Chapter 5
Functions of several variables

Essential reading

(For full publication details, see Chapter 1.)

☞ Anthony and Biggs (1996) Chapters 11, 13 and Sections 21.2, 22.2.

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Further reading

☞ Binmore and Davies (2001) Chapter 3, Sections 3.1 and 3.2; Chapter 4, Section 4.6; and Chapter 6, Sections 6.6 and 6.8. (This book gives a very general approach to the classification of critical points, more complex than is required just for the two-variable case, as discussed in this chapter.)

☞ Bradley (2008) Chapter 7.

☞ Dowling (2000) Chapters 5 and 6.

5.1 Introduction

In this chapter we study how the technique of differentiation, and its applications, can be extended to functions depending on more than one variable. This is one of the most important ideas for the application of mathematics in economics, management, finance, and many other fields.

5.2 Functions of several variables

A function f may be thought of as a 'machine', which accepts an input x and produces an output $f(x)$. In this chapter we look at functions for which the input consists of a pair of numbers (x, y).[1] (The theory extends in an obvious way to the general case when the input consists of n numbers $(x_1, x_2, \ldots, x_n)$.) Such a function is called a *function of two variables*. (The extension to a *function of n variables*, for $n \geq 2$, will be clear. Although the title of this chapter is 'Functions of several variables', we shall mainly work with functions of two variables.) Such functions occur often in economics and other fields in which we might wish to apply mathematical techniques. An important example is the *production function* of a firm, $q(k, l)$, which describes the amount of product the firm produces when using k units of capital and l of labour. Another

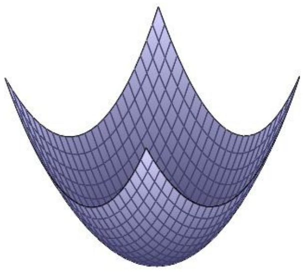
[1]See Anthony and Biggs (1996) Section 11.1.

important class of such functions is the class of *utility functions*. A utility function describes the preferences of a consumer: it enables us to compare the worth to the consumer of different combinations of two goods. These applications will be discussed further later on in this chapter.

5.3 Partial derivatives

Suppose the quantity $Z = f(x, y)$ is a function of two independent variables x, y. (To say that x and y are independent simply means that they do not depend on one another: each may be chosen independently of the other to form an input (x, y) to the function $f(x, y)$.) Then we may think of $Z = f(x, y)$ as defining a surface in 3-dimensional space: we do this by visualising all the points $(x, y, f(x, y))$ in three dimensions. In other words, imagine that for each point (x, y) in the (x, y)-plane, we plot a point at a z-distance $f(x, y)$ above the point. If we do this for all (x, y) we obtain a surface (much in the same way as plotting the points $(x, f(x))$ for a one-variable function f gives the graph of the function, which is a curve in two dimensions).

For example, the following diagram shows part of the surface corresponding to the function $f(x, y) = 2x^2 + 2y^2$. The surface is 'bowl'-shaped, with the bottom of the bowl at coordinates $(0, 0, 0)$.



Activity 5.1 What sort of surface do you think is described by the equation $z = \sqrt{x^2 + y^2}$?

Although curve-sketching (which is sketching the graph of a one-variable function) is important in this course, there is no need for you to be able to describe or sketch such surfaces for functions of two or more variables. However, I have discussed this topic because it might help in your understanding of what follows. (Of course, when dealing with functions of more than two variables, the surfaces we obtain are in more than three dimensions, and here our geometrical intuition is of little use.)

For a fixed $y = y_0$, the rate of change of $Z = f(x, y)$ with respect to x at $x = x_0$ is denoted

$$\frac{\partial f}{\partial x}(x_0, y_0) \text{ or } f_x(x_0, y_0).$$

We then have a function, f_x, which is the derivative of $f(x, y)$ when y is regarded as a constant. This is the *partial derivative with respect to* x.[2] Sometimes the notations $\partial Z/\partial x$ and Z_x are also used for this partial derivative. (Note the 'curly-d', ∂, rather than the normal d one encounters in the notation df/dx for the derivative of a function, $f(x)$, of one variable.)

So what does the partial derivative mean? If you imagined yourself walking across the surface $z = f(x, y)$, passing through the point (x_0, y_0) and heading in a direction parallel to the x-axis (so that your y-value remains at y_0), then $f_x(x_0, y_0)$ would be the instantaneous rate at which the height of the surface increased. That is, $f_x(x_0, y_0)$ is the instantaneous rate of change of the function f when we keep y fixed at y_0 and change x. (Thus, it is the derivative of the single-variable function $f(x, y_0)$.)

We can define $f_y = \partial f/\partial y$ similarly. $\partial f/\partial x, \partial f/\partial y$ are sometimes denoted f_1, f_2. Thus, the notations

$$\frac{\partial f}{\partial x},\ f_x,\ \frac{\partial Z}{\partial x},\ Z_x,\ f_1$$

are all notations for the partial derivative with respect to x of $Z = f(x, y)$. Partial derivatives of important functions in economics often have special names. For instance, if $q(k, l)$ is the production function of a firm, then $\partial q/\partial l$ is known as the *marginal product of labour*.

Calculating partial derivatives is only slightly more difficult than calculating standard derivatives. To calculate the partial derivative of a function $f(x, y)$ with respect to x, you just treat y as if it were a fixed number and differentiate with respect to x.

Example 5.1 Let $f(x, y) = x^2y + 5xy^3 + y^2$. Then

$$\frac{\partial f}{\partial x} = 2xy + 5y^3 \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 + 15xy^2 + 2y.$$

Activity 5.2 Suppose $f(x, y) = x^3y - \dfrac{x}{y}$. Find the partial derivatives of f.

Of course, these definitions can be extended to functions of more than two variables, defining the partial derivative with respect to each variable.

We may go on to define the partial derivatives with respect to x and y of the functions f_x and f_y, obtaining the *second-order* partial derivatives

$$f_{xx},\quad f_{xy},\quad f_{yx},\quad f_{yy}.$$

These are also denoted by

$$\frac{\partial^2 f}{\partial x^2},\quad \frac{\partial^2 f}{\partial x \partial y},\quad \frac{\partial^2 f}{\partial y \partial x},\quad \frac{\partial^2 f}{\partial y^2}$$

or by

$$f_{11},\quad f_{12},\quad f_{21},\quad f_{22}.$$

[2] See Anthony and Biggs (1996) Section 11.2.

For all suitably well-behaved functions (and we shall only encounter such functions) $f_{xy} = f_{yx}$.

The derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are often called *first-order* derivatives. But we shall mainly continue simply to call them the partial derivatives.

Example 5.2 Suppose

$$Z = f(x, y) = x^2y + y^3x.$$

Then

$$Z_x = 2xy + y^3, \quad Z_y = x^2 + 3y^2x,$$

$$Z_{xx} = 2y, \quad Z_{xy} = Z_{yx} = 2x + 3y^2, \quad Z_{yy} = 6yx.$$

Activity 5.3 Find the second-order derivatives of the function $f(x, y) = x^3y - \dfrac{x}{y}$. (This was the function of the preceding activity.)

Activity 5.4 Find all the partial derivatives and second-order partial derivatives of the function $f(x, y) = x^{3/4}y^{1/4}$.

5.4 The chain rule

Sometimes a function of one variable is defined with reference to a function of two variables. For instance, suppose that the production level q of a firm depends on capital k and labour l through the function $q(k, l)$. Suppose also that both k and l change over a period of time in some known way, so that we have formulas for $k(t)$ and $l(t)$, where t is a parameter measuring time. For example, we might have

$$k(t) = 3 + 2t, \quad l(t) = 10 - 0.2t,$$

which means that k increases linearly while l decreases linearly, as functions of time. If we know the production function q in terms of k and l, then we can also work out the level of production in terms of t, and so we can see how the production level will change as a result of changing t. For example, if q is the function $q(k, l) = kl$, and k and l change as above, we get the formula

$$kl = (3 + 2t)(10 - 0.2t) = 30 + 19.4t - 0.4t^2,$$

for the output in terms of t.

