Examiners' commentaries 2015 Mock

MT105a Maths 1

Comments on specific questions in Section A

Question 1

The intersection of the curves $y = x^2 - x - 2$ and $y = x - 4$ corresponds to the solution of $x^2 - x - 2 = x - 4$. This is equivalent to $x^2 - 2x + 2 = 0$, which has no solution. We can see this either by observing that its discriminant is

$$\Delta = (-2)^2 - 4(2) = -4 < 0,$$

or by noting that it is equivalent to $(x - 1)^2 = -1$.

Figure 1 shows the details that are required in the sketch of the two curves. Specifically:

- The orientation of the parabola (upwards) and the straight line (positive slope).
- All points of intersection with the $x-$axis and the $y-$axis.
- All critical points (which is simply the minimum of the parabola).

Note that *all four quadrants* are required to show the full behaviour of these curves.

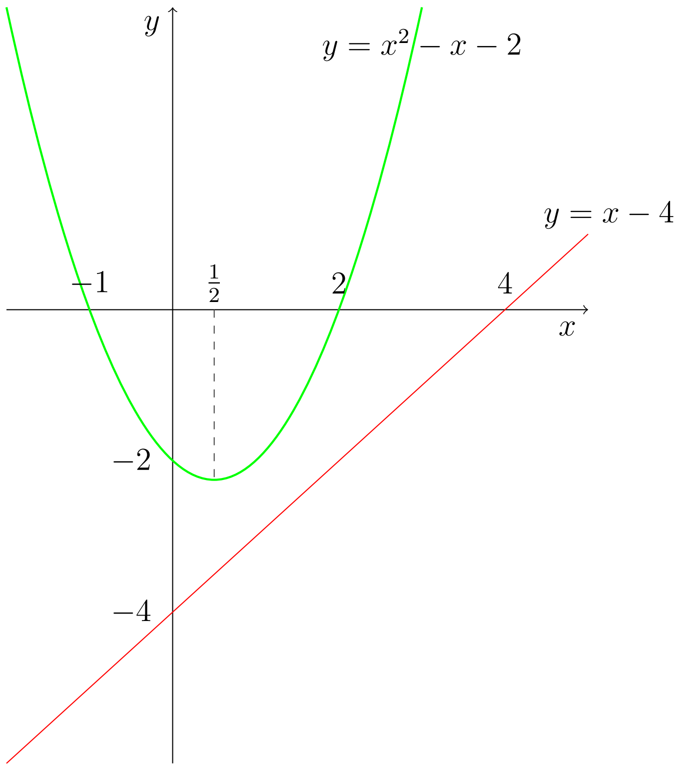

Figure 1: The curves $y = x^2 - x - 2$ and $y = x - 4$.

The graphs of $f(x) = x^2 - x - 2$ and $g(x) = x - a$ intersect if and only if there is a solution to $x^2 - x - 2 = x - a$. This equation is equivalent to

$$x^2 - 2x + (a - 2) = 0,$$

which has solutions if and only if the discriminant Δ satisfies

$$0 \leq \Delta = (-2)^2 - 4(1)(a - 2) = 12 - 4a,$$

i.e. if and only if $a \leq 3$.

There will be exactly one solution when $\Delta = 0$; that is, when $a = 3$.

Question 2

The first condition yields

$$-5 = f(1) = a(1)^2 + b(1) + \frac{c}{(1)} = a + b + c.$$

Noting that f has the derivative $f'(x) = 2ax + b - \frac{c}{x^2}$, the second condition then gives

$$-1 = f'(1) = 2a(1) + b - \frac{c}{(1)^2} = 2a + b - c.$$

Finally, we note that

$$\int_1^2 f(x)\,dx = \left[\frac{a}{3}x^3 + \frac{b}{2}x^2 + c\ln x\right]_1^2 = \frac{7a}{3} + \frac{3b}{2} + c\ln 2.$$

Setting this equal to $\ln 2 - 4$ and multiplying by 6 yields the third and final equation.

