18.905 Notes

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1 December 04, 2020

Recall that if X is a topological space, we defined the Čech cohomology of a closed subset $K \subseteq X$ to be

$$\check{H}^*(K) = \coprod_{K \subseteq U} H^*(U)/\sim,$$

where U ranges over the set of open neighborhoods of K, where we consider two classes to be the same if one is an inclusion of the other.

If $L \subseteq K$ is a pair of closed subsets of X, then we can form

$$\check{H}^*(K, L) := \coprod_{\substack{L \subseteq V \\ \cap \quad \cap \\ K \subseteq U}} H^*(U, V)/\sim$$

where (U, V) ranges over all pairs $Y \subseteq U$ such that V is an open neighborhood of L and U is an open neighborhood of K.

The theorems that we expect to hold from standard cohomology indeed hold.

1 **Theorem** (LES)

Let (K, L) be a closed pair in X. Then there is a natural long exact sequence

$$\cdots \longrightarrow \check{H}^p(K, L) \longrightarrow \check{H}^p(K) \longrightarrow \check{H}^p(L) \xrightarrow{\delta} \check{H}^{p+q}(K, L) \longrightarrow \cdots$$

2 **Theorem** (Excision)

Suppose A and B are compact subsets of a Hausdorff space X. Then the inclusion

$$(B, A \cap B) \subseteq (A \cup B, A)$$

induces an isomorphism

$$\check{H}^p(A \cup B, A) \cong \check{H}^p(B, A \cap B)$$

for all p.

Before we had a cap produce for a closed subspace $K \subseteq X$

$$\cap \colon \check{H}^p(K) \otimes_{\mathbb{Z}} H_n(X, X - K) \to H_{n-p}(X, X - K).$$

We can extend this to a **fully relative cap product**.

3 **Definition**

Let $L \subseteq K$ be a pair of closed subspaces of X. Then there is a map

$$\cap \colon \check{H}^p(K, L) \otimes_{\mathbb{Z}} H_n(X, X - K) \to H_{n-p}(X, X - K).$$

Then, this fully relative cap produce commutes with Mayer-Vietoris sequence.

4 **Theorem**

Let A and B be compact subsets of a Hausdorff space X. Let $x_{A\cup B} \in H_n(X, X - A\cup B)$ be a homology class. This gives us canonical restrictions

$$x_A \in H_n(X, X - A), x_B \in H_n(X, X - B), \text{ and } x_{A\cap B} \in H_n(X, X - A\cap B).$$

Then there is a map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(A\cup B) \arrow[r] \arrow[d, "-\cap x_{A\cup B}"] & \check{H}^p(A)\oplus H^p(B) \arrow[r] \arrow[d, "(-\cap x_A)\oplus(-\cap x_B)"] & \check{H}^p(A\cap B) \arrow[r, "\delta"] \arrow[d, "-\cap x_{A\cap B}"] & \check{H}^{p+1}(A\cup B) \arrow[r] \arrow[d, "-\cap x_{A\cup B}"] & \cdots \\
\cdots \arrow[r] & H_{n-p}(X, X - A\cup B) \arrow[r] & H_{n-p}(X, X-A)\oplus H_{n-p}(X, X-B) \arrow[r] & H_{n-p}(X, X - A\cap B) \arrow[r, "\delta"] & H_{n-p-1}(X, X - A\cup B) \arrow[r] & \cdots
\end{tikzcd}$$

1.1 Poincaré duality, again

Let M be an n-dimensional manifold and let K be a compact subset. Recall that

$$H_n(M, M-K) \to \Gamma(K; o_M) = \{f : K \to o_M \mid K \xrightarrow{f} o_M \xrightarrow{\pi} K = \mathrm{id}\}.$$

A $\mathbb{Z}$-orientation along a closed subset K is a section of o_M over K (i.e. an element of $\Gamma(K; o_M)$) that restricts to a generator of $H_n(M, M - \{x\})$ for every $x \in K$. The corresponding class in $H_n(M, M-K)$ is called the **fundamental class along K**, denoted $[M]_K$.

If $L \subseteq K$ is an inclusion of compact subsets of M, then the map

$$H_n(M, M-K) \to H_n(M, M-L)$$

sends $[M]_K$ to a fundamental class $[M]_L$. In particular, an inclusion of subsets gives a restriction of the orientation.

