18.905 Notes

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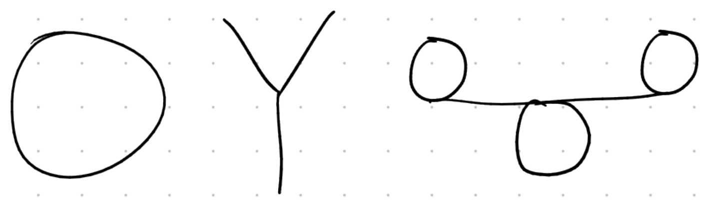
1 September 02, 2020

The first half of the course is on **homology**, where we want to study space by asking how many "holes" a space has.

For example, a circle, the letter Y, and a line segment with three circles attached to it have 1, 0, and 3 holes respectively.

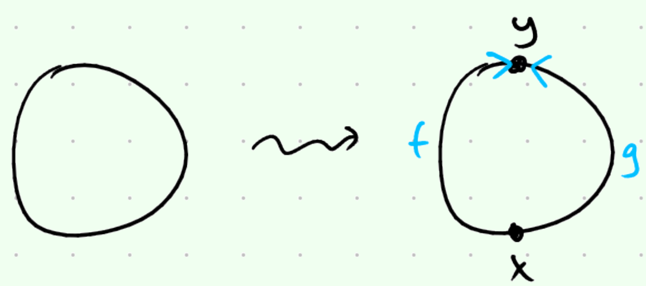


The fundamental group is one way of measuring the number of holes, but we will present a different way of understanding the number of holes in these spaces with the first homology group H_1.

Let's first provide an algorithm of how to find the homology of a space.

1 **Example** (Homology of the circle)

First, we transform the space into a more graph-theoretic structure by adding some vertices, and then making the edges directed, like so:



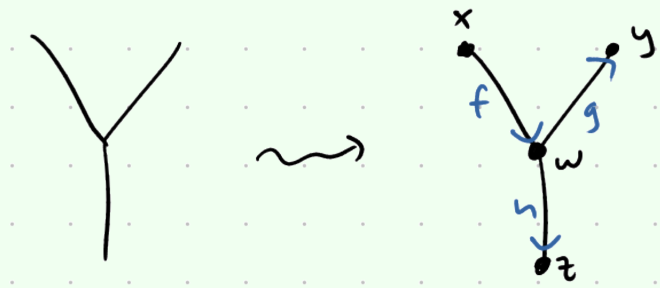
Let $\mathbb{Z}\{f,g\}$ denote the free abelian group generated by f and g. Consider the group homomorphism

$$\begin{aligned}\partial\colon \mathbb{Z}\{f,g\} &\to \mathbb{Z}\{x,y\}\\ f &\mapsto y-x\\ g &\mapsto y-x,\end{aligned}$$

where we mapped each edge to their "target minus source". Note that the kernel of ∂ is the integer multiples of $f-g$. In particular, $\ker\partial$ is generated by *one* element, so there is *one* hole. This kernel is called the first homology group H_1 of the circle.

2 **Example** (Homology of the letter Y)

Again, transform the space into a combinatorial gadget by assigning vertices and directing edges:



We consider the homomorphism

$$\begin{aligned} \partial \colon \mathbb{Z}\{f, g, h\} &\to \mathbb{Z}\{x, y, z, w\} \\ f &\mapsto w - x \\ g &\mapsto y - w \\ h &\mapsto z - w. \end{aligned}$$

The only element that gets sent to 0 by ∂ is 0, so $\ker \partial$ is the free abelian group on $\varnothing$. Therefore, there are no holes.

The kernel we have been computing is the first homology group, and the number of generators corresponds to the number of holes.

This algorithm involves many arbitrary decisions. Why is it giving the same answer each time? How does this generalize to larger dimensional spaces?

2 September 04, 2020

Last time, we computed the homology group of a combinatorial object, a directed graph. However, there are two problems that arise:

1. We want to handle higher dimensional objects.
2. We want to handle topological spaces and not just combinatorial objects.

Let's try to work on the first issue.

3 **Definition**

A **semisimplicial set** X is a sequence of sets $X_0, X_1, X_2, \dots$ and functions

$$\begin{aligned} d_0, d_1 &: X_1 \longrightarrow X_0 \\ d_0, d_1, d_2 &: X_2 \longrightarrow X_1 \\ &\vdots \\ d_0, d_1, d_2, \dots, d_n &: X_n \longrightarrow X_{n-1} \\ &\vdots \end{aligned}$$

that satisfy the **simplicial identities**

$$d_i d_j = d_{j-1} d_i \quad \text{whenever } i < j.$$

These sets X_n are called the ***n*-simplices**, and the maps d_i are called the **face maps**.

4 **Example**

A semisimplicial set with $\varnothing = X_2 = X_3 = \cdots$, etc. is just a directed graph

$$\begin{aligned} X_0 &= \{\text{vertices}\} \\ X_1 &= \{\text{edges}\} \end{aligned}$$

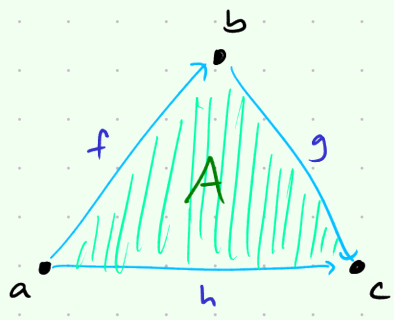
with

$$d_0, d_1 : X_1 \longrightarrow X_0,$$

with d_0 mapping an edge to the "target" and d_1 mapping an edge to the "source".

5 **Example**

If we have the following combinatorial structure:



This represents the semisimplicial set with

$$\begin{aligned} X_0 &= \{a, b, c\} \\ X_1 &= \{f, g, h\} \\ X_2 &= \{A\} \\ X_3 = X_4 &= \varnothing. \\ &\vdots \end{aligned}$$

Then we have the functions

$$d_0, d_1 \colon X_1 \longrightarrow X_0 \qquad \text{and} \qquad d_0, d_1, d_2 \colon X_2 \longrightarrow X_1$$

where

$$\begin{aligned} d_0 f &= b & d_1 f &= a & d_0 A &= g \\ d_0 g &= c & d_1 g &= b & d_1 A &= h \\ d_0 h &= c & d_1 h &= a & d_2 A &= f. \end{aligned}$$

To see why d_0, d_1, d_2 act on A the way they do, note that the simplicial identities require

$$d_0 d_2 A = d_1 d_0 A.$$

The element in X_0 that must equal these two values must be a source and a target, so it must be b. The values of $d_0 A$, $d_1 A$, and $d_2 A$ follow.

Note that it's important that the edges go forward from $a \longrightarrow b \longrightarrow c$ and backwards from $c \longleftarrow a$. Otherwise, the simplicial identity would not be able to uniquely define d_0, d_1, d_2 on A.

2.1 Homology

If X is a semisimplicial set, then let $S_n(X)$ be the abelian group of **singular n-chains**, the free abelian group generated by the set X_n of n-simplices.

6 **Definition**

For $n \geq 1$, the **boundary operators** are group homomorphisms

$$\partial_n \colon S_n(X) \longrightarrow S_{n-1}(X)$$

defined by the generators $\sigma \in X_n$ where

$$\sigma \mapsto \sum_{k=0}^{n} (-1)^k d_k \sigma.$$

We also define $\partial_0 : S_0(X) \to 0$ to be the zero homomorphism.

7 **Definition**

Suppose X is a semisimplicial set. The group of ***n*-cycles** in X, denoted by $Z_n(X)$ is the kernel of ∂_n.

The group of ***n*-boundaries** in X, denoted $B_n(X)$, is the image of ∂_{n+1}.

8 **Exercise** (On homework)

$B_n(X)$ is a subgroup of $Z_n(X)$, which is in turn a subgroup of $S_n(X)$.

We can therefore define the ***n*th singular homology group** of X,

$$H_n(X) = Z_n(X)/B_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}},$$

which is an abelian group because the quotient of an abelian group is abelian. This intuitively measures the number of n-dimensional holes in X.

2.2 Topological spaces

9 **Question**

How do we get a semisimplicial set out of a topological space?

The basic tool we will use are simplices.

10 **Definition**

For $n \geq 0$, the **standard *n*-simplex** Δ^n is the subspace of $\mathbb{R}^{n+1}$ that is the convex hull of the standard basis $\{e_0, \ldots, e_n\}$ in $\mathbb{R}^{n+1}$. In particular,

$$\Delta^n = \left\{ \sum t_i e_i \;\middle|\; \sum t_i = 1, t_i \geq 0 \right\} \subseteq \mathbb{R}^{n+1}.$$

The t_i are sometimes called **barycentric coordinates**.

11 **Example**

$\Delta^1 \subseteq \mathbb{R}^2$ is the line segment from $(1,0)$ to $(0,2)$.

$\Delta^2 \subseteq \mathbb{R}^3$ is the triangle with vertices $\{(1,0,0), (0,1,0), (0,0,1)\}$.

12 **Definition**

For a topological space X, a **singular *n*-simplex in *X*** is a continuous map $\sigma : \Delta^n \to X$. We will often drop the adjective "singular". Let $\mathrm{Sing}_n(X)$ be the set of all n-simplices in X.

We can assemble these into a semisimplicial set. In particular, there are maps $d_i : \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n-1}(X)$ that come from "forgetting" the basis element e_i. In particular, we have the function $d_1 : \Delta^2 \to \Delta^1$ that takes the segment (Δ^1) that does not include the vertex corresponding to 1. So given $\sigma\colon \Delta^2 \to X \in \mathrm{Sing}_2(X)$, we can restrict σ to $d_1\Delta^2 = \Delta^1$, giving a map $\Delta^1 \to X \in \mathrm{Sing}_1(X)$.

$$\begin{aligned} d_1\colon\ \mathrm{Sing}_2(X) &\to \mathrm{Sing}_1(X) \\ \sigma &\to \sigma|_{d_1\Delta^2} \end{aligned}$$

Therefore, $\mathrm{Sing}(X)$, the collection of all of the Sing_n sets, is a semisimplicial set.

13 **Remark**

For more intuition, look at Hatcher's treatment of Δ-complexes (another name for semisimplicial set).

14 **Definition**

If X is a topological space, we can define

$$S_n(X) = S_n(\mathrm{Sing}\,X), \qquad Z_n(X) = Z_n(\mathrm{Sing}\,X), \qquad B_n(X) = B_n(\mathrm{Sing}\,X), \qquad \text{and} \qquad H_n(X) = H_n(\mathrm{Sing}\,X).$$

Overall, we are transforming geometric objects into combinatorial objects, and then in turn transforming them into algebraic objects so that they are easier to study.

$$\begin{array}{ccccc} \text{Topological spaces} & \xrightarrow{\mathrm{Sing}} & \text{Semisimplicial sets} & \xrightarrow{H_n} & \text{Abelian groups} \\ \text{(geometric)} & & \text{(combinatorial)} & & \text{(algebraic)} \end{array}$$

3 September 09, 2020

Note that if $X \to Y$ is a continuous map of topological spaces, and $\sigma\colon \Delta^n \to X$ is a continuous function, then the composition of them $\Delta^n \to X \to Y$ is also continuous. Therefore, a map $X \to Y$ induces a map $\mathrm{Sing}_n(X) \to \mathrm{Sing}_n(Y)$.

3.1 Category theory

15 **Definition**

A **category** C consists of:

- A class $\mathrm{Obj}\, C$ of **objects** in C.
- For every pair of objects $X, Y \in \mathrm{Obj}\, C$, a set of **morphisms** $\mathrm{Hom}_C(X, Y)$.
- For every object $X \in \mathrm{Obj}\, C$, an identity morphism $1_X \in \mathrm{Hom}_C(X, X)$.
- For every triple of objects $X, Y, Z \in \mathrm{Obj}\, C$, a composition operation

$$\mathrm{Hom}_C(X, Y) \times \mathrm{Hom}_C(Y, Z) \to \mathrm{Hom}_C(X, Z)$$

  written $(f, g) \mapsto g \circ f$.

These data are required to satisfy two properties:

- $1_Y \circ f = f$ for all $f \in \mathrm{Hom}_C(X, Y)$ and $f \circ 1_Y = f$ for all $f \in \mathrm{Hom}_C(Y, X)$.
- Composition is associative, i.e.

$$(h \circ g) \circ f = h \circ (g \circ f)$$

  whenever these operations are defined.

Note that we say that $\mathrm{Obj}\, C$ is a class instead of a set to prevent logical paradoxes, such as talking about the set of all sets. We can just think about these as sets, and our intuition would work perfectly well.

16 **Example**

- Set is the category of sets. Obj(Set) is the class of all sets and morphisms are functions: if X, Y are two sets, then $\mathrm{Hom}_{\mathsf{Set}}(X, Y)$ is the set of all functions $X \to Y$.
- Ab is the category of abelian groups. $\mathrm{Hom}_{\mathsf{Ab}}(A, B)$ refers to the set of all group homomorphisms $A \to B$.
- $\mathsf{Vect}_{\mathbb{R}}$ is the category of real vector spaces. Morphisms are linear transformations.
- Top is the category of topological spaces. $\mathrm{Hom}_{\mathsf{Top}}(X, Y)$ is the set of continuous maps $X \to Y$.

A note on notation: If C is a category, sometimes we say $X \in C$ to mean $X \in \mathrm{Obj}\, C$ and $f\colon X \to Y$ to mean $f \in \mathrm{Hom}_C(X, Y)$. We also may sometimes say map instead of morphism.

17 **Definition**

A morphism $f\colon X \to Y$ in a category C is called an **isomorphism** if there exists a map $g\colon Y \to X$ such that

$$g \circ f = 1_X, \qquad \text{and} \qquad f \circ g = 1_Y.$$

18 **Example**

A morphism in Set is an isomorphism if and only if it is a bijection.

A morphism in Ab is an isomorphism if and only if it is a group isomorphism.

A morphism in Top is the same thing as a homeomorphism.

19 **Proposition**

If $f: X \to Y$ is an isomorphism in a category $\mathcal{C}$, then the inverse $g: Y \to X$ is unique.

Proof. Suppose $g, g': Y \to X$ are two inverses of $f: X \to Y$. Then

$$g = g \circ 1_Y = g \circ (f \circ g') = (g \circ f) \circ g' = 1_X \circ g' = g'.$$

□

20 **Definition**

Given categories $\mathcal{C}, \mathcal{D}$, a **functor** $F: \mathcal{C} \to \mathcal{D}$ consists of:

- An assignment $F\colon \operatorname{Obj} \mathcal{C} \to \operatorname{Obj} \mathcal{D}$, and
- for all $X, Y \in \operatorname{Obj} \mathcal{C}$, a function

$$F\colon \operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y)).$$

These are required to satisfy the following two properties:

- For all $X \in \operatorname{Obj} \mathcal{C}$, $F(1_X) = 1_{F(X)}$.
- For all composable pairs of morphisms f, g in $\mathcal{C}$,

$$F(g \circ f) = F(g) \circ F(f).$$

21 **Example**

For each $n \geq 0$, there are the functors

$$\begin{aligned} \operatorname{Sing}_n\colon \mathrm{Top} &\to \mathrm{Set} \\ X &\to \{\sigma\colon \Delta^n \to X\} \\ S_n\colon \mathrm{Top} &\to \mathrm{Ab}. \end{aligned}$$

There is a huge category Cat of categories. In particular, the objects of Cat are categories, and the morphisms are functors. Warning: If $\mathcal{C}, \mathcal{D}$ are two categories, then $\operatorname{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathcal{D})$ might be a class and not a set. In particular, we can compose functors.

22 **Example**

There is a functor

$$\mathrm{Free}\colon \mathrm{Set} \to \mathrm{Ab}$$

that maps a set to the free abelian group generated by the set. In particular, $S_n = \mathrm{Free} \circ \operatorname{Sing}_n\colon \mathrm{Top} \to \mathrm{Ab}$.

If $f: X \to Y$ is a map in Top, then there is a diagram for each $0 \leq i \leq n$

$$\begin{tikzcd} \operatorname{Sing}_n(X) \arrow[r, "d_i"] \arrow[d, "\operatorname{Sing}_n(f)"'] & \operatorname{Sing}_{n-1}(X) \arrow[d, "\operatorname{Sing}_{n-1}(f)"] \\ \operatorname{Sing}_n(Y) \arrow[r, "d_i"] & \operatorname{Sing}_{n-1}(Y) \end{tikzcd}$$

that commutes. In category theory, one can generalize this kind of diagram.

23 **Definition**

Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A **natural transformation** $\Theta\colon F \to G$ consists of maps $\Theta_X\colon F(X) \to G(X)$ for each $X \in \mathcal{C}$ such that for all $f\colon X \to Y$, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\Theta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\Theta_Y"] & G(Y) \end{tikzcd}$$

24 **Example**

Our previous example becomes the following: Suppose $n \geq 1$ and $0 \leq i \leq n$. Then there is a natural transformation

$$d_i\colon \operatorname{Sing}_n \to \operatorname{Sing}_{n-1},$$

where $\operatorname{Sing}_n, \operatorname{Sing}_{n-1}$ are functors $\mathrm{Top} \to \mathrm{Set}$.

25 **Definition**

A natural transformation $\Theta\colon F \to G$ is called a **natural isomorphism** if each map Θ_X is an isomorphism for all $X \in \operatorname{Obj} \mathcal{C}$.

Suppose $\mathcal{C}, \mathcal{D}$ are categories and $\operatorname{Obj} \mathcal{C}$ is a set (as opposed to a class). Then there is another category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ of functors where the morphisms are natural transformations of functors. Note that if $\operatorname{Obj} \mathcal{C}$ is not a set, then some sets of morphisms would be a class instead of a set, while the definition requires a set.

4 September 11, 2020

Today we will finish putting semisimplicial sets into a category theory viewpoint.

Suppose $\mathcal{C}$ is a category. Then a new category we can consider is the **opposite category** $\mathcal{C}^{\text{op}}$ with

- $\text{Obj}(\mathcal{C}^{\text{op}}) = \text{Obj}(\mathcal{C})$
- For $X, Y \in \text{Obj}(\mathcal{C}^{\text{op}})$, $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$.

In particular, if $f \in \text{Hom}_{\mathcal{C}}(Y, X)$, we use f^{op} to represent the corresponding object of $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y)$. The composition law is

$$(f \circ g)^{\text{op}} = g^{\text{op}} \circ f^{\text{op}}.$$

A good way to think about the opposite category is through functors. As a category itself, $\mathcal{C}^{\text{op}}$ is essentially the same as $\mathcal{C}$, but a functor $F\colon \mathcal{C}^{\text{op}} \to \mathcal{D}$ can be thought of as a function that still takes objects in $\mathcal{C}$ to objects in $\mathcal{D}$, but takes maps $c \to c'$ in $\mathcal{C}$ to maps $F(c') \to F(c)$ in $\mathcal{D}$.

26 **Example**

Recall the category $\text{Vect}_{\mathbb{R}}$ of real vector spaces. Every vector space V has a dual $V^* = \underline{\text{Hom}}_{\text{Vect}_{\mathbb{R}}}(V, \mathbb{R})$, where the underline indicates that the Hom is a vector space itself.

Note that if $f\colon W \to V$ is a linear map, then the corresponding map $f^*\colon V^* \to W^*$ in the opposite category maps

$$f^*\colon (\underbrace{g\colon V \to \mathbb{R}}_{\in V^*}) \mapsto (\underbrace{g \circ f\colon W \to V \to \mathbb{R}}_{\in W^*}).$$

In particular, we have the functor $(\,)^*\colon \text{Vect}_{\mathbb{R}} \to \text{Vect}_{\mathbb{R}}^{\text{op}}$.

From last time, recall that for categories $\mathcal{C}, \mathcal{D}$ where $\mathcal{C}$ has a set of objects, $\text{Fun}(\mathcal{C}, \mathcal{D})$ is a category where the objects are the set of functors $\mathcal{C} \to \mathcal{D}$, and morphisms are natural transformations of functors. In particular, we can compose natural transformations.

4.1 Semisimplicial sets

27 **Definition**

Let Δ_{inj} be the category with objects

$$\text{Obj}\,\Delta_{\text{inj}} = \{[0], [1], [2], \ldots\},$$

and morphisms

$$\text{Hom}_{\Delta_{\text{inj}}}([a], [b]) = \left\{\begin{matrix}\text{injective functions } f\colon \{0, 1, \ldots, a\} \to \{0, 1, \ldots, b\} \text{ that} \\ \text{preserve order, i.e. } f(x) < f(y) \text{ whenever } x < y\end{matrix}\right\}.$$

For instance, there are three maps in $\text{Hom}_{\Delta}([1], [2])$, which are injective order preserving functions

$$\{0, 1\} \to \{0, 1, 2\}.$$

They are determined by their image $(f(0), f(1))$, which could be $(0, 1)$, $(0, 2)$, or $(1, 2)$.