In matrix form, the system of equations becomes

$$\begin{pmatrix} 41 & 1 & 1 \\ 2 & 1 & -1 \\ 14 & 9 & 6\ln 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 6\ln 2 - 24 \end{pmatrix}.$$

The augmented matrix associated with this system of equations is then

$$\begin{pmatrix} 1 & 1 & 1 & -5 \\ 2 & 1 & -1 & -1 \\ 14 & 9 & 6\ln 2 & 6\ln 2 - 24 \end{pmatrix}.$$

Under row operations, this becomes:

$$\begin{pmatrix} 1 & 1 & 1 & -5 \\ 0 & 1 & 3 & -9 \\ 0 & -5 & 6\ln 2 - 14 & 6\ln 2 + 46 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & -5 \\ 0 & 1 & 3 & -9 \\ 0 & 0 & 6\ln 2 + 1 & 6\ln 2 + 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & -5 \\ 0 & 1 & 3 & -9 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Backwards substitution then yields

$$c = 1, \quad b = -12, \quad a = 6.$$

Question 3

This looks more difficult than it is. Where candidates are likely to go wrong is in handling the denominator term x^a when differentiating; in particular, in handling the power a. If you are unsure as to how this behaves under differentiation, then it would be helpful to rewrite f as $x^{-a}ye^{2y}$.

The required derivatives are then

$$f_x = -ax^{-a-1}ye^{2y} \qquad f_y = x^{-a}(1+2y)e^{2y}$$

$$f_{xx} = -a(-a-1)x^{-a-2}ye^{2y} \qquad f_{yy} = 4x^{-a}(1+y)e^{2y}$$

The next part of the question clearly expects you to replace f, f_{xx} and f_{yy} with the expressions you have been given or have found above. While it might not be immediately clear how to derive the result asked for once we have made this substitution, our first task should be to *simplify* the resulting expression. Thus:

$$\begin{aligned}
0 &= yx^2 f_{xx} - 3y f_{yy} + 12 f \\
&= yx^2 \frac{a(a+1)ye^{2y}}{x^{a+2}} - 3y\frac{4(1+y)e^{2y}}{x^a} + 12\frac{ye^{2y}}{x^a} && \text{(Substitution)} \\
&= yx^2 \frac{a(a+1)ye^{2y}}{x^{a+2}} - 3y\frac{4(1+y)e^{2y}}{x^a} + 12\frac{ye^{2y}}{x^a} && \text{(Cancelling)} \\
&= \frac{ye^{2y}}{x^a}\Big[a(a+1)y - 12(1+y) - 12\Big] && \text{(Factorising)} \\
&= \frac{ye^{2y}}{x^a}\Big(a(a+1)y - 12y\Big) && \text{(Cancelling)} \\
&= \frac{y^2e^{2y}}{x^a}\Big(a^2 + a - 12\Big) && \text{(Factorising } y\text{)} \\
&= \frac{y^2e^{2y}}{x^a}(a+4)(a-3). && \text{(Factorising)}
\end{aligned}$$

This will equal 0 if and only if $a = -4$ or $a = +3$, which ultimately is what the question asks for!

Question 4

The partial derivatives are

$$f_x = 8x - 4y + 2xy^2, \qquad f_y = -4x + 2y + 2x^2y.$$

At a critical point, both of these must be 0, so we solve

$$\begin{aligned}
0 &= 8x - 4y + 2xy^2, \\
0 &= -4x + 2y + 2x^2y,
\end{aligned}$$

While we can solve these by using one equation to write one variable in terms of the other, and then substituting into the remaining equation, a little consideration suggests an easier approach. By dividing the first equation by 2, we have:

$$\begin{aligned} 0 &= \quad 4x - 2y + xy^2, \\ 0 &= -4x + 2y + 2x^2y. \end{aligned}$$

Here, adding the result equations together will cancel several terms, leaving:

$$0 = xy^2 + 2x^2y = xy(2x + y).$$

The only way this product (of three components) will be equal to zero is if at least one of the components is equal to zero. Thus, this yields *three* possible conditions to check: *either* $x = 0$ *or* $y = 0$ *or* $y = -2x$.

- If $x = 0$ then $y = 0$ (from either original equation) so $(0, 0)$ is a critical point.
- If $y = 0$ then $x = 0$ (from either original equation) so we get $(0, 0)$ again.
- If $y = -2x$ then substituting into, say, the second equation gives

  $$0 = -4x - 4x - 4x^3 = x(1 + x^2),$$

  so $x = 0$ (as $1 + x^2 = 0$ would require $x^2 = -1$) and then $y = 0$. We therefore get, once again, $(0, 0)$. (The same result is obtained from the first equation.)

So $(0, 0)$ is the *only* critical point. (Note that it is not enough here to observe that $(0, 0)$ is a critical point: you must show that it is the *only* one.)

To classify this point, we must first calculate the second order derivatives. From f_x we have

$$f_{xx} = 8 + 2y^2, \qquad f_{xy} = -4 + 4xy;$$

while from f_y we have

$$f_{yy} = 2 + 2x^2, \qquad f_{yx} = -4 + 4xy,$$

where $f_{xy} = f_{yx}$ as expected.