Also, we have a cap product

$$\cap \colon \check{H}^p(K, L) \otimes_{\mathbb{Z}} H_n(M, M-K) \to H_{n-p}(M-L, M-K).$$

5 **Theorem** (Poincaré duality)

Let M be an n-dimensional manifold and let $L \subseteq K$ be a pair of compact subspaces. Assume we have a $\mathbb{Z}$-orientation along K with fundamental class $[M]_K$. Then the map

$$-\cap [M]_K \colon \check{H}^p(K, L) \to H_{n-p}(M-L, M-K)$$

is an isomorphism.

Sketch. Read Miller's notes for more details.

1. Prove the theorem for $M = \mathbb{R}^n$, where K and L are compact convex subsets.
2. Prove for $M = \mathbb{R}^n$ where K and L are a finite union of compact convex subsets of $\mathbb{R}^n$.
3. Prove for $M = \mathbb{R}^n$ where K and L are any compact subsets of $\mathbb{R}^n$.
4. Prove for arbitrary M where K and L are finite unions of compact Euclidean subspace of M.
5. Prove for arbitrary M where K and L are arbitrary compact subspaces. $\square$

1.1.1 Applications

6 **Theorem**

Let M be an n-dimensional manifold and K be a compact subset. A $\mathbb{Z}$-orientation along K determines $[M]_K \in H_n(M, M-K)$ and capping with it gives an isomorphism

$$\check{H}^{n-p}(K) \longrightarrow H_p(M, M-K).$$

7 **Corollary** (Alexander duality)

Suppose K is a compact subset of $\mathbb{R}^n$. The composite

$$\check{H}^{n-p}(K) \xrightarrow{\cong} H_p(\mathbb{R}^n, \mathbb{R}^n - K) \xrightarrow{\partial} \tilde{H}_{p-1}(\mathbb{R}^n - K)$$

is an isomorphism.

8 **Example** (Jordan curve theorem)

Suppose $n = 2$. Let K be a closed loop in $\mathbb{R}^2$ with $\check{H}^1(K) \cong \mathbb{Z}$. (This will almost always be the case except for pathological embeddings of the circle into $\mathbb{R}^2$.)

Then $\tilde{H}_0(\mathbb{R}^2 - K) \cong \check{H}^1(K) \cong \mathbb{Z}$, so $H_0(\mathbb{R}^2 - K) \cong \mathbb{Z} \oplus \mathbb{Z}$.

In other words, $\mathbb{R}^2 - K$ has two path components.

9 **Warning**

The analog of this in $\mathbb{R}^3$ is false. In particular, there are maps $f\colon S^2 \to \mathbb{R}^3$ where $\mathbb{R}^3 - \operatorname{im} f$ does not give two path components. The counterexample is called the Alexander horned sphere.

2 December 07, 2020

The next two days will be about the future, i.e. what exists in algebraic topology outside of this class. In this course, we focused on four topics:

- category theory
- homological algebra (chAb, $D(\mathbb{Z})$, Tor, Ext)
- homotopy theory (study of Ho(Top))
- algebraic topology (study of Top, and in particular manifolds)

2.1 Category theory

We studied products and pushouts, which are special cases of **limits** and **colimits**. We also studied curring isomorphisms, which are special cases of **adjunctions**. Since the early 2000s, people also like to study **higher categories** or **∞-categories**, where there are

1. objects,
2. morphisms,
3. 2-morphisms, or morphisms between morphisms,

and these can keep going on.

10 **Example**

$H\mathbb{Z}$-modules are the ∞-category of chain complexes, where

- objects are chain complex of abelian groups,
- morphisms are chain maps between complexes,
- 2-morphisms are chain homotopies between chain maps.

11 **Example**

$\mathcal{S}$ is the ∞-category of homotopy theory, where

- objects are topological spaces,
- morphisms are continuous maps, and
- 2-morphisms are homotopies.

We can make an equivalent ∞-category with a combinatorial definition with simplicial sets.

12 **Example**

Cat_2 is the ∞-category of categories, where

- objects are categories,
- morphisms are functors,

- 2-morphisms are natural transformations.

To learn more, Google `Kerodon`.

2.2 Homological algebra

Homological algebra a developed much more in algebraic geometry (i.e. 18.726).

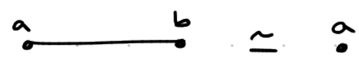
In particular, there are things called sheaf Hom, sheaf Ext, sheaf cohomology, etc.