28 **Claim**

A semisimplicial set is a functor $\Delta_{\text{inj}}^{\text{op}} \to \mathsf{Set}$, i.e. an element of $\operatorname{Fun}(\Delta_{\text{inj}}^{\text{op}}, \mathsf{Set})$.

Let's unpack this. A semisimplicial is a sequence of sets

$$X_0, X_1, X_2, \ldots$$

with maps:

$$\begin{aligned} d_0, d_1 \colon X_1 &\to X_0 \\ d_0, d_1, d_2 \colon X_2 &\to X_1 \\ &\vdots \end{aligned}$$

such that $d_i d_j = d_{j-1} d_i$ when $i < j$.

We are looking for a functor $F \colon \Delta_{\text{inj}}^{\text{op}} \to \mathsf{Set}$. We have

$$\operatorname{Obj} \Delta_{\text{inj}}^{\text{op}} = \operatorname{Obj} \Delta_{\text{inj}} = \{[0], [1], \ldots\}.$$

The most natural way to do this is to map $F([n])$ to X_n in the semisimplicial set. To get the d_i maps, consider that

$$\operatorname{Hom}_{\Delta_{\text{inj}}^{\text{op}}}([2],[1]) = \operatorname{Hom}_{\Delta_{\text{inj}}}([1],[2]) = \left\{ \begin{matrix} \text{injective order preserving maps} \\ \{0,1\} \to \{0,1,2\} \end{matrix} \right\}.$$

Recall that there are three of these. Let $f_0 \colon \{0,1\} \to \{0,1,2\}$ be the map with image $\{1,2\}$, i.e. the one that does not contain 0. In general, let $f_k \colon \{0,1\} \to \{0,1,2\}$ be the map with image $\{0,1,2\} \setminus \{k\}$. Then, we can map $F(f_k^{\text{op}})$ to d_k in the semisimplicial set.

It turns out that the combinatorics of these injective morphisms translates directly into the simplicial identity $d_i d_j = d_{j-1} d_i$.

29 **Remark**

$\operatorname{Fun}(\Delta_{\text{inj}}^{\text{op}}, \mathsf{Set})$ forms a category of semisimplicial sets where the morphisms are natural transformations.

30 **Theorem**

There are functors

$$\operatorname{Sing} \colon \mathsf{Top} \to \operatorname{Fun}(\Delta_{\text{inj}}^{\text{op}}, \mathsf{Set})$$

$$S_n, Z_n, B_n, H_n \colon \operatorname{Fun}(\Delta_{\text{inj}}^{\text{op}}, \mathsf{Set}) \to \mathsf{Ab}.$$

31 **Corollary**

There are functors

$$S_n, Z_n, B_n, H_n \colon \mathsf{Top} \to \mathsf{Ab}.$$

Note that this corollary was deduced from the fact that Cat is a category.

32 **Definition**

Let Fil (standing for filtered) denote the category with one object for each nonnegative integer, and where $\operatorname{Hom}(a,b)$ is empty if $a < b$, and $\operatorname{Hom}(a,b)$ has a unique element if $a \geq b$.

Note that the data of a functor $\mathsf{Fil} \to \mathsf{Ab}$ is a sequence of abelian groups with maps between them

$$A_0 \xleftarrow{\partial_1} A_1 \xleftarrow{\partial_2} A_2 \xleftarrow{\partial_3} A_3 \xleftarrow{\partial_4} \cdots$$

33 **Definition**

A **nonnegative chain complex** of abelian groups is a functor $\mathsf{Fil} \to \mathsf{Ab}$ with the property that $\partial_{i-1} \circ \partial_i = 0$ for all $i \geq 2$.

Note that there is a category $\mathsf{chAb}_{\geq 0}$ of nonnegative chain complexes where the morphisms are natural transformations of functors $\mathsf{Fil} \to \mathsf{Ab}$. More explicitly, a map of nonnegative chain complexes is a diagram

$$\begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[l, "\partial_1"'] \arrow[d] & A_2 \arrow[l, "\partial_2"'] \arrow[d] & A_3 \arrow[l, "\partial_3"'] \arrow[d] & \cdots \arrow[l, "\partial_4"'] \\
B_0 & B_1 \arrow[l, "\partial_1"] & B_2 \arrow[l, "\partial_2"] & B_3 \arrow[l, "\partial_3"] & \cdots \arrow[l, "\partial_4"]
\end{tikzcd}$$

where all the squares commute.

34 **Claim**

There is a functor

$$\begin{aligned} S_* \colon \mathsf{Fun}(\Delta_{\mathrm{inj}}^{\mathrm{op}}, \mathsf{Set}) &\to \mathsf{chAb}_{\geq 0} \\ X &\mapsto \left(S_0(X) \xleftarrow{\partial_1} S_1(X) \xleftarrow{\partial_2} S_2(X) \xleftarrow{\partial_3} S_3(X) \xleftarrow{\partial_4} \cdots \right) \end{aligned}$$

mapping a semisimplicial set to the chain complex of singular n-chains, which are the free abelian groups generated by the n-simplices X_n.

35 **Theorem**

There are functors $Z_n, B_n, H_n \colon \mathsf{chAb}_{\geq 0} \to \mathsf{Ab}$.

In summary, the $H_n \colon \mathsf{Top} \to \mathsf{Ab}$ is a composite of

$$\mathsf{Top} \xrightarrow{\mathrm{Sing}} \mathsf{Fun}(\Delta_{\mathrm{inj}}^{\mathrm{op}}, \mathsf{Set}) \xrightarrow{S_*} \mathsf{chAb}_{\geq 0} \xrightarrow{H_n = \ker(\partial_n)/\mathrm{im}(\partial_{n+1}) = Z_n/B_n} \mathsf{Ab}$$

5 September 14, 2020

We can extend the definition of nonnegative chain complexes to all of the integer.

36 **Definition**

A (not necessarily nonnegative) **chain complex** is a sequence of abelian groups and maps:

$$\cdots \xrightarrow[\partial_2]{} A_1 \xrightarrow[\partial_1]{} A_0 \xrightarrow[\partial_0]{} A_{-1} \xrightarrow[\partial_{-1}]{} \cdots$$

There are functors $\mathrm{chAb}_{\geq 0} \longrightarrow \mathrm{chAb}$ that just sends

$$\cdots \xrightarrow[\partial_2]{} A_2 \xrightarrow[\partial_1]{} A_1 \xrightarrow[\partial_0]{} A_0$$

to

$$\cdots \xrightarrow[\partial_2]{} A_2 \xrightarrow[\partial_1]{} A_1 \xrightarrow[\partial_0]{} A_0 \xrightarrow[\partial_{-1}]{} 0 \xrightarrow[0]{} \cdots$$

Therefore, we can similarly define functors $Z_n, B_n, H_n \colon \mathrm{chAb} \longrightarrow \mathrm{Ab}$ analogously to how we did before.

5.1 Homology

5.1.1 Of a point

Let $X = *$ be the one point topological space. Then

$$\mathrm{Sing}_n(X) = \{\text{continuous functions } \Delta^n \longrightarrow *\} = \{a_n\},$$

where a_n is the unique continuous map $\Delta^n \longrightarrow *$. Then, the map S_* gives

$$S_*(X) = \left(\mathbb{Z}\{a_0\} \xleftarrow[\partial_1]{} \mathbb{Z}\{a_1\} \xleftarrow[\partial_2]{} \mathbb{Z}\{a_2\} \xleftarrow[\partial_3]{} \cdots \right)$$

where

$$\partial_n(a_n) = \sum_{k=0}^{n} (-1)^k (d_k a_n) = \sum_{k=0}^{n} (-1)^k a_{n-1} = a_{n-1} \sum_{k=0}^{n} (-1)^k = \begin{cases} a_{n-1} & k \text{ is even} \\ 0 & k \text{ is odd.} \end{cases}$$

In summary, $S_*(*)$ is isomorphic, in $\mathrm{chAb}_{\geq 0}$, to

$$\mathbb{Z} \xleftarrow[\partial_1 = 0]{} \mathbb{Z} \xleftarrow[\partial_2 = \mathrm{id}]{} \mathbb{Z} \xleftarrow[\partial_3 = 0]{} \mathbb{Z} \xleftarrow[\partial_4 = \mathrm{id}]{} \mathbb{Z} \longleftarrow \cdots$$

We can then compute

$$\begin{aligned} H_0(*) &\cong \ker(\partial_0)/\mathrm{im}(\partial_1) \cong \mathbb{Z}/0 \cong \mathbb{Z} \\ H_1(*) &\cong \ker(\partial_1)/\mathrm{im}(\partial_2) \cong \mathbb{Z}/\mathbb{Z} \cong 0 \\ H_2(*) &\cong \ker(\partial_2)/\mathrm{im}(\partial_3) \cong 0/0 \cong 0 \\ &\vdots \end{aligned}$$

In particular,

$$H_n(*) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

5.1.2 Of stars

37 **Definition**

A subset $X \subseteq \mathbb{R}^n$ is **star-shaped** with respect to $b \in X$ if for every $x \in X$,

$$\{tb + (1-t)x \mid t \in [0,1]\} \subseteq X.$$

Intuitively, this means that the segment from x to any other point is contained within X.

38 **Theorem**

Suppose X is star-shaped with respect to $b \in X$. Then for all $n \in \mathbb{Z}$, we have $H_n(X) \cong H_n(*)$.

To approach this, we will hop over to some algebra with chAb.

Digression to chAb

Let C_*, D_* be chain complexes and $f_0, f_1 \colon C_* \to D_*$ be two chain maps. A **chain homotopy** $h \colon f_0 \simeq f_1$ is a collection of homomorphisms $h \colon C_n \to D_{n+1}$ such that $\partial h + h\partial = f_1 - f_0$.

To visualize this, consider the (not commutative) diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] & C_n \arrow[r] \arrow[dl, "h"'] & C_{n-1} \arrow[r] \arrow[dl, "h"'] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

There are two ways to get from $C_n \to D_n$ using h, namely ∂h and $h\partial$. The sum of these is $f_1 - f_0$. We say that f_0 and f_1 are **chain homotopic** if there exists some chain homotopy $h \colon f_0 \simeq f_1$.

39 **Lemma** (Homotopy invariance on homology)

Suppose $f, g \colon C_* \to D_*$ are chain homotopic. Then $H_n(f), H_n(g) \colon H_n(C_*) \to H_n(D_*)$ are equal.

Proof. If f, g are chain homotopic, then we know

$$f = \partial h + h\partial + g$$

for some $h \colon C_n \to D_{n+1}$. Let $c \in Z_n(C_*) = \ker \partial_n$. Then,

$$H_n(f)([c]) = H_n(\partial h + h\partial + g)([c]) = H_n(\partial hc + h\underbrace{\partial c}_{=0} + gc) = H_n(\underbrace{\partial hc}_{\in \operatorname{im}\partial_{n+1}}) + H_n(gc) = H_n(g)([c])$$

where the brackets indicate the class of the element of the homology group. Therefore, $H_n(f) = H_n(g)$. □

Here, we're omitting the indices on the ∂s, where some of them should be ∂_n and others should be ∂_{n+1}. This makes it faster to write equations, since there is a unique choice that makes sense.

Proof (theorem 38). Suppose X is star shaped with respect to $b \in X$. Then the map $X \to *$ induces a chain map $\varepsilon \colon S_*(X) \to S_*(*)$ and the map $b \colon * \to X$ induces a chain map $\eta \colon S_*(*) \to S_*(X)$.

We will show that for all n, $H_n(\varepsilon)$ and $H_n(\eta)$ are inverse isomorphisms of abelian groups. Then, that means that $H_n(S_*(X))$ and $H_n(S_*(*))$ are isomorphic. First note that

$$H_n(\varepsilon) \circ H_n(\eta) = H_n(\varepsilon \circ \eta) = H_n(1_{S_*(X)}) = 1_{H_n(S_*(*))}.$$

It remains to analyze $H_n(\eta) \circ H_n(\varepsilon) = H_n(\eta \circ \varepsilon)$. We will prove that $\eta \circ \varepsilon \colon S_*(X) \to S_*(X)$ is chain homotopic to the identity, from which the result will follow by lemma 39. The chain homotopy $h \colon S_q(X) \to S_{q+1}(X)$ will send simplices to simplices.

In particular, for $\sigma \in \mathrm{Sing}_q(X)$, define $h(\sigma)\colon \Delta^{q+1} \to X$ by

$$h(\sigma)(t_0, t_1, \dots, t_{q+1}) = \begin{cases} b & t_0 = 1 \\ t_0 b + (1 - t_0)\sigma\left(\frac{t_1}{1-t_0}, \frac{t_2}{1-t_0}, \dots, \frac{t_{q+1}}{1-t_0}\right) & \text{otherwise.} \end{cases}$$

If σ embeds Δ^q into X, then $h\sigma$ embeds Δ^{q+1} into X, where indices are shifted up by 1 and the extra coordinate t_0 gets mapped linearly to b. Since X is star shaped, the entire simplex Δ^{q+1} embedding is contained within X.

Note that we have $d_0(h(\sigma)) = \sigma$, and for $i > 0$,

$$d_i(h(\sigma)) = h(d_{i-1}(\sigma)).$$

Therefore,

$$\begin{aligned} \partial(h(\sigma)) &= d_0 h\sigma - d_1 h\sigma + d_2 h\sigma - \cdots \\ &= \sigma - hd_0\sigma + hd_1\sigma - \cdots \\ &= \sigma - h(d_0\sigma - d_1\sigma + \cdots) \\ &= \sigma - h\partial\sigma. \end{aligned}$$

We conclude that $\partial h + h\partial = 1 - \eta\varepsilon$ because TODO. Therefore, $H_n(\eta \circ \varepsilon) = 1_{H_n(S_*(*))}$, and $H_n(X) \cong H_n(*)$, as desired. □

6 September 16, 2020

40 **Question**

Given two maps $f, g\colon X \to Y$ in Top, when are $S_*(f)$ and $S_*(f)$ chain homotopic? When they are, we learn that $H_n(f) = H_n(g)$.

41 **Definition**

A **homotopy** h between $f, g\colon X \to Y$ is a continuous map $h\colon X \times [0,1] \to Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$. We say that f and g are **homotopic** if there exists a homotopy from f to g.

Often we think of the second variable as time, and we can think of h as a family of maps that interpolates from f to g, such that at $t = 0$, we have the map f and at $t = 1$, we have g.

42 **Theorem**

If f and g are homotopic, then $S_*(f)$ and $S_*(g)$ are chain homotopic. Hence, $H_n(f) = H_n(g)$ as maps in Ab.

The proof of this is on PSet 2.

Suppose $h_{12}\colon X \times [0,1] \to Y$ is a homotopy from f_1 to f_2, and $h_{12}\colon X \times [0,1] \to Y$ is a homotopy from f_2 to f_3. Then we can define $h_{13}\colon X \times [0,1] \to Y$ where

$$h_{13}(x,t) = \begin{cases} h_{12}(x,2t) & 0 \le t \le 1/2 \\ h_{23}(x,2t-1) & 1/2 \le t \le 1. \end{cases}$$

This h_{13} is a homotopy from f_1 to f_3, implying that homotopy equivalence is transitive. Indeed, if we define $f \simeq g$ if f and g are homotopic, then $\simeq$ is an equivalence relation on the set $\mathrm{Hom}_{\mathrm{Top}}(X,Y)$.

Furthermore, there is a category Ho(Top) called the homotopy category where

$$\mathrm{Obj}\,\mathrm{Ho}(\mathrm{Top}) = \mathrm{Obj}\,\mathrm{Top}$$

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top})}(X,Y) = \mathrm{Hom}_{\mathrm{Top}}(X,Y)/\simeq.$$

In particular, the maps in the homotopy category Ho(Top) are homotopy classes of maps in Top.

Note that there is a canonical functor Top $\to$ Ho(Top), namely sending a topological space to itself, and sending a continuous map to the equivalence class containing that continuous map under the homotopy relation.

A consequence of the theorem is that there is a diagram of functors

$$\begin{tikzcd} \mathrm{Top} \arrow[rr, "H_n"] \arrow[dr] & & \mathrm{Ab} \\ & \mathrm{Ho}(\mathrm{Top}) \arrow[ur] & \end{tikzcd}$$

that commutes. In particular, we claim that there exists a unique functor Ho(Top) $\to$ Ab.

43 **Aside on fundamental groups**

The fundamental group is similar in that we can form the diagram

$$\begin{tikzcd} \mathsf{Top}_* \arrow[rr, "\pi_1"] \arrow[dr] & & \mathsf{Grp} \\ & \mathsf{Ho}(\mathsf{Top}) \arrow[ur] & \end{tikzcd}$$

where Top_* is the category of topological spaces with a base point.

44 **Definition**

A continuous map $f\colon X \to Y$ of topological spaces is called a **homotopy equivalence** if it maps to an isomorphism under the functor $\mathsf{Top} \to \mathsf{Ho}(\mathsf{Top})$, i.e. there exists a map $g\colon Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$.

Two spaces X, Y are **homotopy equivalent** if there exists a homotopy equivalence $f\colon X \to Y$.

More explicitly, f is a homotopy equivalence if and only if there exists an inverse $g\colon Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. Note that we only require these compositions to be homotopic to the identity, as opposed to equal to the identity, which describes homeomorphism. An important distinction to make is that maps can be homotopic, but spaces are homotopy equivalent.

45 **Example**

The maps $f\colon \mathbb{R} \to *$ and $g\colon * \to \mathbb{R}$ where $f\colon x \mapsto *$ and $g\colon * \mapsto 0$ compose to $f \circ g\colon * \mapsto *$ and $g \circ f\colon x \mapsto 0$, which are homotopic to their respective identities.

Therefore, $\mathbb{R}$ and $*$ are homotopy equivalent, but not homeomorphic.

46 **Fact**

The thing to remember here is that homology cannot tell the difference between homotopy equivalent spaces or homotopic maps. In particular, homology is an invariant on a weaker notion of equivalence, namely homotopy equivalence.

It's useful to get a geometric feeling for when two non-homeomorphic spaces are homotopy equivalent. The following way is a simple way to check for homotopy equivalence a lot of the time.

47 **Definition**

An inclusion $A \hookrightarrow X$ is a **deformation retraction** provided that there is a map $h\colon X \times [0,1] \to X$ such that

1. $h(x, 0) = x$ for all $x \in X$,
2. $h(x, 1) \in A$ for all $x \in X$,
3. $h(a, t) = a$ for all $a \in A$ and $t \in [0,1]$.

Intuitively, we can again think of the second coordinate t as time, where at $t = 0$, we have the space X, and as time goes on, we retract everything into A, while not moving anything in A throughout the process.

48 **Example**

$S^{n-1} \subset \mathbb{R}^n - \{0\}$ is a deformation retraction. In particular, we can take each point x in $\mathbb{R}^n - \{0\}$ and move it to the point on S^{n-1} that intersects the ray from the origin passing through x.

One way to prove that two spaces X and Y are homotopy equivalent is to exhibit a common deformation retraction:

$$X \hookleftarrow A \hookrightarrow Y$$

We know that if $A \simeq B$ are homotopy equivalent spaces, then they have the same homology. Last class, we proved that star-shaped regions are homotopy equivalent to a point. Our next goal is to answer the following question:

49 **Question**

If $A \subseteq X$ is the inclusion of a subspace, how does $H_n(A)$ compare to $H_n(X)$?

50 **Claim**

If $A \subseteq X$ is the inclusion of a subspace, $S_*(A) \to S_*(X)$ is a subcomplex.

51 **Definition**

A **subcomplex** $C_* \subseteq D_*$ is a diagram

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & C_{-1} & \longrightarrow & C_{-2} & \longrightarrow & \cdots \\
 & & \cap\!\!| & & \cap\!\!| & & \cap\!\!| & & \cap\!\!| & & \cap\!\!| & & \\
\cdots & \longrightarrow & D_2 & \longrightarrow & D_1 & \longrightarrow & D_0 & \longrightarrow & D_{-1} & \longrightarrow & D_{-2} & \longrightarrow & \cdots
\end{array}$$

where the $\subseteq$s are inclusion maps and all squares commute.

When we have an inclusion, then we can take the quotient.

52 **Construction**

Suppose $C_* \subseteq D_*$ is a subcomplex. The quotient complex D_*/C_* has groups $(D_*/C_*)_n = D_n/C_n$, where the differential $\partial_n \colon D_n/C_n \to D_{n-1}/C_{n-1}$ is well defined because if two classes in D_n differ by a class in C_n, then their boundaries differ by a class in C_{n-1}. This is the same as the condition that all the squares commute.

