Examiners' commentaries 2016

MT105a Mathematics 1

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2015–16. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the subject guide (2011). You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements – if none are available, please use the contents list and index of the new edition to find the relevant section.

General remarks

Learning outcomes

At the end of this half course and having completed the Essential reading and activities you should have:

- used the concepts, terminology, methods and conventions covered in the half course to solve mathematical problems in this subject
- the ability to solve unseen mathematical problems involving understanding of these concepts and application of these methods
- seen how mathematical techniques can be used to solve problems in economics and related subjects.

Showing your working

We start by emphasising that you should **always** include your working. This means two things. First, you should not simply write down the answer in the examination script, but you should explain the method by which it is obtained. Second, you should include rough working (even if it is messy!). The examiners want you to get the right answers, of course, but it is more important that you prove you know what you are doing: that is what is really being examined. We also stress that if you have not completely solved a problem, you may still be awarded marks for a partial, incomplete, or slightly wrong, solution; but, if you have written down a wrong answer and nothing else, no marks can be awarded. So it is certainly in your interests to include all your workings.

Covering the syllabus and choosing questions

You should ensure that you have covered the syllabus in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable for questions on these topics. There are no formal options in this course: you should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, **any** topic could appear in the examination questions.

Expectations of the examination paper

Every examination paper is different. You should not assume that your examination will be almost identical to the previous year's: for instance, just because there was a question, or a part of a question, on a certain topic last year, you should not assume there will be one on the same topic this year. Each year, the examiners want to test that candidates know and understand a number of mathematical methods and, in setting an examination paper, they try to test whether the candidate does indeed know the methods, understands them, and is able to use them, and not merely whether they vaguely remember them. Because of this, every year there are some questions which are likely to seem unfamiliar, or different, from previous years' questions. You should **expect** to be surprised by some of the questions. Of course, you will only be examined on material in the syllabus, so all questions can be answered using the material of the course. There will be enough, routine, familiar content in the examination so that a candidate who has achieved competence in the course will pass, but, of course, for a high mark, more is expected: you will have to demonstrate an ability to solve new and unfamiliar problems.

Answer the question

Please do read the questions carefully. You might be asked to use specific methods, even when others could be used. The purpose of the examination is to test that you know certain methods, so the examiners might occasionally ask you to use a specific technique. In such circumstances, only limited partial credit can be given if you do not use the specified technique. It is also worth reading the question carefully so that you do not do more than is required (because it is unlikely that you would get extra marks for doing so). For instance, if a question asked you only to find the critical points of a function, but not their natures, then you should not determine their natures. Be careful to read all questions carefully because, although they may look like previous examination questions on first glance, there can be subtle differences.

Graph sketching

Some examinations in this subject ask you to sketch the graph of a function. Any sketching of graphs should be done in the answer book. Graph paper is not needed. Indeed, as we have mentioned often in the *Examiners' commentaries*, the plotting of points in order to graph a function is not the correct approach. A sketch of the graph of a function should indicate its shape, its position with respect to the axes, and its intercepts on those axes: it need not be drawn to scale. Graph paper is not necessary for this.

Calculators

You are reminded that calculators are **not** permitted in the examination for this course, under any circumstances. The examiners know this, and so they set questions that do not require a calculator. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise this course.

Examination revision strategy

Many candidates are disappointed to find that their examination performance is poorer than they expected. This may be due to a number of reasons. The *Examiners' commentaries* suggest ways of addressing common problems and improving your performance. One particular failing is '**question spotting**', that is, confining your examination preparation to a few questions and/or topics which have come up in past papers for the course. This can have serious consequences.

We recognise that candidates may not cover all topics in the syllabus in the same depth, but you need to be aware that the examiners are free to set questions on **any aspect** of the syllabus. This means that you need to study enough of the syllabus to enable you to answer the required number of examination questions.

The syllabus can be found in the Course information sheet in the section of the VLE dedicated to each course. You should read the syllabus carefully and ensure that you cover sufficient material in preparation for the examination. Examiners will vary the topics and questions from year to year and may well set questions that have not appeared in past papers. Examination papers may legitimately include questions on any topic in the syllabus. So, although past papers can be helpful during your revision, you cannot assume that topics or specific questions that have come up in past examinations will occur again.

If you rely on a question-spotting strategy, it is likely you will find yourself in difficulties when you sit the examination. We strongly advise you not to adopt this strategy.

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Comments on specific questions – Zone A

Candidates should answer all **EIGHT** questions: all **SIX** questions of Section A (60 marks in total) and **BOTH** questions from Section B (20 marks each).

Section A

Answer all **six** questions from this section (60 marks in total).

Question 1

A monopoly has fixed costs of 20 and marginal cost function $3q^2 + 4$. The demand equation for its product is $p + q = 20$.

Determine the profit function in terms of q.

Hence find the production level that maximises the profit.

Reading for this question

See Chapter 3 of the subject guide for related reading.

Approaching the question

We have:

$$TC = \int MC \, \mathrm{d}q = q^3 + 4q + c$$

where, since $TC(0) = FC = 20$, we have $c = 20$.