13 **Idea**
$\{x^2 + y^2 - 1 = 0\} \subseteq \mathbb{R}^2$ is an algebraic way of describing the circle. Can one come up with a purely algebraic algorithm for calculating, starting with a set of polynomials with real coefficients, the homology of the set of those solutions?
What happens if one applies these algorithms to polynomials with coefficients in a number field?

2.3 Homotopy theory

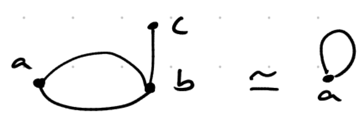
This is "the study of the equals sign", or better yet, "the study of the isomorphism sign".

Suppose there is a CW-complex with two objects and a path. This can be thought of as two object, and the path between them tells us that $a = b$. In particular, we can identify the two objects, and think of this as a single object.



On the other hand, if we have two objects with two paths between them, i.e. they are equal in two different ways. In particular, there are two different equalities between them. We can use one path to identify them, but then we have the other equality left. This is the same as a single object with a nontrivial automorphism.

Similarly, if we have an extra object c that is uniquely identified with b, then we identify b and c, and then it reduces to the previous case.



Note that these are just homotopy equivalences of graphs. In the spirit of higher category theory, we can also think about equalities between equalities.

This is equivalent to saying $D^2 \simeq *$.

In general, homotopy theory is about objects, equalities between objects, 2-equalities between equalities, etc.

2.4 Algebraic topology

One example of a question we ask in algebraic topology is: "Can we classify all n-dimensional compact manifolds up to homomorphism? What about smooth manifolds up to diffeomorphism?"

For example, surfaces (2-dimensional compact manifolds) are classified. For example, $\mathbb{Z}$-oriented surfaces are classified by their genus. In particular, it is determined by its first homology group.

However, when we increase the dimension, the manifolds get very complicated, so we can ask the question about manifolds with particularly simple homology.

14 **Theorem** (Kervaire-Milnor, 1961)

Classification of all compact simply connect n-dimensional manifolds M with

$$H^q(M) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0, n \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 5$. For lower dimensions, we have known $n = 2$ since around 1900, we know $n = 3$ due to Perelman in 2003, and $n = 4$ is still unknown.

We can also ask if we can classify simply connected and compact $2n$-dimensional manifolds M where $H_0(M) \cong \mathbb{Z}$ and

$$H_1(M) \cong H_2(M) \cong \ldots \cong H_{n-1}(M) \cong 0.$$

Every simply connected space is $\mathbb{Z}$-orientable, so it follows that

$$H_{n+1}(M) \cong H_{n+2}(M) \cong \ldots \cong H_{2n}(M) \cong \mathbb{Z}.$$

$H_n(M)$ can be anything. This is classified in all dimensions $2n$ except $2n = 4, 24, 126$.

- $n \equiv 6 \mod 8$ was proved by Wall in 1962
- $n \equiv 5 \mod 8$ was proved by Brown and Peterson in 1966
- $n \equiv 3 \mod 8$ was proved by Browder in 1969
- $n \equiv 2 \mod 8$ was proved by Schultz in 1972
- $n \equiv 1 \mod 8$ was proved by Stolz in 1985
- $n \equiv 7 \mod 8$ and $n \neq 63$ was done by Hill-Hopkins-Ravenel in 2009. At the time it was proved, Hopkins was a professor at MIT and Hill was a graduate student at MIT. He is now at UCLA and is the head of the LGBTQ society of mathematicians.
- $n \equiv 0 \mod 4$ and $n \neq 12$ was done by Burkland-Senger-Hahn in 2019. Burkland and Senger are grad students at MIT, and Hahn is the lecturer of this course! Adela Zhang is thinking about extending this to when two homology groups are nonzero.

2.5 More homotopy theory

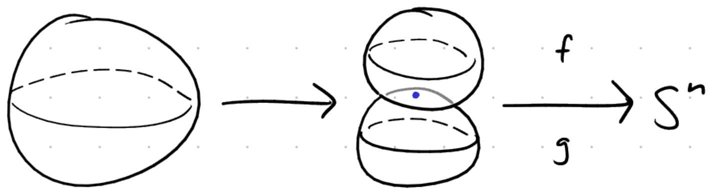
15 **Question**

If m and n are positive integers, how many maps are there from $S^m \to S^n$ up to homotopy? The set of maps $S^m \to S^n$ up to homotopy is denoted $\pi_m S^n$.