7 September 18, 2020

53 **Definition**

If $A \subseteq X$ is a subspace, let $H_n(X, A)$ be the nth homology of the quotient chain complex $S_*(X)/S_*(A)$.

The general strategy for computing $H_m(X)$ is roughly the following:

1. Find a slightly simpler space $A \subseteq X$.
2. Relate $H_m(A)$, $H_m(X)$, and $H_m(X, A)$.
3. Relate $H_m(X, A)$ to $X_m(X/A)$.

Before we make steps 2 and 3 precise, it is useful to introduce a new category.

54 **Definition**

There is a category Top_2 with

$$\operatorname{Obj}\mathrm{Top}_2 = \begin{Bmatrix} \text{pairs } (X, A) \text{ where } X \text{ is a topological space} \\ \text{and } A \subseteq X \text{ is a subspace} \end{Bmatrix}$$

$$\operatorname{Hom}_{\mathrm{Top}_2}((X, A), (Y, B)) = \{\text{continuous maps } f: X \to Y \text{ where } f(A) \subseteq B\}$$

There is, for each $m \geq 0$ a functor

$$H_m: \mathrm{Top}_2 \to \mathrm{Ab}.$$

There is also a functor

$$\mathrm{Top} \to \mathrm{Top}_2$$

sending X to the pair $(X, \varnothing)$. This lets us view Top as a subcategory of Top_2.

7.1 Some algebra

55 **Definition**

Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence of abelian groups with maps between them. We say this sequence is **exact** at B if $\ker g = \operatorname{im} f$.

56 **Example**

- The sequence $0 \longrightarrow A \xrightarrow{f} B$ is exact if and only if f is injective.
- The sequence $A \xrightarrow{f} B \longrightarrow 0$ is exact if and only if f is surjective.

57 **Definition**

A longer sequence is said to be **exact** if it is exact at every three-term subsequence.

58 **Example**

The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is exact. Indeed, at the first $\mathbb{Z}$, multiplication by 2 is injective; at the second $\mathbb{Z}$, the kernel of the projection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is the same as the image of the multiplication by 2 map; and at the $\mathbb{Z}/2\mathbb{Z}$, the projection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is surjective.

In general, an exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called a **short exact sequence**.

59 **Example**

$$S_*(\varnothing) = \left(\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \right)$$

is exact.

$S_*(*)$ is not exact. However, it is away from $S_0(*)$.

60 **Theorem** (Five lemma)

Suppose

$$\begin{tikzcd}
A_4 \arrow[r, "d"] \arrow[d, "f_4"] & A_3 \arrow[r, "d"] \arrow[d, "f_3"] & A_2 \arrow[r, "d"] \arrow[d, "f_2"] & A_1 \arrow[r, "d"] \arrow[d, "f_1"] & A_0 \arrow[d, "f_0"] \\
B_4 \arrow[r, "d"'] & B_3 \arrow[r, "d"'] & B_2 \arrow[r, "d"'] & B_1 \arrow[r, "d"'] & B_0
\end{tikzcd}$$

is a commutative diagram in Ab. Suppose that both rows are exact and that f_0, f_1, f_3, f_4 are isomorphisms. Then f_2 is also an isomorphism.

Proof. We will show that f_2 is surjective. The proof method we will use is called **diagram chasing**, which typically involves pointing at a diagram. To simulate the experience in text, we will use bullet points, as we often make seemingly unrelated claims that follow one after another.

We will show surjectivity of f_2 first.

- Let $b_2 \in B_2$.
- Since f_1 is surjective, there exists $a_1 \in A_1$ such that $f_1 a_1 = db_2$.
- By commutativity of the diagram, we have $f_0 da_1 = df_1 a_2$.
- By exactness, $d^2 = 0$, so $f_0 da_1 = df_1 a_1 = ddb_2 = 0$.
- Since f_0 is an isomorphism, $da_1 = 0$.
- By exactness, $a_1 \in \ker d = \operatorname{im} d$ (where the ds have different indices), so there exists $a_2 \in A_2$ such that $da_2 = a_1$.
- By commutativity, $df_2 a_2 = f_1 da_2 = f_1 a_1 = db_2$.
- Since d is a homomorphism, $d(b_2 - f_2 a_2) = 0$.
- By exactness, there exists some $b_3 \in B_3$ such that $db_3 = b_2 - f_2 a_2$.
- By commutativity, $df_3 = f_2 d$, so $f_2 da_3 = b_2 - f_2 a_2$.
- Rearranging, $b_2 = f_2(a_2 + da_3)$, so f_2 is surjective.

Now we will show that f_2 is injective, i.e. $\ker f_2 = 0$.

- Let $a_2 \in A_2$ such that $f_2 a_2 = 0$, and therefore $df_2 a_2 = 0$. We want to show $a_2 = 0$.
- By commutativity, we have $f_1 da_2 = 0$.
- Since f_1 is injective, $da_2 = 0$.

- By exactness, there exists $a_3 \in A_3$ such that $da_3 = a_2$.
- By commutativity, $df_3a_3 = f_2da_3 = f_2a_2 = 0$.
- By exactness, $f_3a_3 \in \ker d$, so there exists $b_4 \in B_3$ such that $db_4 = f_3a_3$.
- Since f_4 is surjective, there exists $a_4 \in A_4$ such that $f_4a_4 = b_4$, and furthermore $df_4a_4 = db_4 = f_3a_3$.
- By commutativity, $f_3da_4 = df_4a_4 = f_3a_3$.
- Rearranging, $f_3(a_3 - da_4) = 0$.
- Since f_3 is injective, $a_3 = da_4$.
- Applying d to both sides and using injectivity, $da_3 = d^2a_4 = 0$, as desired. □

61 **Definition**

A **short exact sequence of chain complexes** is a diagram

$$0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0$$

or equivalently

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the rows are chain complexes and the vertical maps are short exact sequences of abelian groups.

62 **Example**

Suppose $A \subseteq X$ is an inclusion of topological spaces. Then

$$0 \longrightarrow S_*(A) \longrightarrow S_*(X) \longrightarrow S_*(X)/S_*(A) \longrightarrow 0$$

is a short exact sequence of chain complexes. This is easy to check because there is a natural injection $S_*(A) \to S_*(X)$ and a natural surjection $S_*(X) \to S_*(X)/S_*(A)$.

63 **Theorem**

If

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

is a short exact sequence of chain complexes, then there is a "long" exact sequence of homology groups

$$\cdots \longrightarrow H_{n+1}(B) \xrightarrow{H_{n+1}(g)} H_{n+1}(C) \longrightarrow H_n(A) \xrightarrow{H_n(f)} H_n(B) \xrightarrow{H_n(g)} H_n(C) \longrightarrow H_{n-1}(A) \xrightarrow{H_{n-1}(f)} H_{n-1}(B) \longrightarrow \cdots$$

where the interesting maps are the ones $H_{n+1}(C) \to H_n(A)$.

In particular, there is a relationship between the homology of C and A, with the degree lowered by 1. This is a version of what is called the "snake lemma".

The proof is again by diagram chasing, so we will defer it to the homework. It is also proved in this clip from a movie.

64 **Corollary**

Suppose $A \subseteq X$ is an inclusion of spaces. Then there is a long exact sequence

$$\cdots \longrightarrow H_{n+1}(X) \longrightarrow H_{n+1}(X, A) \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

The point of this is that there exists a long exact sequence, even though the degree lowering maps are mysterious and unknown. In particular, to express this relationship, we have to use all of the homologies at once, as opposed to a homology in a fixed degree.

65 **Question**

How do we compute $H_n(X, A)$?

Simple case: If $b \in X$, i.e. a pair $(X, \{b\})$, what is $H_m(X, \{b\})$?

Consider

$$S_*(\{b\}) \longrightarrow S_*(X) \longrightarrow S_*(X)/S_*(\{b\})$$

Note for all $m \geq 0$, the quotient $S_m(X)/S_m(\{b\})$ is the quotient of the free abelian group generated by the m-simplices of X by the group generated by the single m-simplex

$$\Delta^m \longrightarrow * \xrightarrow{b} X$$

This m-simplex is easy to calculate the boundary of. For $m > 0$, the homology $H_m(X) \cong H_m(X, \{b\})$. When $m = 0$, we know from the problem set that $H_0(X) \cong \mathbb{Z}\pi_0 X$. Indeed, there is a direct sum decomposition

$$H_0(X) \cong \mathbb{Z} \oplus H_0(X, \{b\})$$

where the summand $\mathbb{Z}$ corresponds to the path component of b.

8 September 21, 2020

The final piece that we need before we do calculations is the following theorem.

66 **Theorem** (Excision)

Let (X, A) be a pair of spaces, and assume there is a subspace $B \subseteq X$ such that

1. $\overline{A} \subseteq \operatorname{int} B$ and
2. $A \longrightarrow B$ is a deformation retract.

Then for all m there is an isomorphism

$$H_m(X, A) \cong H_m(X/A, *) \cong \begin{cases} H_m(X/A) & m > 0 \\ \ker(\mathbb{Z}\pi_0 X \longrightarrow \mathbb{Z}) & m = 0. \end{cases}$$

The condition that there is a subspace B is not particularly restrictive, as it almost always exists. Once this condition is satisfied, this theorem says that relative homology is the same thing as the absolute homology of a quotient, as seen from the snake lemma.

67 **Definition**

Suppose $U \subseteq A \subseteq X$. Such a triple is said to be **excisive** if $\overline{U} \subseteq \operatorname{Int} A$. The inclusion $(X - U, A - U) \subseteq X, A$ is called an **excision**.

68 **Theorem**

An excision induces an isomorphism

$$H_m(X - U, A - U) \cong H_m(X, A).$$

Before we prove this result, we will first make a definition and summarize what we will have afterwards.

69 **Definition**

Two maps of pairs $f, g\colon (X, A) \longrightarrow (Y, B)$ are **homotopic** if there exists a $h\colon X \times [0, 1] \longrightarrow Y$ such that

1. $h(x, 0) = f$,
2. $h(x, 1) = g$,
3. $h(a, t) \in B$ for all $a \in A$ and $t \in [0, 1]$.

This allows us to define $\mathrm{Ho}(\mathrm{Top}_2)$ analogous to how we defined $\mathrm{Ho}(\mathrm{Top})$.

8.0.1 Summary

We have:

- A sequence of functors $H_n\colon \mathrm{Top}_2 \longrightarrow \mathrm{Ab}$ for all $n \in \mathbb{Z}$.
- A sequence of natural transformations $\partial\colon H_n(X, A) \longrightarrow H_{n-1}(A) := H_{n-1}(A, \varnothing)$ from the snake lemma.

such that:

1. For any pair (X, A), the sequence

$$\cdots \longrightarrow H_{q+1}(X,A) \xrightarrow{\partial} H_q(A) \longrightarrow H_q(X) \longrightarrow H_q(X,A) \xrightarrow{\partial} \cdots$$

is exact.
2. If $f_0, f_1 \colon (X,A) \to (Y,B)$ are homotopic, then $H_n(f_0) = H_n(f_1)$.
3. Excisions induce homology isomorphisms.
4. (Dimension axiom) The groups

$$H_n(*) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

5. Let X_i be a collection of spaces indexed by I. Then

$$H_m\left(\coprod_{i \in I} X_i\right) \cong \bigoplus_{i \in I} H_m(X_i).$$

70 **Theorem** (Eilenberg-Steenrod)
These facts characterize the H_m functors. In other words, any other functor Top $\to$ Ab satisfying these properties would be isomorphic to H_m.

71 **Remark**
A set of functors $E_n \colon \mathrm{Top}_2 \to \mathrm{Ab}$ satisfying all the axioms except for 4 is called an **extraordinary homology theory**.

If we modify the dimension axiom so that the homology of a point some data that is not the integers, we can still compute things.

72 **Question**
Can one define such things that are computable but contain more refined than that seen by H_m?

This is an active area of research, namely asking which data are useful. Examples include K-theory, bordism theory, topological modular fans, topological automorphic forms, and Morava E-theories.

8.1 Calculations

Rather than proving excision right now, we can start to do some calculations.

Let's calculate $H_m(S^1)$ for various $m \in \mathbb{Z}$. Consider $S^1 \cong [0,1]/\{0,1\}$. The long exact sequence associated to the pair $([0,1], \{0,1\})$ is

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(\{0,1\}) \arrow[r] & H_2([0,1]) \arrow[r] & H_2([0,1],\{0,1\}) \arrow[dll, "\delta"'] \\
& H_1(\{0,1\}) \arrow[r] & H_1([0,1]) \arrow[r] & H_1([0,1],\{0,1\}) \arrow[dll, "\delta"'] \\
& H_0(\{0,1\}) \arrow[r] & H_0([0,1]) \arrow[r] & H_0([0,1],\{0,1\}) \arrow[r] & \cdots
\end{tikzcd}$$

We can start computing some of the elements in this chain.

- $H_2(\{0,1\}) \cong H_2(\{0\}) \sqcup H_2(\{1\}) \cong 0$
- $H_2([0,1]) \cong 0$ because the interval is contractable.
- This means that the map $H_2(\{0,1\}) \to H_2([0,1])$ is 0.
- Similarly, we have $H_1(\{0,1\}) \cong H_2([0,1]) \cong 0$.
- By exactness, we have that the maps in the chain $H_2([0,1]) \to H_2([0,1],\{0,1\}) \to H_1(\{0,1\})$ are the zero maps.

- From the dimension axiom and additivity, $H_0(\{0,1\}) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- $H_0([0,1]) \cong \mathbb{Z}$ because $[0,1]$ is contractable.

This gives us the updated diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[dll, rounded corners, "\delta=0"'] & & \\
& 0 \arrow[r, "0"] & 0 \arrow[r] & H_1([0,1],\{0,1\}) \arrow[dll, rounded corners, "\delta"'] & & \\
& \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "f"] & \mathbb{Z} \arrow[r, "g"] & H_0([0,1],\{0,1\}) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Since H_0 measures path components, the map $f\colon \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}$ maps $(1,0) \mapsto 1$ and $(0,1) \mapsto 1$. In particular, it maps the two elements representing the path components of $\{0,1\}$ to the single path component of $[0,1]$. By exactness and the first isomorphism theorem, $H_1([0,1],\{0,1\})$ is isomorphic to $\ker f = \mathbb{Z}$. Similarly, we know that $\operatorname{im} f = \mathbb{Z} = \ker g$ by exactness, so g is the zero map. This means that $H_0([0,1],\{0,1\}) \cong 0$.

Overall, we have deduced from this calculation that

$$H_0([0,1],\{0,1\}) \cong 0 \qquad H_1([0,1],\{0,1\}) \cong \mathbb{Z} \qquad H_2([0,1],\{0,1\}) \cong 0.$$

By excision, we have for $m > 0$

$$H_m([0,1],\{0,1\}) = H_m(S^1,*).$$

This gives

$$H_0(S^1) = \mathbb{Z} \oplus \tilde{H}_0(S^1) \cong \mathbb{Z} \qquad H_1(S^1) = \mathbb{Z} \qquad H_2(S^1) = 0.$$

What this means geometrically is that S^1 has no 2-dimensional holes, one 1-dimensional hole, and one path component.

If we extend the exact sequence to the left, we can continue to calculate

$$H_m(S^1) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

As a more general fact, we have

$$H_m(S^q) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0,q \\ 0 & \text{otherwise.} \end{cases}$$

73 **Corollary**

If $q \neq r$, S^q and S^r are not homotopy equivalent.

Sketch. They would have the same homology groups otherwise □

74 **Corollary**

If $q \neq r$, $\mathbb{R}^q$ and $\mathbb{R}^r$ are not homeomorphic.

Proof. Suppose they were. Then $\mathbb{R}^q - \{0\}$ would be homeomorphic to $\mathbb{R}^r - \{0\}$. However, there are deformation retracts

$$S^{q-1} \hookrightarrow \mathbb{R}^q - \{0\}$$
$$S^{r-1} \hookrightarrow \mathbb{R}^r - \{0\},$$

meaning that S^{q-1} and S^{r-1} are homotopy equivalent, which is a contradiction. □

75 **Theorem** (Brouwer fixed point theorem)

Let $n \geq 2$. If $f\colon D^n \to D^n$ is continuous, then there is a point $x \in D^n$ such that $f(x) = x$.

Proof. Suppose there were no fixed point. Then one can define the continuous map $g\colon D^n \to S^{n-1}$ that maps x to the point where the ray connecting $f(x)$ to x hits the boundary.

If $x \in S^{n-1} \subseteq D(n)$ is on the boundary, $g(x) = x$. Then, we have the maps

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[rr, bend right, "\mathrm{id}"'] & D^n \arrow[r, "g"] & S^{n-1} \end{tikzcd}$$

Applying the functor H_{n-1}, we have

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & 0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$

which is a contradiction. □

9 September 23, 2020

Recall theorem 68, namely that if $U \subseteq A \subseteq X$ with $\overline{U} \subseteq \operatorname{Int}(A)$, then the inclusion $(X-U, A-U) \subseteq (X, A)$ is called an excision. Then, the last Eilenberg-Steenrod axiom states that excisions induces homology equivalence, i.e. $H_m(X-U, A-U) \cong H_m(X, A)$.

The key to excision will be the "locality principle".

76 **Definition**

Let X be a topological space. A family $\mathcal{A}$ of subsets of X is a **cover** if X is the union of the interiors of $A \in \mathcal{A}$.

77 **Definition**

If $\mathcal{A}$ is a cover of X, an n-simplex $\sigma\colon \Delta^n \to X$ is **$\mathcal{A}$-small** if the image of σ is entirely contained in a single element of $\mathcal{A}$.

Note that if $\sigma\colon \Delta^n \to X$ is $\mathcal{A}$-small, then so is $d_i\sigma$ for $0 \le i \le n$. This means that we can form a semisimplicial set $\operatorname{Sing}^{\mathcal{A}}(X)$ with n-simplices that are the $\mathcal{A}$-small n-simplices in X. Furthermore, a chain complex $S_*^{\mathcal{A}}(X)$ can be formed.

78 **Theorem** (Locality principle)

The inclusion $S_*^{\mathcal{A}}(X) \subseteq S_*(X)$ induces isomorphisms on homology groups.

To show this, we want to show that a cycle in $Z_n(X)$ is equivalent to a cycle that is the sum of smaller simplices, i.e is $\mathcal{A}$-small. We can do this by adding boundaries.

For intuition, if we have some $\sigma\colon \Delta^1 \to X$, we can consider two other simplices f, g such that $f - g - \sigma = 0$ in $H_1(X)$, as shown in fig. 9.1. If we let the endpoints of f and g be the midpoint of σ, then we will have $\sigma = f - g$, modulo ∂. If we repeat this process, we will eventually have simplices that are $\mathcal{A}$-small.

To make this precise, we will construct a natural transformation

$$\$\colon S_n \to S_n.$$

In other words, for any $X \in \mathsf{Ab}$, we will construct a map $\$\colon S_n(X) \to S_n(X)$. To do so, we need to say what it does to a generator $\sigma\colon \Delta^n \to X$.

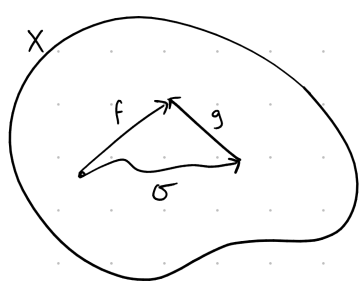
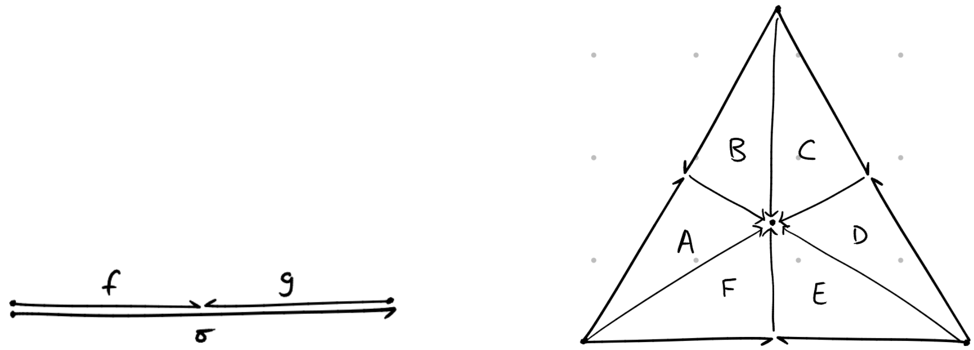


Figure 9.1: a 1-simplex in X



(a) $\$(1_{\Delta^1})$ (b) $\$(1_{\Delta^2})$

Figure 9.2: Diagrams specifying \$

Consider the naturality square

$$\begin{tikzcd} S_n(\Delta^n) \arrow[r, "\$"] \arrow[d, "S_n(\sigma)"'] & S_n(\Delta^n) \arrow[d, "S_n(\sigma)"] \\ S_n(X) \arrow[r, "\$"'] & S_n(X) \end{tikzcd}$$

which commutes if \$ is natural. There is a special n-simplex

$$1_{\Delta^n} : \Delta^n \to \Delta^n.$$

$1_{\Delta^n} \in S_n(\Delta^n)$ is one of the generators of the free abelian group. Note that $S_n(\sigma)(1_{\Delta^n}) = \sigma \in S_n(X)$. Naturality says

$$\$(\sigma) = S_n(\sigma)(\$(1_{\Delta^n})).$$

Therefore, if we specify $\$(1_{\Delta^n})$, then we specify \$.

