151163A - Financial Econometrics

III. Nonstationary Processes: Unit Root and Unit Root Test

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ARIMA(p, d, q) Model

Consider the ARMA(p, q) process

$$(1 - a_1 L - \cdots - a_p L^p)(X_t - \mu_X) = (1 - b_1 L - \cdots - b_q L^q)\varepsilon_t$$

where $\varepsilon_t \overset{\text{iid}}{\sim} (0, \sigma^2)$. Factoring the lag polynomial for the AR part,

$$(1 - r_1 L) \ldots (1 - r_p L)(X_t - \mu_X) = (1 - b_1 L - \cdots - b_q L^q)\varepsilon_t.$$

This process is weakly stationary if $|r_j| < 1$ for all $j = 1, \ldots, p$.

What if $r_1 = \cdots = r_d = 1$?

- If $r_j = 1$ for some j, then the ARMA process is not weakly stationary.
- Since r_j are the roots of the lag polynomial, we say X_t has d *unit roots* if $r_1 = \cdots = r_d = 1$?

Integrated ARMA process

Let $\Delta = (1 - L)$, and suppose that $r_1 = \cdots = r_k = 1$, while $|r_j| < 1$ for $j = d + 1, \ldots, p$. Then,

$$\Delta^d (X_t - \mu_X)$$
$$= \underbrace{(1 - r_{d+1} L)^{-1} \ldots (1 - r_p L)^{-1} (1 - b_1 L - \cdots - b_q L^q) \varepsilon_t}_{\text{this is weakly stationary ARMA}(p-d,q)\text{ process}}.$$

We call X_t an ARIMA$(p - d, d, q)$ process.

Definition

More generally, we call X_t an $I(d)$ process, or $X_t \sim I(d)$, if X_t is weakly stationary after first differencing for d times.

Suppose Y_t is a weakly stationary process, and suppose also that $X_t = \Delta^d Y_t$, then $X_t \sim I(d)$. Moreover, $Y_t \sim I(0)$.

Random Walk Model

Random walk process

Consider the following AR(1) process

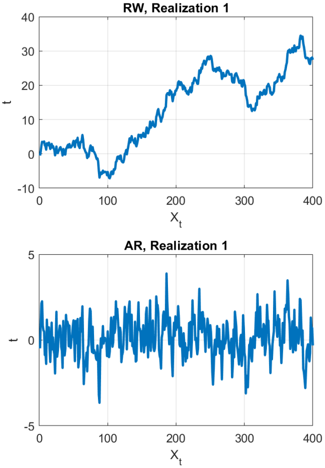
$$X_t = X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right).$$

Suppose the stochastic process X_t starts at time 0 from X_0, then

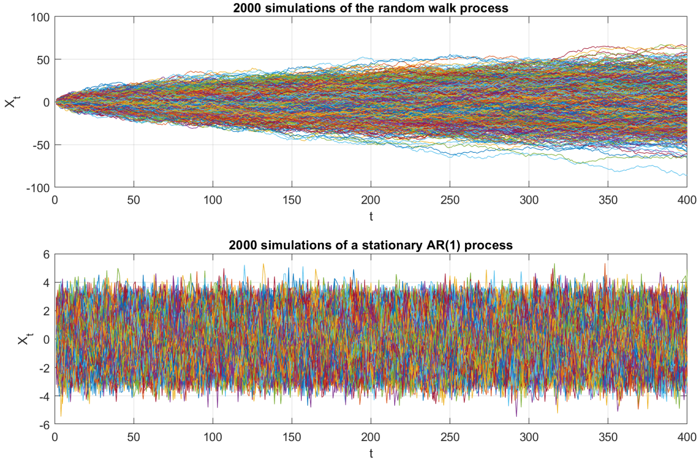
$$\begin{aligned}
X_1 &= X_0 + \varepsilon_1 & \mathbb{E}\left[X_1\right] &= X_0, \;\; \mathrm{var}(X_1) = \sigma^2 \\
&\vdots & &\vdots \\
X_t &= X_0 + \varepsilon_1 + \cdots + \varepsilon_t & \mathbb{E}\left[X_t\right] &= X_0, \;\; \mathrm{var}(X_t) = t\sigma^2
\end{aligned}$$

We say a unit root process has a *stochastic process*.

