Examiners' commentaries 2016 Mock

MT105a Maths 1

Comments on specific questions

Question 1

> See Chapters 2 and 3 of the subject guide.
>
> A common source of errors in this question is confusion between *marginal* and *average* quantities; be sure you understand how these are used in the solutions below.

Note that the question gives you the *marginal* revenue, which we must integrate to obtain the total revenue:

$$TR(q) \;=\; \int MR(q)\;dq \;=\; \int (9-q)\;dq \;=\; 9q - \frac{q^2}{2} + c.$$

As $0 = TR(0)$, since nothing is being sold, we conclude $c = 0$.

Total cost is the sum of variable cost and fixed cost, i.e.

$$TC(q) = VC(q) + FC,$$

where variable cost is

$$VC(q) \;=\; q \times AVC(q) \;=\; q \times \left(1 + \frac{q}{2}\right) \;=\; q + \frac{q^2}{2}.$$

As $FC = 12$, we then have

$$TC(q) = \frac{q^2}{2} + q + 12.$$

So the profit, as a function of q, is

$$\Pi(q) \;=\; TR(q) - TC(q) \;=\; -q^2 + 8q - 12.$$

To maximise profit, we first set $0 = \Pi'(q) = 8 - 2q$, which yields $q = 4$.[1]

To see that a maximum of Π does indeed occur when $q = 4$, we note that $\Pi''(4) = -2 < 0$.[2]

The maximum profit is $\Pi(4) = 4$.

[1] You might think that an alternative method here is to solve the equation $MR = MC$ in order to maximise the profit. But this does not fully answer the question because, although it will indeed lead to the value $q = 4$, it will not result in an explicit expression for the profit function in terms of q, something the question clearly asks for. Nor, without extra work, will it show that the value $q = 4$ does indeed *maximise* profit. If you took this approach, you would still get some marks, but not all of them. So, this is a case where a careful reading of the question is important.

[2] In fact, $\Pi''(q) = -2 < 0$ for all q.

Question 2

> This involves the standard techniques of Chapter 6 of the subject guide. But it is important to realise that the given equations are not in the form to which the standard method of row operations can apply. Explicitly, the standard methods for solving linear systems assume that the equations take the form $A\mathbf{x} = \mathbf{b}$ where A is a matrix, $\mathbf{x}$ is the vector of unknowns (variables) and $\mathbf{b}$ is a vector whose entries are constants (that is, numbers). The system in the question has to be re-written in this standard format before solving.

Rearranging the equations in standard format yields the following system of equations

$$\begin{aligned} 2x - y + 5z &= 128, \\ x - 3y + 2z &= 29, \\ y - z &= -28. \end{aligned}$$

The augmented matrix is then

$$(A \mid \mathbf{b}) = \begin{pmatrix} 2 & -1 & 5 & 128 \\ 1 & -3 & 2 & 29 \\ 0 & 1 & -1 & -28 \end{pmatrix}.$$

We then use row operations to reduce. Note that some very simple row swaps very easily gives us our leading 1s.

$$(A \mid \mathbf{b}) \longrightarrow \begin{pmatrix} 1 & -3 & 2 & 29 \\ 0 & 1 & -1 & -28 \\ 2 & -1 & 5 & 128 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 & 2 & 29 \\ 0 & 1 & -1 & -28 \\ 0 & 5 & 1 & 70 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & -55 \\ 0 & 1 & -1 & -28 \\ 0 & 0 & 6 & 210 \end{pmatrix}.$$

So $x = -20$, $y = 7$ and $z = 35$.

Question 3

> This is a fairly standard type of question, using the material in Chapter 5 of the subject guide. Candidates often have problems with notation in this question, with many using incorrect, and at times incomprehensible, ways to denote partial derivatives. Make sure you are familiar with the accepted notations or you may lose marks in the exam.