- For Δ^0, we define $\$(1_{\Delta^0}) = 1_{\Delta^0}$, meaning that if we have Δ^0, meaning that we do not need to subdivide a point to make it more local.
- For Δ^1, we let $\$(1_{\Delta^1}) = f - g \in S_1(\Delta^1)$, as given by fig. 9.2.
- For Δ^2, we let $\$(1_{\Delta^2}) = A - B + C - D + E - F \in S_1(\Delta^2)$, also as given by fig. 9.2.

If we calculate the boundary of these expressions, we get the same thing as what we would have if we took the boundary of Δ^1 or Δ^2. We think of these expressions as equivalent under homology, but more local.

More generally, to define $\$(1_{\Delta^n})$, we let b denote the center of mass of Δ^n. We subdivide the boundary of Δ^n according to \$ in one lower dimension, and then connect everything to b. In particular,

$$\$1_{\Delta^n} = b * \$(\partial 1_{\Delta^n}),$$

where $b * - : S_{n-1}(\Delta^n) \to S_n(\Delta^n)$ is the cone construction for star shaped regions defined for star-shaped regions.

79 **Theorem**

For any topological space X,

$$\$: S_*(X) \to S_*(X)$$

is a chain map. Furthermore, it is chain homotopic to the identity.

Proof. Note that we have $\partial \$ 1_{\Delta^n} = \$ \partial 1_{\Delta^n}$ from the definition of the construction of \$. It follows by naturailty of \$ that if $\sigma : \Delta^n \to X$ is any n-simplex,

$$\partial \$ \sigma = \partial \$ (S_n(\sigma)(1_{\Delta^n})) = S_n(\sigma)(\partial \$ 1_{\Delta^n}) = S_n(\sigma)(\$ \partial 1_{\Delta^n}) = \$ \partial \sigma.$$

Therefore, $\$$ is a chain map.

To show that $\$$ is homotopic to the identity, we must define a chain homotopy

$$T\colon S_n(X) \longrightarrow S_{n+1}(X)$$

from $\$$ to $1_{S_*(X)}$. Again by naturality, it suffices to define $T_{1_{\Delta^n}}$. In particular, we define $T(1_{\Delta^0}) = 0$ and for $n > 0$ define

$$T(1_{\Delta^n}) = b * (\$1_{\Delta^n} - 1_{\Delta^n} - T\partial 1_{\Delta^n}) \in S_{n+1}(\Delta^n).$$

□

Example

To give some intuition, $T(1_{\Delta^1}) \in S_2(\Delta^1)$ is the squashed Δ^2 we drew in fig. 9.2. In particular, T specifies how some simplex is related to its subdivision.

Suppose that $\mathcal{A}$ is a cover of X. We want to show that $S_*^{\mathcal{A}}(X) \subseteq S_*(X)$ is a homology isomorphism by repeatedly applying $\$$.

80 **Lemma**

Let $\mathcal{A}$ be a cover of Δ^n. Then for any $\sigma \in S_n(\Delta^n)$, there exists $k \in \mathbb{N}$ such that $\$^k\sigma = \underbrace{\$\$\ldots\$}_{k \text{ times}}\sigma$ is $\mathcal{A}$-small.

This is a consequence of the following lemma:

81 **Lemma** (Lebesgue covering)

Let M be a compact metric space, and let $\mathcal{U}$ be an open cover of M. Then there is some $\varepsilon > 0$ such that for all $x \in M$, $B_\varepsilon(x) \subseteq U$ for some $U \in \mathcal{U}$.

In particular, we apply this with $M = \Delta^n$ and $\mathcal{U} = \{\operatorname{Int} A \mid A \in \mathcal{A}\}$. More specifically, Δ^n has a specific diameter, and as we subdivide it, the diameter gets smaller until it is less than ε.

Here is a more general lemma:

82 **Lemma**

Let $\mathcal{A}$ be a cover of X and $\sigma \in S_n(X)$. There exists $k \in \mathbb{N}$ such that $\$^k\sigma$ is $\mathcal{A}$-small.

The reason why we're able to do this with general spaces that aren't necessarily compact is because we are only working with compact subsets of X at one instance.

Proof. Assume without loss of generality that $\sigma\colon \Delta^n \to X$. We can consider $\sigma^{-1}(\operatorname{Int} A)$ for $A \in \mathcal{A}$, which together form an open cover of Δ^n. Then we can apply the previous lemma. □

10 September 25, 2020

Let's give some more intuition about the dimension-lowering connecting map in the long exact sequence.

Let $A \subseteq X$ be a pair of spaces such that

$$H_m(A, X) \cong H_m(X/A, *) \cong H_m(X/A)$$

for $m > 0$. We have the long exact sequence

$$\cdots \longrightarrow H_m(A) \longrightarrow H_m(X) \longrightarrow H_m(X, A) \cong H_m(X/A) \xrightarrow{\partial} H_{m-1}(A) \longrightarrow H_{m-1}(X) \longrightarrow \cdots$$

83 **Question**

What is the geometric intuition of ∂?

Consider some class $c \in S_m(X)$ not in $Z_m(X)$, i.e. $\partial c \neq 0$. Suppose that the image of c under the quotient map $S_m(X) \longrightarrow S_m(X/A)$ is a cycle. Then $\partial c \in S_{m-1}(X)$ is actually in the image of $S_{m-1}(A) \hookrightarrow S_{m-1}(X)$, and $\partial c \in Z_{m-1}(A)$. Therefore, ∂c represents and element of $H_{m-1}(A)$.

84 **Example**

Suppose $X = [0, 1]$ and $A = \{0, 1\}$. Then $X/A \cong S^1$. Consider $\sigma\colon \Delta^1 \longrightarrow X \in S_1(X)$ where $\sigma\colon e_0 \mapsto 0, e_1 \mapsto 1$. In $S_*(X)$, $\partial\sigma = \{1\} - \{0\} \neq 0 \in S_0(X)$. However, under the quotient $X \longrightarrow X/A$, σ is sent to an element in $Z_1(X/A)$. Therefore, σ represents a class in $H_1(X, A) \cong H_1(S^1)$, but its boundary $\partial\sigma = \{1\} - \{0\}$ is a class in $H_0(A)$. Symbolically,

$$\begin{gathered} \cdots \longrightarrow H_1(S^1) \longrightarrow H_0(\{0,1\}) \longrightarrow \cdots \\ \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \\ \sigma \longrightarrow (-1, 1). \end{gathered}$$

10.1 Excision

Recall that there is a chain homotopy T from \$ to $1_{S_*(X)}$.

85 **Corollary**

There is a natural chain homotopy T_k from $\$^k$ to $1_{S_*(X)}$.

86 **Corollary**

If $\sigma \in Z_n(X)$, then $\$^k\sigma$ is equivalent to σ modulo boundaries.

We will now prove the locality principle.

Theorem (Locality principle)

Let $\mathcal{A}$ be a cover of X. Then $S_*^{\mathcal{A}}(X) \subseteq S_*(X)$ induces an isomorphism $H_n^{\mathcal{A}}(X) \leftrightarrow H_n(X)$.

Proof. First we will show surjectivity. Suppose we have some $c \in Z_n(X)$. We want to find an $\mathcal{A}$-small element of $Z_n(X)$ equivalent to c modulo boundaries. We just use $\$^k c$ for a sufficiently large k, and the Lebesgue covering lemma tells us this works.

To show injectivity, we can show that the kernel is 0. In particular, suppose $c \in Z_n^{\mathcal{A}}(X)$ is an $\mathcal{A}$-small cycle, and $c = \partial b$ for some $b \in S_{n+1}(X)$. We wish to find some $b' \in S_{n+1}^{\mathcal{A}}(X)$ such that $\partial b' = c$. We can select k such that $\$^k b$ is $\mathcal{A}$-small. Note that

$$\begin{aligned}
\partial(\$^k b) - c &= \partial((\$^k - 1_{S_{n+1}(X)})(b)) \\
&= \partial((\partial T_k + T_k \partial) b) \\
&= \partial\partial T_k b + \partial(T_k \partial b) \\
&= 0 + \partial(T_k c).
\end{aligned}$$

Rearranging,

$$c = \partial(\$^k b - T_k c).$$

We finish by the following:

> **Lemma**
> If $c \in S_n(X)$ is $\mathcal{A}$-small, then $T_k c \in S_{n+1}(X)$ is also $\mathcal{A}$-small.

Lemma proof. Without loss of generality assume that $c = \sigma$ is an $\mathcal{A}$-small n-simplex. We can represent it as a composite $\Delta^n \xrightarrow{\sigma'} A \xrightarrow{i} X$ where $A \in \mathcal{A}$, where σ' denotes the map $\Delta^n \to A$ and i is the inclusion of A into X. Note that $\sigma = S_n(i)(\sigma')$, so $T_k\sigma = (S_{n+1}(i))(T_k\sigma')$ by the naturailty of T_k. We know that $T_k\sigma$ is the image of some $T_k\sigma' \in S_{n+1}(A)$, so $T_k\sigma$ is $\mathcal{A}$-small. □

This concludes the proof of the locality principle. □

Now we can prove excision.

> **Theorem** (Excision)
> Suppose $U \subseteq A \subseteq X$ is excisive, i.e. $\overline{U} \subseteq \operatorname{Int} A$. Then the map $(X - U, A - U) \to (X, A)$ induces homology isomorphisms.

Proof. Note that the condition $\overline{U} \subseteq \operatorname{Int} A$ is equivalent to

$$\operatorname{Int} A \cup \operatorname{Int}(X - U) = X.$$

Let $B := X - U$. Then $\{A, B\}$ is a cover of X. We can rewrite $(X - U, A - U) = (B, A \cap B)$, so we want to show that the inclusion

$$S_*(B, A \cap B) \hookrightarrow S_*(X, A)$$

induces homology isomorphisms. Note the diagram of a map of short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & S_*(A) \arrow[r] \arrow[d, equal] & S_*^{\mathcal{A}}(X), \arrow[r] \arrow[d, hook] & S_*^{\mathcal{A}}(X)/S_*^{\mathcal{A}}(A) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & S_*(A) \arrow[r] & S_*(X) \arrow[r] & \underset{=S_*(X)/S_*(A)}{S_*(X, A)} \arrow[r] & 0
\end{tikzcd}$$

We can obtain a map of long exact sequences of homology groups

$$\begin{tikzcd}
\cdots \arrow[r] & H_m(A) \arrow[r] \arrow[d] & H_m^{\mathcal{A}}(X) \arrow[r] \arrow[d] & H_m(S_*^{\mathcal{A}}(X)/S_*(X)) \arrow[r, "\partial"] \arrow[d] & H_{m-1}(A) \arrow[r] \arrow[d] & H_{m-1}^{\mathcal{A}}(X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_m(A) \arrow[r] & H_m(X) \arrow[r] & H_m(S_*(X)/S_*(X)) \arrow[r, "\partial"] & H_{m-1}(A) \arrow[r] & H_{m-1}^{\mathcal{A}}(X) \arrow[r] & \cdots
\end{tikzcd}$$

By locality, the $H_m^{\mathcal{A}}(X) \to H_m(X)$ and $H_{m-1}^{\mathcal{A}}(X) \to H_{m-1}(X)$ maps are isomorphisms. By the five-lemma,

$$H_m(S_*^{\mathcal{A}}(X)/S_*(X)) \to H_m(X, A)$$

is also an isomorphism. Therefore, $S_*^{\mathcal{A}}(X)/S_*(A) \to S_*(X)/S_*(A)$ induces homology isomorphisms $H_m(S_*(X)/S_*(A)) \cong H_m(X,A)$. Then observe

$$\begin{aligned} S_n^{\mathcal{A}}(X)/S_n(A) &= \frac{S_n(A)+S_n(B)}{S_n(A)} \\ &= \frac{S_n(B)}{S_n(A)\cap S_n(B)} \\ &= \frac{S_n(B)}{S_n(A\cap B)} \\ &= \frac{S_n(X-U)}{S_n(A-U)}. \end{aligned}$$

□

87 **Corollary**

Let (X,A) be a pair of spaces and let $B \subseteq X$ be a subspace such that

1. $\overline{A} \subseteq \operatorname{Int} B$ and
2. $A \to B$ is a deformation retract.

Then for all m, $H_m(X,A) \to H_m(X/A,*)$ is an isomorphism.

Proof. Consider the commutative diagram in Top_2

$$\begin{tikzcd} (X,A) \arrow[r, "i"] \arrow[d] & (X,B) \arrow[d] & (X-A, B-A) \arrow[l, "j"'] \arrow[d, "k"] \\ (X/A,*) \arrow[r, "\bar{i}"] & (X/A, B/A) & (X/A-*, B/A-*) \arrow[l, "\bar{j}"'] \end{tikzcd}$$

We can check that each labeled arrow induces a homology isomorphism:

- k is a homeomorphism in Top_2.
- j is an excision
- i is a homology isomorphism by item 2 and the homotopy invariance of homology.
- $\bar{j}$ is an excision.
- $\bar{i}$ is also a deformation detract, obtained from the given retract $B \times I \to B$ by quotienting out by A.

Since all of the maps along the edge induce an homology isomorphism, the map on the left also induces a homology isomorphism. □

This concludes the proof of all of the Eilenberg-Steenrod axioms.

11 September 28, 2020

Here is a summary of what we have:

88 **Theorem** (Eilenberg-Steenrod axioms)

There are functors $H_n \colon \mathrm{Top}_2 \to \mathrm{Ab}$. If X is a topological space, we write $H_n(X) \coloneqq H_n(X, \varnothing)$.

There are natural transformations $\partial \colon H_n(X, A) \to H_{n-1}(A)$ where:

- If $f_0, f_1 \colon (X, A) \to (Y, B)$ are homotopic, then $H_n(f_0)H_n(f_1)$.
- Excisions in Top_2 induce isomorphisms on H_n.
- For any pair (X, A), the sequence
$$\cdots \longrightarrow H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\partial} \cdots$$
is exact.
- $H_n\left(\coprod_{i \in I} X_i\right) = \bigoplus_{i \in I} H_n(X_i)$.
- (Dimension axiom) $H_n(*) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$

An application of excision is that, in favorable conditions, $H_n(X, A) \cong H_n(X/A, *) \cong H_n(X, A)$ if $n > 0$.

11.1 Mayer-Vietoris sequence

One more practical tool for computing the homology of topological spaces is the **Mayer-Vietoris sequence**.

Suppose X is a space and $\mathcal{A} = \{A, B\}$ is a cover of X. Let us label the inclusions

$$\begin{aligned} i &\colon A \cap B \hookrightarrow A \\ j &\colon A \cap B \hookrightarrow B \\ k &\colon A \hookrightarrow X \\ l &\colon B \hookrightarrow X. \end{aligned}$$

89 **Theorem** (Mayer-Vietoris)

There is a long exact sequence

$$\cdots \longrightarrow H_{n+1}(X) \xrightarrow{\partial} H_n(A \cap B) \xrightarrow{H_n(i) \oplus H_n(j)} H_n(A) \oplus H_n(B) \xrightarrow{H_n(k) - H_k(l)} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow \cdots$$

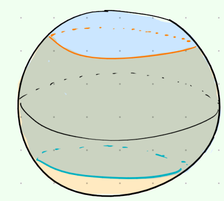
Proof. Consider the short exact sequence of chain complexes

$$0 \longrightarrow S_*(A \cap B) \xrightarrow{S_*(i,j)} S_*(A) \oplus S_*(B) \xrightarrow{S_*(k) - S_*(l)} S_*^{\mathcal{A}}(X) \longrightarrow 0$$

By the locality principle, $H_n(S_*^{\mathcal{A}}(X)) \cong H_n(X)$, so we obtain the Mayer-Vietoris sequence from the long exact sequence associated to a short exact sequence of chain complexes. $\square$

90 **Example**

Consider the sphere S^2, with the cover $\mathcal{A} = \{A, B\}$, where



$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z > -1/2\},$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid z < 1/2\},$$

as in the diagram to the right.

We have the Mayor-Vietoris sequence

$$H_2(A) \oplus H_2(B) \longrightarrow H_2(S^2) \xrightarrow{\partial} H_1(A \cap B) \longrightarrow H_1(A) \oplus H_1(B)$$

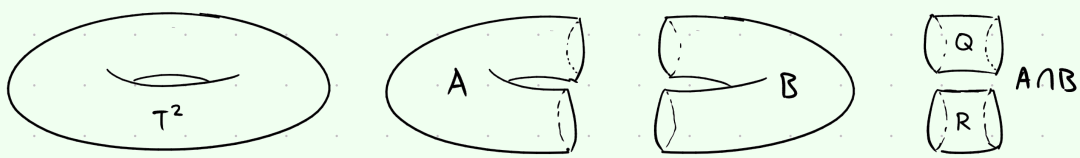
We know that A and B are homotopy equivalent to a point, so $H_2(A) \oplus H_2(B) \cong 0 \oplus 0 \cong 0$. Similarly, we know that $H_1(A) \oplus H_1(B) \cong 0$. We also know that $A \cap B$ is homotopy equivalent to S^1, so $H_1(A \cap B) = \mathbb{Z}$. Therefore, the exact sequence is

$$0 \longrightarrow H_2(S_2) \xrightarrow{\partial} \mathbb{Z} \longrightarrow 0$$

Exactness says that ∂ is both injective and surjective, so $H_2(S^2) \cong \mathbb{Z}$.

91 **Example**

Let $T^2 \cong S^1 \times S^1$ be the torus, and let $\mathcal{A} = \{A, B\}$ be the cover as defined by the image:



We have the long exact sequence

$$H_2(A) \oplus H_2(B) \to H_2(T^2) \xrightarrow{\partial} H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(T^2) \xrightarrow{\partial} H_0(A \cap B) \to H_0(A) \oplus H_0(B)$$

Note that A and B are homeomorphic to an annulus, and therefore they are homotopy equivalent to S^1. This simplifies our sequence to

$$0 \to H_2(T^2) \xrightarrow{\partial} H_1(A \cap B) \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(T^2) \xrightarrow{\partial} H_0(A \cap B) \to \mathbb{Z} \oplus \mathbb{Z}$$

Since $A \cap B$ has two path components, $H_0(A \cap B) = \mathbb{Z} \oplus \mathbb{Z}$. Also, since $A \cap B \simeq S^1 \sqcup S^1$, we know $H_1(A) \oplus H_1(B) \cong \mathbb{Z} \oplus \mathbb{Z}$.

$$0 \to H_2(T^2) \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(T^2) \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

This is not enough to understand the homology groups of the torus, so we will need to look at some of the maps. Consider the map $g \colon H_0(A \cap B) \to H_0(A) \oplus H_0(B)$. This maps the path component Q, represented by $(1, 0) \in \mathbb{Z} \oplus \mathbb{Z} \cong H_0(A \cap B)$, to $(1, 1)$ because $Q \subseteq A, B$. Similarly $R \mapsto (1, 1)$. Therefore, $g \colon (x, y) \mapsto (x + y, x + y)$.

Now consider $f \colon H_1(A \cap B) \to H_1(A) \oplus H_1(B)$. To think about this, we can consider the natural chain

$$Q \to A \cap B \to A \sqcup B \to A,$$

and applying H_1 gives

$$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}.$$

This maps $1 \mapsto (1, 0) \mapsto f(1, 0) \mapsto 1$, because $Q \to A$ is a deformation retract. Similarly, $Q \to B$ is a deformation retract, so $f(1, 0) = (1, 1)$. If we replace Q with R, we also get $f(0, 1) = (1, 1)$. Therefore, $f \colon (x, y) \mapsto (x + y, x + y)$.

