1. Find the first five terms of the power series representing each function, and find the radius of convergence

a) $f(x) = \dfrac{1}{1+3x}$

Answer.

$$\frac{1}{1-(-3x)} = \sum_{n=0}^{\infty} (-3x)^n$$
$$= 1 - 3x + 9x^2 - 27x^3 + 81x^4 + \dots$$

$\square$

b) $g(x) = \dfrac{x}{(1-x)(1-2x)}$
(hint: use partial fractions)

Answer.

$$\frac{x}{(1-x)(1-2x)} = \frac{1}{1-2x} - \frac{1}{1-x} = \sum_{n=0}^{\infty} (2x)^n - \sum_{n=0}^{\infty} x^n$$
$$= \sum_{n=0}^{\infty} (2^n - 1)x^n$$
$$= 0 + x + 3x^2 + 7x^3 + 15x^4 + \dots$$

$\square$

2. Find the radius of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$$

Answer. Use the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(n+1)!x^{n+1}}{(n+1)^{n+1}} \frac{n^n}{n!x^n} \\
&= \lim_{n\to\infty} \frac{(n+1)n^n x}{(n+1)(n+1)^n} \\
&= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} x \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n x \\
&= \lim_{n\to\infty} \frac{x}{\left(\frac{n+1}{n}\right)^n} \\
&= \lim_{n\to\infty} \frac{x}{\left(1+\frac{1}{n}\right)^n} = \frac{x}{e}
\end{aligned}$$

Finally, the ratio test says that we converge absolutely when this is less than one, so we get

$$\left|\frac{x}{e}\right| < 1 \text{ and so } |x| < e$$

□

Answer: e

3. True or False?

Suppose that the interval of convergence of $\sum_{n=1}^{\infty} c_n x^n$ is $[-4, 2)$.

a) $\sum_{n=1}^{\infty} c_n$ converges Answer: True

b) $\sum_{n=1}^{\infty} (-c_n 3^n)$ converges Answer: False