FINM 32000: Homework 6

Due Friday May 10, 2024 at 11:59pm

Problem 1

Let $\mathbf{S}$ be the column vector with components $S^{[1]}, S^{[2]}$, where the stock prices $S^{[j]}$ have risk-neutral dynamics

$$\mathrm{d}S_t^{[j]} = rS_t^{[j]}\mathrm{d}t + \sigma_{[j]}S_t^{[j]}\mathrm{d}W_t^{[j]} \qquad j = 1, 2$$

with risk-free interest rate $r = 0.05$, and constant volatilities $\sigma_{[1]} = 0.3$, $\sigma_{[2]} = 0.2$.

The time-0 prices are $S_0^{[1]} = 100$, $S_0^{[2]} = 110$. The $\mathbb{P}$-Brownian motions $W^{[1]}$ and $W^{[2]}$ have correlation $\rho = 0.8$.

(a) Let $\mathbf{X}$ be the column vector with components $X^{[1]}, X^{[2]}$ where $X^{[j]} := \log S^{[j]}$. Find the covariance matrix of $\mathbf{X}_T$.

Hint: One approach is to manually fill in the covariance matrix, using relationships such as $\mathrm{Cov}(W_T^{[1]}, W_T^{[2]}) = 0.8T$ in combination with the volatilities.

Another approach is to use matrix multiplication: write $\mathbf{X}_T$ as a nonrandom vector plus $\mathbf{\Sigma W_T}$ where $\mathbf{\Sigma}$ is the nonrandom diagonal matrix with diagonal elements $\sigma_{[1]}, \sigma_{[2]}$, and $\mathbf{W}$ is the random column vector with components $W^{[1]}, W^{[2]}$. Then $\mathrm{Cov}(\mathbf{X}_T) = \mathbb{E}(\mathbf{X}_T\mathbf{X}_T^\top) = \mathbb{E}(\mathbf{\Sigma W_T W_T^\top \Sigma^\top}) = \mathbf{\Sigma}\mathrm{Cov}(\mathbf{W}_T)\mathbf{\Sigma}^\top = T\mathbf{\Sigma}\mathrm{Corr}(\mathbf{W}_T)\mathbf{\Sigma}^\top$.

Consider a basket $H := \frac{1}{2}S^{[1]} + \frac{1}{2}S^{[2]}$ of one-half of a share of each stock.

(b) Using 10000 standard Monte Carlo simulations, estimate the time-0 price C of an option that pays $(H_T - 110)^+$ at time $T = 1.0$. Also give the standard error [the sample standard deviation, divided by the square root of the number of simulations] of your Monte Carlo estimate.

You may either use a random number generator that produces normals with a given covariance matrix (which you found in (a)), or alternatively use a random number generator that produces independent normals which you then transform to introduce correlation.

In either approach, each of the 10000 simulations should use just *one* $\mathbb{R}^2$-valued random vector $\boldsymbol{Z}$ of simulated normal *zero-mean* random variables.

(c) Use 10000 antithetic pairs $(\boldsymbol{Z}, -\boldsymbol{Z})$ to estimate C, together with a standard error (L5.30).

Consider the "geometric basket" $G := \left(S^{[1]}S^{[2]}\right)^{1/2}$.

(d) The random variable $\log G_T$ is normally distributed (because it's a linear transformation of a multivariate normal vector). Show that $\log G_T$ has expectation

$$\frac{1}{2}\log(S_0^{[1]}S_0^{[2]}) + \left(r - \frac{\sigma_{[1]}^2 + \sigma_{[2]}^2}{4}\right)T$$

and variance

$$\frac{\sigma_{[1]}^2 + 2\rho\sigma_{[1]}\sigma_{[2]} + \sigma_{[2]}^2}{4}T.$$

(e) Let C^G be the time-0 price of a geometric basket option paying $(G_T - K)^+$ at time T.

Express C^G in terms of the function C^{BS} defined in FINM 33000 L6. Specifically, fill in the blanks:

$$C^G = C^{BS}(\underline{\quad\quad}, 0, K, T, \underline{\quad\quad}, r, \underline{\quad\quad}).$$

Your answer should be a general formula, in which you have not substituted 0.8 for ρ, etc. (You may *also* do the substitutions, but don't neglect the general formula).

(f) Using a geometric basket option as a control variate, run $M = 10000$ Monte Carlo simulations to estimate C, together with a standard error. Use the control variate estimate $\hat{C}_M^{\mathrm{cv},\hat{\beta}}$ from L6.7 or L6.8. Use the (asymptotically valid) standard error $\hat{\sigma}_M^{\mathrm{cv},\hat{\beta}}/\sqrt{M}$.

See the `ipynb` file.

Problem 2

Let the bank account and non-dividend paying stock have risk-neutral dynamics

$$\begin{aligned} \mathrm{d}B_t &= rB_t\mathrm{d}t & B_0 &= 1 \\ \mathrm{d}S_t &= rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t & S_0 &> 0 \end{aligned}$$

where $\sigma > 0$ and W is a $\mathbb{P}$-Brownian motion.

Consider a K-strike T-expiry vanilla call option, and let C denote its time-0 price.

(a) Let $S_0 = 100$, $\sigma = 0.2$, $r = 0.02$, $K = 150$, $T = 1$.

Run 100000 ordinary Monte Carlo simulations to estimate C, together with a standard error.

(b) Suppose that we sample from a new probability measure $\mathbb{P}^*$, under which W now has constant drift λ instead of drift 0. Thus $W_t = W_t^* + \lambda t$ where W^* is a standard $\mathbb{P}^*$-BM.

Find the $\mathbb{P}^*$-expectation $\mathbb{E}^*S_T$ in terms of S_0, r, σ, λ, and T.

Calculate λ such that $\mathbb{E}^*S_T = 165$.

(Why did we choose 165? The picture in L6.17 shows that the optimal distribution from which to sample will have a mean that is greater than the strike K. So let's choose 10% higher than K. This will not be optimal, but we expect that it will be an improvement over ordinary Monte Carlo. There are more systematic ways to determine a reasonable drift adjustment, not utilized here.)

(c) Run 100000 importance sampling simulations, using the specific drift adjustment calculated in (b), to estimate C, together with a standard error. Be aware that your zero-mean normal random draws, here, simulate increments of W^* not W.

Each simulation should require only one number to be generated by `rng.normal`.

See the `ipynb` file.

Comment: of course, we do not need Monte Carlo to price a call under GBM. However, suppose you wanted to price a deep OTM option under an intractable *stochastic volatility* model, using importance sampling. You could still use a similar approach.