The firm is a monopoly so the selling price in terms of its production is, from the demand equation, $p = 20 - q$. So $TR = q(20 - q) = 20q - q^2$.

Then:

$$\Pi = TR - TC = q(20 - q) - (q^3 + 4q + 20) = 16q - q^2 - q^3 - 20.$$

To maximise Π, we set $\Pi' = 0$, giving $16 - 2q - 3q^2 = 0$, so:

$$(3q + 8)(q - 2) = 0.$$

Since $q \geq 0$, the only relevant critical point is $q = 2$ and this is a maximum since $\Pi''(q) = -2 - 6q$ is negative when $q = 2$.

Question 2

Express the following system of equations in matrix form, and solve it using row operations.

$$\begin{aligned} \boldsymbol{4x - y - z} &= \boldsymbol{4,} \\ \boldsymbol{2x - 3y - z} &= \boldsymbol{4,} \\ \boldsymbol{2x - 5y + z} &= \boldsymbol{8.} \end{aligned}$$

Reading for this question

The recommended method for solving linear equations using row operations can be found in Chapter 6 of the subject guide. It is known by several names: the row operation method, the Gauss–Jordan method, the row-reduction method, and so on.

Approaching the question

The system expressed in matrix form is:

$$\begin{pmatrix} 4 & -1 & -1 \\ 2 & -3 & -1 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 8 \end{pmatrix}.$$

(The system 'expressed in matrix form' is not the augmented matrix.)

The standard matrix method approach is now to reduce the augmented matrix to reduced form. Here is one way. (There are others, equally valid.)

$$\begin{pmatrix} 4 & -1 & -1 & 4 \\ 2 & -3 & -1 & 4 \\ 2 & -5 & 1 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & -1 & -1 & 4 \\ 4 & -6 & -2 & 8 \\ 4 & -10 & 2 & 16 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & -1 & -1 & 4 \\ 0 & -5 & -1 & 4 \\ 0 & -9 & 3 & 12 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 4 & -1 & -1 & 4 \\ 0 & 45 & 9 & -36 \\ 0 & -45 & 15 & 60 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & -1 & -1 & 4 \\ 0 & 45 & 9 & -36 \\ 0 & 0 & 24 & 24 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & -1 & -1 & 4 \\ 0 & 5 & 1 & -4 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

It follows, from this last matrix, that:

$$\begin{aligned} 4x - y - z &= 4 \\ 5y + z &= -4 \\ z &= 1. \end{aligned}$$

So, $z = 1$, $y = (1/5)(-8 - z) = -1$ and $x = (1/4)(4 + z + y) = 1$.

This is probably a good point at which to make some general comments about how questions are marked. Clearly, in a question like this, it is easy to get the wrong answer. (Though it should be noted that in this particular question, you can always substitute the values that you have found into the original equations, and this will show whether these are correct or not. So you can tell if you have the wrong answer and, if you have time, you can then re-work the calculation.) Examiners understand that arithmetical errors can be made, especially in the stressful circumstances of an examination. Quite probably, the examiners themselves would make some mistakes if they sat the paper. So, although there are marks for correct calculation, there are also marks for using the right method (even if you make a mistake). So, here, for instance, examiners will award marks if you can indicate that you know how to start to solve the equations (by writing down an augmented matrix); that you know what row operations are; that you know what it is you want to achieve with row operations (the reduced matrix, that is); and that you then know how to work from that reduced matrix to determine the required solutions. There are marks for all these things.

Be sure to understand that only certain types of operations qualify as valid row operations. In particular, a number of candidates make the mistake of thinking that subtracting a fixed constant from each entry of a row is valid. It is not. (And, if you don't know what we mean by that, then you're probably not doing it, which is good!)

Question 3

Use the method of Lagrange multipliers to find the positive values of x and y which maximise

$$\frac{2y}{y+2} + \frac{x}{x+1}$$

subject to the constraint $x + y = 120$.

Reading for this question

The Lagrange multiplier method for constrained optimisation is discussed in Chapter 5 of the subject guide.

Approaching the question

The Lagrangian is:

$$L = \frac{2y}{y+2} + \frac{x}{x+1} - \lambda(x + y - 120).$$

The first-order conditions are:

$$\begin{aligned} L_x &= \frac{1}{(x+1)^2} - \lambda = 0 \\ L_y &= \frac{4}{(y+2)^2} - \lambda = 0 \\ L_\lambda &= -(x + y - 120) = 0. \end{aligned}$$

From the first two equations, eliminating λ, we have:

$$(y+2)^2 = 4(x+1)^2.$$

So, given that $x, y > 0$, $y + 2 = 2(x + 1)$ and hence $y = 2x$. Hence, by the third equation (the constraint), $x + 2x = 120$, so $x = 40$ and $y = 2x = 80$.

Question 4

Determine the following integrals:

$$\int \frac{x}{x^2 + 7x + 10}\,\mathrm{d}x,$$

$$\int \frac{\ln(2x+1)}{\sqrt{2x+1}}\,\mathrm{d}x.$$

Reading for this question

Integration is discussed in Chapter 4 of the subject guide. It can be difficult because it is not always clear which technique will work. The three main techniques are: substitution, parts, and partial fractions. More than one method might work, and some integrals require a combination of methods.

