

1. Find the first five terms of the power series representing each function, and find the radius of convergence

a)  $f(x) = \frac{1}{1+3x}$

*Answer.*

$$\begin{aligned}\frac{1}{1-(-3x)} &= \sum_{n=0}^{\infty} (-3x)^n \\ &= 1 - 3x + 9x^2 - 27x^3 + 81x^4 + \dots\end{aligned}$$

□

b)  $g(x) = \frac{x}{(1-x)(1-2x)}$   
(hint: use partial fractions)

*Answer.*

$$\begin{aligned}\frac{x}{(1-x)(1-2x)} &= \frac{1}{1-2x} - \frac{1}{1-x} = \sum_{n=0}^{\infty} (2x)^n - \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} (2^n - 1)x^n \\ &= 0 + x + 3x^2 + 7x^3 + 15x^4 + \dots\end{aligned}$$

□

2. Find the radius of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$$

*Answer.* Use the ratio test:

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \frac{(n+1)!x^{n+1}}{(n+1)^{n+1}} \frac{n^n}{n!x^n} \\&= \lim_{n \rightarrow \infty} \frac{(n+1)n^n x}{(n+1)(n+1)^n} \\&= \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} x \\&= \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^n x \\&= \lim_{n \rightarrow \infty} \frac{x}{\left( \frac{n+1}{n} \right)^n} \\&= \lim_{n \rightarrow \infty} \frac{x}{\left( 1 + \frac{1}{n} \right)^n} = \frac{x}{e}\end{aligned}$$

Finally, the ratio test says that we converge absolutely when this is less than one, so we get

$$\left| \frac{x}{e} \right| < 1 \text{ and so } |x| < e$$

□

Answer:  $e$

3. True or False?

Suppose that the interval of convergence of  $\sum_{n=1}^{\infty} c_n x^n$  is  $[-4, 2)$ .

a)  $\sum_{n=1}^{\infty} c_n$  converges

Answer: True

b)  $\sum_{n=1}^{\infty} (-c_n 3^n)$  converges

Answer: False