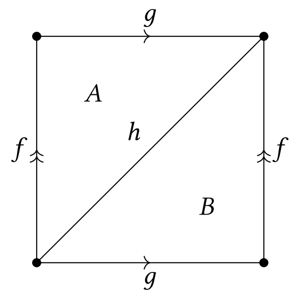


Figure 11.1: Fundamental polygon of torus

This gives the following long exact sequence

$$\begin{array}{ccccccccccccc}
H_2(A)\oplus H_2(B) & \longrightarrow & H_2(T^2) & \longrightarrow & H_1(A\cap B) & \xrightarrow{f} & H_1(A)\oplus H_1(B) & \xrightarrow{r} & H_1(T^2) & \xrightarrow{s} & H_0(A\cap B) & \xrightarrow{g} & H_0(A)\oplus H_0(B) \\
0\oplus H_2(B) & \longrightarrow & H_2(T^2) & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & H_1(T^2) & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} \\
 & & & & (x,y) & \longmapsto & (x+y,x+y) & & & & (x,y) & \longmapsto & (x+y,x+y)
\end{array}$$

Exactness at $H_2(T^2)$ tells us that $H_2(T^2) \hookrightarrow \mathbb{Z}\oplus\mathbb{Z}$ is an injection. Exactness at $H_1(A\cap B) \cong \mathbb{Z}\oplus\mathbb{Z}$ tells us that $H_2(T) \cong \ker f \cong \mathbb{Z}(1,-1) \cong \mathbb{Z}$. Therefore, we have computed $H_2(T^2) \cong \mathbb{Z}$.
To compute $H_1(T^2)$, note that exactness at $H_1(A)\oplus H_1(B)$ tells us that $\ker r = \operatorname{im} f \cong \mathbb{Z}$, and exactness at $H_0(A\cap B)$ tells us that $\ker g = \operatorname{im} s \cong \mathbb{Z}$. Therefore, $H_1(T^2)/\mathbb{Z} \cong \mathbb{Z} \implies H_1(T^2) \cong \mathbb{Z}\oplus\mathbb{Z}$.

A more combinatorial way to think about the torus as a square with the edges identified like in fig. 11.1. This represents a semisimplicial set

$$\mathbb{Z}\{A,B\} \longrightarrow \mathbb{Z}\{f,g,h\} \longrightarrow \mathbb{Z}\{x\}$$

which lets us find the homology more combinatorially. The first map maps $A \mapsto f+g-h$ and $B \mapsto g+f-h$. The second map maps $f,g,h \mapsto 0$, since all four of the points are the same.

Our goal in the next few classes is to show that no matter how we cut up the space.

12 September 30, 2020

92 **Definition**

Let C be a category and let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

be a diagram in C. A **pushout** of this diagram is a commuting square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p_B"] \\ C \arrow[r, "p_C"'] & P \end{tikzcd}$$

such that for any other commuting square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p'_B"] \\ C \arrow[r, "p'_C"'] & P' \end{tikzcd}$$

there is a unique map $p\colon P \to P'$ with $p \circ p_C = p'_C$ and $p \circ p_B = p'_B$.

12.0.1 Pushouts in Top

Suppose $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$ is a diagram in Top. Then the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "p_B"] \\ C \arrow[r, "p_C"'] & P \end{tikzcd}$$

has $P = (B \sqcup C)/(f(a) = g(a) \text{ for all } a \in A)$, where $p_B\colon B \to P$ is the composite of the natural inclusion and the quotient map $B \to B \sqcup C \to P$.

93 **Example**

The pushout of $C \longleftarrow \varnothing \longrightarrow B$ is just the disjoint union $B \sqcup C$.

94 **Example**

The pushout of $* \longleftarrow A \xrightarrow{f} B$ is the quotient $P = B/\operatorname{im} f$.

95 **Definition**

If there is a pushout square

$$\begin{tikzcd} \coprod_{i\in I} S^{n-1} \arrow[r] \arrow[d, hook] & B \arrow[d] \\ \coprod_{i\in I} D^n \arrow[r] & P \end{tikzcd}$$

then we say that P is obtained from B by attaching n-cells.

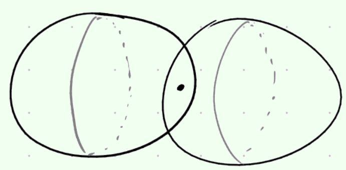
The idea is that we take B, and find copies of S^{n-1} inside B. At those points, we glue copies of D^n such that the boundary of the disks are the copies of S^{n-1} we selected.

96 **Example**

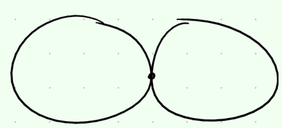
The following pushout

$$\begin{tikzcd} S^1 \sqcup S^1 \arrow[r] \arrow[d] & * \arrow[d] \\ D^2 \sqcup D^2 \arrow[r] & P \end{tikzcd}$$

is obtained from the point by attaching two 2-cells. Geometrically, P looks like this:



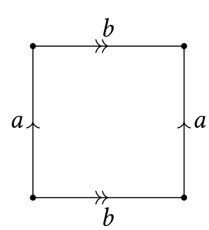
97 **Example**



The example one dimension lower is the figure-8 space, which is obtained from attaching two 1-cells to a point, i.e.

$$\begin{tikzcd} S^0 \sqcup S^0 \arrow[r] \arrow[d] & * \arrow[d] \\ D^1 \sqcup D^1 \arrow[r] & P_8 \end{tikzcd}$$

Another way to describe the figure-8 space P_8 is to say that it is homeomorphic to



where the square is not filled in. Then, we see that there is a continuous map $r\colon S^1 \to P_8$ called $aba^{-1}b^{-1}$. We can glue a 2-cell to P_8

by using r as the boundary.

$$\begin{tikzcd} S^1 \arrow[r, "r"] \arrow[d, hook] & P_8 \arrow[d] \\ D^2 \arrow[r] & P \end{tikzcd}$$

However, this just fills in the square, so what we get is the torus $P = T^2$.

98 **Definition**

A **CW complex** X is a space with a sequence of subspaces

$$\varnothing = \operatorname{Sk}_{-1} X \subseteq \operatorname{Sk}_0 X \subseteq \operatorname{Sk}_1 X \subseteq \operatorname{Sk}_2 X \subseteq \cdots \subseteq X$$

such that

- X is the union of its **skeleta** $\operatorname{Sk}_n X$
- there exist pushout diagrams

$$\begin{tikzcd} \coprod_{i \in I_n} S^{n-1} \arrow[r] \arrow[d, hook] & \operatorname{Sk}_{n-1} X \arrow[d] \\ \coprod_{i \in I_n} D^n \sqcup D^1 \arrow[r] & \operatorname{Sk}_n X \end{tikzcd}$$

  i.e. $\operatorname{Sk}_n X$ is obtained from $\operatorname{Sk}_{n-1} X$ by attaching n-cells.

99 **Remark**

Tomorrow is October so we can start talking about skeletons.

100 **Example**

The torus T^2 can be given the structure of a CW complex with

$$\begin{aligned} \operatorname{Sk}_0 T^2 &= * \\ \operatorname{Sk}_1 T^2 &= P_8 \\ \operatorname{Sk}_2 T^2 &= T \\ \operatorname{Sk}_3 T^2 &= T. \\ &\vdots \end{aligned}$$

101 **Definition**

A CW-complex is **finite dimensional** if $\operatorname{Sk}_n X = X$ for some n. The **dimension** of X is the smallest n for which this is true.

This means that T^2 is two dimensional. However, a generic CW complex may he infinite-dimensional.

102 **Definition**

A CW complex is of **finite type** if each I_n is finite.

103 **Definition**

A CW complex is **finite** if it is finite dimensional and of finite type.

Here are some facts from point set topology that we will not check, but are true.

104 **Lemma**

- Any CW complex is Hausdorff.
- A CW complex X is compact if and only if X is finite.
- Any compact smooth manifold can be given some finite CW complex structure.

Note that we can organize CW complexes into a category CWcomp where the morphisms are diagrams

$$\begin{array}{ccc}
\vdots & & \vdots \\
\cup\!| & & \cup\!| \\
\mathrm{Sk}_{n+1} X & \longrightarrow & \mathrm{Sk}_{n+1} Y \\
\cup\!| & & \cup\!| \\
\mathrm{Sk}_{n} X & \longrightarrow & \mathrm{Sk}_{n} Y \\
\cup\!| & & \cup\!| \\
\vdots & & \vdots \\
\cup\!| & & \cup\!| \\
\mathrm{Sk}_{0} X & \longrightarrow & \mathrm{Sk}_{0} Y
\end{array}$$

There is a functor

$$\mathcal{U}\colon \mathsf{CWcomp} \to \mathsf{Top}$$

that ignores the skeletal structure and just returns the union of the skeleta.

There is also a functor

$$\mathrm{Fun}(\Delta^{\mathrm{op}}_{\mathrm{inj}}, \mathsf{Set}) \to \mathsf{CWcomp}$$

where we consider the simplicial sets geometrically, i.e. if $X_\bullet$ is a simplicial set, X_n is the set of n-cells in the corresponding CW complex, and the d maps give the boundaries of the gluing.

The composite functor

$$\mathrm{Fun}(\Delta^{\mathrm{op}}_{\mathrm{inj}}, \mathsf{Set}) \to \mathsf{CWcomp} \xrightarrow{U} \mathsf{Top}$$

is called the **geometric realization**.

The goal is to understand H_n of a geometric realization. In particular We want to confirm that the homology of a semisimplicial set is the same as its geometric realization.

105 **Example**

S^n can be given a CW structure with one 0-cell and one n-cell. In particular, there is a pushout diagram

$$\begin{array}{ccc}
S^{n-1} & \longrightarrow & * \\
\big\downarrow & & \big\downarrow \\
D^n & \longrightarrow & S^n
\end{array}$$

However, we can form a different CW structure on S^n:

$$\begin{aligned}
\mathrm{Sk}_{-1} S^n &= \varnothing \\
\mathrm{Sk}_0 S_n &= * \sqcup * = S^0 \\
\mathrm{Sk}_1 S_n &= S^1 \\
&\vdots \\
\mathrm{Sk}_n S_n &= S^n \\
\mathrm{Sk}_{n+1} S_n &= S^n \\
&\vdots
\end{aligned}$$

where to get from $\mathrm{Sk}_{k-1} S^n$ to $\mathrm{Sk}_n S^n$, we glue two k-cells as "hemispheres".

Note that there is a CW complex S^∞ which has $\mathrm{Sk}_n S^\infty = S_n$ for all n. Note that S^∞ is of finite type, but is not finite.

13 October 02, 2020

Recall that

$$H_q(S^n) = \begin{cases} \mathbb{Z} & \text{if } q = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

We claim that

$$H_q(S^\infty) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

106 **Proposition**

S^∞ is contractible.

Proof. Since $S^\infty = \bigcup S^n$, a point $x \in S^\infty$ is a sequence $x = (x_0, x_1, x_2, \dots)$ such that

1. $x_n = 0$ for all sufficiently large n,
2. $\sum x_i^2 = 1$.

Consider $* \xrightarrow{f} S^\infty$ that picks out $(1, 0, 0, \dots)$. Also consider the map $S^\infty \xrightarrow{g} *$ that maps every point in S^∞ to $*$.

We will show that $f \circ g \simeq 1_{S^\infty}$ and $g \circ g \simeq 1_*$. The second homotopy is easy because there is only one point that $*$ can go to. To show the first, note that $f \circ g \colon S^\infty \to S^\infty$ sends every point to $(1, 0, 0, \dots)$.

Consider another map

$$\begin{aligned} T \colon S^\infty &\to S^\infty \\ (x_0, x_1, x_2, \dots) &\mapsto (0, x_0, x_1, \dots). \end{aligned}$$

We will show that $f \circ g \simeq T \simeq 1_{S^\infty}$. In particular, the homotopy $T \simeq 1_{S^\infty}$ can be shown by

$$\begin{aligned} h \colon S^\infty \times [0, 1] &\to S^\infty \\ (x, t) &\mapsto \frac{tx + (1-t)Tx}{\|tx + (1-t)Tx\|}. \end{aligned}$$

This is well-defined and continues because $tx + (1-t)Tx$ is never the origin, since the first nonzero coordinate in Tx occurs after the first nonzero coordinate in x.

We can show the homotopy $T \simeq f \circ g$ with the homotopy

$$h(x, t) = \frac{tTx + (1-t)(1, 0, 0, \dots)}{\|tTx + (1-t)(1, 0, 0, \dots)\|}.$$

This is also well-defined because the first coordinate is always positive except for $t = 1$, for which the value is Tx. □

This proof technique is called a swindle, where in infinite dimensions we shift all the coordinates, which works because there are an infinite number of dimensions.

13.1 Real projective space

For some $k \in \mathbb{N}$, $\mathbb{RP}^k$ is **real projective k-space**, defined to be $\mathbb{RP}^k := S^k/(x \sim -x)$.

107 **Example**

$\mathbb{RP}^0$ is $S^0/(x \sim -x)$, which is just $*$.

$\mathbb{RP}^1$ is $S^1/(x \sim -x)$, which is another S^1.

$\mathbb{RP}^2$ is $S^2/(x \sim x)$ is the sphere with antipodal points identified, but is hard to describe in more familiar terms. In fact, this is not embeddable in 3-dimensional space.

Note that the inclusion of the equator $S^k \to S^{k+1}$ is compatible with $x \mapsto -x$. In particular, there are inclusions

$$\varnothing \subseteq \mathbb{RP}^0 \subseteq \mathbb{RP}^1 \subseteq \mathbb{RP}^2 \subseteq \cdots$$

In fact, this is a CW complex:

$$\begin{tikzcd} S^{k-1} \arrow[r] \arrow[d, hook] & \mathbb{RP}^{k-1} \arrow[d] \\ D^k \arrow[r] & \mathbb{RP}^k \end{tikzcd}$$

where $S^{k-1} \to \mathbb{RP}^{k-1}$ is the quotient map, $S^{k-1} \to D^k$ is the boundary of D^k, where we think of D^k as the upper hemisphere of S^k.

Then, we can define $\mathbb{RP}^\infty = \bigcup \mathbb{RP}^n$, which is a finite type, but not a finite CW complex. Unlike S^∞, $\mathbb{RP}^\infty$ is not contractible.

13.2 Homology of CW complexes

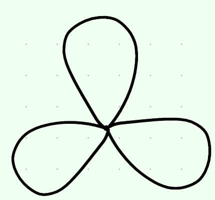
Before we start taking about homologies, let's introduce one more space.

108 **Definition**

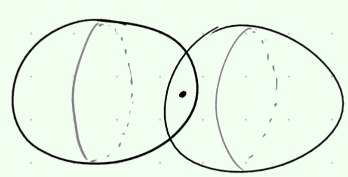
A **wedge of k-spheres** is a space, denoted by $\bigvee_{i \in I} S^k$, consisting of $\|I\|$ different k-spheres all meeting at a single point.

109 **Example**

The figure eight is a wedge of two 1-spheres, the wedge of three 1-spheres is



and the wedge of two 2-spheres is



More formally, we can say $\bigvee_{i \in I} S^k = \coprod_{i \in I} S^k / \coprod_{i \in I} *$.

We can ask what are the reduced homology groups

$$\tilde{H}_q\Big(\bigvee_{i \in I} S^k\Big) = H_q\Big(\bigvee_{i \in I} S^k, *\Big) = H_q\Big(\coprod_{i \in I} S^k, \coprod_{i \in I} *\Big).$$

We consider the long exact sequence of the pair $\left(\coprod_{i\in I} S^k, \coprod_{i\in I} *\right)$.

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(\coprod_{i\in I} *) \arrow[r] & H_q(\coprod_{i\in I} S^k) \arrow[r] & H_q(\coprod_{i\in I} S^k, \coprod_{i\in I} *) \arrow[dll, "\delta"'] & \\
& H_{q-1}(\coprod_{i\in I} *) \arrow[r] & H_{q-1}(\coprod_{i\in I} S^k) \arrow[r] & H_{q-1}(\coprod_{i\in I} S^k, \coprod_{i\in I} *) \arrow[r] & \cdots
\end{tikzcd}$$

Note that

$$H_q\Big(\coprod_{i\in I} S^k\Big) \cong \begin{cases} \bigoplus_{i\in I} \mathbb{Z} & \text{if } q = 0, k \\ 0 & \text{otherwise} \end{cases}$$

$$H_q\Big(\coprod_{i\in I} *\Big) \cong \begin{cases} \bigoplus_{i\in I} \mathbb{Z} & \text{if } q = 0 \\ 0 & \text{if } q \neq 0. \end{cases}$$

with yields

$$\tilde{H}_q\Big(\bigvee_{i\in I} S^k\Big) \cong \begin{cases} \bigoplus_{i\in I} \mathbb{Z} & \text{if } q = k \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose X is a CW complex. We want to find the relationship between $H_q(\mathrm{Sk}_{k-1} X)$ and $H_q(\mathrm{Sk}_k X)$.

Consider the long exact sequence of the pair $(\mathrm{Sk}_k X, \mathrm{Sk}_{k-1} X)$:

$$\cdots \longrightarrow H_{q+1}(\mathrm{Sk}_k X, \mathrm{Sk}_{k-1} X) \longrightarrow H_q(\mathrm{Sk}_{k-1} X) \longrightarrow H_q(\mathrm{Sk}_k X) \longrightarrow H_q(\mathrm{Sk}_k X, \mathrm{Sk}_{k-1} X) \longrightarrow \cdots$$

Note that there is a pushout square

$$\begin{tikzcd}
\coprod_{i\in I_k} S^{k-1} \arrow[r] \arrow[d] & \mathrm{Sk}_{k-1} X \arrow[d] \\
\coprod_{i\in I_k} D^k \arrow[r] & \mathrm{Sk}_k X
\end{tikzcd}$$

This pushout diagram says that

$$\mathrm{Sk}_k X \big/ \mathrm{Sk}_{k-1} X \cong \coprod_{i\in I_k} D^k \Big/ \coprod_{i\in I_k} S^{k-1} \cong \bigvee_{i\in I_k} S^k,$$

which means

$$H_q(\mathrm{Sk}_k X, \mathrm{Sk}_{k-1} X) = H_q(\mathrm{Sk}_k X/\mathrm{Sk}_{k-1} X, *) = \tilde{H}_q\Big(\bigvee_{i\in I_k} S^k\Big) = \begin{cases} \bigoplus_{i\in I_k} \mathbb{Z} & \text{if } k = q \\ 0 & \text{otherwise.} \end{cases}$$

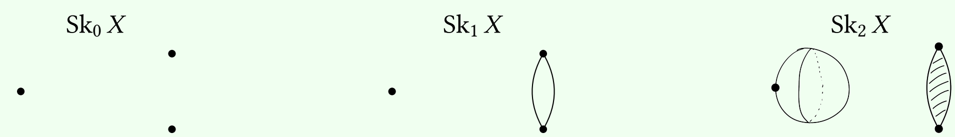
14 October 05, 2020

110 **Proposition**

Suppose X is a CW complex and $k, q, \geq 0$, then $H_q(\operatorname{Sk}_k X) \cong 0$ for $k < q$ and $H_q(\operatorname{Sk}_k X) \cong H_q(X)$ for $k > q$.

111 **Example**

If $q = 0$, then $H_0(\operatorname{Sk}_k X) \cong H_0(X)$ for $k > 0$. For example, if we had the following skeleta,



note that attaching a k-cell ($k > 1$) to objects already present cannot change the number of path components, which is exactly what the proposition says.

If $q = 1$, then we have

- $H_1(\operatorname{Sk}_0 X) \cong 0$,
- $H_1(\operatorname{Sk}_1 X)$ is something,
- $H_1(X) \cong H_1(\operatorname{Sk}_2 X) \cong H_1(\operatorname{Sk}_3 X) \cong \cdots$

In particular, we have $H_1(\operatorname{Sk}_1 X) \cong \mathbb{Z}$, and $H_1(\operatorname{Sk}_2 X) \cong 0$.

Intuitively, when we add the k-dimensional cells, we are adding cycles. But when we compute homologies, we mod out by the boundaries, which are added when $(k+1)$-cells are added. In particular, once $(k+1)$-cells are added, higher dimensional cells will not affect H_k.

In general, we have

- $H_q(\operatorname{Sk}_{q-1} X) \cong 0$,
- $H_q(\operatorname{Sk}_q X)$ surjects onto $H_q(X)$, and
- $H_q(\operatorname{Sk}_{q+1} X) \cong H_q(X)$.

Proof. To compare $H_q(\operatorname{Sk}_{k-1} X)$ and $H_q(\operatorname{Sk}_k X)$, we can use the long exact sequence of the pair $(\operatorname{Sk}_k X, \operatorname{Sk}_{k-1} X)$.