Approaching the question

For the first integral, since $x^2 + 7x + 10 = (x + 2)(x + 5)$, we can use partial fractions. We know that for some A and B, we will have:

$$\frac{x+1}{x^2+7x+10} = \frac{A}{x+2} + \frac{B}{x+5}.$$

Finding A and B, we see that:

$$\frac{x+1}{x^2+7x+10} = \frac{1}{3}\left(\frac{-1}{x+2} + \frac{4}{x+5}\right)$$

and so the integral is:

$$-\frac{1}{3}\ln|x+2| + \frac{4}{3}\ln|x+5| + c.$$

For the second integral, we could make the substitution $u = 2x + 1$. The integral becomes:

$$I = \frac{1}{2}\int \frac{\ln u}{\sqrt{u}}\,\mathrm{d}u.$$

We then use integration by parts:

$$\begin{aligned} I &= \sqrt{u}\ln u - \int \sqrt{u}\frac{1}{u}\,\mathrm{d}u \\ &= \sqrt{u}\ln u - \int \frac{1}{\sqrt{u}}\,\mathrm{d}u \\ &= \sqrt{u}\ln u - 2\sqrt{u} + c \\ &= \sqrt{2x+1}\ln(2x+1) - 2\sqrt{2x+1} + c. \end{aligned}$$

Question 5

Suppose that a is a positive number and that the function f is given by

$$f(x,y) = x^4 - 2a^2x^2 - y^2 + 1.$$

Find the critical points of f. For each critical point of f, determine whether it is a local minimum, local maximum, or a saddle point.

Reading for this question

This question uses the material in Chapter 5 of the subject guide.

Approaching the question

It is important in answering questions like this to use a correct notation for partial derivatives. We can write f_x instead of $\frac{\partial f}{\partial x}$ and f_{xx} instead of $\frac{\partial^2 f}{\partial x^2}$, and so on, but it is a bad idea to invent your own notation!

In order to find the critical points of the function, we solve $f_x = f_y = 0$. This is:

$$4x^3 - 4a^2 x = 0 \quad \text{and} \quad -2y = 0.$$

The second equation shows $y = 0$ and the first shows $x(x^2 - a^2) = 0$, giving solutions $x = 0$ and $\pm a$. So the critical points are:

$$(0, 0), \quad (a, 0) \quad \text{and} \quad (-a, 0).$$

The second derivatives are:

$$f_{xx} = 12x^2 - 4a^2, \quad f_{xy} = 0 \quad \text{and} \quad f_{yy} = -2.$$

At $(0, 0)$, we have $f_{xx} = -4a^2 < 0$ and $H = f_{xx}f_{yy} - f_{xy}^2 = 8a^2 > 0$, so this is a local maximum. At each of the other two critical points, we have $H = f_{xx}f_{yy} - f_{xy}^2 = 8a^2(-2) = -16a^2 < 0$, so these are both saddle points.

Question 6

Suppose the function f is given by

$$\boldsymbol{f(x, y) = \sin(2x - y) + e^y \ln(x^2 + 1).}$$

Show that

$$\boldsymbol{\frac{\partial^2 f}{\partial x \partial y} + 2\frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} = 0.}$$

Reading for this question

Partial differentiation is discussed in Chapter 5 of the subject guide.

Approaching the question

We have:

$$\begin{aligned}
f_x &= 2\cos(2x - y) + e^y \frac{2x}{x^2 + 1} \\
f_y &= -\cos(2x - y) + e^y \ln(x^2 + 1) \\
f_{xy} &= 2\sin(2x - y) + e^y \frac{2x}{x^2 + 1} \\
f_{yy} &= -\sin(2x - y) + e^y \ln(x^2 + 1).
\end{aligned}$$

Then:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} + 2\frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} &= 2\sin(2x - y) + e^y \frac{2x}{x^2 + 1} - 2\sin(2x - y) + 2e^y \ln(x^2 + 1) \\
&\quad -2\cos(2x - y) - e^y \frac{2x}{x^2 + 1} + 2\cos(2x - y) - 2e^y \ln(x^2 + 1) \\
&= 0.
\end{aligned}$$

Section B

Answer **both** questions from this section (20 marks each).

Question 7

(a) **A firm is the only producer of two goods, X and Y. The demand equations for X and Y are given by**

$$\begin{aligned} x &= 600 - p_X - p_Y, \\ y &= 450 - \frac{1}{2}p_X - p_Y, \end{aligned}$$

where x and y are the quantities of X and Y demanded (respectively) and p_X, p_Y are (respectively) the prices of X and Y. The firm's joint total cost function (that is, the cost of producing x of X and y of Y) is

$$x^2 + 3xy + y^2 + 100.$$

Find an expression in terms of x and y for the profit function.

Hence determine the quantities x and y that maximise the profit, and find the corresponding prices p_X, p_Y.

Reading for this question

Partial differentiation is discussed in Chapter 5 of the subject guide.

