# Mock commentary 2014

## MT105a Mathematics 1

## Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2013–14. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

### Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the subject guide (2011). You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refers to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements — if none are available, please use the contents list and index of the new edition to find the relevant section.

## Specific comments on questions

#### Question 1

The values of $x$ where the curves intersect are the solutions of $f(x) = g(x)$. Now,

$$f(x) = g(x) \iff x^4 + 2x^3 + 2x^2 + 2 = -x^4 + 2x^3 + 18x^2 + 20 \iff 2x^4 - 16x^2 - 18 = 0,$$

so we have the equation $x^4 - 8x^2 - 9 = 0$. We don't have a general method for solving polynomial equations of degree 4. However, this equation can be viewed is a quadratic in $x^2$, for it is $(x^2)^2 - 8(x^2) - 9 = 0$.

Now we can use the formula for the solutions of a quadratic, or we can use factorisation to find $x^2$:

$$(x^2 - 9)(x^2 + 1) = 0,$$

so $x^2 = 9$. (Note that we cannot have $x^2 = -1$.) So $x = 3, -3$ are the two required values.

#### Question 2

The recommended method for solving linear equations can be found in Chapter 6 of the subject guide. It is known by several names: the row operation method, the Gauss-Jordan method, the row-reduction method, and so on. Many students like to use a different method, not covered in the subject guide, especially 'Cramer's rule'. It is acceptable to do so, but our view is that the row operations method is easier and less prone to error. But, whatever method you use, it has to be a *matrix* method, as the question makes explicit: manipulation of the equations will not suffice.

In matrix form, the equations are given by

$$\begin{pmatrix} 2 & -3 & -1 \\ 2 & 3 & 1 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 3 \end{pmatrix}.$$

The augmented matrix is

$$\begin{pmatrix} 2 & -3 & -1 & 0 \\ 2 & 3 & 1 & 8 \\ 1 & -2 & 3 & 3 \end{pmatrix}.$$

Using row operations to reduce (and there are many ways this can be done), we have

$$\begin{pmatrix} 2 & -3 & -1 & 0 \\ 2 & 3 & 1 & 8 \\ 1 & -2 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 2 & -3 & -1 & 0 \\ 4 & 0 & 0 & 8 \\ 1 & -2 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 2 & -3 & -1 & 0 \\ 1 & -2 & 3 & 3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & -1 & -4 \\ 0 & -2 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 11 & 11 \end{pmatrix}.$$

So, using back-substitution, we have

$$z = 1,$$

$$y = (4-1)/3 = 1,$$

$$x = 2.$$

This is probably a good point at which to make some general comments about how questions are marked. Clearly, in a question like this, it is easy to get the wrong answer. (Though it should be noted that in this particular question, you can always substitute the values that you have found into the original equations, and this will show whether these are correct or not. So you can tell if you have the wrong answer and, if you have time, you can then re-work the calculation.) Examiners understand that arithmetical errors can be made, especially in the stressful circumstances of an examination. Quite probably, the examiners themselves would make some mistakes if they sat the paper. So, although there are certainly some marks for correct calculation, there are many marks for using the right method (even if you make a mistake). So, here, for instance, examiners will award marks if you can indicate that you know how to start to solve the equations (by writing down an augmented matrix); that you know what row operations are; that you know what it is you want to achieve with row operations (the reduced matrix, that is); and that you then know how to work from that reduced matrix to determine the required solutions. There are marks for all these things.

Be sure to understand that only certain types of operations qualify as valid row operations. In particular, a number of students make the mistake of thinking that subtracting a fixed constant from each entry of a row is valid. It is not. (And, if you don't know what we mean by that, then you're probably not doing it, which is good!)

## Question 3

Chapter 3 of the subject guide gives the required background material.

We need to solve $f'(x) = 0$, finding $f'(x)$ is quite tricky. One way to tackle this is to notice that $f(x) = x^x$ gives us

$$\ln\Big(f(x)\Big) = \ln(x^x) = x \ln x$$

and so, using the chain rule on the left-hand side and the product rule on the right-hand side, we have

$$\frac{f'(x)}{f(x)} = 1 + \ln x$$

and so, as $f(x) = x^x$, this gives us

$$f'(x) = (1 + \ln x)x^x.$$

Now, we see that

$$f' = 0 \Longrightarrow (1 + \ln x)x^x = 0 \Longrightarrow \ln x = -1 \Longrightarrow x = e^{-1},$$

as $x^x \neq 0$ for all $x > 0$. Thus, we have one critical point and this occurs when $x = e^{-1}$.

