].$$

As $t \to \infty$, $(2/3)^t \to 0$ and so $S_t \to 2$.

Answers to Sample examination/practice questions

Answer to question 7.1

We know that the sum to infinity is given by the formula $a/(1-x)$ and that $y_1 = ax$. Therefore, the given information is

$$\frac{a}{1-x} = 3 \qquad \text{and} \qquad ax = \frac{2}{3}.$$

From the first equation, $a = 3(1-x)$ and the second equation then gives $3(1-x)x = 2/3$, from which we obtain the quadratic equation $9x^2 - 9x + 2 = 0$. This has the two solutions $x = 2/3$ and $x = 1/3$. The corresponding values of the first term a can then be found, using $a = 3(1-x)$, to be 1 and 2, respectively. So, as suggested by the question, there are two geometric progressions that have the required sum to infinity and second term.

Answer to question 7.2

(a) After 1 year, at the beginning of the second, the amount A_1 in the account is

$$A_1 = A_0\left(1+\frac{i}{100}\right) + F,$$

because the initial amount A_0 has attracted interest at a rate of $i/100$ and F has been added. Similar considerations show that

$$\begin{aligned} A_2 &= \left(1+\frac{i}{100}\right)A_1 + F \\ &= \left(1+\frac{i}{100}\right)\left[A_0\left(1+\frac{i}{100}\right)+F\right] + F \\ &= A_0\left(1+\frac{i}{100}\right)^2 + F\left(1+\frac{i}{100}\right) + F, \end{aligned}$$

and

$$\begin{aligned} A_3 &= \left(1+\frac{i}{100}\right)A_2 + F \\ &= \left(1+\frac{i}{100}\right)\left[A_0\left(1+\frac{i}{100}\right)^2 + F\left(1+\frac{i}{100}\right)+F\right] + F \\ &= A_0\left(1+\frac{i}{100}\right)^3 + F\left(1+\frac{i}{100}\right)^2 + F\left(1+\frac{i}{100}\right) + F. \end{aligned}$$

In general, if we continued, we could see that A_n is given by

$$A_0\left(1+\tfrac{i}{100}\right)^n + \underbrace{F\left(1+\tfrac{i}{100}\right)^{n-1} + F\left(1+\tfrac{i}{100}\right)^{n-2} + \cdots + F\left(1+\tfrac{i}{100}\right) + F}_{n \text{ terms}}.$$

Now, looking at the n terms involving F, we use the formula for the sum of geometric progression to get

$$F\frac{1-\left(1+\frac{i}{100}\right)^n}{1-\left(1+\frac{i}{100}\right)} = \frac{100F}{i}\left[\left(1+\frac{i}{100}\right)^n - 1\right],$$

so that

$$A_n = A_0\left(1+\frac{i}{100}\right)^n + \frac{100F}{i}\left[\left(1+\frac{i}{100}\right)^n - 1\right].$$

For (b), we us the formula just obtained, with $A_0 = 10000$, $i = 10$, $F = 5000$ and $n = 5$, and we see that

$$\begin{aligned} A_5 &= 10000\left(1+\frac{10}{100}\right)^5 + \frac{100(5000)}{10}\left[\left(1+\frac{10}{100}\right)^5 - 1\right] \\ &= 10000\,(1.1)^5 + 50000\left[(1.1)^5 - 1\right] \\ &= 46630.60, \end{aligned}$$

dollars.

Now, for the balance to be at least 80000 dollars after N years, we need $A_N \geq 80000$ which means

$$10000\,(1.1)^N + 50000\left[(1.1)^N - 1\right] \geq 80000.$$

This is equivalent, after a little manipulation, to

$$60000(1.1)^N \geq 130000,$$

or $(1.1)^N \geq 13/6$. To solve this, we can take logarithms and see that we need

$$N\ln(1.1) \geq \ln(13/6),$$

so

$$N \geq \frac{\ln(13/6)}{\ln(1.1)},$$

as required.

Answer to question 7.3

We use the fact that if the interest is paid in m equally spaced instalments, then the total after one year is $1000\left(1+\frac{r}{m}\right)^m$, where $r = 0.1$ and $m = 1, 4, 12, 365$ in the four cases. Therefore the answers to the first four parts of the problem are as follows:

(a) $1000\,(1+0.1) = 1100.$

(b) $1000\left(1+\dfrac{0.1}{4}\right)^4 = 1000(1.025)^4.$

(c) $1000\left(1+\dfrac{0.1}{12}\right)^{12}.$

(d) $1000\left(1+\dfrac{0.1}{365}\right)^{365}.$

For the last part, we use the fact that under continuous compounding at rate r, an amount P grows to Pe^r after one year, so the answer here is $1000e^{0.1}$.

Answer to question 7.4

Note that $1/2^{2t} = 1/4^t = (1/4)^t$, so this is a geometric series where the common ratio is $1/4$. The first term is 1, and there are t terms, so

$$S_t = \frac{1 - (1/4)^t}{1 - (1/4)} = \frac{4}{3}\left[1 - \left(\frac{1}{4}\right)^t\right].$$

As $t \to \infty$, $(1/4)^t \to 0$ and so $S_t \to 4/3$.