Approaching the question

The profit function is given by:

$$\Pi = x\,p_X + y\,p_Y - TC.$$

We want to get this as a function of x and y, so we need to find p_X and p_Y in terms of x and y. Subtracting the second equation from the first:

$$x - y = 150 - \frac{1}{2}p_X$$

so:

$$p_X = 300 - 2x + 2y.$$

Then:

$$p_Y = 600 - p_X - x = 300 + x - 2y.$$

So:

$$\Pi = x\,p_X + y\,p_Y - TC = 300x + 300y - 3x^2 - 3y^2 - 100.$$

We solve $\Pi_x = 0$ and $\Pi_y = 0$, giving:

$$\Pi_x = 300 - 6x = 0 \quad \text{and} \quad \Pi_y = 300 - 6y = 0.$$

So $x = y = 50$.

We have $\Pi_{xx} = -6 < 0$ and $\Pi_{xx}\Pi_{yy} - \Pi_{xy}^2 = (-6)(-6) - 0^2 > 0$, so this does indeed maximise the profit.

The corresponding prices are $p_X = 300$ and $p_Y = 250$.

(b) **Suppose the function f is given by $f(x) = e^x - x$. Prove that, for $x \geq 0$, $f'(x) \geq 0$. Hence show that, for all $x \geq 0$,**

$$e^x \geq 1 + x.$$

Now suppose the function g is given by

$$g(x) = \mathrm{e}^x - x - \frac{x^2}{2}.$$

Show that $g'(x) \geq 0$ for all $x \geq 0$. Hence show that, for all $x \geq 0$,

$$\mathrm{e}^x \geq 1 + x + \frac{x^2}{2}.$$

Now use a similar method (considering an appropriate function h and its derivative) to show that, for all $x \geq 0$,

$$\mathrm{e}^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

Reading for this question

Differentiation is discussed in Chapter 3 of the subject guide.

Approaching the question

This question was not well done. Some (using knowledge from other courses, such as **MT105b Mathematics 2**) attempted to solve it using Taylor's theorem, but that is not the way to do it. The question indicates what to do at each stage. All that is needed is basic calculus and an understanding of what it means for a function to be increasing.

We have $f'(x) = \mathrm{e}^x - 1$. Since $\mathrm{e}^x \geq 1$ for $x \geq 0$, $f'(x) \geq 0$ for all $x \geq 0$. The function f is therefore increasing for all $x \geq 0$ and so $f(x) \geq f(0) = 1$ for all $x \geq 0$. That is, $\mathrm{e}^x - x \geq 1$ and so $\mathrm{e}^x \geq 1 + x$.

We have $g'(x) = \mathrm{e}^x - (1 + x)$. By what we have just shown, $g'(x) \geq 0$ for $x \geq 0$. Hence, $g(x) \geq g(0) = 1$ and so:

$$\mathrm{e}^x \geq 1 + x + \frac{x^2}{2}.$$

Let:

$$h(x) = \mathrm{e}^x - x - \frac{x^2}{2} - \frac{x^3}{6}.$$

Then, for $x \geq 0$, we have:

$$h'(x) = \mathrm{e}^x - \left(1 + x + \frac{x^2}{2}\right) \geq 0$$

and hence, for $x \geq 0$, $h(x) \geq h(0) = 1$, and so:

$$\mathrm{e}^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

Question 8

(a) **The number of fish in a lake is 10000 at the start of 2016. Each year, 5% of the fish who were alive at the start of the year die and 1000 new fish are born. Find an expression, in as simple a form as possible, for the number of fish in the lake N years after the start of 2016. Describe what happens to the number of fish in the lake in the long run.**

Reading for this question

Chapter 7 of the subject guide gives the required background material.

Approaching the question

Let y_N be the population N years after the start of 2016. Then we have $y_0 = 10000$, and:

$$\begin{aligned} y_1 &= (0.95)10000 + 1000 \\ y_2 &= (0.95)\left((0.95)10000 + 1000\right) + 1000 = (0.95)^2 10000 + (0.95)1000 + 1000 \\ y_3 &= (0.95)y_2 + 1000 = (0.95)^3 10000 + (0.95)^2 1000 + (0.95)1000 + 1000 \end{aligned}$$

and, in general:

$$y_N = (0.95)^N 10000 + (0.95)^{N-1} 1000 + \cdots + (0.95)1000 + 1000.$$

Simplifying:

$$y_N = (0.95)^N 10000 + 1000 \left(\frac{1 - (0.95)^N}{1 - 0.95} \right).$$

This is:

$$10000(0.95)^N + 20000(1 - (0.95)^N) = 20000 - 10000(0.95)^N.$$

This increases with N, converging to 20000.

(b) Suppose the quantity y is defined as a function of x through the equation

$$x^2y^3 - 6x^3y^2 + 2xy = 1.$$

Find the value of y when $x = 1/2$. [You might find it useful to note that $y^3 - 3y^2 + 4y - 4 = (y - 2)(y^2 - y + 2)$.]

Find a general expression for the derivative $y'(x) = \dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Hence determine the value of $y'(1/2)$.