The first-order partial derivatives of f are $f_x = 2x \ln y$ and $f_y = \dfrac{x^2}{y} - 1 - \ln y$. The first-order conditions (FOCs) are given by setting these equal to zero, i.e.

$$0 = 2x \ln y, \qquad 0 = \frac{x^2}{y} - 1 - \ln y.$$

The first of these equations yields either $x = 0$ or $\ln y = 0$, that is $x = 0$ or $y = 1$, where we must analyse these cases *separately*.

(a) If $x = 0$ then the second FOC becomes $0 = -1 - \ln y$, and implies that $\ln y = -1$, i.e. $y = e^{-1}$. Thus there is a critical point at $(0, e^{-1})$.

(b) If $y = 1$ then the second FOC becomes $0 = x^2 - 1$, i.e. $x = \pm 1$. We therefore have two critical points at $(1, 1)$ and $(-1, 1)$.

To determine the nature of the 3 critical points we have found, we use the second-derivative test. We have

$$f_{xx} = 2 \ln y, \qquad f_{xy} = \frac{2x}{y} = f_{yx}, \qquad f_{yy} = -\frac{x^2}{y^2} - \frac{1}{y}.$$

- At $(0, e^{-1})$, we have $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -e$.
  Thus $f_{xx} < 0$ and $D = 2e > 0$ and $(0, e^{-1})$ is therefore a *local maximum*.
- At $(1, 1)$, we have $f_{xx} = 0$, $f_{xy} = 2$, $f_{yy} = -2$.
  So $D = -4 < 0$ and $(1, 1)$ is therefore a *saddle point*.
- At $(-1, 1)$, we have $f_{xx} = 0$, $f_{xy} = -2$, $f_{yy} = 0$.
  So $D = -4 < 0$ and $(-1, 1)$ is therefore a *saddle point*.

Question 4

> This question on differentiation uses material from Chapters 3 and 5 of the subject guide. In particular, candidates must be familiar with the product rule and chain rule for differentiation. A common mistake made here is the use of incorrect and ambiguous notation: it is vital that you can distinguish between the first-order partial derivatives of U.

The required partial derivatives are:

$$\frac{\partial W}{\partial x} = e^{x-4y}\left(U + \frac{\partial U}{\partial x}\right),$$

$$\frac{\partial W}{\partial y} = e^{x-4y}\left(-4U + \frac{\partial U}{\partial y}\right),$$

$$\frac{\partial^2 W}{\partial x^2} = e^{x-4y}\left(U + 2\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2}\right).$$

Substituting these into

$$\frac{\partial W}{\partial y} = \frac{\partial^2 W}{\partial x^2} - 2\frac{\partial W}{\partial x} - 3W,$$

yields

$$e^{x-4y}\left(-4U + \frac{\partial U}{\partial y}\right) = e^{x-4y}\left(U + 2\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2}\right) - 2e^{x-4y}\left(U + \frac{\partial U}{\partial x}\right) - 3e^{x-4y}U.$$

To simplify things, we first cancel the exponential term, so that the left-hand side becomes

$$L.H.S. = -4U + \frac{\partial U}{\partial y},$$

and the right-hand side becomes

$$\begin{aligned} R.H.S. &= \left(U + 2\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2}\right) - 2\left(U + \frac{\partial U}{\partial x}\right) - 3U \\ &= (U - 2U - 3U) + \left(2\frac{\partial U}{\partial x} - 2\frac{\partial U}{\partial x}\right) + \frac{\partial^2 U}{\partial x^2} \\ &= -4U + \frac{\partial^2 U}{\partial x^2}. \end{aligned}$$

Cancelling the $-4U$ term from both sides gives the desired result.

Question 5

> Part (a) of this question requires integration, which is covered in Chapter 4 of the subject guide, while part (b) you need to know the meaning of an arithmetic progression, as discussed in Chapter 7 of the subject guide.