Indeed, as $x$ passes through $e^{-1}$, $f'(x)$ changes sign from negative to positive, because for $x < e^{-1}$, $\ln x < -1$ and for $x > e^{-1}$, $\ln x > -1$ ($x^x$ is, of course, always positive for $x > 0$). Thus, our critical point is a local minimum.

Alternatively, we can use the second-derivative test as

$$f''(x) = \left(\frac{1}{x}\right) x^x + (1 + \ln x)^2 x^x,$$

if we use the product rule and our earlier answer for $f'(x)$ which is the derivative of $x^x$. At $x = e^{-1}$, we have $\ln x = -1$ and so the second term is zero, leaving us with

$$f''(e^{-1}) = e(e^{-1})^{e^{-1}} > 0,$$

and so, once again, we see that our critical point is a local minimum.

## Question 4

This involves techniques from Chapter 5 of the Subject Guide. The partial derivatives are

$$f_x = -2x + 2y^2, \quad f_y = 4y - 8y^3 + 4xy.$$

We solve $f_x = f_y = 0$. We have (from $f_x = 0$) $x = y^2$. Substituting into $f_y = 0$ gives $4y - 8y^3 + 4y^3 = 0$, which is $4y(1 - y^2) = 0$. So $y = 0, 1, -1$. Then $x = 0, 1, 1$, respectively. So there are three critical points: $(0, 0), (1, 1), (1, -1)$.

The second derivatives are

$$f_{xx} = -2, f_{xy} = 4y, f_{yy} = 4 - 24y^2 + 4x.$$

At $(0, 0)$: $f_{xx}f_{yy} - f_{xy}^2 = -8 < 0$, so this is a saddle point.

At $(1, 1)$, $f_{xx}f_{yy} - f_{xy}^2 = (-2)(-16) - (4)^2 > 0$ and $f_{xx} = -2 < 0$, so it's a maximum.

At $(1, -1)$, $f_{xx}f_{yy} - f_{xy}^2 = (-2)(-16) - (-4)^2 > 0$ and $f_{xx} = -2 < 0$, so it's a maximum.

## Question 5

Integration is discussed in Chapter 4 of the subject guide. It can be difficult because it is not always clear which technique will work. The three main techniques are: substitution, parts, and partial fractions.

**(a)** The integral here is

$$I = \int x^{-2} \ln(x + 1)\, \mathrm{d}x.$$

We start by using integration by parts:

$$\begin{aligned} I &= -x^{-1} \ln(x + 1) + \int x^{-1} \frac{1}{x + 1}\, \mathrm{d}x \\ &= -x^{-1} \ln(x + 1) + \int \frac{1}{x(x + 1)}\, \mathrm{d}x \end{aligned}$$

Now we use partial fractions:

$$I = -x^{-1} \ln(x + 1) + \int \left(\frac{1}{x} - \frac{1}{x + 1}\right) \mathrm{d}x = -x^{-1} \ln|x + 1| + \ln|x| - \ln|x + 1| + c.$$

**(b)** We start by using substitution $u = \sin x$ so that $\mathrm{d}u = \cos x \,\mathrm{d}x$. This gives us

$$I = \int \frac{\cos x}{(1-\sin x)(2+\sin x)}\,\mathrm{d}x = \int \frac{\mathrm{d}u}{(1-u)(u+2)}.$$

Now we use partial fractions.

$$I = \frac{1}{3}\int\left(\frac{1}{1-u}+\frac{1}{u+2}\right)\mathrm{d}u = \frac{1}{3}(-\ln|1-u|+\ln|u+2|)+c = \frac{1}{3}(-\ln|1-\sin x|+\ln|2+\sin x|)+c.$$

**(c)** We start by using substitution $u = \sqrt{x}$ so that $\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$. This gives us

$$I = \int e^{\sqrt{x}}\,\mathrm{d}x = 2\int ue^u\,\mathrm{d}u.$$

Now we can use integration by parts:

$$I = 2ue^u - 2\int e^u\,\mathrm{d}u = 2ue^u - 2e^u + c = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + c.$$

## Question 6

Chapter 7 of the subject guide gives the required background material.