More generally, suppose we are given a function f of two variables (x, y), both of which are themselves functions of t. We can think of this situation as defining a *composite function* $F(t) = f(x(t), y(t))$. In the case of a single variable we have a rule, the 'composite function rule', or chain rule, which enables us to work out the derivative of a composite function. There is a similar rule here, (also) known as the *chain rule*:

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Sometimes, in this context, we call dF/dt the *total derivative* of F with respect to t (to distinguish it from the partial derivatives of F with respect to x and y).

Example 5.3 Suppose, as above, $f(x, y) = xy$, $x(t) = 3 + 2t$ and $y(t) = 10 - 0.2t$. Then, using the chain rule,

$$\frac{dF}{dt} = y \times 2 + x \times (-0.2) = 2(10 - 0.2t) + (-0.2)(3 + 2t) = 19.4 - 0.8t.$$

We can check the result explicitly, because we know (from above) that $F(t) = 30 + 19.4t - 0.4t^2$ and hence $dF/dt = 19.4 - 0.8t$, which is of course the same answer.

Activity 5.5 Suppose that $f(x, y) = x^2y$ and that $x(t) = 2 + 3t$ and $y(t) = t^2 + 1$. If $F(t) = f(x(t), y(t))$, find the derivative dF/dt.

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5.5 Implicit partial differentiation

An equation $g(x, y) = c$ can, in some cases, be solved to give 'y as a function of x'. For example, if $g(x, y)$ is $x^2 - y$ then the equation $g(x, y) = 0$ is

$$x^2 - y = 0 \quad \text{which gives} \quad y = x^2.$$

In general, we say that an equation $g(x, y) = 0$ defines y *implicitly* as a function of x if there is a function $y(x)$ which satisfies the equation for a range of values of x.[3] It is often difficult or impossible to solve the equation $g(x, y) = c$ and find a formula for $y(x)$. But, we can often still find the derivative dy/dx, even if we don't have an explicit expression for y in terms of x. In fact, dy/dx can be found simply in terms of the partial derivatives of g, by the formula

$$\frac{dy}{dx} = -\frac{\partial g/\partial x}{\partial g/\partial y}.$$

Be careful not to forget the minus sign!

Example 5.4 Suppose the quantity y is defined as a function of x through the equation

$$x^2y^3 - 6x^3y^2 + 2xy = 1.$$

Let's find a general expression for the derivative dy/dx. The equation defining y implicitly as a function of x is of the form $g(x, y) = 1$ where $g(x, y) = x^2y^3 - 6x^3y^2 + 2xy$. According to the formula given above,

$$\frac{dy}{dx} = -\frac{\partial g/\partial x}{\partial g/\partial y}.$$

Now,

$$\frac{\partial g}{\partial x} = 2xy^3 - 18x^2y^2 + 2y \quad \text{and} \quad \frac{\partial g}{\partial y} = 3x^2y^2 - 12x^3y + 2x,$$

[3] See Anthony and Biggs (1996) Sections 12.1 and 12.2.

so

$$\frac{dy}{dx} = \frac{18x^2y^2 - 2xy^3 - 2y}{3x^2y^2 - 12x^3y + 2x}.$$

Suppose we wanted to calculate this derivative when $x = 1/2$. Clearly we first need to find the corresponding value of y. Putting $x = 1/2$ into the defining equation $x^2y^3 - 6x^3y^2 + 2xy = 1$, we obtain

$$\frac{1}{4}y^3 - \frac{3}{4}y^2 + y = 1,$$

or

$$y^3 - 3y^2 + 4y - 4 = 0.$$

This equation factorises as

$$(y - 2)(y^2 - y + 2) = 0.$$

(Check this!) The quadratic $y^2 - y + 2$ has negative discriminant and so has no zeroes. It follows that when $x = 1/2$, $y = 2$. Substituting these values into the expression for dy/dx, we see that the derivative when $x = 1/2$ is 6.

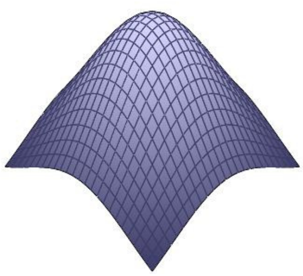
The theory can be extended. Suppose that $g(x, y, z) = c$ defines z implicitly as a function of x and y. Then

$$\frac{\partial z}{\partial x} = -\frac{\partial g/\partial x}{\partial g/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial g/\partial y}{\partial g/\partial z}.$$

5.6 Optimisation

We now study the maximisation and minimisation of a function of two variables.[4] As we have seen, we can think of the equation $z = f(x, y)$ as the equation of a surface in three dimensions. Earlier in this chapter we plotted the function $f(x, y) = 2x^2 + 2y^2$. We saw that the resulting surface is bowl-shaped and it has a lowest point at $(0, 0, 0)$. This is an example of a *local minimum*.

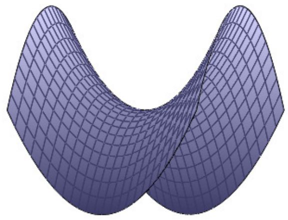
Here is another example. The function $f(x, y) = e^{-(x^2+y^2)}$ has a surface of the following shape:



[4]See Anthony and Biggs (1996) Chapter 13.

It has a local maximum when $x = y = 0$. (The maximum is at the top of the hill on the surface.)

The local maxima and minima of a function $f(x, y)$ occur at points where the partial derivatives $\partial f/\partial x, \partial f/\partial y$ are both equal to 0. Such points are called *critical points* or *stationary points*. A critical point which is neither a local maximum nor a local minimum is a *saddle point*. There are various types of saddle point. For example, consider the function $f(x, y) = x^2 - y^2$. The surface described by this is indeed 'saddle-shaped', with a saddle point at $(0, 0)$, as the following diagram shows.



We do not analyse saddle points in detail in this course, but in the applications of optimisation, it is important to be able to be sure that a critical point is a maximum or minimum and not a saddle point.

Having determined the critical points, one then uses the following test to determine whether a critical point is a local maximum, a local minimum or a saddle point. (In other words, we determine its nature.) Note that when applying this test, all the derivatives are evaluated at the critical point.

Suppose that (a, b) is a critical point of f.

- If $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$ and $\dfrac{\partial^2 f}{\partial x^2} < 0$, it is a maximum.
- If $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$ and $\dfrac{\partial^2 f}{\partial x^2} > 0$, it is a minimum.
- If $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 < 0$, it is a saddle point.

This is much more complicated than the corresponding one-variable test, in which we have a maximum if $d^2f/dx^2 < 0$ at the critical point, and a minimum if $d^2f/dx^2 > 0$. It is not enough just to check the sign of $\partial^2 f/\partial x^2$ or $\partial^2 f/\partial y^2$ (or both): we also need to check that

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0,$$

which you will sometimes see written as

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2,$$

You should note that when

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0,$$

this test fails to classify the nature of the critical point. In this case, some other technique must be used. For example, if $f(x, y) = x^3 - y^3$ then f has a critical point at $(0, 0)$, but the test fails to classify it. (In fact, however, it can be seen that $(0, 0)$ is a saddle point, for $f(x, 0) = x^3$ and this takes both negative and positive values in any small region around the point $(0, 0)$. Since $f(0, 0) = 0$, this means that $(0, 0)$ is neither a maximum nor a minimum.)

Example 5.5 Consider

$$f(x, y) = 160x - 3x^2 - 2xy - 2y^2 + 120y - 18.$$

Let us find the critical points of $f(x, y)$ and determine their nature. Now,

$$\frac{\partial f}{\partial x} = 160 - 6x - 2y \quad \text{and} \quad \frac{\partial f}{\partial y} = -2x - 4y + 120.$$

At a critical point, both of these must be 0. So we solve

$$6x + 2y = 160 \quad \text{and} \quad 2x + 4y = 120,$$

obtaining $x = y = 20$. So there is precisely one critical point: the point $(20, 20)$. To determine its nature, consider the second-order partial derivatives. We have

$$\frac{\partial^2 f}{\partial x^2} = -6, \quad \frac{\partial^2 f}{\partial x \partial y} = -2 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = -4.$$

(Here, these values are constants. Were they functions of x, y, we would now substitute $x = y = 20$ to obtain the values of the second-order partial derivatives at the critical point.) So we have

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} < 0.$$

Therefore the point $(20, 20)$ gives a maximum of $f(x, y)$, and this maximum value is $f(20, 20) = 2782$.