Part (a)

Making the substitution $u = \ln x$ yields

$$\int \frac{x^{-1}dx}{(1+\ln x)\ln x} = \int \frac{du}{(1+u)u},$$

where, using the method of partial fractions, we have

$$\begin{aligned} &= \int \frac{1}{u}du - \int \frac{1}{1+u}du \\ &= \ln u - \ln(1+u) + c. \end{aligned}$$

Reverting back from u to x, we have

$$\int \frac{x^{-1}dx}{(1+\ln x)\ln x} = \ln(\ln x) - \ln(1+\ln x) + c.$$

Part (b)

For this question, you need to know the meaning of an arithmetic progression, as discussed in Chapter 7 of the subject guide.

The nth term of the sequence takes the form $a + (n - 1)d$ so the given conditions give rise to the following two equations:

$$a + d = 7, \qquad a + 12d = 10a.$$

Solving these yields $d = 3$ and $a = 4$.

Question 6

> Chapter 7 of the subject guide gives the required background material.

Let y_n be the balance after the n^{th} deposit.

The balance after the first deposit, i.e. y_n when $n = 1$, is

$$y_1 = D.$$

The balance after the second deposit, i.e. y_n when $n = 2$, is

$$y_2 = D(1 + r) + D.$$

The balance after the third deposit,, i.e. y_n when $n = 3$, is

$$y_3 = D(1 + r^2) + D(1 + r) + D.$$

Spotting the pattern, the balance after the N^{th} deposit, i.e. y_N, is

$$\begin{aligned} y_N &= D(1 + r)^{N-1} + D(1 + r)^{N-2} + \ldots + D(1 + r) + D \\ &= D\left[1 + (1 + r) + \ldots + (1 + r)^{N-2} + (1 + r)^{N-1}\right] \end{aligned}$$

This is a G.P. with N terms; thus, using the GP formula, this is

$$\begin{aligned} &= D\frac{(1 + r)^N - 1}{(1 + r) - 1} \\ &= \frac{D}{r}\left[(1 + r)^N - 1\right], \end{aligned}$$

as required.

Question 7

> Part (a) of this question primarily concerns the Lagrange multiplier method, discussed in Chapter 5 of the subject guide, while part (b) requires integration, which is covered in Chapter 4 of the subject guide.

Part (a)

The Lagrangean is

$$\mathcal{L}(x, y, \lambda) = \left(x^{-2} + y^{-2}\right)^{1/2} + \lambda(x + y - \sqrt{2}).$$

The first order conditions are therefore

$$0 = \mathcal{L}_x = -\frac{1}{x^3}\left(x^{-2} + y^{-2}\right)^{-1/2} + \lambda,$$

$$0 = \mathcal{L}_y = -\frac{1}{y^3}\left(x^{-2} + y^{-2}\right)^{-1/2} + \lambda,$$

$$0 = \mathcal{L}_\lambda = x + y - \sqrt{2}.$$

Eliminating λ, we have

$$\lambda = \frac{1}{x^3}\left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-1/2} = \frac{1}{y^3}\left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-1/2},$$

which yields $x = y$. Substituting into $\mathcal{L}_\lambda = 0$, we then have $x = y = 1/\sqrt{2}$.

The minimum value is then

$$\sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} = \sqrt{4} = 2.$$

Part (b)

The increase in total cost is found as the integral

$$I = \int_1^2 MC(q)dq = \int_1^2 qe^q + \frac{q}{q^2 + 4q + 3}dq = \int_1^2 qe^q\, dq + \int_1^2 \frac{q}{q^2 + 4q + 3}\, dq.$$

That is, $I = J + K$, where

$$J = \int_1^2 qe^q\, dq, \qquad K = \int_1^2 \frac{q}{q^2 + 4q + 3}\, dq.$$

Using integration by parts

$$J = \int_1^2 qe^q dq = [qe^q]_1^2 - \int_1^2 e^q dq = [qe^q - e^q]_1^2 = e^2.$$