With the usual notation, the sum to infinity is $a/(1-r)$ and the second term is $ar$. So

$$\frac{a}{1-r} = 20, \quad ar = 5.$$

So $a = 20(1-r)$ and

$$20(1-r)r = 5,$$

giving

$$20r^2 - 20r + 5 = 0, \ \ 4r^2 - 4r + 1 = 0$$

(or an equivalent equation involving $a$ if $r$ has been eliminated instead). It follows that

$$(2r-1)^2 = 0,$$

and hence $r = 1/2$ The corresponding value of $a$ is $a = 10$.

## Question 7

**(a)** This is a very standard type of question, using the material in Chapter 5 of the subject guide. The profit is given by

$$\Pi = xp_X + yp_Y - TC$$

and we want to get this as a function of $x$ and $y$, so we need to find $p_X$ and $p_Y$ in terms of $x$ and $y$. We have

$$p_X = 200 - x, \ \ p_Y = 100 - y.$$

So,

$$\Pi = xp_X + yp_Y - TC = 200x + 100y - 3x^2 - 2y^2 - 2xy.$$

The partial derivatives are

$$\Pi_x = 200 - 6x - 2y, \ \ \Pi_y = 100 - 4y - 2x.$$

We now solve $\Pi_x = 0$, $\Pi_y = 0$, obtaining $x = 30$ and $y = 10$. We have $\Pi_{xx} = -6 < 0$ and $\Pi_{xx}\Pi_{yy} - \Pi_{xy}^2 = (-6)(-4) - (-2)^2 > 0$, so this does indeed maximise the profit.

**(b)** These questions always seem to cause more difficulty than they ought to. Chapter 7 of the subject guide gives the background and some examples. Essentially, what you have to do is: identify the sequence that you are trying to find; determine the first few values; spot a general expression for the sequence; and simplify that expression. It is also possible to attack such problems using general techniques for solving difference equations, but that is not expected (as such techniques are not part of the 05a syllabus, though they are in 05b).

It is, more than ever, important to read questions such as this carefully: they might well be subtly different from any 'standard' ones that you are used to.

His initial balance is \$200 and for the $K$ years that we are dealing with in the first part of the question, the interest rate is 5% per annum. To find his balance $K$ years after he opens the account (just after he has made the new deposit of \$200) we start by noticing that, in dollars, he has

- a balance of
$$200(1.05) + 200,$$
*one* year after opening the account;
- a balance of
$$200(1.05)^2 + 200(1.05) + 200,$$
*two* years after opening the account;
- a balance of
$$200(1.05)^3 + 200(1.05)^2 + 200(1.05) + 200,$$
*three* years after opening the account.

At this point, we should be able to see the pattern and this gives him a balance of

$$200(1.05)^K + 200(1.05)^{K-1} + \cdots + 200(1.05) + 200,$$

$K$ years after opening the account. This is a geometric series with $K+1$ terms and so, writing it as

$$200\left(1 + 1.05 + \cdots + 1.05^{K-1} + 1.05^K\right),$$

we can use the formula for its sum to get

$$200\frac{1 - 1.05^{K+1}}{1 - 1.05} = 4000(1.05^{K+1} - 1).$$

Let's call this $B$ and this, in dollars, is his balance $K$ years after opening the account in a nice simple form.

In the next part of the question, we start with a balance of $B$ dollars and we are considering the next $N$ years during which the interest rate is now just 4% per annum. As such, to find his balance $K+N$ years after opening the account (just after he has made the new deposit of \$200) we start by noticing that, in dollars, he has

- a balance of
$$B(1.04) + 200,$$
$K$ plus *one* years after opening the account;
- a balance of
$$B(1.04)^2 + 200(1.04) + 200,$$
$K$ plus *two* years after opening the account;
- a balance of
$$B(1.04)^3 + 200(1.04)^2 + 200(1.04) + 200,$$
$K$ plus *three* years after opening the account.

