Numerically Solving the Lane-Emden Equation for Sphere of Isothermal Perfect Gas: The Bonnor-Ebert Profile

By considering the requirement of hydrostatic equilibrium for a self-gravitating spherically symmetric isothermal mass of a perfect gas, one can show that the equilibrium state is described by the solution to a second-order ODE, a case of the Lane-Emden equation:

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\ln(\rho/\rho_0)\right] = -\frac{4\pi G\rho_0}{v_{th}^2}\exp[\ln(\rho/\rho_0)],$$

where ρ_0 is the central density and v_{th} is the thermal speed in the gas. This is derived from the equation for hydrostatic equilibrium when the perfect gas equation of state is used with temperature held constant.

If we define $r_0 \equiv v_{th}/[4\pi G\rho_0]^{1/2}$, and $\psi \equiv \ln(\rho/\rho_0)$, with $x \equiv r/r_0$, the ODE is equivalent to

$$\frac{1}{x^2}\frac{d}{dx}\left[x^2\frac{d}{dx}\psi\right] = -\exp[\psi]. \tag{1}$$

Also by integrating the equation for hydrostatic equilibrium, we get $\Phi - \Phi_0 = -v_{th}^2\ln(\rho/\rho_0)$ so that $\psi(x) = -[\Phi(r) - \Phi_0]/v_{th}^2$ is the scaled gravitational potential.

We can now easily show that the second-order ODE for ψ as a function of x is equivalent to two coupled first-order ODE's,

$$\frac{d\psi}{dx} = \frac{y}{x^2}$$

and

$$\frac{dy}{dx} = -x^2\exp[\psi]$$

We start with the first of the coupled ODEs and take it as a definition. Then, immediately substituting this into equation (1) we get

$$\frac{1}{x^2}\frac{d}{dx}\left[x^2\frac{y}{x^2}\right] = \frac{1}{x^2}\frac{dy}{dx} = -\exp[\psi].$$

We rewrite this to obtain the second of the coupled ODEs:

$$\frac{dy}{dx} = -x^2\exp[\psi]$$

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Now, assuming the profile is non-singular (i.e. the solution is not the unique solution with a density singularity at the sphere center, and an exponential density distribution, which is scale independent), the boundary conditions at the center are $y = 0$ and $\psi = 0$. Starting with these boundary conditions, we are ready to use Matlab to integrate these equations from $x = 0$ to $x = 10$. *Note: to avoid the singularity at $x = 0$, one may start the integration at very small x.* We want to plot $-\psi$ and $\rho/\rho_0 = \exp(\psi)$ as functions of x.