Reading for this question

Implicit differentiation is discussed in Section 5.5 of the subject guide.

Approaching the question

Suppose $x = 1/2$. Put $x = 1/2$ into the defining equation, $x^2y^3 - 6x^3y^2 + 2xy = 1$. We obtain:

$$\frac{1}{4}y^3 - \frac{3}{4}y^2 + y = 1$$

or:

$$y^3 - 3y^2 + 4y - 4 = 0.$$

Using the hint given, this equation factorises as:

$$(y - 2)(y^2 - y + 2) = 0.$$

The quadratic $y^2 - y + 2$ has negative discriminant and so has no zeroes. It follows that when $x = 1/2$, $y = 2$.

The equation defining y implicitly as a function of x is of the form $g(x, y) = 1$ where $g(x, y) = x^2y^3 - 6x^3y^2 + 2xy$. So:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial g/\partial x}{\partial g/\partial y}.$$

Now:

$$\frac{\partial g}{\partial x} = 2xy^3 - 18x^2y^2 + 2y \quad \text{and} \quad \frac{\partial g}{\partial y} = 3x^2y^2 - 12x^3y + 2x.$$

So:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{18x^2y^2 - 2xy^3 - 2y}{3x^2y^2 - 12x^3y + 2x}.$$

Substituting the values $x = 1/2$ and $y = 2$ into the expression for $\mathrm{d}y/\mathrm{d}x$, we see that the derivative when $x = 1/2$ is 6.

Alternatively, we could simply argue (with the same conclusion) that:

$$2xy^3 + 3x^2y^2y' - 18x^2y^2 - 12x^3yy' + 2y + 2xy' = 0.$$

Solving for y' gives:

$$y' = \frac{18x^2y^2 - 2xy^3 - 2y}{3x^2y^2 - 12x^3y + 2x}.$$

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Comments on specific questions – Zone B

Candidates should answer all **EIGHT** questions: all **SIX** questions of Section A (60 marks in total) and **BOTH** questions from Section B (20 marks each).

Section A

Answer all **six** questions from this section (60 marks in total).

Question 1

A monopoly has fixed costs of 10 and marginal cost function $6q^2 + 8$. The demand equation for its product is $p + 2q = 40$.

Determine the profit function in terms of q.

Hence find the production level that maximises the profit.

Reading for this question

See Chapter 3 of the subject guide for related reading.

Approaching the question

We have:

$$TC = \int MC \, \mathrm{d}q = 2q^3 + 8q + c$$

where, since $TC(0) = FC = 10$, we have $c = 10$.

The firm is a monopoly so the selling price in terms of its production is, from the demand equation, $p = 40 - 2q$. So $TR = q(40 - 2q) = 40q - 2q^2$.

Then:

$$\Pi = TR - TC = q(40 - 2q) - (2q^3 + 8q + 10) = 32q - 2q^2 - 2q^3 - 10.$$

To maximise Π, we set $\Pi' = 0$, giving $32 - 4q - 6q^2 = 0$, so:

$$(3q + 8)(q - 2) = 0.$$

Since $q \geq 0$, the only relevant critical point is $q = 2$ and this is a maximum since $\Pi''(q) = -2 - 6q$ is negative when $q = 2$.

Question 2

Express the following system of equations in matrix form, and solve it using row operations.

$$\begin{aligned} \mathbf{4x - y - z} &= \mathbf{8}, \\ \mathbf{2x - 3y - z} &= \mathbf{8}, \\ \mathbf{2x - 5y + z} &= \mathbf{16}. \end{aligned}$$

Reading for this question

The recommended method for solving linear equations using row operations can be found in Chapter 6 of the subject guide. It is known by several names: the row operation method, the Gauss–Jordan method, the row-reduction method, and so on.

Approaching the question

The system expressed in matrix form is:

$$\begin{pmatrix} 4 & -1 & -1 \\ 2 & -3 & -1 \\ 2 & -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ 16 \end{pmatrix}.$$

(The system 'expressed in matrix form' is not the augmented matrix.)

The standard matrix method approach is now to reduce the augmented matrix to reduced form. Here is one way. (There are others, equally valid.)

$$\begin{pmatrix} 4 & -1 & -1 & 8 \\ 2 & -3 & -1 & 8 \\ 2 & -5 & 1 & 16 \end{pmatrix} \to \begin{pmatrix} 4 & -1 & -1 & 8 \\ 4 & -6 & -2 & 16 \\ 4 & -10 & 2 & 32 \end{pmatrix} \to \begin{pmatrix} 4 & -1 & -1 & 8 \\ 0 & -5 & -1 & 8 \\ 0 & -9 & 3 & 24 \end{pmatrix}$$

$$\to \begin{pmatrix} 4 & -1 & -1 & 8 \\ 0 & 45 & 9 & -72 \\ 0 & -45 & 15 & 120 \end{pmatrix} \to \begin{pmatrix} 4 & -1 & -1 & 8 \\ 0 & 45 & 9 & -72 \\ 0 & 0 & 24 & 48 \end{pmatrix} \to \begin{pmatrix} 4 & -1 & -1 & 8 \\ 0 & 5 & 1 & -8 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

It follows, from this last matrix, that:

$$\begin{aligned} 4x - y - z &= 8 \\ 5y + z &= -8 \\ z &= 2. \end{aligned}$$

So, $z = 2$, $y = (1/5)(-8 - z) = -2$ and $x = (1/4)(4 + z + y) = 2$.