At $(0, 0)$

$$f_{xx} = 8, \quad f_{yy} = 2, \quad f_{xy} = -4.$$

We have $f_{xx}f_{yy} - f_{xy}^2 = 0$, so *the second derivative test fails.*

What can we do now? This is difficult, for there are no set rules about what to do when the test fails, and to answer this question you will have to experiment a little.

We could consider the behaviour of the function around the critical point $(0, 0)$, where we note that $f(0, 0) = 0$. After a few calculations, we might start seeing that $f(x, y)$ is positive, i.e. strictly greater than 0, at any point we consider, suggesting that the function might be non-negative, only taking the value 0 at our critical point. By looking at the function, we may begin to recognise the familiar form of a quadratic in x and y, and in fact, we find that

$$f(x, y) = (y - 2x)^2 + x^2y^2 \;\geq\; 0 = f(0, 0).$$

Question 5

These types of questions can be difficult because it is not always clear which of the three main techniques will work. However, the more practice you get with these types of questions, the more intuition you will develop in identifying the best approach: do not be afraid to try something and fail here, as long as you can identify when you have gone wrong.

On that point, note that each of these three methods do not avoid integration, but merely exchange one integral for another. Thus, the first thing you should do immediately after applying any of these methods is ask yourself whether your new integral is any easier than the original one. If not, then *do not proceed any further with it* and try something else.

Part (a)

While it is possible to use parts on this integral, it proves to be far simpler to use substitution. Here we shall try the simple substitution $w = x^2 - 1$. Then $dw = 2xdx$ and

$$I = \frac{1}{2}\int (w+1)\ln w \, dw$$

Before we proceed, comparison with our initial integral does indeed reveal an easier integral to perform. While we still have a product of two functions, the log function is a much simpler $\ln u$, and this is multiplied by a linear function, rather than a cubic. Thus, we shall proceed with this approach. Using integration by parts, the integral then equals

$$\begin{aligned}
&= \frac{1}{2}\left(\frac{w^2}{2} + w\right)\ln w - \frac{1}{2}\int \left(\frac{w^2}{2} + w\right)\frac{1}{w}\, dw \\
&= \left(\frac{w^2}{4} + \frac{w}{2}\right)\ln w - \frac{w^2}{8} - \frac{w}{2} + c \\
&= \left(\frac{(x^2-1)^2}{4} + \frac{(x^2-1)}{2}\right)\ln(x^2-1) - \frac{(x^2-1)^2}{8} - \frac{x^2-1}{2} + c.
\end{aligned}$$

Part (b)

We rewrite the integrand in a form which is more intuitive for integration, specifically

$$J = \int x e^{-x/2}\, dx$$

While we can approach this using substitution, here we use parts to get

$$= -2xe^{-x/2} + 2\int e^{-x/2}\, dx.$$

Once again, before proceeding, we compare our integrals and note that our new integral *is* indeed simpler to consider. Not only are we considering the integral of a product but we have actually computed the remaining integral in our application of parts! We can then easily write down the final step:

$$= -2xe^{-x/2} - 4e^{-x/2} + c.$$

Part (c)

$$K = \int \frac{x}{x^2 + 5x + 4}\, dx$$

Before we begin, our immediate reaction to a quadratic expression is to ask whether we can factorise it and, if so, what is its factorisation. In fact, we have

$$= \int \frac{x}{(x+4)(x+1)}\, dx$$

Using partial fractions, the integral then becomes

$$\begin{aligned} &= \int \frac{4/3}{x+4} - \frac{1/3}{x+1}\, dx \\ &= \frac{4}{3}\ln|x+4| - \frac{1}{3}\ln|x+1| + c. \end{aligned}$$

Question 6

Many candidates immediately launch into using sequence notation to answer this question, without fully defining, or perhaps understanding, what their sequences represent. While sequence notation provides a useful shorthand for this problem, it is not necessary, and here we shall answer the question without using it at all.

For the first part, we are asked to determine the population of fleas N years after the start of 2015. Here, the year 2015 is not important, rather the number of years that have elapsed since then (e.g. 0 years, 1 year, 2 years, etc). To derive a general expression, as asked, let us therefore consider the number of fleas are 0 years, 1 year, 2 years, etc, have elapsed, and see if a pattern emerges. Thus, we construct the following table:

Years after start of 2015	Number of fleas
0	10000
1	$10000(0.95) + 2000$
2	$\left[10000(0.95) + 2000\right] \times 0.95 + 2000$
	$= 10000(0.95)^2 + 2000(0.95) + 2000$
3	$\left[10000(0.95)^2 + 2000(0.95) + 2000\right] \times 0.95$
	$= 10000(0.95)^3 + 2000(0.95)^2 + 2000(0.95) + 2000$

In trying to derive the general expression, we identify the following pattern:

- Each expression contains a term involving 10000.