Activity 5.6 Show that the function

$$f(x, y) = 6 + 4x - 3x^2 + 4y + 2xy - 3y^2,$$

has one critical point, and classify the critical point.

Activity 5.7 Let us consider three of the functions we used as examples earlier. Find and classify the nature of the critical points of the following functions:

- $f(x, y) = 2x^2 + 2y^2$

- $g(x, y) = e^{-(x^2+y^2)}$
- $h(x, y) = x^2 - y^2$.

Here is a more difficult example.

Example 5.6 Find the critical points of the function

$$f(x, y) = x^4 + 2x^2y + 2y^2 + 2y,$$

and determine, for each, whether it is a local maximum, a local minimum, or a saddle point.

The partial derivatives are

$$f_x = 4x^3 + 4xy \quad \text{and} \quad f_y = 2x^2 + 4y + 2.$$

We solve $f_x = 0$ and $f_y = 0$. Now, $f_x = 0$ means $x(x^2 + y) = 0$, so $x = 0$ or $y = -x^2$. Suppose $x = 0$. Then, from $f_y = 0$ we have $4y + 2 = 0$, so $y = -1/2$. Now suppose $y = -x^2$. From $f_y = 0$ we have $2x^2 - 4x^2 + 2 = 0$, which means $x = \pm 1$. So the critical points are $(0, -1/2)$, $(1, -1)$ and $(-1, -1)$. The second derivatives are

$$f_{xx} = 12x^2 + 4y, \quad f_{xy} = 4x \quad \text{and} \quad f_{yy} = 4.$$

At $(0, -1/2)$,$f_{xx}f_{yy} - f_{xy}^2 < 0$, so this is a saddle point.

At $(1, -1)$, $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ so this is a local minimum.

At $(-1, -1)$, $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ so this is a local minimum.

It is easy to make mistakes in such examples and not find all of the critical points. For instance, we might note that the fact that $f_x = 0$ means $4x^3 = -4xy$ and hence, cancelling x, $x^2 = -y$. But you have to be careful: we can only cancel x if $x \neq 0$. So the possibility $x = 0$ has to also be considered. This is why we argue, correctly, that $f_x = 0$ means $x(x^2 + y) = 0$, so we have the two possibilities: $x = 0$ or $y = -x^2$.

Note that in this example we have used the notation f_{xx} and so on. This is often easier and quicker to write than the $\partial^2 f/\partial x^2$ notations.

5.7 Applications of optimisation

There are very many problems in management, economics and other areas which concern the optimisation of functions of several variables, as the following examples illustrate.

Example 5.7 A data processing company employs both senior and junior programmers. A particular large project will cost

$$C(x, y) = 2000 + 2x^3 - 12xy + y^2,$$

dollars, where x and y represent the number of junior and senior programmers used respectively. How many employees of each kind should be assigned to the project in order to minimise its cost? What is this minimum cost?

To minimise the cost, we set $C_x = 0$ and $C_y = 0$, obtaining $6x^2 - 12y = 0$ (or $x^2 - 2y = 0$) and $-12x + 2y = 0$. From the second equation, we have $y = 6x$. Substituting this into the equation $x^2 - 2y = 0$, we get $x^2 - 2(6x) = x(x - 12) = 0$, so $x = 12$ (or 0) and $y = 72$ (or 0). Ignoring the $(0, 0)$ case for the moment and checking the nature of the critical point $x = 12, y = 72$ we find that

$$\frac{\partial^2 C}{\partial x^2} = 12x = 144, \quad \frac{\partial^2 C}{\partial x \partial y} = -12 \quad \text{and} \quad \frac{\partial^2 C}{\partial y^2} = 2.$$

Thus, at the point $(12, 72)$,

$$\frac{\partial^2 C}{\partial x^2}\frac{\partial^2 C}{\partial y^2} - \left(\frac{\partial^2 C}{\partial x \partial y}\right)^2 = (144)(2) - 144^2 > 0 \quad \text{and} \quad \frac{\partial^2 C}{\partial x^2} > 0.$$

Hence we do have a minimum at $x = 12, y = 72$. When $x = 12$ and $y = 72$, the cost is

$$C = 2000 + 3456 - 10368 + 5184 = 272.$$

(It is clear that $(0, 0)$ is not the required solution, since there the cost is 2000, which is larger than this value.)

We now describe the problem of maximising profit for a firm making two products, X and Y. Generally, if p_X and p_Y are the selling prices of one unit of X and one unit of Y, then the total revenue obtained by producing amounts x and y is

$$TR(x, y) = xp_X + yp_Y.$$

There are a number of ways in which the prices p_X and p_Y may be related to the quantities x and y: they could be fixed constants, for instance, or both could depend on both of x and y (which would be the case if the goods were related, for example if they were CDs and cassettes).[5] The *joint total cost function* $TC(x, y)$ will tell us how much it costs the manufacturer to produce x units of X and y of Y. Then, the profit function is

$$\Pi(x, y) = TR(x, y) - TC(x, y) = xp_X + yp_Y - TC(x, y),$$

and we maximise this function of x and y using the techniques described above.[6]

Example 5.8 Suppose that a firm is the only firm producing X and Y (in other words, it has a monopoly on the goods) and that the demand for X is given by

$$x = 2 - 2p_X + p_Y,$$

and the demand for Y is given by

$$y = 13 + p_X - 2p_Y.$$

[5] See Anthony and Biggs (1996) Chapter 13, for a discussion.
[6] See Anthony and Biggs (1996) Chapter 13, for many examples.

(Note that if the price of X is fixed and the price of Y is increased, then the demand for X will rise and the demand for Y will fall. This is the behaviour one might expect if X and Y are two different types of chocolate bar, for instance.) Suppose also that the joint total cost function is $TC(x, y) = 5 + x^2 - xy + y^2$. We may rearrange the equations to find expressions for p_X and p_Y. Multiplying the first equation by 2 and adding it to the second, we obtain

$$2x + y = 2(2 - 2p_X + p_Y) + 13 + p_X - 2p_Y = 17 - 4p_X + p_X = 17 - 3p_X,$$

from which we get p_X as a function of x and y:

$$p_X(x, y) = (17 - 2x - y)/3.$$

Using this expression for p_X, together with the first equation, we can obtain a similar expression for p_Y:

$$p_Y = x - 2 + 2p_X = x - 2 + \frac{2}{3}(17 - 2x - y) = \frac{1}{3}(28 - x - 2y).$$

The profit function in this case is

$$\begin{aligned}\Pi(x, y) &= xp_X + yp_Y - TC(x, y) \\ &= \frac{x}{3}(17 - 2x - y) + \frac{y}{3}(28 - x - 2y) - (5 + x^2 - xy + y^2) \\ &= -5 + \frac{17}{3}x + \frac{28}{3}y - \frac{5}{3}x^2 - \frac{5}{3}y^2 + \frac{1}{3}xy,\end{aligned}$$

and we would now maximise this profit in the manner described above.

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Activity 5.8 Finish the problem started in this example. That is, find the values of x and y that maximise the profit function

$$\Pi(x, y) = -5 + \frac{17}{3}x + \frac{28}{3}y - \frac{5}{3}x^2 - \frac{5}{3}y^2 + \frac{1}{3}xy.$$

5.8 Constrained optimisation

Suppose that $f(x, y)$ has to be minimised or maximised *subject to the constraint* $g(x, y) = 0$. This means we want to find the maximum (or minimum) value of the function f at points (x, y) which satisfy the condition $g(x, y) = 0$. Then we may use the method of *Lagrange multipliers*.[7] To find the optimal (maximal or minimal) points of $f(x, y)$ subject to $g(x, y) = 0$, we first find the critical points of the three-variable function

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y).$$

[7] See Anthony and Biggs (1996) Chapters 21 and 22.