On the homework, we proved $\pi_3 S^2$ has more than one element. In 18.906, we will see that

1. If $m < n$, all maps $S^m \to S^n$ are homotopic, i.e. $\pi_m S^n$ has one element.
2. If $m = n$, then $\pi_m S^m \cong \mathbb{Z}$. Maps $S^m \to S^m$ are determined up to homotopy by their degrees.
3. If $m > n$, then unless $m = 2n - 1$, $\pi_m S^n$ is finite.

A natural question to ask is how large is this set? Moreover, note that $\pi_m S^n$ is not just a set, but a group in the following way: If we have two maps $f, g\colon S^n \to S^n$, we first ensure (by homotopy) that the south pole of f is mapped to the same point as the north pole of g. Then, $f * g$ is the map depicted in the following diagram, where we first pinch the equator, and then we map the top half by f and the bottom half by g.



3 December 09, 2020

3.1 Homotopy groups of spheres

Recall that $\pi_m S^n$ is the group of continuous maps from S^m to S^n up to homotopy.

16 **Theorem** (Freudenthal)

For $n \geq k+2$, $\pi_{n+k} S^n$ is independent of n. This group is denoted $\pi_k \mathbb{S}$.

17 **Example**

$\pi_3 S^2 \cong \mathbb{Z}$, $\pi_4 S^3 \cong \mathbb{Z}/2\mathbb{Z}$, $\pi_5 S^4 \cong \mathbb{Z}/2\mathbb{Z}$, …, so $\pi_1 \mathbb{S} \cong \mathbb{Z}/2\mathbb{Z}$.

The first couple of stable groups are

- $\pi_1 \mathbb{S} \cong \mathbb{Z}/2\mathbb{Z}$
- $\pi_2 \mathbb{S} \cong \mathbb{Z}/2\mathbb{Z}$
- $\pi_3 \mathbb{S} \cong \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
- $\pi_4 \mathbb{S} \cong 0$
- $\pi_5 \mathbb{S} \cong 0$
- $\pi_6 \mathbb{S} \cong \mathbb{Z}/2\mathbb{Z}$
- $\pi_7 \mathbb{S} \cong \mathbb{Z}/16\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$
- $\pi_8 \mathbb{S} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

18 **Question**

How can we use homology to understand these groups?

An element of $\pi_7 \mathbb{S}^n$ is a map

$$f\colon S^{7+n} \to S^n$$

for $n \gg 0$. Then $H_*(f)\colon H_*(S^{7+n}) \to H_*(S^n)$ is going to trivial no matter what f is.

19 **Definition**

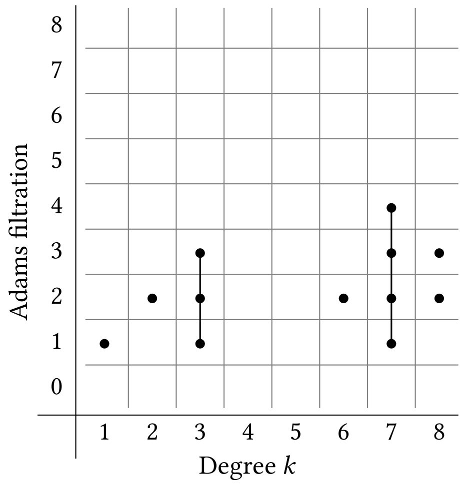
A map of spheres $S^m \to S^n$ has **$\mathbb{F}_p$-Adams filtration** at least k if it can be factored as a composite

$$S^m \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3 \xrightarrow{f_4} \cdots \xrightarrow{f_k} X_k = S^n$$

such that each $H_*(f_i; \mathbb{F}_p)$ is trivial.

Intuitively, this means that it is hard to figure out what is going on when viewed through homology, because we can factor the map but still don't understand what the individual maps are.

