**Mathematics 1 mock examination paper (2015)**

**UNIVERSITY OF LONDON** **MT105a Mock**

**BSc degrees and Diplomas for Graduates in Economics, Management, Finance and the Social Sciences, the Diplomas in Economics and Social Sciences and Access Route**

**Mathematics 1**

Time allowed : 2 hours

Candidates should answer all **EIGHT** questions: all **SIX** of Section A (60 marks in total) and **BOTH** questions from Section B (20 marks each). **Candidates are strongly advised to divide their time accordingly.**

Calculators may not be used for this paper.



**PLEASE TURN OVER**

## SECTION A

Answer all **six** questions from this section (60 marks in total).

**1.** Show that the curve with equation $y = x^2 - x - 2$ does not intersect the line with equation $y = x - 4$. Sketch the two curves on the same diagram.

For which values of the number $a$ will the curve with equation $y = x^2 - x - 2$ intersect the line with equation $y = x - a$?

For which particular value of $a$ will there be precisely one point at which the curve and the line intersect?

**2.** The function $f$ is defined for $x > 0$ by

$$f(x) = ax^2 + bx + \frac{c}{x} \ ,$$

for some constants $a, b, c$. If $f(1) = -5$, $f'(1) = -1$ and $\displaystyle\int_1^2 f(x)\,dx = \ln 2 - 4$, show that the following system of equations holds for $a$, $b$ and $c$:

$$a + b + c = -5 \ , \qquad 2a + b - c = -1 \ , \qquad 14a + 9b + (6\ln 2)c = 6\ln 2 - 24 \ .$$

Express this system of equations in matrix form.

Write down the augmented matrix for this system of equations and **use row operations** to determine the values of $a$, $b$ and $c$.

**3.** The function $f(x, y)$ is defined for $x, y > 0$ by

$$f(x, y) = \frac{ye^{2y}}{x^a},$$

where $a$ is a fixed real number.

Find expressions for the partial derivatives

$$\frac{\partial f}{\partial x} \ , \qquad \frac{\partial f}{\partial y} \ , \qquad \frac{\partial^2 f}{\partial x^2} \ , \qquad \frac{\partial^2 f}{\partial y^2} \ .$$

Determine the values of $a$ for which the function will satisfy the equation

$$yx^2\frac{\partial^2 f}{\partial x^2} - 3y\frac{\partial^2 f}{\partial y^2} + 12f = 0 \ .$$

**4.** Show that the function $f$ given by

$$f(x, y) = y^2 - 4xy + 4x^2 + x^2y^2,$$

has just one critical point.

Show that this critical point is a local minimum of $f$.

**5.** Determine the following integrals:

(a) $\displaystyle\int x^3 \ln(x^2-1)dx$ ,

(b) $\displaystyle\int \frac{x}{\sqrt{e^x}}dx$ ,

(c) $\displaystyle\int \frac{x}{x^2+5x+4}dx$ .

**6.** The number of fleas in a carpet is 10,000 at the start of 2015. Each year, 5% of the fleas die and 2,000 are born. Find an expression, in as simple a form as possible, for the number of fleas $N$ years after the start of 2015.

What happens to the number of fleas in the long run?

Find an expression for the minimum number of years required for the population of fleas to have doubled since the start of 2015.

**SECTION B**

All **both** questions from this section (20 marks each).

**7.** (a) Find the critical points of the function $f$ given by

$$f(x) = x^3 e^{-x} .$$

Determine the nature of each critical point.

(b) A firm is the only producer of two goods, $X$ and $Y$. The demand quantities $x$ and $y$ for $X$ and $Y$ (respectively) and the corresponding prices $p_X$ and $p_Y$ are related as follows:

$$-p_X + p_Y + 13 - 3x = 0 , \qquad p_X - 4p_Y + 26 - 3y = 0 .$$

The firm's joint total cost function (that is, the cost of producing $x$ of $X$ and $y$ of $Y$) is $14x + 7y$. Find an expression in terms of $x$ and $y$ for the profit function. Determine the quantities $x$ and $y$ that maximise the profit.

**8.** A consumer has utility function

$$u(x, y) = \frac{xy}{x + y} ,$$

for two goods, $X$ and $Y$. Here, $x > 0$ denotes the amount of $X$ consumed and $y > 0$ the amount of $Y$ consumed. Each unit of $X$ costs $p$ dollars, each unit of $Y$ costs $q$ dollars, and the consumer has a budget for $X$ and $Y$ of $M$ dollars.

Use the Lagrange multiplier method to find the quantities $x^\star$ of $X$ and $y^\star$ of $Y$ the consumer will consume in order to maximise his utility subject to the budget constraint. (Your answers will depend on $p$, $q$ and $M$.)

Show that

$$x^\star + y^\star = \frac{M}{\sqrt{p}\sqrt{q}} .$$

Hence find the optimal value, $\lambda^\star$, of the Lagrange multiplier, simplifying your expression as much as possible.

Suppose that $V = u(x^\star, y^\star)$ is the maximum achievable utility. Find an explicit expression for $V$ in terms of $p$, $q$ and $M$, simplifying your answer as much as possible.

Find $\dfrac{\partial V}{\partial M}$ and verify that it is equal to $\lambda^\star$.

Find $\dfrac{\partial V}{\partial p}$.

Use this to find an approximate expression for the change in $V$ if the price of $X$ increases by one dollar.



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