This is probably a good point at which to make some general comments about how questions are marked. Clearly, in a question like this, it is easy to get the wrong answer. (Though it should be noted that in this particular question, you can always substitute the values that you have found into the original equations, and this will show whether these are correct or not. So you can tell if you have the wrong answer and, if you have time, you can then re-work the calculation.) Examiners understand that arithmetical errors can be made, especially in the stressful circumstances of an examination. Quite probably, the examiners themselves would make some mistakes if they sat the paper. So, although there are marks for correct calculation, there are also marks for using the right method (even if you make a mistake). So, here, for instance, examiners will award marks if you can indicate that you know how to start to solve the equations (by writing down an augmented matrix); that you know what row operations are; that you know what it is you want to achieve with row operations (the reduced matrix, that is); and that you then know how to work from that reduced matrix to determine the required solutions. There are marks for all these things.

Be sure to understand that only certain types of operations qualify as valid row operations. In particular, a number of candidates make the mistake of thinking that subtracting a fixed constant from each entry of a row is valid. It is not. (And, if you don't know what we mean by that, then you're probably not doing it, which is good!)

Question 3

Use the method of Lagrange multipliers to find the positive values of x and y which maximise

$$\frac{2x}{x+2}+\frac{y}{y+1}$$

subject to the constraint $x+y=120$.

Reading for this question

The Lagrange multiplier method for constrained optimisation is discussed in Chapter 5 of the subject guide.

Approaching the question

The Lagrangian is:

$$L=\frac{2x}{x+2}+\frac{y}{y+1}-\lambda(x+y-120).$$

The first-order conditions are:

$$\begin{aligned} L_x &= \frac{4}{(x+2)^2}-\lambda=0 \\ L_y &= \frac{1}{(y+1)^2}-\lambda=0 \\ L_\lambda &= -(x+y-120)=0. \end{aligned}$$

From the first two equations, eliminating λ, we have:

$$(x+2)^2=4(y+1)^2.$$

So, given that $x,y>0$, $x+2=2(y+1)$ and hence $x=2y$. Hence, by the third equation (the constraint), $2y+y=120$, so $y=40$ and $x=2y=80$.

Question 4

Determine the following integrals:

$$\int \frac{\boldsymbol{x}}{\boldsymbol{x^2+7x+12}}\,\mathbf{d}\boldsymbol{x},$$

$$\int \frac{\mathbf{ln}(\boldsymbol{x}+\mathbf{1})}{\sqrt{\boldsymbol{x}+\mathbf{1}}}\,\mathbf{d}\boldsymbol{x}.$$

Reading for this question

Integration is discussed in Chapter 4 of the subject guide. It can be difficult because it is not always clear which technique will work. The three main techniques are: substitution, parts, and partial fractions. More than one method might work, and some integrals require a combination of methods.

Approaching the question

For the first integral, since $x^2+7x+12=(x+3)(x+4)$, we can use partial fractions. We know that for some A and B, we will have:

$$\frac{x}{x^2+7x+12}=\frac{A}{x+3}+\frac{B}{x+4}.$$

Finding A and B, we see that:

$$\frac{x}{x^2+7x+12}=\frac{-3}{x+3}+\frac{4}{x+4}$$

and so the integral is:

$$-3\ln|x+3|+4\ln|x+4|+c.$$

For the second integral, we could make the substitution $u=x+1$. The integral becomes:

$$I=\int \frac{\ln u}{\sqrt{u}}\,\mathrm{d}u.$$

We then use integration by parts:

$$\begin{aligned} I &= 2\sqrt{u}\ln u-2\int \sqrt{u}\frac{1}{u}\,\mathrm{d}u \\ &= 2\sqrt{u}\ln u-2\int \frac{1}{\sqrt{u}}\,\mathrm{d}u \\ &= 2\sqrt{u}\ln u-4\sqrt{u}+c \\ &= 2\sqrt{x+1}\ln(x+1)-4\sqrt{x+1}+c. \end{aligned}$$

Question 5

Suppose that $\boldsymbol{a}$ is a positive number and that the function $\boldsymbol{f}$ is given by

$$\boldsymbol{f(x,y)=y^4-8a^2y^2-x^2+1.}$$

Find the critical points of $\boldsymbol{f}$. For each critical point of $\boldsymbol{f}$, determine whether it is a local minimum, local maximum, or a saddle point.

Reading for this question

This question uses the material in Chapter 5 of the subject guide.

Approaching the question

It is important in answering questions like this to use a correct notation for partial derivatives. We can write f_x instead of $\frac{\partial f}{\partial x}$ and f_{xx} instead of $\frac{\partial^2 f}{\partial x^2}$, and so on, but it is a bad idea to invent your own notation!