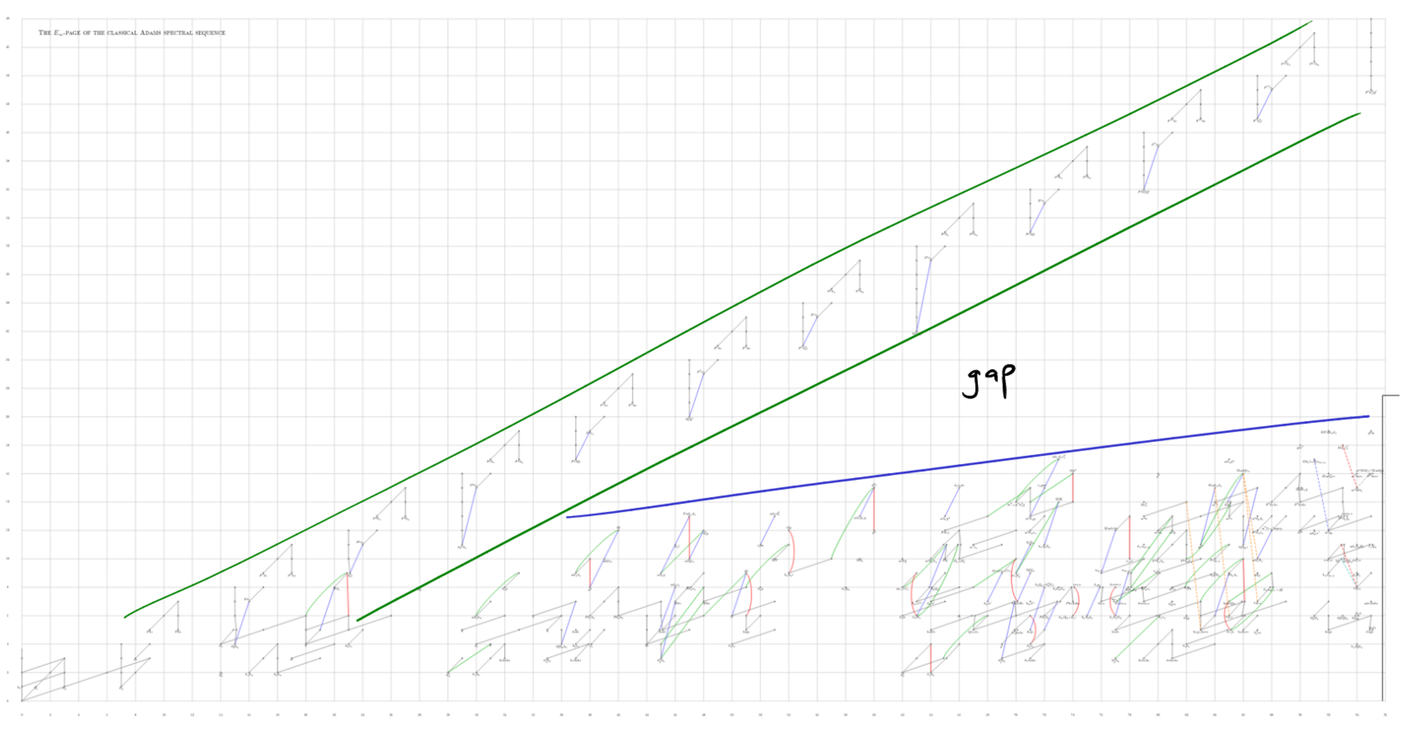
One way that we can think about it graphically is with a table like the following



- $\pi_1\mathbb{S}_{(2)} = \mathbb{Z}/2\mathbb{Z}$
- $\pi_2\mathbb{S}_{(2)} = \mathbb{Z}/2\mathbb{Z}$
- $\pi_3\mathbb{S}_{(2)} = \mathbb{Z}/8\mathbb{Z}$
- $\pi_4\mathbb{S}_{(2)} = 0$
- $\pi_5\mathbb{S}_{(2)} = 0$
- $\pi_6\mathbb{S}_{(2)} = \mathbb{Z}/2\mathbb{Z}$
- $\pi_7\mathbb{S}_{(2)} = \mathbb{Z}/16\mathbb{Z}$
- $\pi_8\mathbb{S}_{(2)} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

Here, we're localizing at the prime 2. In particular, since $\pi_3\mathbb{S} = \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, we have $\pi_3\mathbb{S}_{(2)} \cong \mathbb{Z}/8\mathbb{Z}$. To read the table, we look at each column individually. The number of dots tells us what the group is, and the connections tell us how large the powers of the primes are. The placement of the first dot tells us what the Adams filtration is. For instance, for degree 3, we can interpret the table as saying that $\frac{1}{2}$ of the elements in the homotopy groups are in Adams filtration 1, $\frac{1}{4}$ of the elements are in Adams filtration 2, and $\frac{1}{8}$ of the elements are in Adams filtration 3.

When we extend the chart, we start to become limited due to computation power. Ten years ago, we were able to compute the chart up to around degree ~ 50, and now we have the chart up to degree 96. This chart is on page 8 of `Adamschart.pdf`.



When we look at the chart, we see that there is a simple collection of points with larger slope than the rest of the points, indicated with green. There is also a chaotic pattern of points below it, with a gap in between.