The function L is known as the *Lagrangean* (sometimes spelt Lagrangian) and λ is known as the *Lagrange multiplier*. (Some texts use $f + \lambda g$ rather than $f - \lambda g$, but there are good reasons to use $f - \lambda g$, and this is the approach we recommend.) In other words, we find the points at which the *first-order conditions*

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \lambda} = 0,$$

are satisfied. Then the theory of Lagrange multipliers asserts that the required optimal points of f, subject to the constraint, are to be found among these critical points.

Example 5.9 Consider the function f (mentioned earlier)

$$f(x, y) = 160x - 3x^2 - 2xy - 2y^2 + 120y - 18.$$

Let us find the maximum value of f subject to the constraint

$$x + y = 34.$$

We write this constraint as $g(x, y) = x + y - 34 = 0$. Consider then the Lagrangean

$$\begin{aligned} L(x, y, \lambda) &= f(x, y) - \lambda g(x, y) \\ &= 160x - 3x^2 - 2xy - 2y^2 + 120y - 18 - \lambda(x + y - 34). \end{aligned}$$

We have

$$\begin{aligned} \frac{\partial L}{\partial x} &= 160 - 6x - 2y - \lambda, \\ \frac{\partial L}{\partial y} &= -2x - 4y + 120 - \lambda, \\ \frac{\partial L}{\partial \lambda} &= -(x + y - 34). \end{aligned}$$

The point (x, y, λ) is a critical point of L if and only if

$$\begin{aligned} 160 - 6x - 2y - \lambda &= 0, \\ -2x - 4y + 120 - \lambda &= 0, \\ x + y - 34 &= 0. \end{aligned}$$

(Notice that $\partial L/\partial\lambda = 0$ recovers the constraint $g(x, y) = 0$.) To solve these, we adopt a strategy that very often works: we eliminate λ from the first two equations, determining a relationship between x and y which we then substitute into the third equation. Explicitly, we have from the first equation that $\lambda = 160 - 6x - 2y$, and from the second equation we have $\lambda = -2x - 4y + 120$. These two expressions for λ must be equal, so $160 - 6x - 2y = -2x - 4y + 120$, so that $y = 2x - 20$. Then the third equation becomes $x + (2x - 20) - 34 = 0$, or $3x = 54$. Hence $x = 18$ and $y = 2x - 20 = 16$. So we get a constrained maximum value of f of $f(18, 16) = 2722$. Earlier, we saw that if no constraint is imposed, then the maximum is at the point $(20, 20)$. But that point fails to satisfy the constraint in this problem.

Activity 5.9 Use the Lagrange multiplier method to find the values of x, y which minimise $x^2 + y^2$ subject to the constraint $x + y = 1$.

5.9 Applications of constrained optimisation

Constrained optimisation is very useful in management, economics and finance. Two standard types of problem in economics are utility maximisation subject to a budget constraint, and problems concerning the output of a firm and its capital/labour costs.

Suppose a consumer likes to consume two goods, X and Y. A utility function $u(x, y)$ is a way of deciding between alternative *bundles* (that is, combinations) of the two goods. For example, if $u(21, 5) > u(20, 7)$, then the consumer would prefer to have the bundle consisting of 21 of X and 5 of Y rather than that comprising 20 of X and 7 of Y. (Usually, this is all that utility functions tell us. They enable us to rank bundles; that is, to determine whether one bundle is preferable to another. In general, we should not, for example, infer from a fact such as $u(21, 5) = 2u(20, 7)$ that the bundle $(21, 5)$ is 'twice as good' as $(20, 7)$.) The consumer's basic problem is to find the 'best' (that is, highest utility-giving) bundle that he or she can afford. Supposing they have a budget M for X and Y and that the prices of X and Y are p_X and p_Y, then the consumer can only afford bundles (x, y) satisfying $xp_X + yp_Y \leq M$. We assume that the quantities can be bought in fractional amounts, so that we don't need to consider only values of x and y that are whole numbers. It's clear that if the consumer really regards X and Y as 'goods' (rather than 'bads'!) then he or she should spend all of their budget. So the consumer wants to maximise $u(x, y)$ subject to the budget constraint $xp_X + yp_Y = M$. This is now a standard constrained optimisation problem.

Example 5.10 Suppose there are two goods with prices $p_X = 2$ and $p_Y = 5$, the income is $M = 40$, and the utility function is

$$u(x, y) = x^{1/3}y^{1/2}.$$

The budget constraint is

$$2x + 5y = 40,$$

and the Lagrangean is

$$L(x, y, \lambda) = x^{1/3}y^{1/2} - \lambda(2x + 5y - 40).$$

We have to solve the three equations

$$\frac{1}{3}x^{-2/3}y^{1/2} - 2\lambda = 0, \; \frac{1}{2}x^{1/3}y^{-1/2} - 5\lambda = 0 \text{ and } 2x + 5y = 40,$$

for the three unknowns x, y, λ. We employ our standard strategy of using the first two equations to eliminate λ and find a relationship between x and y. From the first two equations we get

$$\lambda = \frac{1}{6}x^{-2/3}y^{1/2} \quad \text{and} \quad \lambda = \frac{1}{10}x^{1/3}y^{-1/2}.$$

Equating these two different expressions for λ, we clearly have, in particular, that

$$\frac{1}{6}x^{-2/3}y^{1/2} = \frac{1}{10}x^{1/3}y^{-1/2}.$$

This does not look particularly simple, but it easily reduces to the simple equation $y = 3x/5$. Substituting this in the budget constraint gives

$$2x + 5\left(\frac{3}{5}x\right) = 40, \quad \text{that is} \quad 5x = 40.$$

From this we get the optimum values

$$x^* = 8 \quad \text{and} \quad y^* = 24/5.$$

The corresponding value of λ is $\lambda^* = \sqrt{1/120}$. We don't really need this to answer the problem, but as we shall see soon, it can be useful.

5

Now let us give an example of the type of constrained optimisation problem encountered when one considers a firm.

Example 5.11 A firm's weekly output is given by the production function $q(k,l) = k^{3/4}l^{1/4}$, and the unit costs for capital and labour are $v = 1$ and $w = 5$ per week, so that the total cost incurred in using k units of capital and l of labour is $k + 5l$. Find the minimum cost of producing a weekly output of 5000 and the corresponding values of k and l.

The problem to be solved is the constrained optimisation problem

$$\text{minimise} \quad k + 5l \quad \text{subject to} \quad k^{3/4}l^{1/4} = 5000.$$

The Lagrangean for the problem is

$$L(k,l,\lambda) = k + 5l - \lambda(k^{3/4}l^{1/4} - 5000),$$

and the optimal values of k and l are the solutions to the three equations

$$\frac{\partial L}{\partial k} = 1 - \frac{3}{4}\lambda k^{-1/4}l^{1/4} = 0,$$
$$\frac{\partial L}{\partial l} = 5 - \frac{1}{4}\lambda k^{3/4}l^{-3/4} = 0,$$
$$\frac{\partial L}{\partial \lambda} = k^{3/4}l^{1/4} - 5000 = 0.$$

The first two equations imply that

$$\lambda = \frac{4}{3}k^{1/4}l^{-1/4} = 20k^{-3/4}l^{3/4},$$

which simplifies to $l = k/15$. Substituting this information into the third equation gives

$$k^{3/4}\left(\frac{1}{15}\right)^{1/4}k^{1/4} = 5000 \quad \text{so that} \quad k = 5000(15)^{1/4}.$$

Then, $l = k/15 = 5000(15)^{-3/4}$ and the minimum cost is

$$k + 5l = 5000(15)^{1/4} + 5(5000)(15)^{-3/4} = 100000(15)^{-3/4},$$

which is approximately 13120.

Here is another type of problem concerning a firm, which this time involves not capital and labour costs, but raw material costs.

Example 5.12 A firm manufactures a good from two raw materials, X and Y. The quantity of its good which is produced from x units of X and y of Y is given by $Q(x, y) = x^{1/4}y^{3/4}$. If the firm spends no more than \$1280 each week on the raw materials, what is its maximum possible weekly production, given that one unit of X costs \$16 and one unit of Y costs \$1?