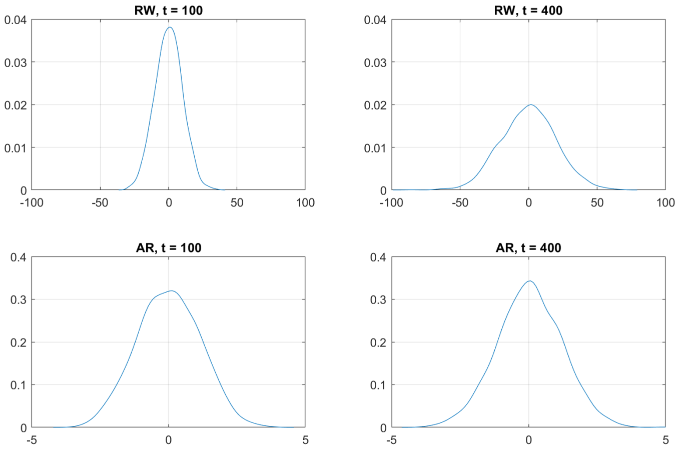
Random walk process



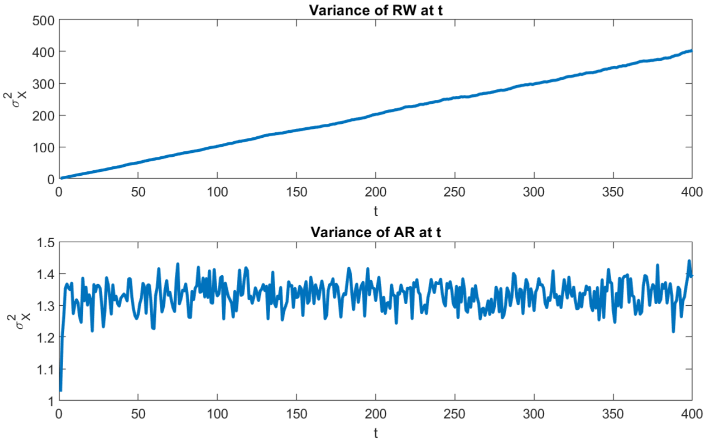
Random walk process



Random walk process



Random walk process



Random walk process with an intercept

Consider the following AR(1) process with an intercept

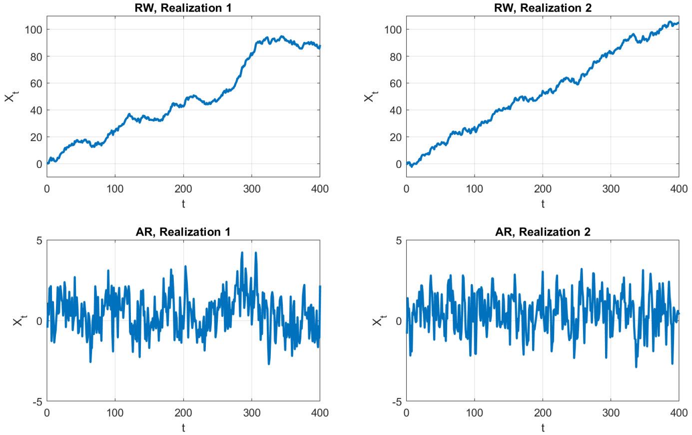
$$X_t = a_0 + X_{t-1} + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right).$$