In order to find the critical points of the function, we solve $f_x = f_y = 0$. This is:

$$-2x = 0 \quad \text{and} \quad 4y^3 - 16a^2y = 0.$$

The first equation shows $x = 0$ and the second shows $y(y^2 - 4a^2) = 0$, giving solutions $y = 0$ and $\pm 2a$. So the critical points are:

$$(0,0), \quad (0,2a) \quad \text{and} \quad (0,-2a).$$

The second derivatives are:

$$f_{xx} = -2, \quad f_{xy} = 0 \quad \text{and} \quad f_{yy} = 12y^2 - 16a^2.$$

At $(0,0)$, we have $f_{xx} = -2 < 0$ and $H = f_{xx}f_{yy} - f_{xy}^2 = 32a^2 > 0$, so this is a local maximum. At each of the other two critical points, we have $H = f_{xx}f_{yy} - f_{xy}^2 = -64a^2 < 0$, so these are both saddle points.

Question 6

Suppose the function f is given by

$$\boldsymbol{f(x,y) = \cos(2y - x) + e^x \ln(y^2+1).}$$

Show that

$$\boldsymbol{\frac{\partial^2 f}{\partial x \partial y} + 2\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial y} - 2\frac{\partial f}{\partial x} = 0.}$$

Reading for this question

Partial differentiation is discussed in Chapter 5 of the subject guide.

Approaching the question

We have:

$$\begin{aligned}
f_x &= \sin(2y-x) + e^x \ln(y^2+1) \\
f_y &= -2\sin(2y-x) + e^x \frac{2y}{y^2+1} \\
f_{xy} &= 2\cos(2y-x) + e^x \frac{2y}{y^2+1} \\
f_{xx} &= -\cos(2y-x) + e^x \ln(y^2+1).
\end{aligned}$$

Then:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} + 2\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial y} - 2\frac{\partial f}{\partial x} &= 2\cos(2y-x) + e^x \frac{2y}{y^2+1} - 2\cos(2y-x) + 2e^x \ln(y^2+1) \\
&\quad +2\sin(2y-x) - e^x \frac{2y}{y^2+1} - 2\sin(2y-x) - 2e^x \ln(y^2+1) \\
&= 0.
\end{aligned}$$

Section B

Answer **both** questions from this section (20 marks each).

Question 7

(a) **A firm is the only producer of two goods, X and Y. The demand quantities x and y for X and Y (respectively) and the corresponding prices p_X and p_Y are related by the equations**

$$\begin{aligned} -p_X + p_Y + 13 - 3x &= 0, \\ p_X - 4p_Y + 26 - 3y &= 0. \end{aligned}$$

The firm's joint total cost function (that is, the cost of producing x of X and y of Y) is $14x + 7y$.

Find an expression in terms of x and y for the profit function.

Hence determine the quantities x and y that maximise the profit.

Reading for this question

Partial differentiation is discussed in Chapter 5 of the subject guide.

Approaching the question

The profit function is given by:

$$\Pi = x\,p_X + y\,p_Y - TC.$$

We want to get this as a function of x and y, so we need to find p_X and p_Y in terms of x and y. Adding the two equations:

$$-3p_Y + 39 - 3x - 3y = 0$$

so:

$$p_Y = 13 - x - y.$$

Then:

$$p_X = p_Y + 13 - 3x = 26 - 4x - y.$$

So:

$$\Pi = x\,p_X + y\,p_Y - TC = 12x + 6y - 4x^2 - y^2 - 2xy.$$

We solve $\Pi_x = 0$ and $\Pi_y = 0$, giving:

$$\Pi_x = 12 - 8x - 2y = 0 \quad \text{and} \quad \Pi_y = 6 - 2y - 2x = 0.$$

So $x = 1$ and $y = 2$.

We have $\Pi_{xx} = -8 < 0$ and $\Pi_{xx}\Pi_{yy} - \Pi_{xy}^2 = (-8)(-2) - (-2)^2 > 0$, so this does indeed maximise the profit.

(b) **Suppose the function f is given by $f(x) = \mathrm{e}^x - x$. Prove that, for $x \geq 0$, $f'(x) \geq 0$. Hence show that, for all $x \geq 0$,**

$$\mathrm{e}^x \geq 1 + x.$$

Now suppose the function g is given by

$$g(x) = \mathrm{e}^x - x - \frac{x^2}{2}.$$

Show that $g'(x) \geq 0$ for all $x \geq 0$. Hence show that, for all $x \geq 0$,

$$\mathrm{e}^x \geq 1 + x + \frac{x^2}{2}.$$

Now use a similar method (considering an appropriate function h and its derivative) to show that, for all $x \geq 0$,

$$\mathrm{e}^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

Reading for this question

Differentiation is discussed in Chapter 3 of the subject guide.

Approaching the question

This question was not well done. Some (using knowledge from other courses, such as **MT105b Mathematics 2**) attempted to solve it using Taylor's theorem, but that is not the way to do it. The question indicates what to do at each stage. All that is needed is basic calculus and an understanding of what it means for a function to be increasing.