The problem here is to maximise $Q(x, y)$ subject to the constraint that the amount spent on raw materials is at most \$1280. Clearly, the optimal values of x and y will satisfy the constraint $16x + y = 1280$. The Lagrangean is

$$L(x, y, \lambda) = x^{1/4}y^{3/4} - \lambda(16x + y - 1280),$$

and the equations to solve are

$$\frac{\partial L}{\partial x} = \frac{1}{4}x^{-3/4}y^{3/4} - 16\lambda = 0,$$

$$\frac{\partial L}{\partial y} = \frac{3}{4}x^{1/4}y^{-1/4} - \lambda = 0,$$

$$\frac{\partial L}{\partial \lambda} = 1280 - 16x - y = 0.$$

As in the previous example, we eliminate λ from the first two equations to obtain a relationship between the key variables x and y. From the first equation,

$$\lambda = \frac{1}{64}x^{-3/4}y^{3/4},$$

and from the second,

$$\lambda = \frac{3}{4}x^{1/4}y^{-1/4}.$$

We therefore have

$$\frac{1}{64}x^{-3/4}y^{3/4} = \frac{3}{4}x^{1/4}y^{-1/4},$$

which, on moving all the y terms to the left and the x terms to the right (by cross-multiplication), simplifies to $y = 48x$. Then, the third equation implies $64x = 1280$, so that $x = 20$ and $y = 48x = 960$. The maximum quantity is therefore $Q(20, 960) = (20)^{1/4}(960)^{3/4}$.

5.10 The meaning of the Lagrange multiplier

The Lagrange multiplier has a useful interpretation in many applications. Formally, let us suppose that the constrained optimisation problem is to maximise a function $f(x, y)$ subject to a constraint of the form $h(x, y) = a$ where a is a constant. Then, provided f and h are 'well-behaved', the value of the Lagrange multiplier is the rate of change of the maximum value of f with respect to a. Explicitly, if $x^*(a)$ and $y^*(a)$ are the optimising values of f when the constraint is $h(x, y) = a$, then the maximum value of f is $f^*(a) = f(x^*(a), y^*(a))$, and it turns out that the value $\lambda^*(a)$ of the Lagrange multiplier satisfies:

$$\lambda^*(a) = \frac{\partial f^*}{\partial a}.$$

An interesting example of this occurs in the problem of the consumer maximising his or her utility.[8] Here, the problem is to maximise a *utility function*, $u(x_1, x_2)$, subject to a budget constraint $p_1x_1 + p_2x_2 = M$. The utility of the optimum bundle $\mathbf{x}^* = (x_1^*, x_2^*)$ is $u(x_1^*, x_2^*)$. Since each x_i^* is a function of the prices p_1, p_2 and the income M, so also is the optimal utility. Using the notation $x_i^* = q_i(p_1, p_2, M)$, we have

$$u(\mathbf{x}^*) = u(q_1(p_1, p_2, M), q_2(p_1, p_2, M)) = V(p_1, p_2, M),$$

say. The function V is called the *indirect utility function.* It specifies the individual consumer's optimal utility when the prices are p_1, p_2 and the income is M. The partial derivative $\partial V/\partial M$ is the *marginal utility of income.* It tells us what change in optimal utility will result from a small change in income, given that prices remain constant. The value, λ^*, of the Lagrange multiplier satisfies:

$$\lambda^* = \frac{\partial V}{\partial M}.$$

Example 5.13 We'll now work through an example in detail to show that the above theory works. Suppose a consumer has utility function $u(x_1, x_2) = x_1^{1/3}x_2^{1/2}$, and budget constraint $p_1x_1 + p_2x_2 = M$. Then the Lagrangean is

$$L(x_1, x_2, \lambda) = x_1^{1/3}x_2^{1/2} - \lambda(p_1x_1 + p_2x_2 - M).$$

The equations we need to solve are

$$\frac{\partial L}{\partial x_1} = \frac{1}{3}x_1^{-2/3}x_2^{1/2} - \lambda p_1 = 0,$$

$$\frac{\partial L}{\partial x_2} = \frac{1}{2}x_1^{1/3}x_2^{-1/2} - \lambda p_2 = 0,$$

$$\frac{\partial L}{\partial \lambda} = -(p_1x_1 + p_2x_2 - M) = 0.$$

From the first two equations, we have

$$\lambda = \frac{1}{3p_1}x_1^{-2/3}x_2^{1/2} = \frac{1}{2p_2}x_1^{1/3}x_2^{-1/2},$$

[8]See Anthony and Biggs (1996) Sections 22.4 and 22.5.

which gives

$$x_2 = \frac{3p_1}{2p_2} x_1.$$

By the third equation, $p_1 x_1 + p_2 x_2 = M$, so

$$p_1 x_1 + p_2 \left(\frac{3p_1}{2p_2} x_1 \right) = M,$$

and hence

$$\frac{5p_1}{2} x_1 = M \quad \text{so that} \quad x_1 = \frac{2M}{5p_1}.$$

Thus, the optimising values of x_1, x_2 are

$$x_1^* = q_1(p_1, p_2, M) = \frac{2M}{5p_1} \quad \text{and} \quad x_2^* = q_2(p_1, p_2, M) = \frac{3p_1}{2p_2} x_1^* = \frac{3M}{5p_2}.$$

The indirect utility function is

$$V(p_1, p_2, M) = (x_1^*)^{1/3} (x_2^*)^{1/2} = \left(\frac{2M}{5p_1} \right)^{1/3} \left(\frac{3M}{5p_2} \right)^{1/2} = \frac{2^{1/3} 3^{1/2}}{5^{5/6}} \frac{M^{5/6}}{p_1^{1/3} p_2^{1/2}}.$$

It follows that

$$\frac{\partial V}{\partial M} = \frac{5}{6} \frac{2^{1/3} 3^{1/2}}{5^{5/6}} \frac{M^{-1/6}}{p_1^{1/3} p_2^{1/2}}.$$

According to the theory presented above, this should equal the Lagrange multiplier. Now, by the equations arising from the first-order conditions, we have

$$\lambda^* = \frac{1}{3p_1} (x_1^*)^{-2/3} (x_2^*)^{1/2},$$

and you can verify for yourself that this is exactly the same as $\partial V / \partial M$.

How would we use this theory? Well, having worked through a particular constrained optimisation problem, say a utility maximisation problem, this interpretation of the Lagrange multiplier can help us estimate the change in the maximum utility if a small change in income is made. For example, suppose there are two goods with prices $p_1 = 2, p_2 = 5$, and the utility function is $x_1^{1/3} x_2^{1/2}$ (as above). When the income is $M = 40$, working through a Lagrangean calculation (as we did earlier in this chapter), we would find that the maximum utility is $u(8, 24/5)$, which is about 4.38, and that the value of the Lagrange multiplier is $\sqrt{1/120}$. Now suppose the consumer's income rises to 42. What would the new maximum utility be? We could work through a complete Lagrangean calculation again, but there is no need if we are happy with an approximate answer. A good approximate answer is obtained by using the fact that $\lambda^* = \sqrt{1/120}$. Since

$$\lambda^* = \frac{\partial V}{\partial M},$$

and M is increased from 40 to 42, the change in M is $\Delta M = 2$, and the change in maximum utility is given approximately as:

$$\Delta V \simeq \lambda^* \Delta M = \sqrt{1/120} \times 2,$$

which is approximately 0.18. So when the income increases from 40 to 42 the maximum utility increases approximately from 4.38 to 4.56.

Learning outcomes

At the end of this chapter and the relevant reading, you should be able to:

- explain the concept of a function of many variables
- calculate partial derivatives
- use the chain rule for partial differentiation to find total derivatives
- find and classify stationary/critical points
- solve optimisation and constrained optimisation problems
- be able to interpret the meaning of the Lagrange multiplier, λ

You do not need to know how to verify the nature of a critical point obtained using the Lagrange multiplier method. (Thus, if, for example, a question asks you to maximise subject to a constraint and there turns out to be just one point satisfying the Lagrangean first-order conditions, then you may assume that the point is indeed a maximum.)