Suppose the stochastic process X_t starts at time 0 from X_0, then

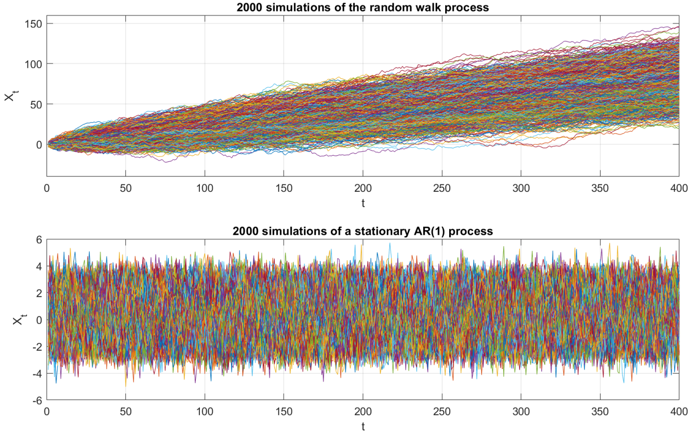
$$\begin{aligned} X_1 &= a_0 + X_0 + \varepsilon_1 \\ &\vdots \\ X_t &= t \cdot a_0 + X_0 + \varepsilon_1 + \cdots + \varepsilon_t \end{aligned}$$

Therefore, $\mathbb{E}\left[X_t\right] = t \cdot a_0 + X_0$, $\operatorname{var}(X_t) = t\sigma^2$.

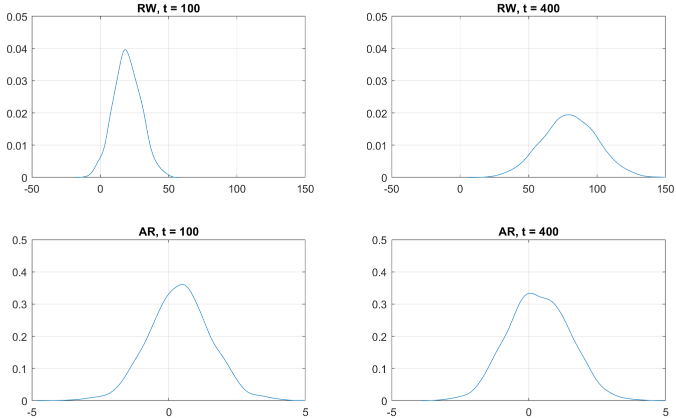
Random walk process with drift



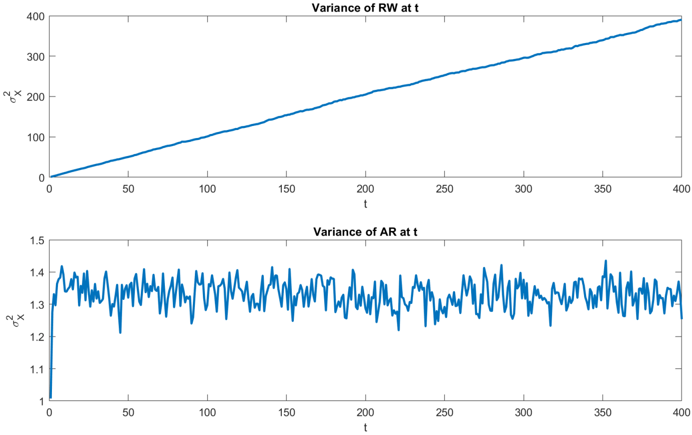
Random walk process with drift



Random walk process with drift



Random walk process with drift



Consider the following AR(1) process with a time trend

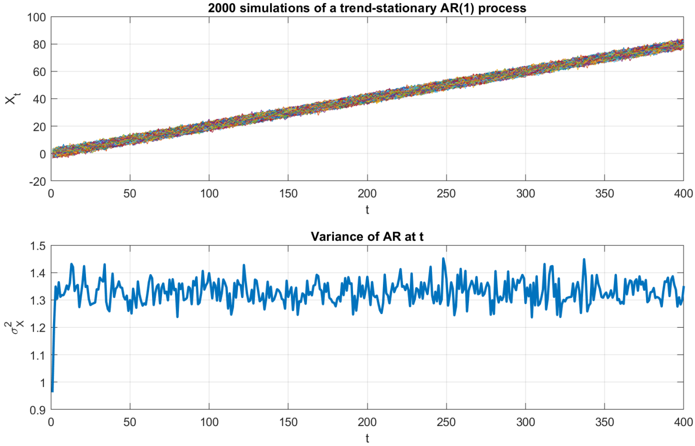
$$(1 - a_1 L)(X_t - \mu_X - \gamma \cdot t) = \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} \left(0, \sigma^2\right).$$