We have $f'(x) = \mathrm{e}^x - 1$. Since $\mathrm{e}^x \geq 1$ for $x \geq 0$, $f'(x) \geq 0$ for all $x \geq 0$. The function f is therefore increasing for all $x \geq 0$ and so $f(x) \geq f(0) = 1$ for all $x \geq 0$. That is, $\mathrm{e}^x - x \geq 1$ and so $\mathrm{e}^x \geq 1 + x$.

We have $g'(x) = \mathrm{e}^x - (1 + x)$. By what we have just shown, $g'(x) \geq 0$ for $x \geq 0$. Hence, $g(x) \geq g(0) = 1$ and so:

$$\mathrm{e}^x \geq 1 + x + \frac{x^2}{2}.$$

Let:

$$h(x) = \mathrm{e}^x - x - \frac{x^2}{2} - \frac{x^3}{6}.$$

Then, for $x \geq 0$, we have:

$$h'(x) = \mathrm{e}^x - (1 + x + \frac{x^2}{2}) \geq 0$$

and hence, for $x \geq 0$, $h(x) \geq h(0) = 1$, and so:

$$\mathrm{e}^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

Question 8

(a) **On the first day of 2016 there are 1000 online book retailers in a particular country. During each subsequent year, the number of new such retailers grows by 30, but by the end of the year 2% of all the online book retailers that were in business at the start of the year will have closed down. Find an expression, in terms of N (and in as simple a form as possible) for the number of online book retailers N years after the first of January 2016. What happens to the number of such retailers in the long run?**

Reading for this question

Chapter 7 of the subject guide gives the required background material.

Approaching the question

Let y_N be the number of booksellers N years after the first day of 2016. Then we have $y_0 = 1000$, and:

$$\begin{aligned}
y_1 &= (0.98)1000 + 30 \\
y_2 &= (0.98)\left((0.98)1000 + 30\right) + 30 = (0.98)^2 1000 + (0.98)30 + 30 \\
y_3 &= (0.98)y_2 + 30 = (0.98)^3 1000 + (0.98)^2 30 + (0.98)30 + 30
\end{aligned}$$

and, in general:

$$y_N = (0.98)^N 1000 + (0.98)^{N-1} 30 + (0.98)^{N-2} 30 + \cdots + (0.98)30 + 30.$$

Simplifying:

$$\begin{aligned} y_N &= (0.98)^N 1000 + (0.98)^{N-1} 30 + (0.98)^{N-2} 30 + \cdots + (0.98)30 + 30 \\ &= (0.98)^N 1000 + \frac{30(1-(0.98)^N)}{1-0.98}. \end{aligned}$$

This is:

$$1000(0.98)^N + 1500(1-(0.98)^N) = 1500 - 500(0.98)^N.$$

This increases with N, converging to 1500.

(b) Suppose the quantity y is defined as a function of x through the equation

$$\boldsymbol{x^2y^3 - 6x^3y^2 + 2xy = 1.}$$

Find the value of y when $x = 1/2$. [You might find it useful to note that $y^3 - 3y^2 + 4y - 4 = (y-2)(y^2 - y + 2)$.]

Find a general expression for the derivative $y'(x) = \dfrac{dy}{dx}$.

Hence determine the value of $y'(1/2)$.

Reading for this queston

Implicit differentiation is discussed in Section 5.5 of the subject guide.

Approaching the question

Suppose $x = 1/2$. Put $x = 1/2$ into the defining equation, $x^2y^3 - 6x^3y^2 + 2xy = 1$. We obtain:

$$\frac{1}{4}y^3 - \frac{3}{4}y^2 + y = 1$$

or:

$$y^3 - 3y^2 + 4y - 4 = 0.$$

Using the hint given, this equation factorises as:

$$(y-2)(y^2 - y + 2) = 0.$$

The quadratic $y^2 - y + 2$ has negative discriminant and so has no zeroes. It follows that when $x = 1/2$, $y = 2$.

The equation defining y implicitly as a function of x is of the form $g(x, y) = 1$ where $g(x, y) = x^2y^3 - 6x^3y^2 + 2xy$. So:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial g/\partial x}{\partial g/\partial y}.$$

Now:

$$\frac{\partial g}{\partial x} = 2xy^3 - 18x^2y^2 + 2y \quad \text{and} \quad \frac{\partial g}{\partial y} = 3x^2y^2 - 12x^3y + 2x.$$

So:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{18x^2y^2 - 2xy^3 - 2y}{3x^2y^2 - 12x^3y + 2x}.$$

Substituting the values $x = 1/2$ and $y = 2$ into the expression for $\mathrm{d}y/\mathrm{d}x$, we see that the derivative when $x = 1/2$ is 6.

Alternatively, we could simply argue (with the same conclusion) that:

$$2xy^3 + 3x^2y^2y' - 18x^2y^2 - 12x^3yy' + 2y + 2xy' = 0.$$

Solving for y' gives:

$$y' = \frac{18x^2y^2 - 2xy^3 - 2y}{3x^2y^2 - 12x^3y + 2x}.$$