In particular, there are a simple group of homotopy groups that are hard to detect with homology, and then there is a messy collection of homotopy groups that are easier to detect (but still quite hard) with homology.

There is indeed a good understanding of the maps corresponding to very high Adams filtration, and this region is called the v_1-periodic homotopy groups of spheres. Then there seems to be a gap, and an open problem is whether this gap continues, and how big it is. Another natural question is whether we can understand the non-v_1-periodic stuff.

20 **Definition**

An **extraordinary (co)homology theory** E_* is a functor

$$E_*\colon \mathrm{Ho}(\mathrm{Top}) \to \text{graded Ab groups}$$

satisfying all of the Eilenberg-Steenrod axioms except the dimension axiom.

They can directly tell, sometimes, the differences between various maps $f\colon S^m \to S^n$. In particular, $E_*(f)\colon E_*(S^m) \to E_*(S^n)$ could be nontrivial, because the homologies of a sphere do not have to be concentrated in one degree.

The most important example is $E_* = KO_*$, called **topological K-theory**. This theory sees the v_1-periodic part of $\pi_*\mathbb{S}$. It has a geometric definition in terms of vector bundles, which assemble to define K-theory. It also has a more algebraic/combinatorial definition.

21 **Question**

How do we make extraordinary (co)homology theories that detect other elements in $\pi_*\mathbb{S}$?

22 **Definition** (Idea)

An **$\mathbb{E}_\infty$-ring** is a cohomology theory E^* valued in graded commutative rings.

For example, we could have $E^*(X) = H^*(X;R)$, where R is a commutative ring. We can also have $E^*(X) = KO^*(X)$.

Given an $\mathbb{E}_\infty$-ring E^*, we can extract a classical ring, which is $E^*(*)$. For example, the classical ring underlying $H^*(-;R)$ is $H^*(*;R) \cong R$ in degree 0. On the other hand, the classical ring underlying $KO^*(-)$ is $KO^*(*)$, which is 8-periodic. (Bott peridicity)

The main idea of algebraic homotopy theory is that we should develop all of commutative algebra replacing rings with $\mathbb{E}_\infty$-rings. In particular, wherever we have concepts about rings like PIDs or Nakayama's lemma, we should try to find an analogous theorem about $\mathbb{E}_\infty$-rings. This is what Waldhausen called "brave new algebra", which is today sometimes called "higher algebra", "spectral algebra", or "derived algebra". A lot of the results we get are the same, but sometimes there are difference.

23 **Example**

In classical algebra or number theory, people study elliptic curves. To imitate the theory of elliptic curves in $\mathbb{E}_\infty$-rings, we are led to the notion of **elliptic cohomology theories**.

One intersecting thing that doesn't happen in classical number theory is that there is a universal elliptic cohomology theory. In particular, there is an elliptic cohomology theory that captures the information of all other elliptic cohomology theories. This is a new phenomenon that we cannot understand with regular rings. This universal theory is called **topological modular forms** (tmf).

We can try to understand the underlying classical ring by looking at the underlying classical ring $\mathrm{tmf}^*(*) \otimes_{\mathbb{Z}} \mathbb{Q}$, and it turns out to be the classical ring of modular forms.

One can keep going into things like automorphic forms or abelian varieties and try to construct analogs in this land of brave new algebra. $\mathbb{E}_\infty$-rings are about commutative algebra, but we can also talk about associative rings, which are called $\mathbb{E}_1$- rings. When we do this, we get a lot of analogs, and oftentimes we get more objects that we can construct compared to classical ring theory, and a lot of the time they are universal. This is the topic of study in **chromatic homotopy theory**, which assembles the homotopy groups $\pi_*\mathbb{S}$ out of things detected by $\mathbb{E}_\infty$-rings. Each $\mathbb{E}_\infty$-ring has a chromatic height that roughly measures how much of the homotopy group it can see.

- Height 0 is ordinary homology,
- Height 1 is topological K-theory,
- Height 2 is TMF, and so on.

Furthermore, we have the chromatic convergence theorem, which says that each element in $\pi_*\mathbb{S}$ is detected at some chromatic height.

Another big open question in this subject is whether there is a geometric construction of TMF. For example, we can understand ordinary homology through cycles and maps of simplices, and we can understand K-theory with vector bundles. Physicists tell us that it should be related to the Dirac operator on the space of loops in X. However, nobody has been able to do this mathematically in a well defined way.