Assuming $|a_1| < 1$, then the model can be written as

$$X_t = \mu_X + \gamma \cdot t + \sum_{j=0}^{\infty} a_1^j \varepsilon_{t-j}.$$

Therefore, $\mathbb{E}\left[X_t\right] = \mu_X + t \cdot \gamma + X_0$, $\operatorname{var}(X_t) = \sigma^2/(1 - a_1^2)$. We call the component $t \cdot \gamma$ *deterministic trend.*

Trend-stationary process

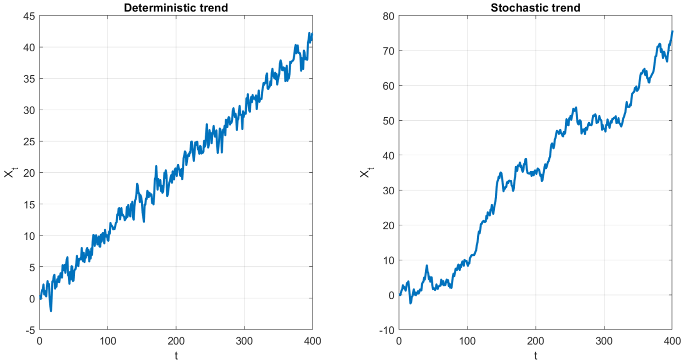


Stochastic trend vs deterministic trend

A stochastic trend and a deterministic trend are different in the following aspects:

1. The mean of a process with a *deterministic* trend increases with time; while that with a *stochastic* trend does not.
2. The variance of a process with a *stochastic* trend increases with time; while that with a *deterministic* trend does not.
3. A process with a *stochastic* trend can be made stationary by first-differencing; while that with a *deterministic* trend can be made stationary by linear regression.

Stochastic trend vs deterministic trend



Unit Root Test

Consider the AR(1) model

$$X_t = a_1 X_{t-1} + \varepsilon_t, \qquad X_0 = 0,\ \varepsilon \overset{\text{iid}}{\sim} \left(0, \sigma^2\right).$$

The unit root test is equivalent to testing the hypothesis

$$H_0 : a_1 = 1, \qquad H_1 : a_1 < 1$$

The least-square estimators of the model is given by

$$\widehat{a}_1 = \frac{\sum_{t=2}^{T} X_{t-1} X_t}{\sum_{t=2}^{T} X_{t-1}^2}, \qquad \widehat{\sigma}^2 = \frac{\sum_{t=2}^{T} (X_t - \widehat{a}_1 X_t)^2}{T-2}$$

The Dickey-Fuller test statistic is

$$\mathrm{DF} = \frac{\widehat{a}_1 - 1}{\mathrm{std}(\widehat{a}_1)} = \frac{\sum_{t=2}^{T} X_{t-1} \varepsilon_t}{\sum_{t=2}^{T} X_{t-1}^2}$$

- The Dickey-Fuller test statistic converges to a function of the Brownian motion as $T \to \infty$.
- The Dickey-Fuller test statistic converges to different values if we include the intercept or a time trend in the model.
- Under either cases, simulation is used to obtain the critical value of the test.

If one is interested in finding the presence of a unit root in an AR(p) process, one may consider estimating

$$\Delta X_t = \beta X_{t-1} + \sum_{j=1}^{p-1} \phi_j \Delta X_{t-j} + \varepsilon_t$$

and testing the null hypothesis $H_0 : \beta = 0$ versus $H_1 : \beta < 0$. This is called the *augmented Dickey-Fuller test*.

To determine the order of integration of a time series X_t, we may follow the steps below:

1. Test the level of the series for a unit root.
    - If the null is rejected, stop and conclude that $X_t \sim I(0)$.
    - If the null is not rejected, conclude that X_t is at least $I(1)$ and move to the next step.
2. Test the first difference of the series for a unit root.
    - If the null is rejected, stop and conclude that $X_t \sim I(1)$.
    - If the null is not rejected, conclude that X_t is at least $I(2)$ and move to the next step.

After determining the integration order d of X_t, take the first difference d times and analyze the properties of $\Delta^d X_t$.