Sample examination/practice questions

Question 5.1

Suppose that a firm has production function $q(k,l) = Ak^{\alpha}l^{1-\alpha}$ where $A > 0$ and $0 < \alpha < 1$. Show that the marginal product of labour $\partial q/\partial l$ is positive, and that it is a decreasing function of l when k is fixed.

Question 5.2

A monopoly manufactures two goods, X and Y, with demand functions

$$x = 12 - p_X \quad \text{and} \quad y = 18 - p_Y.$$

The firm's cost function is $C(x,y) = x^2 + y^2 + 2xy$. Find the maximum profit achievable, and the quantities produced of each of X and Y in order to achieve this.

Question 5.3

A firm manufactures two products, X and Y, and sells these in related markets. Suppose that the firm is the only producer of X and Y and that the inverse demand functions for X and Y are

$$p_X = 13 - 2x - y \quad \text{and} \quad p_Y = 13 - x - 2y.$$

Determine the production levels that maximise profit, given that the cost function is $C(x,y) = x + y$.

Question 5.4

Use the technique of Lagrange multipliers to find the values of x and y which maximise the function $3\sqrt{x} + 4\sqrt{y}$, subject to the constraint $x + y = 100$.

Question 5.5

A firm manufactures a good from two raw materials, X and Y. The quantity of the good which is produced from x units of X and y of Y is given by

$$Q(x, y) = (\sqrt{x} + 2\sqrt{y})^2.$$

Each unit of X costs the firm \$2 and each unit of Y costs \$1. Find the minimum cost of producing 100 units of the manufactured good.

Question 5.6

A consumer buys two goods, X and Y. The price of one unit of X is \$1 and the price of one unit of Y is \$16. The consumer's utility function, which describes how she values x units of X and y units of Y, is given by

$$u(x, y) = x^{3/4}y^{1/4}.$$

She has a budget of \$1280 in total each year to spend on X and Y. Using the method of Lagrange multipliers, find the values of x, y which will maximise the consumer's utility function $u(x, y)$ subject to the constraint on her budget. Use the value of the Lagrange multiplier to estimate the increase in the maximum obtainable utility if the consumer's budget for the goods rises to \$1282.

Question 5.7

A firm has production function $q(k, l) = 50k^{2/3}l^{1/3}$, and unit capital and labour costs of 6 and 4, respectively, so that the total cost incurred when using k units of capital and l of labour is $6k + 4l$. What is the maximum weekly output achievable if the firm spends no more than 1000 a week on capital and labour?

Question 5.8

A student has a part-time job in a restaurant. For this she is paid \$8 per hour. Her utility function for earning \$$I$ and spending S hours studying is

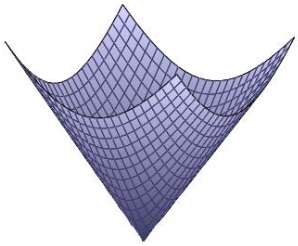
$$u(I, S) = I^{1/4}S^{3/4}.$$

(The utility function is a measure of the 'usefulness' or 'worth' to the student of a certain combination of money and study time.) The total amount of time she spends each week working in the restaurant and studying is 100 hours. How should she divide up her time in order to maximise her utility?

Answers to activities

Feedback to activity 5.1

Think about what would happen if we were to slice the surface parallel to the (x, y)-plane at a height $c > 0$ above the plane. Then we would have the set of points with z-coordinate c. But this means, since $z = \sqrt{x^2 + y^2}$, that all the points (x, y, c) in this cross-section would satisfy $\sqrt{x^2 + y^2} = c$, so $x^2 + y^2 = c^2$. This last equation is the equation of a circle, centred on the origin and of radius c. So slicing through the surface at $z = c$ and examining the shape of the section obtained, it is a circle of radius c. It follows that the surface is a *cone*. It would look something like the following.



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This is quite tricky. However, you do not need to be able to determine the shape of surfaces. I've included this activity simply to help you think about them geometrically.

Feedback to activity 5.2

We have $f(x, y) = x^3 y - xy^{-1}$, so

$$\frac{\partial f}{\partial x} = 3x^2 y - y^{-1} = 3x^2 y - \frac{1}{y},$$

$$\frac{\partial f}{\partial y} = x^3 - x\left(-\frac{1}{y^2}\right) = x^3 + \frac{x}{y^2}.$$

Feedback to activity 5.3

We already have the first-order derivatives. We calculate the second-order derivatives as follows. First,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(3x^2 y - y^{-1}\right) = 6xy,$$

and

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(x^3 + xy^{-2}\right) = -2xy^{-3} = -2\frac{x}{y^3}.$$

We can calculate the remaining second-order derivative in two ways: we can either differentiate $\partial f/\partial x$ with respect to y, or we can differentiate $\partial f/\partial y$ with respect to x. We need only do one of these, but let's just check they both give the same answer.

$$\frac{\partial}{\partial y}\left(3x^2 y - \frac{1}{y}\right) = 3x^2 + \frac{1}{y^2},$$

and

$$\frac{\partial}{\partial x}\left(x^3 + \frac{x}{y^2}\right) = 3x^2 + \frac{1}{y^2},$$

so (as expected) they are equal, and

$$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 + \frac{1}{y^2} = \frac{\partial^2 f}{\partial y \partial x}.$$

Feedback to activity 5.4

We find that

$$\frac{\partial f}{\partial x} = \frac{3}{4}x^{-1/4}y^{1/4} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{1}{4}x^{3/4}y^{-3/4}.$$

Then,

$$\frac{\partial^2 f}{\partial x^2} = -\frac{3}{16}x^{-5/4}y^{1/4}, \qquad \frac{\partial^2 f}{\partial y^2} = -\frac{3}{16}x^{3/4}y^{-7/4},$$

and

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{3}{16}x^{-1/4}y^{-3/4} = \frac{\partial^2 f}{\partial y \partial x}.$$

Feedback to activity 5.5

By the chain rule,

$$\begin{aligned}\frac{dF}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= 2xy(3) + x^2(2t) \\ &= 6(2+3t)(t^2+1) + 2t(2+3t)^2 \\ &= 36t^3 + 36t^2 + 26t + 12.\end{aligned}$$

Of course, one could also directly find an expression for F in terms of t and differentiate this (giving the same answer).

Feedback to activity 5.6

We have $f(x, y) = 6 + 4x - 3x^2 + 4y + 2xy - 3y^2$ and to find the critical point(s) we solve

$$\begin{aligned}\frac{\partial f}{\partial x} &= 4 - 6x + 2y = 0, \\ \frac{\partial f}{\partial y} &= 4 + 2x - 6y = 0.\end{aligned}$$

We therefore have to solve simultaneously the equations $6x - 2y = 4$ and $6y - 2x = 4$. The first says $y = 3x - 2$ which, using the second, means that $6(3x - 2) - 2x = 4$ or $16x = 16$. Therefore $x = 1$, and $y = 3x - 2 = 1$. So there is a single critical point, $(1, 1)$. To determine its nature we need the second-order derivatives. We have

$$\frac{\partial^2 f}{\partial x^2} = -6, \quad \frac{\partial^2 f}{\partial x \partial y} = 2 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = -6,$$

and it is clear that

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} < 0,$$

so the critical point is a local maximum.

Feedback to activity 5.7

To find the critical points of f we solve

$$\frac{\partial f}{\partial x} = 4x = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 4y = 0,$$

so there is just one critical point, $(0, 0)$. The second-order derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 4, \quad \frac{\partial^2 f}{\partial x \partial y} = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = 4,$$

so

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} > 0,$$

hence $(0, 0)$ is a minimum. (Actually, this is clear anyway, without using any fancy calculus: we know that $x^2 \geq 0$ and equals 0 only when $x = 0$, and similarly for y^2, so $f(x, y) \geq 0$ for all (x, y), and $f(x, y) = 0$ only when $(x, y) = (0, 0)$, so we see that this gives a minimum. Easy as this is, the point here is to demonstrate the technique, which will work when matters are less obvious.)

