151163A - Financial Econometrics

IIb. Stationary Processes: Autoregressive (AR) Model

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AR(1) model

An AR(1) model reads

$$X_t = a_0 + a_1 X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right). \tag{1}$$

Conditional on X_{t-1},

$$\mathbb{E}\left[X_t | X_{t-1}\right] = a_0 + a_1 X_{t-1}, \qquad \operatorname{var}(X_t | X_{t-1}) = \sigma^2.$$

AR(1) model

By repeated substitutions of Eq.(1),

$$
\begin{aligned}
X_t &= a_0 + a_1 X_{t-1} + \varepsilon_t \\
&= a_0 + a_1[a_0 + a_1(X_{t-2} - \mu_X) + \varepsilon_{t-1}] + \varepsilon_t \\
&= (1 + a_1)a_0 + a_1^2(X_{t-2} - \mu_X) + a_1\varepsilon_{t-1} + \varepsilon_t \\
&= \dots \\
&= \lim_{J\to\infty} a_1^J X_{t-J} + a_0 \sum_{j=0}^{\infty} a_1^j + \sum_{j=0}^{\infty} a_1^j \varepsilon_{t-j}.
\end{aligned}
$$

Therefore, an AR(1) process is also a linear process if the summation converges and if its variance is finite.

AR(1) model

If $|a_1| < 1$,

- the mean is

$$\mathbb{E}\left[X_t\right] = a_0 \sum_{j=0}^{\infty} a_1^j = \frac{a_0}{1 - a_1}$$

- the variance is

$$\gamma_X(0) = \sigma^2 \sum_{j=0}^{\infty} a_1^{2j} = \frac{\sigma^2}{1 - a_1^2}$$

AR(1) model

If $|a_1| < 1$,

- the autocovariance is

$$\gamma_X(h) = \sigma^2 \sum_{j=0}^{\infty} a_1^j \cdot a_1^{j+h} = \frac{a_1^h \sigma^2}{1 - a_1^2}$$

- the autocorrelation is

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = a_1^h.$$

The above results imply that

1. μ_X exists if $a_1 \neq 1$.
2. $\mu_X = 0$ if and only if $a_0 = 0$.
3. $\gamma_X(h)$ exists if $|a_1| < 1$.
4. If μ_X and $\gamma_X(h)$ exist, then X_t is weakly stationary.

Therefore, an AR(1) process is weakly stationary *if and only if* $-1 < a_1 < 1$, or $|a_1| < 1$. We can write an AR(1) process as

$$X_t = \mu_X + \sum_{j=0}^{\infty} a_1^j \varepsilon_{t-j}.$$

Using the property of the weak stationarity of X_t, we can find its moments more easily. Taking expectation on both sides of Eq.(1),

$$\begin{aligned}\mu_X = \mathbb{E}\left[X_t\right] &= a_0 + a_1 \mathbb{E}\left[X_{t-1}\right] \\ &= a_0 + a_1 \mu_X\end{aligned}$$

Therefore,

$$\mu_X = \frac{a_0}{1 - a_1}.$$

Substitute $a_0 = \mu_X(1 - a_1)$ into Eq.(1),

$$X_t - \mu_X = a_1(X_{t-1} - \mu_X) + \varepsilon_t. \tag{2}$$

Taking the square and then the expectation of Eq.(2),

$$\mathbb{E}\left[(X_t - \mu_X)^2\right] = \gamma_X(0) = a_1^2\gamma_X(0) + \sigma^2$$
$$\implies \gamma_X(0) = \frac{\sigma^2}{1 - a_1^2}.$$

Multiplying $(X_{t-h} - \mu_X)$ to Eq.(2) and taking expectation,

$$\gamma_X(h) = \begin{cases} a_1\gamma_X(1) + \sigma^2 & \text{if } h = 0 \\ a_1\gamma_X(h-1) & \text{if } h > 0 \end{cases}$$

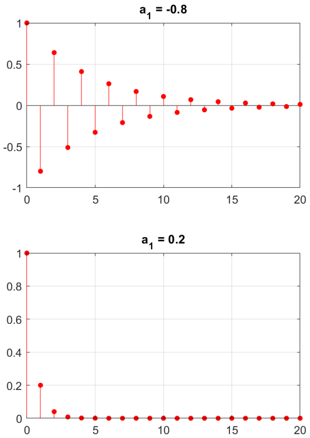
Using the fact that $\gamma_X(h) = \gamma_X(-h)$, we have

$$\gamma_X(0) = \frac{\sigma^2}{1 - a_1^2}, \qquad \gamma_X(h) = a_1\gamma_X(h-1) = a_1^h\gamma_X(0).$$