The key idea is that

$$H_q(\operatorname{Sk}_k X, \operatorname{Sk}_{k-1} X) \cong \tilde{H}_q(\operatorname{Sk}_k X / \operatorname{Sk}_{k-1} X) \cong \tilde{H}_q\Big(\bigvee_{i \in I_k} S^k\Big) \cong \begin{cases} \bigoplus_{i \in I_k} \mathbb{Z} & \text{if } q = k \\ 0 & \text{otherwise.} \end{cases}$$

The long exact sequence of pairs gives $H_q(\operatorname{Sk}_k X) \cong 0$ if $k < q$ because for sufficiently small k, $H_q(\operatorname{Sk}_k X) \cong H_q(\operatorname{Sk}_{k-1} X)$. The long exact sequence also gives $H_q(\operatorname{Sk}_k X) \cong H_q(\operatorname{Sk}_{k+1} X)$ if $k > q$. It remains to check that

$$H_q(\operatorname{Sk}_k X) \simeq H_q(X)$$

if $k = q$. Note that Δ^q and Δ^{q+1} are compact, so anything in $H_q(X)$ has to be represented by a finite linear combination of simplices, which by compactness has to be contained in some finite skeleton. Knowing Δ^q is compact tells us that $H_q(X) \to H_{q+1(X)}$ is surjective, and knowing that Δ^{q+1} is compact tells us that the boundary relations are present as well. $\square$

14.1 Cellular homology

Now we will introduce a tool to compute homology groups.

112 **Definition**

Suppose X is a CW complex. Let $C_n(X) = C_n^{\text{cell}}(X)$ denote

$$H_n(\operatorname{Sk}_n X, \operatorname{Sk}_{n-1} X) \simeq \tilde{H}_n\Big(\bigvee_{i \in I_n} S^n\Big),$$

the free abelian group on the set of n-cells of X.

113 **Definition**

For each $n \geq 0$, we can define the map

$$d\colon C_{n+1}(X) \to C_n(X)$$

to be the composite

$$C_{n+1}(X) = H_{n+1}(\operatorname{Sk}_{n+1} X, \operatorname{Sk}_n X) \xrightarrow{\partial} H_n(\operatorname{Sk}_n X) \longrightarrow H_n(\operatorname{Sk}_n X, \operatorname{Sk}_{n-1} X) = C_n(X)$$

114 **Theorem**

The maps $d\colon C_{n+1}(X) \to C_n(X)$ make $C_*(X)$ into a chain complex (i.e. $d \circ d = 0$). The homology of this chain complex is isomorphic to the homology of X. This is called **cellular homology**.

Consider the sequence of functors

$$\operatorname{Fun}(\Delta_{\text{inj}}^{\text{op}}, \mathsf{Set}) \xrightarrow{F} \mathsf{CWcomp} \xrightarrow{U} \mathsf{Top}$$

i.e. the geometric realization of a semisimplicial set. Then $C_n^{\text{cell}}(F(X)) = \mathbb{Z}\operatorname{Sing}_n X = S_n(X)$ and $C_*^{\text{cell}}(F(X)) = S_*(X)$. Thus, the theorem says that the semisimplicial homology of X agrees with the singular homology of the geometric realization of X.

115 **Example** (Homology of S^n)

Recall that S^n has a very simple CW complex structure that requires one 0-cell and one n-cell. Since $C_l^{\text{cell}}(X)$ is the free abelian group on the set of l-cells, we have

$$C_l^{\text{cell}}(S^n) = \begin{cases} \mathbb{Z} & \text{if } l = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

For e.g. $n = 2$, we have the chain complex $C_*(S^2)$ is isomorphic to

$$\mathbb{Z} \xleftarrow{d} 0 \xleftarrow{d} \mathbb{Z} \xleftarrow{d} 0 \xleftarrow{d} 0 \xleftarrow{d} \cdots$$

Since most of these groups are 0, we can easily compute:

$$H_0(S^2) \cong \mathbb{Z} \qquad H_1(S^2) \cong 0 \qquad H_2(S^2) \cong \mathbb{Z} \qquad H_3(S^2) \cong 0,$$

i.e. $H_q(S^2) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$

116 **Remark**

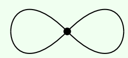
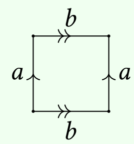
If a CW complex only has even dimensional cells, then

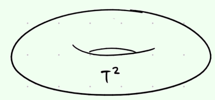
$$H_q(X) \cong \begin{cases} \bigoplus_{i \in I_q} \mathbb{Z} & q \text{ is even} \\ 0 & q \text{ is odd.} \end{cases}$$

117 **Example** (Torus)

The torus T^2 has a CW structure

$$\mathrm{Sk}_0\, T^2 = *$$

$\mathrm{Sk}_1\, T^2 =$  $=$ 

$\mathrm{Sk}_2\, T^2 =$ 

where the 2-skeleton is obtained by attaching one 2-cell via the map $S^1 \longrightarrow \mathrm{Sk}_1\, T^2$ which is $b^{-1}a^{-1}ba$.

We can compute the homology group from the cellular chain complex

$$\mathbb{Z}\{x\} \xleftarrow{d_1} \mathbb{Z}\{a, b\} \cong \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{d_2} \mathbb{Z}\{u\} \longleftarrow 0 \longleftarrow \cdots$$

Note that d_1 has to be zero because the boundaries of a and b are the same point. To compute d_2, note that the boundary of the 2-cell u is glued on via $b^{-1}a^{-1}ba$. Therefore,

$$d_2 u = -b - a + b + a = 0.$$

This gives the expected homology groups.

Proof ($C_*^{cell}(X)$ agrees with singular homology). Consider the diagram

$$\begin{tikzcd}
 & C_{n+1}^{\mathrm{cell}}(X) = H_{n+1}(\mathrm{Sk}_{n+1} X, \mathrm{Sk}_n X) \arrow[d, "\partial_n"] \arrow[dr, "d"] & & H_{n-1}(\mathrm{Sk}_{n-2} X) = 0 \arrow[d] \\
0 = H_{n+1}(\mathrm{Sk}_{n-1} X) \arrow[r] & H_n(\mathrm{Sk}_n X) \arrow[r, "j_n"] \arrow[d] & H_n(\mathrm{Sk}_n X, \mathrm{Sk}_{n-1} X) \arrow[r, "\partial_{n-1}"] \arrow[dr, "d"] & H_{n-1}(\mathrm{Sk}_{n-1} X) \arrow[d, "j_{n-1}"] \\
 & H_n(\mathrm{Sk}_{n+1} X) \arrow[d] & & H_{n-1}(\mathrm{Sk}_{n-1} X, \mathrm{Sk}_{n-2} X) \\
 & 0 = H_n(\mathrm{Sk}_{n+1} X, \mathrm{Sk}_n X) & &
\end{tikzcd}$$

All of the rows and columns are exact. The theorem claims that $d \circ d = 0$, and $\ker d / \operatorname{im} d$ agrees with singular homology.

To show that $d \circ d = 0$, note that $d \circ d$ is the same as following the blue arrows. However, the horizontal sequence is exact, so the composite of the two blue horizontal maps is zero.

To show that $\ker d / \operatorname{im} d$ computes singular homology, note that j_{n-1} is injective from exactness. Therefore, $\ker d = \ker \partial_{n-1}$, and furthermore $\ker d = \ker \partial_{n-1} = \operatorname{im} j_n$ by exactness of the horizontal sequence. We also have j_n is injective from exactness, so $\operatorname{im} j_n \cong H_n(\mathrm{Sk}_n X)$.

Therefore,

$$\frac{\ker d}{\operatorname{im} d} = \frac{H_n(\mathrm{Sk}_n X)}{\operatorname{im} \partial_n} = H_n(\mathrm{Sk}_{n+1} X) = H_n(X).$$

□

15 October 07, 2020

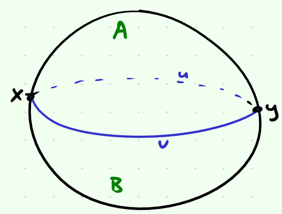
Last time we saw that the cellular chain complex $C_*^{\text{cell}}(X)$ of a CW complex X computes the singular homology.

118 **Example** (S^2)

Let's consider a less-minimal cell structure of S^2 where

$$\text{Sk}_0\, S^2 = S^0 \qquad \text{Sk}_1\, S^2 = S^1 \qquad \text{Sk}_2\, S^2 = S^2,$$

depicted below.



Then

$$C_*^{\text{cell}}(S^2) = \left(\mathbb{Z}\{x, y\} \xleftarrow{d} \mathbb{Z}\{u, v\} \xleftarrow{d} \mathbb{Z}\{A, B\} \longleftarrow \cdots \right)$$

Note that when we are computing the homology maps, we only care about the image of the d maps. In particular, when we are working with semisimplicial sets, we cared about the direction that the edges went. Here we can make an arbitrary choice of direction to help us compute the d maps. In particular, if we select $x \xrightarrow{u} y$ and $x \xrightarrow{v} y$, then

$$du = y - x, \qquad dv = y - x.$$

Then

$$H_0(S^2) = \frac{\ker d}{\operatorname{im} d} = \frac{\mathbb{Z}\{x, y\}}{\mathbb{Z}\{y - x\}} \cong \mathbb{Z}.$$

To determine dA and dB, note that we can add some maps to the pushout diagram of the 2-skeleton

$$\begin{tikzcd} S^1 \sqcup S^1 \arrow[r, "f"] \arrow[d, hook] & \text{Sk}_1\, S^2 = S^1 \arrow[d] \\ D^2 \sqcup D^2 \arrow[r] & \text{Sk}_2\, S^2 = S^2 \end{tikzcd}$$

Note that the map f determines $\text{Sk}_2\, S^2$ as the pushout. In particular, consider the map

$$S^1 \sqcup S^1 \longrightarrow \text{Sk}_1\, S^2 \longrightarrow \text{Sk}_1\, S^2 / \text{Sk}_0\, S^2 = S^1 \vee S^1$$

Taking H_1, this gives a map

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

that gives the information of dA and dB, since the $\mathbb{Z}$s on the left represent A and B, and the $\mathbb{Z}$s on the right represent u and v. In particular, we can again arbitrarily assign

$$dA = u - v, \qquad dB = u - v.$$

And indeed, when we compute the homologies from this sequence, we get the expected result.

Moreover, we have that a generator of $H_2(S^2)$ in this CW structure is given by $A - B$. This is because we know $dA = dB$, since the attaching maps $S^1 \to \mathrm{Sk}_1 S^1$ defining A and B are the same.

15.0.1 Homotopy equivalence

Note that homology can be used to prove that topological spaces are not homotopy equivalent. In particular, this is just a consequence of homology being an invariant under homology.

A less obvious statement is that homology can also be used to prove that continuous maps are not homotopic. Homology is a functor, so it takes maps in Top to maps in Ab. So, if two continuous maps are sent to different maps of abelian groups, then the two continuous maps cannot be homotopic.

119 **Definition**

If $f\colon S^n \to S^n$ is a continuous map, then the **degree** of f, denoted $\deg f$, is the value of 1 under the group homomorphism $H_n(f)\colon \mathbb{Z} \to \mathbb{Z}$.

If $f, g\colon S^n \to S^n$ are two continuous maps that have different degrees, then f and g are hot homotopic, as otherwise $H_n(f) = H_n(g)$.

120 **Lemma**

Suppose $f, g\colon S^n \to S^n$. Then

$$\deg(g \circ f) = (\deg g)(\deg f).$$

Proof. Consider the diagram

$$S^n \xrightarrow{f} S^n \xrightarrow{g} S^n$$

When we apply H_n, we get

$$\begin{array}{llll} \mathbb{Z} \xrightarrow{H_n(f)} & \mathbb{Z} & \xrightarrow{\quad H_n(g) \quad} & \mathbb{Z} \\ 1 \longmapsto & \deg f & & \\ & 1 & \longmapsto & \deg g \\ 1 & \longmapsto & & (\deg f)(\deg g) \end{array}$$

□

121 **Corollary**

Suppose $f\colon S^n \to S^n$ is a homeomorphism. Then $\deg f$ is either 1 or -1.

122 **Example**

Consider the map $f\colon S^2 \to S^2$ that is a reflection about a plane. Note that f is also a map of CW complexes. If we select the skeleta such that the plane of reflection is the equator, then we know that A and B have the same boundary, and the map

$$S^1 \sqcup S^1 \to \mathrm{Sk}_1 S^2 \to \mathrm{Sk}_1 S^2/\mathrm{Sk}_0 S^2 \cong S^1 \vee S^1$$

sends the boundaries of A and B to the same set. In particular, we know that $dA = dB$, so

$$A - B \in \ker d = H_2^{\mathrm{cell}}(S^2) \cong \mathbb{Z}.$$

Therefore, $A - B$ is a generator of $H_2^{\text{cell}}(S^2)$. In particular, f is the map between the cellular complexes

$$\begin{tikzcd} \mathbb{Z}\{A,B\} \arrow[r] \arrow[d, "\substack{A\mapsto B\\ B\mapsto A}"] & \mathbb{Z}\{u,v\} \arrow[r] \arrow[d, "\substack{u\mapsto u\\ v\mapsto v}"] & \mathbb{Z}\{x,y\} \arrow[d, "\substack{x\mapsto x\\ y\mapsto y}"] \\ \mathbb{Z}\{A,B\} \arrow[r] & \mathbb{Z}\{u,v\} \arrow[r] & \mathbb{Z}\{x,y\} \end{tikzcd}$$

This means $f\colon A - B \mapsto B - A$, so $H_2(f)\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by -1. Therefore, f has degree -1.

Note that this argument also works for S^n in general, so no reflection $S^n \to S^n$ is homotopic to the identity.

123 **Corollary**

The degree of a rotation of S^n is 1.

This is because any rotation can be written as the composition of two reflections.

124 **Example**

- The degree of $S.toS^1$ where $x \mapsto -x$ is 1 because it is a 180° rotation.
- We can find he degree of $S^2 \to S^2$ where $x \mapsto -x$ by considering the map as the composite of three reflections

$$(a,b,c) \mapsto (-a,b,c) \mapsto (-a,-b,c) \mapsto (-a,-b,-c).$$

  Therefore, this map has degree -1.
- More generally, the antipodal map $S^n \to S^n$ mapping $x \mapsto -x$ is the composite of $n+1$ reflections, so the map has degree $(-1)^{n+1}$.

16 October 09, 2020

16.0.1 Summary

If X is a CW complex, then $C_*^{\mathrm{cell}}(X)$ is a chain complex with homology groups computing the singular homology groups $H_q(X)$. The nth group $C_n^{\mathrm{cell}}(X)$ is the free abelian group on the set I_n of n-cells. To compute the differential $d\colon C_n^{\mathrm{cell}}(X) \to C_{n-1}^{\mathrm{cell}}(X)$, we consider the diagram

$$\begin{tikzcd} \coprod_{i\in I_n} S^{n-1} \arrow[r] \arrow[d] & \mathrm{Sk}_{n-1} X \arrow[r] \arrow[d] & \mathrm{Sk}_{n-1} X/\mathrm{Sk}_{n-2} X \cong \bigvee_{i\in I_{n-1}} S^{n-1} \\ \coprod_{i\in I_n} D^n \arrow[r] & \mathrm{Sk}_n X & \end{tikzcd}$$

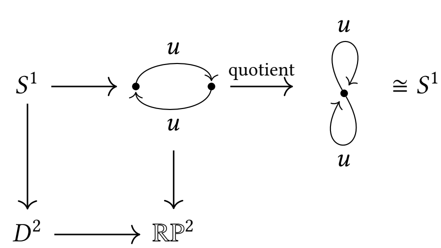
Applying H_{n-1} to the top row, $d\colon C_n^{\mathrm{cell}}(X) \to C_{n-1}^{\mathrm{cell}}(X)$.

If $f\colon X \to Y$ is a map in Top, then to compute $H_q(f)\colon H_q(X) \to H_q(Y)$, We can place CW structures on X and Y so that f induces a map $C_*^{\mathrm{cell}}(X) \to C_*^{\mathrm{cell}}(Y)$.

16.0.2 Homology of real projective space

Recall $\mathbb{RP}^n := S^n/(x \sim -x)$. To compute the homology of this, we may try to do this by coming up with a semisimplicial model for this space, but it is not obvious how to do this.

However, we do have a CW structure on $\mathbb{RP}^n$, with one k-cell in each dimension $0 \le k \le n$. We can try to determine $H_q(\mathbb{RP}^2)$. Consider the pushout square



Taking H_1 of the top row, we have a map $\mathbb{Z} \to \mathbb{Z}$. More specifically, the 2-cell gets mapped to u^2, so the map is $1 \mapsto 2$. We have the chain $C_*^{\mathrm{cell}}(\mathbb{RP}^2)$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}\{A\} \arrow[r] & \mathbb{Z}\{u\} \arrow[r] & \mathbb{Z}\{x\} \\ & A \arrow[r, mapsto] & 2u & \\ & & u \arrow[r, mapsto] & 0 \end{tikzcd}$$

This gives

$$H_0(\mathbb{RP}^2) \cong \mathbb{Z}, \qquad H_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}, \qquad H_2(\mathbb{RP}^2) \cong 0.$$

This is a bit weird. Usually our homologies are free abelian groups. Here, this means that there is some loop that is not a boundary, but when we go around it twice, it is a boundary.

Now let's consider $C_*^{\mathrm{cell}}(\mathbb{RP}^n)$ in general. Let's try to think about this inductively. If we understand $C_*^{\mathrm{cell}}(\mathbb{RP}^{n-1})$, can we say anything about $C_*^{\text{cell}}(\mathbb{RP}^n)$? Consider the pushout diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & \mathbb{RP}^{n-1} \arrow[r] \arrow[d] & \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} \cong S^{n-1} \\ D^n \arrow[r] & \mathbb{RP}^n \end{tikzcd}$$

Note that the way that we can think about $\mathbb{RP}^{n-1}/\mathbb{RP}^{n-2}$ is to take $S^{n-1} \vee S^{n-1}$ and quotient by the antipodal map between the two spheres. The differential d maps 1 to $1 + \deg f$, where f is antipodal map $S^{n-1} \to S^{n-1}$. This is because the top half of the first S^{n-1} gets mapped to S^{n-1}, and the bottom half gets mapped to the second S^{n-1}, but under the quotient this is equal to the antipodal map of the identity. Therefore, $d\colon 1 \mapsto 1 + (-1)^n$.

This gives the chain

$$C_*^{\text{cell}}(\mathbb{RP}^n) \cong \left(\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \right),$$

where the number of $\mathbb{Z}$s in the chain is $n+1$. In general, this means for even n,

$$H_q(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 < q < n, \text{odd } q \\ 0 & \text{otherwise} \end{cases}$$

and for odd n,

$$H_q(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & q = 0, n \\ \mathbb{Z}/2\mathbb{Z} & 0 < q < n, \text{odd } q \\ 0 & \text{otherwise.} \end{cases}$$

16.1 Other invariants

Let's talk about other invariants that give strictly less information than homology. The advantage is that they are easier to compute than homology.

125 **Definition**

If X is a finite CW complex, then the **Euler characteristic** is

$$\chi(X) = \sum_k (-1)^k |I_k|.$$

126 **Example**

Consider S^2 with one 0-cell and one 2-cell. The Euler characteristic is $\chi(S^2) = 1 + 1 = 2$.

There is the other CW structure with two 0-cells, 1-cells, and 2-cells. This gives the Euler characteristic $\chi(S^2) = 2 - 2 + 2 = 2$.

127 **Theorem**

The Euler characteristic $\chi(X)$ depends only on the homotopy type of X.

128 **Example**

Consider the torus T^2 with the CW structure from example 117. This gives us the Euler characteristic $\chi(T^2) = 1 - 2 + 1 = 0$.

A corollary of this is that T^2 and S^2 are not homotopy equivalent.

129 **Theorem**

If X is a finite CW complex, then

$$\chi(X) = \sum_k (-1)^k \operatorname{rank} H_k(X).$$

To define rank, recall that abelian groups are equivalent to $\mathbb{Z}$-modules. Since $\mathbb{Z}$ is a PID, there is a classification of finitely generated abelian groups. In particular, every finitely generated abelian group A is isomorphic to

$$\mathbb{Z}^r \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_t\mathbb{Z}$$

where $r \geq 0$ and $n_1, \ldots, n_t \geq 2$. We call r the **rank** of A.

Note that since X is a finite CW complex, we know that $H_k(X)$ is finitely generated for each k.

17 October 13, 2020

130 **Lemma**

Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of finitely generated abelian groups. Then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$.

We will prove a variant of this on the problem set.

Now we can prove $\chi(X) = \sum_k (-1)^k \operatorname{rank} H_k(X)$.