- The expression for the number of fleas t years after the start of 2015 contains $t+1$ terms...
- ... t of which involve 2000.

The general expression for the population after N years is then

$$\underbrace{10000(0.95)^N + \underbrace{2000(0.95)^{N-1} + 2000(0.95)^{N-2} + \cdots + 2000(0.95) + 2000}_{N \text{ terms}}}_{N+1 \text{ terms}}$$

Factorising out the common term of 2000, we have

$$= 10000(0.95)^N + 2000\Big[1 + (0.95) + \cdots + (0.95)^{N-2} + (0.95)^{N-1}\Big]$$

The term in the square brackets is a geometric progression, with initial term 1, comon ratio 0.95 and consisting of N terms (*not* $N-1$). Using our standard formula, we can therefore rewrite this as

$$= 10000(0.95)^N + 2000\left[\frac{1-(0.95)^N}{1-0.95}\right]$$

Which we simplify to get

$$= 40000 - 30000(0.95)^N.$$

Note that with this simplified form, it is easy now to see the long run behaviour: as we consider more and more years after the start of 2015, the population is calculated as 40000 less a smaller and smaller fraction of 30000. That is, in the long run, the population of fleas increases to $40,000$ (though never reaches that limit).

For the rider, we require the least N such that

$$\begin{aligned} 40000 - 30000(0.95)^N \geq 20000 \quad &\Rightarrow \quad 20000 \geq 30000(0.95)^N \\ &\Rightarrow \quad \frac{2}{3} \geq (0.95)^N \\ &\Rightarrow \quad \log\left(\frac{2}{3}\right) \geq N \log 0.95 \\ &\Rightarrow \quad N \geq \frac{\log 2/3}{\log 0/95}. \end{aligned}$$

That is N must be the smallest integer greater than $\log(2/3)/\log(0.95)$. (This actually turns out to be 8, though you are not expected to calculate this in an examination.)

Comments on specific questions in Section B

Question 7

Part (a)

We have
$$f'(x) = 3x^2e^{-x} - x^3e^{-x} = e^{-x}x^2(3-x).$$

For a critical point, $f'(x) = 0$, which only occurs if either $x = 0$ or $x = 3$.

We classify these critical points by looking at the sign of f'. Using the fact that we can write this as a product of 3 simple functions, we construct the following table:

	$x < 0$	$0 < x < 3$	$3 < x$
e^{-x}	$+$	$+$	$+$
x^2	$+$	$+$	$+$
$3 - x$	$+$	$+$	$-$
$f'(x)$	$+$	$+$	$-$

As the slope of f is positive both before and after the critical point $x = 0$, this is an inflexion.

As the slope of f is positive before and negative after $x = 3$, then this must be a maximum.

Part (b)

This is a very standard application of unconstrained optimisation: see Chapter 5 of the subject guide and the related readings.

To express profit as a function of x and y, we need p_X, p_Y in terms of x and y. One way of doing this is to add the two equations to get
$$0 = -3p_Y + 39 - 3x - 3y,$$
which gives us $p_Y = 13 - x - y$. Substituting this into the first equation, we find that $p_X = 26 - 4x - y$.

Thus profit, as a function of x and y, is given by
$$\begin{aligned}\Pi(x,y) &= TR(x,y) - TC(x,y)\\ &= xp_X + yp_Y - (14x + 7y)\\ &= 12x + 6y - 4x^2 - y^2 - 2xy.\end{aligned}$$

To find the values of x and y that maximise the profit, we solve the equations $\Pi_x = 0$ and $\Pi_y = 0$, i.e.

$$0 = \Pi_x = 12 - 8x - 2y,$$
$$0 = \Pi_y = 6 - 2y - 2x.$$

This is easily done and and we find that $x = 1$ and $y = 2$.

The second derivatives are

$$\Pi_{xx} = -8, \qquad \Pi_{xy} = -2 = \Pi_{yx}, \qquad \Pi_{yy} = -2.$$

Since these assure us that

$$\Pi_{xx}\Pi_{yy} - \Pi_{xy}^2 = (-8)(-2) - (-2)^2 = 12 > 0 \quad \text{and} \quad \Pi_{xx} < 0,$$

these values of x and y do indeed maximise the profit.