Again, at this point, we should be able to see the pattern and this gives him a balance of

$$B(1.04)^N + 200(1.04)^{N-1} + \cdots + 200(1.04) + 200,$$

$K$ plus $N$ years after opening the account. This can be written as

$$B(1.04)^N + 200\Big(1 + 1.04 + \cdots + 1.04^{N-1}\Big),$$

and, in this form, we now have a geometric series with $N$ terms in the big brackets. So, using the formula for its sum, we get

$$B(1.04)^N + 200\frac{1-1.04^N}{1-1.04} = B(1.04)^N + 5000(1.04^N - 1).$$

Of course, using our expression for $B$ from above, we now see that, in dollars, his balance is

$$B(1.04)^N + 5000(1.04^N - 1) = 4000(1.05^{K+1} - 1)(1.04)^N + 5000(1.04^N - 1),$$

$K + N$ years after opening the account.

## Question 8

We want to maximise $\left((x+1)^3 + y^3\right)^{1/3}$ subject to $px + qy = M$. The Lagrangian is

$$L = \left((x+1)^3 + y^3\right)^{1/3} - \lambda(px + qy - M).$$

We have

$$L_x = (x+1)^2\left((x+1)^3 + y^3\right)^{-2/3} - \lambda p = 0$$

and

$$L_y = y^2\left((x+1)^3 + y^3\right)^{-2/3} - \lambda q = 0.$$

These imply that

$$\frac{y^2}{q} = \frac{(x+1)^2}{p},$$

So

$$y = \sqrt{\frac{q}{p}}(x+1).$$

The constraint is $px + qy = M$, which means that

$$px + q\left(\sqrt{\frac{q}{p}}(x+1)\right) = M$$

which is

$$px + q\sqrt{\frac{q}{p}}x + q\sqrt{\frac{q}{p}} = M \quad \Longrightarrow \quad (p\sqrt{p} + q\sqrt{q})x = M\sqrt{p} - q\sqrt{q},$$

and hence, solving for $x$ and simplifying, we get

$$x = \frac{\sqrt{p}M - q\sqrt{q}}{p\sqrt{p} + q\sqrt{q}},$$

for $x^*$. Then, using this, we find that

$$y = \sqrt{\frac{q}{p}}(x+1) \quad \Longrightarrow \quad y = \sqrt{\frac{q}{p}}\left(\frac{\sqrt{p}M - q\sqrt{q}}{p\sqrt{p} + q\sqrt{q}} + 1\right) = \sqrt{\frac{q}{p}}\left(\frac{\sqrt{p}M + p\sqrt{p}}{p\sqrt{p} + q\sqrt{q}}\right) = \frac{\sqrt{q}M + p\sqrt{q}}{p\sqrt{p} + q\sqrt{q}},$$

which is $y^*$.

To find an expression for $V$ we note that

$$x^* + 1 = \frac{\sqrt{p}(M+p)}{p\sqrt{p} + q\sqrt{q}} \quad \text{and} \quad y^* = \frac{\sqrt{q}(M+p)}{p\sqrt{p} + q\sqrt{q}},$$

so that

$$(x^*+1)^3 + (y^*)^3 = \frac{\sqrt{p^3}(M+p)^3 + \sqrt{q^3}(M+p)^3}{(\sqrt{p^3}+\sqrt{q^3})^3} = \frac{(M+p)^3}{(\sqrt{p^3}+\sqrt{q^3})^2}.$$

Consequently, we see that

$$V(p,q,M) = u(x^*, y^*) = \left[\frac{(M+p)^3}{(\sqrt{p^3}+\sqrt{q^3})^2}\right]^{1/3} = \frac{M+p}{(\sqrt{p^3}+\sqrt{q^3})^{2/3}},$$

is the required expression for $V$.

Using our equation for $L_x = 0$, we see that $\lambda^*$ is given by

$$\lambda = \frac{(x+1)^2}{p\left((x+1)^3 + y^3\right)^{2/3}},$$

and so, substituting in our values of $x^*$ and $y^*$, we can use what we have just seen to get

$$\lambda^* = \frac{p(M+p)^2/(\sqrt{p^3}+\sqrt{q^3})^2}{p(M+p)^2/(\sqrt{p^3}+\sqrt{q^3})^{4/3}} = \frac{1}{(\sqrt{p^3}+\sqrt{q^3})^{2/3}}.$$

We can then see that, taking our expression for $V$, we find that

$$\frac{\partial V}{\partial M} = \frac{1}{(\sqrt{p^3}+\sqrt{q^3})^{2/3}},$$

and this is indeed the same as our value for $\lambda^*$.