For g, we have

$$\frac{\partial g}{\partial x} = -2xe^{-(x^2+y^2)} = 0 \quad \text{and} \quad \frac{\partial g}{\partial y} = -2ye^{-(x^2+y^2)} = 0,$$

so (since the exponential term is always positive), there is just one critical point, $(0, 0)$. The second-order derivatives are

$$\frac{\partial^2 g}{\partial x^2} = -2e^{-(x^2+y^2)} + 4x^2e^{-(x^2+y^2)} = (4x^2 - 2)e^{-(x^2+y^2)},$$

and, similarly,

$$\frac{\partial^2 g}{\partial y^2} = (4y^2 - 2)e^{-(x^2+y^2)}.$$

Also,

$$\frac{\partial^2 g}{\partial x \partial y} = 4xye^{-(x^2+y^2)}.$$

We now need to evaluate these at the critical point, so we substitute $x = y = 0$, obtaining

$$\frac{\partial^2 g}{\partial x^2}(0, 0) = -2, \quad \frac{\partial^2 g}{\partial x \partial y}(0, 0) = 0 \quad \text{and} \quad \frac{\partial^2 g}{\partial y^2}(0, 0) = -2,$$

so

$$\frac{\partial^2 g}{\partial x^2}\frac{\partial^2 g}{\partial y^2} - \left(\frac{\partial^2 g}{\partial x \partial y}\right)^2 > 0 \quad \text{and} \quad \frac{\partial^2 g}{\partial x^2} < 0,$$

hence $(0, 0)$ is a maximum.

Finally, for the function $h(x, y) = x^2 - y^2$, the critical points are given by

$$\frac{\partial h}{\partial x} = 2x = 0 \quad \text{and} \quad \frac{\partial h}{\partial y} = -2y = 0,$$

so again there is a unique critical point, $(0, 0)$. The second derivatives are

$$\frac{\partial^2 h}{\partial x^2} = 2, \quad \frac{\partial^2 h}{\partial x \partial y} = 0 \quad \text{and} \quad \frac{\partial^2 h}{\partial y^2} = -2,$$

so

$$\frac{\partial^2 h}{\partial x^2}\frac{\partial^2 h}{\partial y^2} - \left(\frac{\partial^2 h}{\partial x \partial y}\right)^2 = -4 < 0,$$

and hence $(0, 0)$ is a saddle point in this case.

Feedback to activity 5.8

It's clear that the problem of maximising Π is the same as that of maximising $f(x, y) = 3\Pi(x, y)$, the advantage of working with f being that the constants involved are simpler. Now,

$$f(x, y) = -15 + 17x + 28y - 5x^2 - 5y^2 + xy.$$

We solve

$$\frac{\partial f}{\partial x} = 17 - 10x + y = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 28 - 10y + x = 0.$$

That is,

$$10x - y = 17 \quad \text{and} \quad -x + 10y = 28.$$

Multiplying the first equation by 10, $100x - 10y = 170$. Adding this to the second equation gives $99x = 198$, so $x = 2$, and $y = 10x - 17 = 3$. There is therefore one critical point, $(2, 3)$. We should check that this does indeed maximise profit. As usual, we use the second derivatives. The second derivatives of f are

$$\frac{\partial^2 f}{\partial x^2}(x, y) = -10, \quad \frac{\partial^2 f}{\partial x \partial y} = 1 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = -10.$$

So,

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} < 0,$$

and hence $(2, 3)$ is a maximum. Therefore the optimal production levels are $x = 2$ and $y = 3$.

Feedback to activity 5.9

The constraint can be written as $x + y - 1 = 0$ and so the Lagrangian is

$$L(x, y, \lambda) = x^2 + y^2 - \lambda(x + y - 1).$$

The first-order conditions are

$$\frac{\partial L}{\partial x} = 2x - \lambda = 0, \quad \frac{\partial L}{\partial y} = 2y - \lambda = 0 \quad \text{and} \quad \frac{\partial L}{\partial \lambda} = 1 - x - y = 0.$$

From the first two, we have $\lambda = 2x = 2y$, so $x = y$. The third equation then gives $x + x = 1$, so $x = 1/2$ and $y = 1/2$.

Answers to Sample examination/practice questions

Answer to question 5.1

We have

$$\frac{\partial q}{\partial l} = A(1 - \alpha)k^{\alpha}l^{-\alpha}.$$

This is clearly positive, since we are given that $A > 0$ and $1 - \alpha > 0$. Furthermore, as l increases then, for fixed k, $l^{-\alpha}$ decreases, since $\alpha > 0$. It follows that the marginal product of labour decreases with l.

There is another way of verifying that it decreases. The rate of change of $\partial q/\partial l$ with respect to l is its derivative, in other words, the *second* derivative $\partial^2 q/\partial l^2$. By the usual rules we get

$$\begin{aligned}\frac{\partial^2 q}{\partial l^2} &= \frac{\partial}{\partial l}\left(A(1-\alpha)k^\alpha l^{-\alpha}\right)\\ &= A(1-\alpha)(-\alpha)k^\alpha l^{-\alpha-1}\\ &= -A\alpha(1-\alpha)k^\alpha l^{-\alpha-1}.\end{aligned}$$

Because $A > 0$, $\alpha > 0$ and $1-\alpha > 0$, this is negative, from which it follows that the marginal product of labour is a decreasing function of l.

Answer to question 5.2

The profit function is

$$\Pi(x,y) = TR - TC = xp_X + yp_Y - (x^2+y^2+2xy).$$

Now, we want this to be written as a function of the variables x and y, so p_X and p_Y have to be rewritten in terms of x and y. Since $p_X = 12 - x$ and $p_Y = 18 - y$, we have

$$\begin{aligned}\Pi(x,y) &= x(12-x) + y(18-y) - (x^2+y^2+2xy)\\ &= 12x + 18y - 2x^2 - 2y^2 - 2xy.\end{aligned}$$

To find the critical points, we solve

$$\frac{\partial \Pi}{\partial x} = 12 - 4x - 2y = 0 \quad \text{and} \quad \frac{\partial \Pi}{\partial y} = 18 - 4y - 2x = 0.$$

That is,

$$2x + y = 6 \quad \text{and} \quad x + 2y = 9.$$

Multiplying the first equation by 2 and subtracting the second shows $3x = 3$, so $x = 1$. Corresponding to this, $y = 4$. There is therefore one critical point, $(1, 4)$. We should check that this does indeed maximise profit. As usual, we use the second derivatives. The second derivatives of Π are

$$\frac{\partial^2 \Pi}{\partial x^2}(x,y) = -4, \quad \frac{\partial^2 \Pi}{\partial x \partial y} = -2 \quad \text{and} \quad \frac{\partial^2 \Pi}{\partial y^2} = -4.$$

So,

$$\frac{\partial^2 \Pi}{\partial x^2}\frac{\partial^2 \Pi}{\partial y^2} - \left(\frac{\partial^2 \Pi}{\partial x \partial y}\right)^2 > 0 \quad \text{and} \quad \frac{\partial^2 \Pi}{\partial x^2} < 0,$$

hence $(1, 4)$ is a maximum. Therefore the optimal production levels are $x = 1$ and $y = 4$, and the maximum achievable profit is $\Pi(1, 4) = 42$.

