Applied Stochastic Process

1 Poisson Process and Renewal Process

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Poisson and Exponential Distributions

Poisson distribution

Suppose $X_n \sim Bin(n, \lambda/n)$. Let $X = \lim_{n\to\infty} X_n$, when $n \to \infty$ and $p \to 0$, we can show that

$$P(X = i) = \mathrm{e}^{-\lambda}\frac{\lambda^i}{i!}, \qquad i = 0, 1, \ldots$$

We say X has the Poisson distribution with parameter λ, or $X \sim Poi(\lambda)$.

Properties

Some properties of the Poisson distribution:

- Moments: $\mathbb{E}[X] = \lambda$, $\text{var}(X) = \lambda$.
- Independent sums: If X_i are independent $Poi(\lambda_i)$, then $X_1 + \cdots + X_n \sim Poi(\lambda_1 + \cdots + \lambda_n)$

Exponential distribution

Suppose $T_n \sim Geo(p_n)$ where $p_n = \lambda/n$ and $T = \lim_{n\to\infty} n^{-1}T_n$, then one can show that

$$F_T(t) = P(T \le t) = 1 - \mathrm{e}^{-\lambda t}, \qquad t \ge 0.$$

We say T is exponentially distributed with parameter λ, or $T \sim exp(\lambda)$. Its probability density function is given by

$$f_T(t) = \begin{cases} \lambda \mathrm{e}^{-\lambda t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

Properties

Some properties of the exponential distribution:

- Moments: $\mathbb{E}\left[T\right] = 1/\lambda$, $\mathrm{var}(T) = 1/\lambda^2$.
- Lack of memory: $P(T > t + s | T > t) = P(T > s)$.
- Exponential races: If $S \sim exp(\lambda)$ and $T \sim exp(\mu)$ are independent, then $\min\{S, T\} \sim exp(\lambda + \mu)$.

Poisson Process

Definition

$\{N(t), t \geq 0\}$ is a Poisson process if

1. $N(0) = 0$
2. $N(t)$ has *stationary* increments, specifically, $N(t+s) - N(s) \sim Poi(\lambda t)$
3. $N(t)$ has *independent* increments, i.e., for $t_0 < t_1 < \cdots < t_n$, $N(t_1) - N(t_0), \ldots, N(t_n) - N(t_{n-1})$ are independent.

Defining $o(\delta)$ as $\lim_{\delta\to 0}\frac{o(\delta)}{\delta}=0$, a Poisson process possess the following properties:

Property

- $N(t)=N(0+t)-N(0)\sim Poi(\lambda t)$
- For some $\delta\to 0$,
  - $P(N(\delta)=0)=1-\lambda\delta+o(\delta)$
  - $P(N(\delta)=1)=\lambda\delta+o(\delta)$
  - $P(N(\delta)=2)=o(\delta)$
- The autocovariance function is $\gamma(t_1,t_2)=\lambda\min\{t_1,t_2\}$

Construction

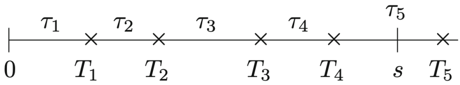
Let $\tau_1, \tau_2, \ldots$ be independent $\exp(\lambda)$ random variables. Let $T_n = \tau_1 + \cdots + \tau_n$, $T_0 = 0$. Then,

$$N(s) = \max\{n : T_n < s\}$$

is a Poisson process with mean λs.

Example: Customer arrivals

Let τ_n be the time between arrivals of customers, so that $T_n = \tau_1 + \cdots + \tau_n$ is the arrival time of the n-th customer. Let $N(s)$ be the number of arrivals by time s. Then, $N(s)$ follows a Poisson process with mean λs. For example,



Then, $N(s) = 4$ when $T_4 \leq s < T_5$.

Renewal Process

Let $\tau_1, \tau_2, \ldots$ be independent $\exp(\lambda)$ random variables. Let $T_n = \tau_1 + \cdots + \tau_n$, $T_0 = 0$. Then,

$$N(s) = \max\{n : T_n < s\}$$

is a Poisson process with mean λs.

Let $X_1, X_2, \ldots$ be independent random variables with some distribution F. Let $S_n = X_1 + \cdots + X_n$, $S_0 = 0$. Then,

$$N(s) = \max\{n : S_n < s\}$$

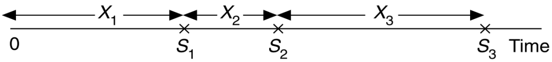
is a *renewal* process.

Example

Suppose that we have an infinite supply of lightbulbs whose lifetimes are independent and identically distributed. Suppose also that we use a single lightbulb at a time, and when it fails we immediately replace it with a new one. Under these conditions, $\{N(t), t \geq 0\}$ is a renewal process when $N(t)$ represents the number of lightbulbs that have failed by time t.

Distribution of $N(t)$

Notice that the number of renewals by time t is greater than or equal to n if and only if the n-th renewal occurs before or at time t, i.e., $N(t) \geq n \iff S_n \leq t$,



Therefore, the distribution of $N(t)$ is related to that of S_n by

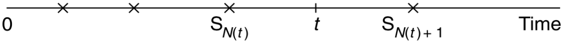
$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n+1) \\ &= P(S_n \leq t) - P(S_{n+1} \leq t) \\ &= F_n(t) - F_{n+1}(t) \end{aligned}$$

where F_n is the distribution of $S_n = \sum_{i=1}^{n} X_i$.

Arrival rate

Notice that:

- $S_{N(t)}$ represents the time of the last renewal *prior to or at* time t; and that
- $S_{N(t)+1}$ represents the time of the first renewal *after* time t,



We can show that, with probaility 1, as $t \to \infty$

$$\frac{N(t)}{t} \to \mu, \text{ where } \mathbb{E}\left[X_i\right] = \frac{1}{\mu}.$$

The number μ is called the *rate* of the renewal process.

Application: $GI/G/1$ Queue

- GI (General input): Time between successive arrivals are independent with distribution F and mean $1/\lambda$.
- G (General service time): The i-th customer requires an amount of service s_i, which is independent with distribution G and mean $1/\mu$.
- 1 (One server)

Application: $GI/G/1$ Queue

Theorem

Suppose $\lambda < \mu$. If the queue starts with some finite number $k \geq 1$ customers who need service, then it will empty out with probability one. That is, the queue is stable. Furthermore, the limiting fraction of time the server is busy is at most λ/μ.