Question 8

This question primarily concerns the Lagrange multiplier method, discussed in chapter 5 of the subject guide.

To determine optimal affordable consumption, the consumer maximises $u(x, y)$ subject to the budget constraint $px + qy = M$. The Lagrangian is

$$L = \frac{xy}{x+y} - \lambda(px + qy - M).$$

Setting the first-order derivatives equal to zero yields:

$$0 = L_x = \frac{y}{x+y} - \frac{xy}{(x+y)^2} - p\lambda = \frac{y^2}{(x+y)^2} - p\lambda, \tag{1}$$
$$0 = L_y = \frac{x}{x+y} - \frac{xy}{(x+y)^2} - q\lambda = \frac{x^2}{(x+y)^2} - q\lambda, \tag{2}$$
$$M = px + qy. \tag{3}$$

Eliminating λ from equations (1) and (2), and rearranging, we see that $px^2 = qy^2$ so that

$$y = \sqrt{\frac{p}{q}}x. \tag{$\dagger$}$$

Substituting into $px + qy = M$ implies $px + \sqrt{pq}x = M$, so $x^\star$, the solution to this (and our Lagrange equations), is given by

$$x^\star = \frac{M}{p + \sqrt{pq}}.$$

Substituting this into ($\dagger$) then gives[1]

[1]Note the symmetry in the solutions for $x^\star$ and $y^\star$, which reflects the underlying symmetry of the problem. *Hint: exchange x with y and p and q throughout to see this.*

$$y^\star = \sqrt{\frac{p}{q}} x^\star = \sqrt{\frac{p}{q}} \frac{M}{p+\sqrt{pq}} = \frac{M}{q+\sqrt{pq}}.$$

Here we have simplified our expression for $y^\star$ in anticipation of the work to follow, but it is something that you should get into the habit of doing anyway.

Now,

$$\begin{aligned}
x^\star + y^\star &= M\left(\frac{1}{p+\sqrt{pq}} + \frac{1}{q+\sqrt{pq}}\right) \\
&= M\left(\frac{1}{\sqrt{p}(\sqrt{p}+\sqrt{q})} + \frac{1}{\sqrt{q}(\sqrt{p}+\sqrt{q})}\right)
\end{aligned}$$

Obtaining a common denominator, we then have

$$\begin{aligned}
&= M\left(\frac{\sqrt{q}+\sqrt{p}}{\sqrt{p}\sqrt{q}(\sqrt{p}+\sqrt{q})}\right) \\
&= \frac{M}{\sqrt{p}\sqrt{q}}.
\end{aligned}$$

The corresponding value of $\lambda^\star$ is

$$\begin{aligned}
\lambda^\star &= \frac{1}{p}(y^\star)^2 \frac{1}{(x^\star + y^\star)^2} \\
&= \frac{1}{p}\left(\frac{tM}{\sqrt{q}(\sqrt{q}+\sqrt{p})}\right)^2 \left(\frac{M}{\sqrt{p}\sqrt{q}}\right)^{-2}
\end{aligned}$$

Here the details are left to the reader to fill in. However, note that we have an M^2 both in the numerator and the denominator, so that we can expect those to cancel. Similar, we have a p in the denominator and a $(\sqrt{p})^2$ in the numerator, cancelling each other out. Proceeding systematically in this manner, we ultimately arrive at

$$\lambda^\star = \frac{1}{(\sqrt{q}+\sqrt{p})^2}.$$

We also have

$$V = \frac{x^\star y^\star}{x^\star + y^\star} = \frac{M}{\sqrt{p}(\sqrt{p}+\sqrt{q})} \frac{M}{\sqrt{q}(\sqrt{q}+\sqrt{p})} \left(\frac{M}{\sqrt{p}\sqrt{q}}\right)^{-1} = \frac{M}{(\sqrt{p}+\sqrt{q})^2}.$$

So

$$\frac{\partial V}{\partial M} = \frac{1}{(\sqrt{p}+\sqrt{q})^2} = \lambda^\star.$$

Finally, we have

$$\frac{\partial V}{\partial p} = -\frac{M}{\sqrt{p}(\sqrt{q}+\sqrt{p})^3}.$$

So if p increases by 1 dollar, V *decreases* by approximately $M/(\sqrt{p}(\sqrt{q}+\sqrt{p})^3)$. This last part relies on us understanding that the derivative of V with respect to p means the rate of change of V when p is changed.