Answer to question 5.3

The profit function is

$$\begin{aligned}\Pi(x,y) &= TR - TC\\ &= xp_X + yp_Y - (x+y)\\ &= x(13-2x-y) + y(13-x-2y) - (x+y)\\ &= 12x + 12y - 2x^2 - 2y^2 - 2xy.\end{aligned}$$

To find the critical points, we solve

$$\frac{\partial \Pi}{\partial x} = 12 - 4x - 2y = 0 \quad \text{and} \quad \frac{\partial \Pi}{\partial y} = 12 - 4y - 2x = 0.$$

That is,

$$2x + y = 6 \quad \text{and} \quad x + 2y = 6.$$

Multiplying the first equation by 2 and subtracting the second shows $3x = 6$, so $x = 2$. Corresponding to this, $y = 2$. There is therefore one critical point, $(2, 2)$. We should check that this does indeed maximise profit. As usual, we use the second derivatives. The second derivatives of Π are

$$\frac{\partial^2 \Pi}{\partial x^2}(x, y) = -4, \quad \frac{\partial^2 \Pi}{\partial x \partial y} = -2 \quad \text{and} \quad \frac{\partial^2 \Pi}{\partial y^2} = -4.$$

So,

$$\frac{\partial^2 \Pi}{\partial x^2}\frac{\partial^2 \Pi}{\partial y^2} - \left(\frac{\partial^2 \Pi}{\partial x \partial y}\right)^2 > 0 \quad \text{and} \quad \frac{\partial^2 \Pi}{\partial x^2} < 0,$$

hence $(2, 2)$ is a maximum. Therefore the optimal production levels are $x = 2$ and $y = 2$

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Answer to question 5.4

The function to be optimised is $3\sqrt{x} + 4\sqrt{y}$ and the constraint equation $g(x, y) = 0$ is $x + y - 100 = 0$. The Lagrangean is therefore

$$L(x, y, \lambda) = 3\sqrt{x} + 4\sqrt{y} - \lambda(x + y - 100).$$

We now solve

$$\frac{\partial L}{\partial x} = \frac{3}{2\sqrt{x}} - \lambda = 0,$$

$$\frac{\partial L}{\partial y} = \frac{2}{\sqrt{y}} - \lambda = 0,$$

$$\frac{\partial L}{\partial \lambda} = 100 - x - y = 0.$$

From the first two equations, we obtain two expressions for λ,

$$\lambda = \frac{3}{2\sqrt{x}} = \frac{2}{\sqrt{y}},$$

so $\sqrt{y} = 4\sqrt{x}/3$ and hence $y = 16x/9$. Now the third equation tells us

$$x + \frac{16x}{9} = 100,$$

or $25x/9 = 100$ and therefore the optimal values of x and y are $x = 36$ and $y = 16(36)/9 = 64$.

Answer to question 5.5

Be careful here about what is the constraint and what is the function to be optimised. Reading the question carefully, you will see that we have to minimise the cost. Now, the cost will be $2x + y$ since X costs \$2 and Y costs \$1 per unit. What's the constraint? We have to produce 100 units, so $Q(x, y) = 100$, which is $(\sqrt{x} + 2\sqrt{y})^2 = 100$. (At this point, you could notice that this is equivalent to the simpler constraint $\sqrt{x} + 2\sqrt{y} = 10$, but let's imagine we haven't been quite that clever and proceed without this observation.) So, the Lagrangean is

$$L(x, y, \lambda) = 2x + y - \lambda\left((\sqrt{x} + 2\sqrt{y})^2 - 100\right).$$

The first-order conditions are

$$\frac{\partial L}{\partial x} = 2 - \lambda\left(\frac{(\sqrt{x} + 2\sqrt{y})}{\sqrt{x}}\right) = 0,$$

$$\frac{\partial L}{\partial y} = 1 - \lambda\left(\frac{2(\sqrt{x} + 2\sqrt{y})}{\sqrt{y}}\right) = 0,$$

$$\frac{\partial L}{\partial \lambda} = 100 - (\sqrt{x} + 2\sqrt{y})^2 = 0.$$

This is more complicated that the previous examples, but we shall employ exactly the same technique, namely using the first two equations to eliminate λ and find a relationship between x and y. From the first two equations, we obtain two expressions for λ,

$$\lambda = \frac{2\sqrt{x}}{\sqrt{x} + 2\sqrt{y}} = \frac{\sqrt{y}}{2(\sqrt{x} + 2\sqrt{y})},$$

which means (cancelling the $\sqrt{x} + 2\sqrt{y}$ factor),

$$2\sqrt{x} = \frac{\sqrt{y}}{2},$$

so $\sqrt{y} = 4\sqrt{x}$ and hence $y = 16x$. Now the third equation tells us

$$100 = (\sqrt{x} + 2\sqrt{y})^2 = (\sqrt{x} + 2\sqrt{16x})^2 = (\sqrt{x} + 8\sqrt{x})^2 = 81x,$$

so $x = 100/81$ and $y = 16x = 1600/81$. The corresponding minimum cost is

$$2x + y = 2\left(\frac{100}{81}\right) + \left(\frac{1600}{81}\right) = \frac{1800}{81} = \frac{200}{9}.$$

Answer to question 5.6

The budget constraint is $x + 16y = 1280$ and so the Lagrangean is

$$L(x, y, \lambda) = x^{3/4}y^{1/4} - \lambda(x + 16y - 1280).$$

The first-order conditions are

$$\frac{\partial L}{\partial x} = \frac{3}{4}x^{-1/4}y^{1/4} - \lambda = 0,$$

$$\frac{\partial L}{\partial y} = \frac{1}{4}x^{3/4}y^{-3/4} - 16\lambda = 0,$$

$$\frac{\partial L}{\partial \lambda} = 1280 - x - 16y = 0.$$

From the first two equations, we obtain two expressions for λ,

$$\lambda = \frac{3}{4}x^{-1/4}y^{1/4} = \frac{1}{64}x^{3/4}y^{-3/4},$$

which means that $y = x/48$. Now the third equation gives

$$x + 16\left(\frac{x}{48}\right) = \frac{4}{3}x = 1280,$$

so $x = 960$ and $y = 960/48 = 20$.

Now, with the optimal values of x, y, the value of λ can be calculated using either of the two expressions we obtained above. From the first of these, we have

$$\lambda = \frac{3}{4}(960)^{-1/4}(20)^{1/4} = \frac{3}{4}\left(\frac{20}{960}\right)^{1/4} = \frac{3}{4(48)^{1/4}}.$$

If the income rises by \$2, then the maximum utility will increase by approximately 2λ, which is $3/2(48)^{1/4}$.

5

Answer to question 5.7

The problem is to maximise $q(k,l) = 50k^{2/3}l^{1/3}$ subject to spending no more than 1000 on capital and labour. Clearly the optimal strategy is to spend all of the 1000 available (since extra capital and extra labour both increase the output q), so the constraint equation is $6k + 4l = 1000$ and the Lagrangean is

$$L(k,l,\lambda) = 50k^{2/3}l^{1/3} - \lambda(6k + 4l - 1000).$$

We now solve

$$\frac{\partial L}{\partial k} = \frac{100}{3}k^{-1/3}l^{1/3} - 6\lambda = 0,$$
$$\frac{\partial L}{\partial l} = \frac{50}{3}k^{2/3}l^{-2/3} - 4\lambda = 0,$$
$$\frac{\partial L}{\partial \lambda} = 1000 - 6k - 4l = 0.$$

From the first two equations, we obtain two expressions for λ,

$$\lambda = \frac{100}{18}k^{-1/3}l^{1/3} = \frac{50}{12}k^{2/3}l^{-2/3},$$

so

$$l^{1/3}l^{2/3} = \frac{18}{100}\cdot\frac{50}{12}k^{1/3}k^{2/3},$$

which simplifies to $l = 3k/4$. The third equation gives

$$6k + 4\left(\frac{3}{4}k\right) = 9k = 1000,$$

so $k = 1000/9$ and $l = (3/4)k = 3000/36 = 250/3$. The corresponding output is

$$50\left(\frac{1000}{9}\right)^{2/3}\left(\frac{250}{3}\right)^{1/3}.$$

Answer to question 5.8

A little care needs to be taken in determining the constraint. Since the number of hours spent working in the restaurant is $I/8$, the income divided by the hourly rate, the constraint 'total time is 100' is

$$I/8 + S = 100.$$

The Lagrangean is therefore

$$L(I, S, \lambda) = I^{1/4}S^{3/4} - \lambda\left(\frac{I}{8} + S - 100\right),$$

and the equations to solve are

$$\begin{aligned}
\frac{\partial L}{\partial I} &= \frac{1}{4}I^{-3/4}S^{3/4} - \frac{\lambda}{8} = 0, \\
\frac{\partial L}{\partial S} &= \frac{3}{4}I^{1/4}S^{-1/4} - \lambda = 0, \\
\frac{\partial L}{\partial \lambda} &= 100 - \frac{I}{8} - S = 0.
\end{aligned}$$

The first two equations, on elimination of λ, yield $I = 8S/3$, and substituting this into the third equation, we obtain $S = 75$. Thus the optimal division of time is 75 hours of study and 25 hours of restaurant work (generating $I = 200$ dollars income).