Proof. Consider, for all $k \geq 0$, there are short exact sequences

$$0 \longrightarrow Z_k^{\text{cell}}(X) \longrightarrow C_k^{\text{cell}}(X) \xrightarrow{\partial} B_{k-1}^{\text{cell}}(X) \longrightarrow 0$$

Indeed, it is exact at $Z_k^{\text{cell}}(X)$ because the map $Z_k^{\text{cell}}(X) \to C_k^{\text{cell}}(X)$ is injective, since the cycles of X are a subset of the chains of X. Moreover, it is exact at $C_k^{\text{cell}}(X)$ because the chains are precisely the elements in kernel of the boundary map ∂. Then, the boundaries are defined do be the image of the boundary map, so ∂ is surjective and the sequence is exact at $B_{k-1}^{\text{cell}}(X)$.

There is another short exact sequence

$$0 \longrightarrow B_k^{\text{cell}}(X) \longrightarrow Z_k^{\text{cell}}(X) \longrightarrow H_k(X) \longrightarrow 0$$

It's exact at $B_k^{\text{cell}}(X)$ because $B_k^{\text{cell}}(X) \to Z_k^{\text{cell}}(X)$ is an injection, which is equivalent to the fact that every boundary is a cycle, i.e. $\partial^2 = 0$. Also, we know that $H_k(X)$ is the quotient of $Z_k^{\text{cell}}(X)$ by $B_k^{\text{cell}}(X)$ by definition, so the sequence is exact at $Z_k^{\text{cell}}(X)$ and $H_k(X)$.

Using these short exact sequences, we know

$$\begin{aligned}
\sum_k (-1)^k |I_k| &= \sum_k (-1)^k \operatorname{rank}(C_k^{\text{cell}}(X)) \\
&= \sum_k (-1)^k (\operatorname{rank}(Z_k^{\text{cell}}(X)) + \operatorname{rank}(B_{k-1}^{\text{cell}}(X))) \\
&= \sum_k (-1)^k (\operatorname{rank}(B_k^{\text{cell}}(X)) + \operatorname{rank}(H_k(X)) + \operatorname{rank}(B_{k-1}^{\text{cell}}(X))).
\end{aligned}$$

Note that the $B_k^{\text{cell}}(X)$ terms telescope, so

$$= \sum_k (-1)^k (\operatorname{rank}(H_k(X))).$$

□

17.1 Homology with coefficients

Another invariant easier to compute than homology is **homology with coefficients**. When defining homology, we applied the free abelian group functor on the simplices of a semisimplicial set. But one does not need to use this specific functor. There are other functors that we can choose.

131 **Definition**

Let R be a commutative ring and X be a semisimplicial set. For any $k \geq 0$, let $S_k(X; R)$, read "S_k of X with coefficients in R", to be the free R-module generated by X_k.

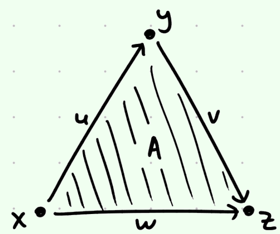
Let R be a commutative ring and X be a topological space. We can similarly define $S_k(X, R)$ to be the free R-module generated by $\mathrm{Sing}_k(X)$.

Note that $S_k(X, \mathbb{Z}) = S_k(X)$ because $\mathbb{Z}$-modules are the same thing as abelian groups.

In both cases, whether we're dealing with semisimplicial sets or topological spaces, the alternating sum of the semisimplicial face maps creates the differential in a chain complex $S_k(X; R)$. In particular, the chain complex is of R-modules where the differentials are R-module maps.

132 **Example**

Consider the following semisimplicial set:



Then $S_*(X; \mathbb{Q})$ is a chain complex of rational vector spaces and linear maps isomorphic to

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \longrightarrow \mathbb{Q} \longrightarrow 0 \longrightarrow \cdots$$

where the nonzero vector spaces have bases $\{A\}$, $\{u, v, w\}$, and $\{x, y, z\}$ respectively. The maps are the maps that we expect

$$\begin{aligned} \mathbb{Q} &\to \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \\ A &\mapsto u + v - w \end{aligned} \qquad \begin{aligned} \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} &\to \mathbb{Q} \\ u &\mapsto y - x \\ v &\mapsto z - y \\ w &\mapsto z - x. \end{aligned}$$

The homology R-modules of $S_*(X; R)$ are denoted by $H_q(X; R)$. Homology with coefficients in a commutative ring satisfies all of the Eilenberg-Steenrod axioms where the dimension axiom is modified to say

$$H_q(*; R) = \begin{cases} R & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

All of the theory we've built up to calculate homology works with homology with coefficients, so if X is a CW complex, we can also calculate $H_q(X; R)$ using the cellular chain complex $C_*^{\mathrm{cell}}(X; R)$.

133 **Example**

Let's consider $\mathbb{RP}^2$ with the CW structure with one 0-cell, one 1-cell, and one 2-cell. We calculated $H_q(\mathbb{RP}^2) = H_q(\mathbb{RP}^2; \mathbb{Z})$ with the cellular chain complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

In particular we calculated

$$H_q(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We can also calculate $H_q(\mathbb{RP}^2; \mathbb{Q})$ with the cellular chain complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Q} \xrightarrow{2} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \longrightarrow 0 \longrightarrow \cdots$$

Then we have the homology

$$H_q(\mathbb{RP}^2) = \begin{cases} \mathbb{Q} & \text{if } q = 0 \\ \mathbb{Z}/2\mathbb{Z} \cong 0 & \text{if } q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We can also calculate $H_q(\mathbb{RP}^2; \mathbb{F}_2)$. We have the chain

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{F}_2 \xrightarrow{2} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \longrightarrow 0 \longrightarrow \cdots$$

This gives the homology

$$H_q(\mathbb{RP}^2; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{if } q = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

134 **Note**

All subsequent material will not be tested until after November 9th.

We have covered the main ideas of homology. However, there are more things we can ask about.

135 **Question**

In what sense is $H_q(X; R)$ easier than $H_q(X; \mathbb{Z})$? Is $H_q(X; R)$ determined by $H_q(X; \mathbb{Z})$?

It turns out that the answer to the second question is yes.

136 **Note**

In applied topology, one is given a giant collection of points in, e.g. $\mathbb{R}^{100}$. Fix a radius R and connect any two points within R of each other, draw a 2-simplex for any three points which are within some diameter R circle, etc. We get a simplicial complex. We can compute the homology, and it turns out that using $\mathbb{Q}$ coefficients as opposed to $\mathbb{Z}$ has much lower computational complexity.

137 **Question**

How do we compute $H_q(X \times Y)$ in terms of $H_q(X)$ and $H_q(Y)$?

138 **Question**

A topological space X has a diagonal map $\Delta \colon X \to X \times X$ mapping $x \mapsto (X, X)$. What can we say about $H_q(\Delta X)$?

139 **Question**

What special features are enjoyed by the homology groups of manifolds as opposed to generic topological spaces?

On the way to answering these questions, we will introduce the purely algebraic functors Tor, Ext, and cohomology.

18 October 14, 2020

18.1 Some algebra

If A and B are abelian groups, and $f, g\colon A \to B \in \mathrm{Hom}_{\mathrm{Ab}}(A, B)$, we can compute $f + g$ and $f - g$. Therefore, we can make the set $\mathrm{Hom}_{\mathrm{Ab}}(A, B)$ into an abelian group, denoted $\underline{\mathrm{Hom}}_{\mathrm{Ab}}(A, B)$.

More generally, if R is a commutative ring and R-mod is the category of R-modules, then $\mathrm{Hom}_{R\text{-mod}}(M, N)$ can be given the structure of an R-module. We denote this R-module $\underline{\mathrm{Hom}}_{R\text{-mod}}(M, N)$.

140 **Example**
Linear maps between two vector spaces over a field form a vector space.

This construction $(M, N) \mapsto \underline{\mathrm{Hom}}_{R\text{-mod}}(M, N)$ is functorial, i.e. it is natural in its inputs. More specifically,

1. If $N \to N'$ is a map of R-modules, then any map $M \to N$ gives a composite map $M \to N \to N'$. There is a R-module map $\underline{\mathrm{Hom}}_{R\text{-mod}}(M, N) \to \underline{\mathrm{Hom}}_{R\text{-mod}}(M, N')$.
2. If $M \to M'$ is an R-module map, then there is a corresponding map, for every N, $\underline{\mathrm{Hom}}_{R\text{-mod}}(M', N) \to \underline{\mathrm{Hom}}_{R\text{-mod}}(M, N)$.

Note that in the first map, it goes from N to N', and in the second map, it goes from M' to M. We can summarize this by saying that there is a functor

$$\begin{aligned} \underline{\mathrm{Hom}}_{R\text{-mod}}\colon (R\text{-mod})^{\mathrm{op}} \times (R\text{-mod}) &\to R\text{-mod} \\ (M, N) &\mapsto \underline{\mathrm{Hom}}_{R\text{-mod}}(M, N) \end{aligned}$$

Some categories $\mathcal{C}$, like R-mod, have **internal Hom**s, which are functors

$$\underline{\mathrm{Hom}}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}.$$

Here, $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ refers to the product category of $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{C}$ in Cat.

For example, the category Set has an internal Hom with

$$\underline{\mathrm{Hom}}_{\mathrm{Set}}(A, B) = \mathrm{Hom}_{\mathrm{Set}}(A, B).$$

If $A, B, C \in$ Set there is a **currying isomorphism**

$$\mathrm{Hom}_{\mathrm{Set}}(A \times B, C) \cong \mathrm{Hom}_{\mathrm{Set}}(A, \mathrm{Hom}_{\mathrm{Set}}(B, C))$$

where a function $f\colon A \to \mathrm{Hom}_{\mathrm{Set}}(B, C)$ is sent to the function $g\colon A \times B \to C$ given by

$$g(a, b) = (f(a))(b).$$

Whenever there is an internal Hom, there is generally a isomorphism of this flavor. The analog of the currying isomorphism in R-mod is the following:

141 **Theorem**

Let R be a commutative ring. There is a functor **tensor product**

$$\otimes_R : R\text{-mod} \times R\text{-mod} \longrightarrow R\text{-mod}$$

such that

$$\underline{\mathrm{Hom}}_{R\text{-mod}}(A \otimes_R B, C) \cong \underline{\mathrm{Hom}}_{R\text{-mod}}(A, \underline{\mathrm{Hom}}_{R\text{-mod}}(B, C))$$

for all R-modules A, B, C. Note that it is customary to write $A \otimes_R B := \otimes_R(A, B)$. Furthermore, the isomorphism is natural in A, B, and C, and this uniquely determines the $\otimes_R$ functor.

142 **Remark**

The functors $\otimes_R$ and $\underline{\mathrm{Hom}}_{R\text{-mod}}$ are **adjoint**. This means that they determine one another.

Sketch. First we will define the functor

$$\otimes_R : R\text{-mod} \times R\text{-mod} \longrightarrow R\text{-mod}$$

and then check that the isomorphism holds.

143 **Definition**

If $A, B \in R$-mod, then $A \otimes_R B$ is the R-module

- generated by the symbols $a \otimes b$ where $a \in A$ and $b \in B$,
- with the relations

$$a \otimes (b + b') = a \otimes b + a \otimes b'$$
$$(a + a') \otimes b = a \otimes b + a' \otimes b$$
$$(ra) \otimes b = a \otimes (rb) = r(a \otimes b)$$

  for all $a, a' \in A$, $b, b' \in B$, and $r \in R$.

We want to show that a map of R-modules $A \otimes_R B \longrightarrow C$ is determined by where it sends the generators $a \otimes b$. Given a map $f : A \otimes_R B \longrightarrow C$, we can determine a function $g : A \longrightarrow \underline{\mathrm{Hom}}_{R\text{-mod}}(B, C)$ by

$$(g(a))(b) = f(a \otimes b).$$

Conversely, given g, we define

$$f(a \otimes b) = (g(a))(b).$$

The relations on $A \otimes_R B$ are designed to ensure that g is an R-module map if and only if f is an R-module map. □

Note that $A \otimes_R B \cong B \otimes_R A$.

18.1.1 Tensor product examples

Suppose $R = \mathbb{Z}$. Let's calculate $A \otimes_{\mathbb{Z}} B$ for various Z-modules A and B.

144 **Example**

The $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ is an abelian group with 8 generators

$$0 \otimes 0, \qquad 0 \otimes 1, \qquad 0 \otimes 2, \qquad 0 \otimes 3, \qquad 1 \otimes 0, \qquad 2 \otimes 1, \qquad 3 \otimes 2, \qquad 4 \otimes 3.$$

We have the relation

$$0 \otimes 2 = (0 \cdot 0) \otimes 2 = 0(0 \otimes 2) = 0.$$

Similarly, $0 \otimes 0$, $0 \otimes 1$, $0 \otimes 3$, and $1 \otimes 0$ are all trivial. Therefore, $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ is generated by

$$1 \otimes 1, \qquad 1 \otimes 2, \qquad 1 \otimes 3.$$

Note that we have

$$\begin{aligned} 1 \otimes 1 + 1 \otimes 1 + 1 \otimes 1 + 1 \otimes 1 &= 1 \otimes 4 = 0 \\ 1 \otimes 1 + 1 \otimes 1 + 1 \otimes 1 &= 1 \otimes 3 \\ 1 \otimes 1 + 1 \otimes 1 &= 1 \otimes 2. \end{aligned}$$

Therefore, the whole R-module is generated by $1 \otimes 1$. Moreover, note

$$1 \otimes 1 + 1 \otimes 1 = (1 + 1) \otimes 1 = 0 \otimes 1 = 0.$$

Therefore, $1 \otimes 1$ has order 2 and

$$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

Let's check an example of the theorem. It states

$$\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/4\mathbb{Z}, \underline{\mathrm{Hom}}_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})).$$

Consider $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$. The elements are the zero map and the identity map. Therefore, the theorem claims

$$\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}).$$

This is indeed true because $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ has two elements that are determined where the morphism sends the generator 1, for which there are two choices.

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145 **Example**

The $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$ has generators

$$0 \otimes 0, \qquad 1 \otimes 0, \qquad 0 \otimes 1, \qquad 1 \otimes 1, \qquad 0 \otimes 2, \qquad 1 \otimes 2.$$

We know that

$$0 \otimes 0 = 0 \otimes 1 = 0 \otimes 2 = 1 \otimes 0 = 0,$$

and

$$1 \otimes 1 + 1 \otimes 1 = 1 \otimes (1+1) = 1 \otimes 2.$$

Therefore, $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$ is generated by $1 \otimes 1$. However, we know

$$1 \otimes 1 + 1 \otimes 1 + 1 \otimes 1 = 1 \otimes 3 = 0$$
$$1 \otimes 1 + 1 \otimes 1 = 1 \otimes 2 = 0,$$

so $1 \otimes 1 = 0$. Therefore, $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} \cong 0$.

146 **Proposition**

If A is an abelian group, then

$$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A/2A.$$

Proof. The $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} A$ is generated by the symbols $1 \otimes a$ for all $a \in A$ with the relation

$$2(1 \otimes a) = 2 \otimes a = 0 \otimes a = 0. \qquad \square$$

147 **Corollary**

In general, we have $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A/nA$.

How about $\mathbb{Z} \otimes_{\mathbb{Z}} A$? We know that if B is an abelian group,

$$\mathrm{Hom}(A \otimes_{\mathbb{Z}} \mathbb{Z}, B) \cong \mathrm{Hom}(A, \underline{\mathrm{Hom}}(\mathbb{Z}, B)).$$

Note that $\underline{\mathrm{Hom}}_{\mathrm{Ab}}(\mathbb{Z}, B) \cong B$, because $\mathbb{Z}$ has one generator. Therefore,

$$\mathrm{Hom}(A \otimes_{\mathbb{Z}} \mathbb{Z}, B) \cong \mathrm{Hom}(A, B),$$

and this suggests that $\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A$. Taking $B = 0$ gives the result, because $\underline{\mathrm{Hom}}(X, 0) \cong X$.

148 **Theorem**

If R is a commutative ring and M is an R-module, then

$$R \otimes_R M \cong M.$$

149 **Definition**

Suppose R is a ring, and A, B, C are R-modules. A **bilinear map** $f: A \times B \to C$ is a function of sets such that

1. $f(a+a', b) = f(a,b) + f(a', b)$,
2. $f(a, b+b') = f(a,b) + f(a,b')$, and
3. $f(ra, b) = rf(a,b) = f(a, rb)$.

150 **Lemma**

Bilinear maps $A \times B \to C$ are in bijection with R-module maps $A \otimes_R B \to C$.

151 **Theorem**

Suppose A, B, C are R-modules. Then

$$(A \oplus B) \otimes_R C \cong (A \otimes_R C) \oplus (B \otimes_R C)$$
$$(a,b) \otimes c \leftrightarrow (a \otimes c, b \otimes c).$$

This means that $(\oplus_R, \oplus_R)$ make R-mod into a **categorical ring**, or a **ring object in categories**. In particular, $\otimes_R$ is associative in the sense that

$$(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C).$$

We now know how to compute $\otimes_{\mathbb{Z}}$ of finitely generated abelian groups.

152 **Example**

$$\begin{aligned}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}) &\cong (\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/6\mathbb{Z}) \otimes (\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/6\mathbb{Z}) \otimes (\mathbb{Z}/3\mathbb{Z} \otimes \mathbb{Z}) \otimes (\mathbb{Z}/3\mathbb{Z} \otimes \mathbb{Z}) \\ &\cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}\end{aligned}$$

The most common abelian groups are finitely generated, so this is enough for us.

We also want to be able to calculate homology in other rings. The rings R we use often in algebraic topology are $\mathbb{Z}$, $\mathbb{Q}$, or a finite field. Since all non-$\mathbb{Z}$ rings are fields, these R-modules are just vector spaces, which are easy to compute the tensor products of.

153 **Example**

In $\mathbb{Q}$-modules, what is $(\mathbb{Q} \oplus \mathbb{Q}) \otimes_{\mathbb{Q}} (\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q})$?

This is a 6-dimensional $\mathbb{Q}$-vector spaces, i.e. $\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$.

19.1 Tensor products of chain complexes

Let's give a topological example to give some intuition on how tensor products of chain complexes will work.

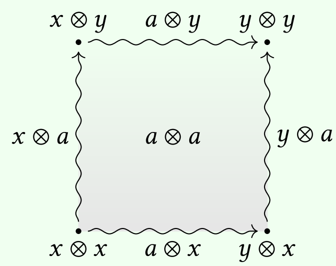
154 **Example**

Consider the interval $I = [0,1] = D^1$ as a CW complex.

$$\underset{x}{\bullet} \xrightarrow{\quad a \quad} \underset{y}{\bullet}$$

$$C_*^{\text{cell}}(I) \cong \left(\begin{array}{ccc} \mathbb{Z}\{a\} & \longrightarrow & \mathbb{Z}\{x,y\} \\ a & \longmapsto & y - x \end{array} \right)$$

There is a product cell decomposition for $I \times I$



We can consider some differentials:

$$\begin{aligned}
\partial(a \otimes x) &= y \otimes x - x \otimes x \\
&= (y - x) \otimes x \\
&= \partial a \otimes x \\
\partial(a \otimes a) &= a \otimes +y \otimes a - a \otimes y - x \otimes a \\
&= (y - x) \otimes a + a \otimes (x - y) \\
&= \partial a \otimes a - a \otimes \partial a.
\end{aligned}$$

155 **Definition**

Let C_* and D_* be two chain complexes of R-modules. Their tensor product $C_* \otimes_R D_*$ is a chain complex with

$$(C_* \otimes_R D_*) = \bigoplus_{p+q=n} C_p \otimes_R D_q,$$

where the differential is given by

$$\partial(c_p \otimes d_q) = (\partial c_p \otimes d_q) + (-1)^p (c_p \otimes \partial d_q).$$

A good property of this construction is that

$$C_*^{\text{cell}}(X) \otimes C_*^{\text{cell}}(Y) \cong C_*^{\text{cell}}(X \times Y).$$

The homology groups of $C_*^{\text{cell}}(X) \otimes C_*^{\text{cell}}(Y)$ will be computable purely in terms of $H_q(X)$ and $H_q(Y)$.

However, a bad property of this construction is that the homology of $C_* \otimes D_*$ is not in general determined by the homology of C_* and the homology of D_*. Moreover, there is no obvious internal Hom in chain complexes.

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If C_* and D_* are chain complexes, then $\mathrm{Hom}(C_*, D_*)$ is not obviously a chain complex.

20.1 Some algebra

Suppose

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of R-modules and M is another R-module. Then, note that the sequence

$$0 \longrightarrow A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0$$

is in general not exact.