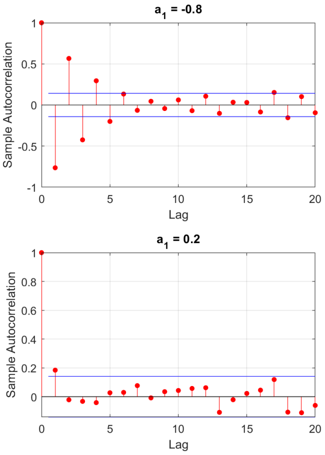
Therefore, the autocorrelation function (ACF) of X_t is

$$\rho_X(h) = a_1\rho_X(h-1) = a_1^h.$$

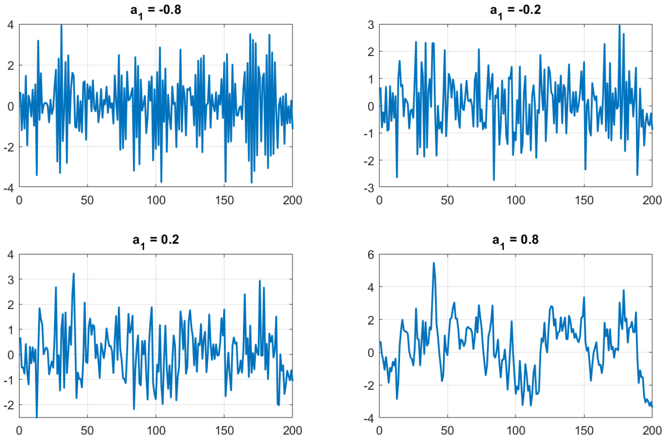
AR(1): Autocorrelation function



AR(1): Sample autocorrelation function



AR(1): Simulation



In general, an AR(p) process is given by

$$X_t = a_0 + a_1 X_{t-1} + \cdots + a_p X_{t-p} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} (0, \sigma^2) \, .$$

Similarly, if X_t is stationary, its mean can be obtained simply as

$$\mu_X = \mathbb{E}\left[X_t\right] = \frac{a_0}{1 - a_1 - \cdots - a_p}.$$

Yule-Walker equations

Substituting $a_0 = (1 - a_1 - \cdots - a_p)\mu_X$, we have

$$X_t - \mu_X = a_1(X_{t-1} - \mu_X) + \cdots + a_p(X_{t-p} - \mu_X) + \varepsilon_t.$$

Multiplying $X_{t-h} - \mu_X$ and taking expectation yield

$$\gamma_X(h) - a_1\gamma_X(h-1) - \cdots - a_p\gamma_X(h-p) = \begin{cases} 0 & \text{if } h > 0 \\ \sigma^2 & \text{if } h = 0 \end{cases}$$

Yule-Walker equations

Dividing $\gamma_X(0)$ from both equations, and noting that $\gamma_X(-h) = \gamma_X(h)$,

$$\begin{aligned} \sigma^2 &= \gamma_X(0)(1 - a_1\rho_X(1) - \cdots - a_p\rho_X(p)) \\ \rho_X(h) &= a_1\rho_X(h-1) + \cdots + a_p\rho_X(h-p), \qquad h > 0 \end{aligned}$$

We can then estimate σ^2 and $a_1, \ldots, a_p$ from the autocorrelations.

Example

Consider a weakly stationary AR(2) process

$$X_t = a_1 X_{t-1} + a_2 X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right).$$

Express a_1 and a_2 as a function of the autocorrelations.

Stationarity of an AR(p) process

Let L be the lag operator, i.e., $LX_t = X_{t-1}$, $L^2 X_t = X_{t-2}$, we can write the AR(p) process as

$$(1 - a_1 L - \cdots - a_p L^p)(X_t - \mu_X) = \varepsilon_t.$$

Theorem

An AR(p) process X_t is weakly stationary if the solutions of the characteristic equation

$$1 - a_1 L - \cdots - a_p L^p = 0$$

are larger than one in absolute value.

Example

Evaluate the stationarity property of the AR(2) process

$$X_t = 0.4X_{t-1} - 0.04X_{t-2} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right).$$