For K, we use partial fractions. Noting that $q^2 + 4q + 3 = (q+1)(q+3)$, we must have

$$\frac{q}{q^2 + 4q + 3} = \frac{A}{q+1} + \frac{B}{q+3},$$

for some A, B. We find that $A = -1/2$ and $B = 3/2$. Thus

$$\begin{aligned} K &= \frac{3}{2}\int_1^2 \frac{dq}{q+3} \ - \ \frac{1}{2}\int_1^2 \frac{dq}{q+1} \\ &= \left[\ \frac{3}{2}\ln(q+3) \ - \ \frac{1}{2}\ln(q+1)\ \right]_1^2 \\ &= \frac{3}{2}\ln\frac{5}{4} \ - \ \frac{1}{2}\ln\frac{3}{2}. \end{aligned}$$

So

$$I = e^2 + \frac{3}{2}\ln\left(\frac{5}{4}\right) - \frac{1}{2}\ln\left(\frac{3}{2}\right).$$

Question 8

> This question primarily concerns the Lagrange multiplier method, discussed in Chapter 5 of the subject guide.

The problem is to minimise $Vk + Wl$ (our objective function) subject to the constraint $k^A l^A = q$. Defining the Lagrangean as

$$\mathcal{L}(k, l, \lambda) = kV + lW + \lambda\left(q - k^A l^A\right),$$

then the first-order conditions are

$$\begin{aligned} 0 &= \mathcal{L}_k = V - A\lambda k^{A-1} l^A, \\ 0 &= \mathcal{L}_l = W - A\lambda k^A l^{A-1}, \\ 0 &= \mathcal{L}_\lambda = q - k^A l^A. \end{aligned}$$

Using the first two equations to eliminate λ, yields

$$\lambda = \frac{k^{A-1} l^A}{V} = \frac{k^A l^{A-1}}{W},$$

so $l = (V/W)k$. Hence, by the third first-order condition (which is equivalent to our constraint),

$$k = \sqrt{\frac{W}{V}}\, q^{\frac{1}{2A}}, \qquad l = \sqrt{\frac{V}{W}}\, q^{\frac{1}{2A}}.$$

The minimum cost is then found by substituting these values into the cost function $Vk + Wl$, where

$$\begin{aligned} V\sqrt{\frac{W}{V}}q^{\frac{1}{2A}} + W\sqrt{\frac{V}{W}}q^{\frac{1}{2A}} &= V\frac{\sqrt{W}}{\sqrt{V}}q^{\frac{1}{2A}} + W\sqrt{\frac{V}{W}}q^{\frac{1}{2A}} \\ &= \sqrt{V}\sqrt{W}q^{\frac{1}{2A}} + \sqrt{W}\sqrt{V}q^{\frac{1}{2A}} \\ &= 2\sqrt{V}\sqrt{W}q^{\frac{1}{2A}}, \end{aligned}$$

as required.

For the final part of the question, note that, when a firm produces q, they will earn a revenue $R(q) = Pq$, while the total cost of producing q is the cost of labour and capital plus the cost of raw materials, i.e.

$$Cq + 2\sqrt{VW}q^{\frac{1}{2A}},$$

(where we disregard any fixed costs, which are not mentioned in the question).

The profit function is therefore

$$\Pi(q) = R(q) - C(q) = Pq - \left(Cq + 2\sqrt{VW}q^{\frac{1}{2A}}\right).$$

The first order condition

$$0 = \Pi'(q) = P - C - \frac{1}{A}\sqrt{VW}q^{\frac{1}{2A}-1}$$

gives

$$q = \left(\frac{A(P-C)}{\sqrt{VW}}\right)^{\frac{2A}{1-2A}}.$$

That this is a maximum can be seen from the fact that

$$\Pi''(q) = -\frac{\sqrt{VW}}{A}\left(\frac{1-2A}{2A}\right)q^{\frac{1}{2A}-2} < 0,$$

as $0 < A < 1/2$.