156 **Example**
Let $R = \mathbb{Z}$ and consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Tensoring with $\mathbb{Z}/2\mathbb{Z}$ yields

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

However, the multiplication by 2 map becomes the zero map when we tensor by $\mathbb{Z}/2\mathbb{Z}$, so the first $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ map becomes 0. However, the zero map is not injective, so the sequence we get is not exact.

157 **Proposition**
If M is an R-module, then the functor

$$- \otimes_R M \colon R\text{-mod} \to R\text{-mod}$$

is **right exact**. In other words, if

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of R-modules, then so is

$$A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0$$

158 **Corollary**
In fact,

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact if and only if

$$A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0$$

is exact for every R-module M.

Sketch. Take $M = R$. ◻

To prove something about tensor products, it is often helpful to think about the tensor products as internal Homs. In particular, we have the related theorem

159 **Proposition**

A sequence of R-modules

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact if and only if for all R-modules N,

$$0 \longrightarrow \underline{\mathrm{Hom}}_{R\text{-mod}}(C, N) \longrightarrow \underline{\mathrm{Hom}}_{R\text{-mod}}(B, N) \longrightarrow \underline{\mathrm{Hom}}_{R\text{-mod}}(A, N)$$

is exact.

Proof (proposition 159 $\implies$ proposition 157). Suppose $A \to B \to C \to 0$ is exact. To prove

$$A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0$$

is exact, by proposition 159 it suffices to check that

$$0 \longrightarrow \underline{\mathrm{Hom}}(C \otimes_R M, N) \longrightarrow \underline{\mathrm{Hom}}(B \otimes_R M, N) \longrightarrow \underline{\mathrm{Hom}}(A \otimes_R M, N)$$

is exact for every N. This is equivalent to checking that

$$0 \longrightarrow \underline{\mathrm{Hom}}(C, \underline{\mathrm{Hom}}(M, N)) \longrightarrow \underline{\mathrm{Hom}}(B, \underline{\mathrm{Hom}}(M, N)) \longrightarrow \underline{\mathrm{Hom}}(A, \underline{\mathrm{Hom}}(M, N))$$

is exact for every N. By proposition 159 again, this is true. ◻

Proof (proposition 159). Suppose $A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is exact. We want to show

$$0 \longrightarrow \underline{\mathrm{Hom}}(C, N) \longrightarrow \underline{\mathrm{Hom}}(B, N) \longrightarrow \underline{\mathrm{Hom}}(A, N)$$

is exact.

Exactness at $\underline{\mathrm{Hom}}(C, N)$ is equivalent to the map $\underline{\mathrm{Hom}}(C, N) \to \underline{\mathrm{Hom}}(B, N)$ being injective. This is indeed true, because ψ is surjective, so the composite $B \to C \to N$ determines a map $C \to N$.

Consider the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "0"'] & B \arrow[r, "\psi"] \arrow[d, "f"] & C \arrow[r] \arrow[dl, "g"] & 0 \\ & N & & \end{tikzcd}$$

Exactness at $\underline{\mathrm{Hom}}(B, N)$ is equivalent to saying that a map $f\colon B \to N$ is restricted to the zero map $A \to B \to N$ if and only if f can be written as a composite $B \to C \to N$. The backward direction is clear because any composite $A \to B \to C \to N$ is zero because $A \to B \to C$ is exact. We want to find a function $g\colon C \to N$ such that $f = g \circ \psi$. If $f \circ \varphi = 0$, then note that $C \cong B/\operatorname{im}\varphi$. We can then assign $g[b] = fb$, where $[b]$ denotes the equivalent class representing b. This gives $f = g \circ \psi$, as desired. ◻

We have now proved that if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact and M is an R-module, then

$$A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0$$

is exact, but

$$0 \longrightarrow A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0$$

is not necessarily exact.

However, if we let $M = R$, then it is exact. Furthermore, if we take $M = R \oplus R$, the sequence will stay exact. In particular, it will become

$$0 \longrightarrow A \oplus A \longrightarrow B \oplus B \longrightarrow C \oplus C \longrightarrow 0$$

In general, if M is a free R-module, then tensoring with M gives an exact sequence and not just a right exact sequence.

160 **Note**

If X is a topological space, then $C_*^{\text{cell}}(X; R)$ is a chain complex of free R-modules.

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161 **Definition**

Suppose C_* and D_* are chain complexes of R-modules. A chain map $f\colon C_* \to D_*$ is a **quasi-isomorphism** if $H_q(f)$ is an isomorphism for all q. Then, one says that C_* and D_* are **quasi-isomorphic**.

162 **Definition**

Suppose M is an R-module. Then a **free resolution** of M is a chain complex C_* of free R-modules and a quasi-isomorphism $C_* \to M$ where we think of M as a chain complex concentrated in degree 0.

163 **Example**

Suppose

$$C_* = \left(\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{5} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \right)$$

$$D_* = \left(\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/5\mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \right)$$

Then the following chain map is a quasi-isomorphism:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "5"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/5\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where the map $\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ is the natural quotient map. Moreover, C_* is a free resolution of $\mathbb{Z}/5\mathbb{Z}$.

164 **Example**

If R is a field, then every module is its own free resolution.

165 **Example**

Suppose $\mathbb{R} = \mathbb{Z}$ and $M = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then we have the free resolution

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where the $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ map maps $(1, 0) \mapsto (3, 0, 0)$ and $(0, 1) \mapsto (0, 0, 2)$.

It turns out that we can always find a two term free resolution by taking a surjection with the degree 0 group, e.g. $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ in the previous example, and then use the degree 1 group to describe the relations.

Note that free resolutions are not unique! In particular,

$$\begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \\
1 & \longmapsto & (0,0,1) & & \\
 & & (1,0,0) & \longmapsto & (3,0,1) \\
 & & (0,1,0) & \longmapsto & (0,0,2) \\
 & & (0,0,1) & \longmapsto & (0,0,0)
\end{array}$$

is also a free resolution of $\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. However, it is not "minimal".

166 **Example**

If $R = \mathbb{Q}[t]/t^2$ and M is the module $\mathbb{Q}$ where t acts by 0, then this has a free resolution, but the smallest possible one is infinite.

167 **Theorem** (Fundamental theorem of homological algebra)

Suppose N and M are R-modules. Let

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[d, "\varepsilon_N"] \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & N
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\cdots \arrow[r] & E_2 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & E_0 \arrow[d, "\varepsilon_M"] \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M
\end{tikzcd}$$

be two free resolutions of N and M respectively. Then any R-module map $f\colon N \to M$ lifts to a chain map $f_*\colon F_* \to E_*$. Furthermore, f_* is unique up to chain homotopy.

168 **Example**

Let $R = \mathbb{Z}$, $N = \mathbb{Z}/2\mathbb{Z}$, and $M = \mathbb{Z}/6\mathbb{Z}$. Consider $f\colon \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ mapping $1 \mapsto 3$. We have the free resolutions with the maps

$$\begin{tikzcd}
F_* = \Big(& \cdots \arrow[r] & 0 \arrow[r] \arrow[dl] & \mathbb{Z} \arrow[r, "2"] \arrow[dl, "1"'] & \mathbb{Z} \arrow[r] \arrow[dl, "3"'] \arrow[dd] & 0 \arrow[r] \arrow[dl] & \cdots \Big) \\
E_* = \Big(\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "6"'] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \Big) & \\
 & & & \mathbb{Z}/6\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \arrow[l, "f"'] & &
\end{tikzcd}$$

This is a chain map f_*, where

$$\begin{tikzcd}
H_0(f_*)\colon H_0(F_*) \arrow[r] \arrow[d, phantom, "\cong"] & H_0(E_*) \arrow[d, phantom, "\cong"] \\
\mathbb{Z}/2\mathbb{Z} \arrow[r, "f"] & \mathbb{Z}/6\mathbb{Z}
\end{tikzcd}$$

This theorem says that if we want to understand maps between modules, it suffices to understand the maps between their free resolutions up to chain homotopy. In particular, they contain the same data, and the map between the modules can be recovered by taking H_0.

Sketch. We can build up a map between the free resolutions inductively, checking at every stage that there is only one choice up to chain homotopy. To start producing a chain map, if we have

$$\begin{tikzcd}
F_0 \arrow[r, "\varepsilon_N"] \arrow[d, dashed, "f_0"] & N \arrow[d, "f"] \\
E_0 \arrow[r, "\varepsilon_M"] & M
\end{tikzcd}$$

we want to produce the dashed f_0 map. Since F_0 is free, say on S_0, for each $s_0 \in S_0$, we can define $f_0(s_0)$ to be any element in E_0 such that

$$\varepsilon_M(f_0 s_0) = f(\varepsilon_N s_0).$$

This is possible because ε_M is surjective, because M is the homology of E_*, which means M is the quotient of E_0 mod the image of the $E_1 \to E_0$ map.

This gives the diagram

$$\begin{tikzcd} \ker \varepsilon_M \arrow[r] \arrow[d, "g_0"] & F_0 \arrow[r, "\varepsilon_N"] \arrow[d, "f_0"] & N \arrow[d, "f"] \\ \ker \varepsilon_N \arrow[r] & E_0 \arrow[r, "\varepsilon_M"'] & M \end{tikzcd}$$

where the map g_0 is unique because $\ker \varepsilon_M \to F_0$ and $\ker \varepsilon_N \to E_0$ are just projections, so f_0 determines g_0.

Now we wish to produce the dashed map in the diagram

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d, dashed, "f_1"] & \ker \varepsilon_M \arrow[r] \arrow[d, "g_0"] & F_0 \arrow[r, "\varepsilon_N"] \arrow[d, "f_0"] & N \arrow[d, "f"] \\ E_1 \arrow[r] & \ker \varepsilon_N \arrow[r] & E_0 \arrow[r, "\varepsilon_M"'] & M \end{tikzcd}$$

However, by exactness of the free resolutions, we know that the $F_1 \to \ker \varepsilon_N$ and $E_1 \to \ker \varepsilon_M$ maps are surjective, so we can apply the same argument as before. □

169 **Definition Sketch**

Let R be a commutative ring, and $\mathrm{ch}(R\text{-mod})$ be the category of chain complexes of R-modules. The **derived category** of R, denoted $\mathrm{D}(R)$, is the category obtained from $\mathrm{ch}(R\text{-mod})$ by formally inverting all quasi-isomorphisms.

In other words, the objects of $\mathrm{D}(R)$ are the objects of $\mathrm{ch}(R\text{-mod})$, but there are many more morphisms than just chain maps. In particular, if $f_*\colon C_* \to D_*$ is a chain map, then there is a formal inverse $g\colon D_* \to C_*$ in $\mathrm{D}(R)$.

The formal construction of $\mathrm{D}(R)$ is beyond the scope of the class because there are set-theoretic technicalities.

In algebraic topology, we don't actually care about $\mathrm{ch}(R\text{-mod})$. We only care about $\mathrm{D}(R)$.

170 **Example**

In $\mathrm{D}(R)$, every object is isomorphic to a chain complex of free modules. Free resolutions are an example of this.

We will state facts about $\mathrm{D}(R)$ and then prove consequences of these facts that can be stated without reference to $\mathrm{D}(R)$. The reason we do so is because it gives motivation for the statements and their proofs.

171 **Example** (Tensor products in $\mathbf{D}(R)$)

If C_* and D_* are two chain complexes, we define the **derived tensor product** $C_* \otimes^{\mathbb{L}} D_*$ as follows:

1. We replace C_* and D_* by quasi-isomorphic chain complexes of free modules C'_* and D'_*.
2. Take $C'_* \otimes_R D'_*$.
3. The result is well defined up to quasi-isomorphism

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If two chain complexes are isomorphic in $\mathrm{D}(R)$, then they have isomorphic homology R-modules.

172 **Fact**

If at least one of C_* and D_* are chain complexes of free R-modules, then $C_* \otimes_R^{\mathbb{L}} D_* \cong C_* \otimes_R D_*$.

Since we haven't described the derived category rigorously, we can't prove this. However, we will show some consequences of this fact.

173 **Definition**

If M and N are two R-modules, then the Tor functor is defined to be $\mathrm{Tor}_i(M, N) := H_i(M \otimes_R^{\mathbb{L}} N)$, where M and N are viewed as chain complexes concentrated in degree 0.

Soon we will prove that $\mathrm{Tor}_i(M, N)$ are well defined. For now, let's do some calculations of them.

174 **Example**

Suppose $R = \mathbb{Z}$. Then we can compute $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/4\mathbb{Z}$ by replacing either $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$ with their free resolution, and then take the tensor product normally.

We can compute this by resolving $\mathbb{Z}/2\mathbb{Z}$, i.e. replacing it with its free resolution.

$$\begin{aligned}
\mathbb{Z}/2\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z}/4\mathbb{Z} &\cong \left(\mathbb{Z} \xrightarrow{2} \mathbb{Z}\right) \otimes \mathbb{Z}/4\mathbb{Z} \\
&\cong \left(\cdots \longrightarrow 0 \longrightarrow \underset{\deg 1}{\mathbb{Z}/4\mathbb{Z}} \xrightarrow{2} \underset{\deg 0}{\mathbb{Z}/4\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots \right)
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathrm{Tor}_0(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) &= \frac{\mathbb{Z}/4\mathbb{Z}}{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z} \\
\mathrm{Tor}_1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) &= \frac{\mathbb{Z}/2\mathbb{Z}}{0} \cong \mathbb{Z}/2\mathbb{Z},
\end{aligned}$$

and $\mathrm{Tor}_i = 0$ for all other i.

We can also compute it by resolving $\mathbb{Z}/4\mathbb{Z}$:

$$\begin{aligned}
\mathbb{Z}/2\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z}/4\mathbb{Z} &\cong \mathbb{Z}/2\mathbb{Z} \otimes \left(\mathbb{Z} \xrightarrow{4} \mathbb{Z}\right) \\
&\cong \left(\mathbb{Z}/4\mathbb{Z} \xrightarrow{4} \mathbb{Z}/4\mathbb{Z}\right)
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathrm{Tor}_0(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) &= \frac{\mathbb{Z}/2\mathbb{Z}}{0} \cong \mathbb{Z}/2\mathbb{Z} \\
\mathrm{Tor}_1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) &= \frac{\mathbb{Z}/2\mathbb{Z}}{0} \cong \mathbb{Z}/2\mathbb{Z},
\end{aligned}$$

and $\mathrm{Tor}_i = 0$ for all other i.

We can also compute this in some more inefficient ways. We can resolve both sides:

$$\mathbb{Z}/2\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z}/4\mathbb{Z} \cong \left(\mathbb{Z}\{a\} \xrightarrow{2} \mathbb{Z}\{b\}\right) \otimes \left(\mathbb{Z}\{c\} \xrightarrow{4} \mathbb{Z}\{d\}\right),$$

where we add generators to make it more clear. Multiplying,

$$\cong \mathbb{Z}\{a \otimes c\} \xrightarrow{\partial_2} \mathbb{Z}\{b \otimes c, a \otimes d\} \xrightarrow{\partial_1} \mathbb{Z}\{b \otimes d\}$$

where

$$\begin{aligned}
\partial(a \otimes c) &= (\partial a \otimes c) + (-1)(a \otimes \partial c) = (2b \otimes c) - (a \otimes 4d) \\
\partial(b \otimes c) &= (\partial b \otimes c) + (b \otimes \partial c) = 4(b \otimes d) \\
\partial(a \otimes d) &= (\partial a \otimes d) + (-1)(a \otimes \partial d) = 2(b \otimes d).
\end{aligned}$$

We can also confirm

$$\partial\partial(a \otimes c) = \partial(2(b \otimes c) - 4(a \otimes d)) = 2 \cdot 4(b \otimes d) - 4 \cdot 2(b \otimes d) = 0.$$

We can then compute H_0 of this complex to be $\frac{\mathbb{Z}\{b \otimes d\}}{\operatorname{im} \partial_1} = \mathbb{Z}/2\mathbb{Z}$, and H_1 to be $\ker \partial_1 / \operatorname{im} \partial_2 = \frac{\mathbb{Z}\{b \otimes c - 2(a \otimes d)\}}{\mathbb{Z}\{2b \otimes c - 4(a \otimes d)\}} \cong \mathbb{Z}/2\mathbb{Z}$. Moreover, we can notice ∂_2 is injective, so all higher order homology groups are zero, as expected.

175 **Example**

To compute $\operatorname{Tor}_i(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$, we have

$$\begin{aligned}
\mathbb{Z}/2\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z}/3\mathbb{Z} &\cong \left(\mathbb{Z} \xrightarrow{2} \mathbb{Z}\right) \otimes \mathbb{Z}/3\mathbb{Z} \\
&\cong \mathbb{Z}/3\mathbb{Z} \xrightarrow{2} \mathbb{Z}/3\mathbb{Z}
\end{aligned}$$

which means

$$\begin{aligned}
\operatorname{Tor}_0(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) &\cong 0 \\
\operatorname{Tor}_1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) &\cong 0.
\end{aligned}$$

176 **Example**

For $\operatorname{Tor}_i(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z})$, note that

$$\mathbb{Z}/3\mathbb{Z} \otimes^{\mathbb{L}} \mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z},$$

where we interpret $\mathbb{Z}/3\mathbb{Z}$ as

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

so

$$\operatorname{Tor}_0(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$$

and $\operatorname{Tor}_i = 0$ for all other i.

177 **Example**

If R is a field, and V and W are vector spaces over R, then $\operatorname{Tor}_0(V, W) \cong V \otimes_R W$ and $\operatorname{Tor}_i(V, W) \cong 0$ for all other i.

We will now prove that the Tor groups are well defined without reference to the derived categories.

*Proof (*Tor *groups are well defined).* Suppose M and N are two R-modules, and suppose F_* and F'_* are two free resolutions of M. We will prove that $F_* \otimes_R N$ and $F'_* \otimes_R N$ have the same homology groups. In fact, we will show that they are chain homotopy equivalent.

By the fundamental theorem of homological algebra, there exists a unique (up to chain homotopy) map $f\colon F_* \longrightarrow F'_*$ lifting 1_M and

$g\colon F'_* \to F_*$ lifting 1_M.

Note that $f \circ g$ is a map from $F'_* \to F'_*$ lifting 1_M, so by the fundamental theorem, $f \circ g$ must be chain homotopic to $1_{F'_*}$. It follows that $(f \circ g) \otimes_R N = (f \otimes N) \circ (g \otimes N)$, which is chain homotopic to $1_{F'_* \otimes N}$.

Similarly, $(g \circ f) \otimes_R N$ is chain homotopic to $1_{F_* \otimes N}$. Therefore, $g \otimes_R N$ and $f \otimes_R N$ are inverses up to chain homotopy. This proves that $F_* \otimes_R N$ and $F'_* \otimes_R N$ are chain homotopy equivalent. □

22.0.1 On internal Homs

178 **Definition**

If F_* is a chain complex of free R-modules and N is an R-module, considered as a chain concentrated in degree 0. Then we may form a new chain complex $\underline{\mathrm{Hom}}_{\mathrm{D}(R)}(F_*, N)$ with

$$\underline{\mathrm{Hom}}(F_*, N)_n = \underline{\mathrm{Hom}}_{R\text{-mod}}(F_{-n}, N).$$

179 **Example**

If $R = \mathbb{Z}$,

$$F_* = \Big(\cdots \longrightarrow 0 \longrightarrow \underset{\deg 1}{\mathbb{Z}} \xrightarrow{4} \underset{\deg 0}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots \Big)$$

and $N = \mathbb{Z}/2\mathbb{Z}$. Then

$$\underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(F_*, N) \cong \Big(\cdots \longrightarrow 0 \longrightarrow \underset{\deg 0}{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\partial} \underset{\deg -1}{\mathbb{Z}/2\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots \Big)$$

The degree 0 R-module is $\underline{\mathrm{Hom}}(F_0, N) \cong \underline{\mathrm{Hom}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, and the degree 1 R-module is $\underline{\mathrm{Hom}}(F_1, N) \cong \mathbb{Z}/2\mathbb{Z}$. We can see that the differential ∂ exists because given a map in $\underline{\mathrm{Hom}}(F_0, N)$, we can compose it with the 4 map to get a map in $\underline{\mathrm{Hom}}(F_1, N)$.

180 **Definition**

If M and N are R-modules, then $\underline{\mathrm{Hom}}_{\mathrm{D}(R)}(M, N)$ can be computed by replacing M with a free resolution and then applying the above construction.

181 **Definition**

If M and N are R-modules, then the **Ext functor** is

$$\begin{aligned}\mathrm{Ext}^i_R(M, N) &= H_{-i}(\underline{\mathrm{Hom}}_{\mathrm{D}(R)}(M, N)) \\ &= H_{-i}(\underline{\mathrm{Hom}}_{\mathrm{D}(R)}(F_*, N)),\end{aligned}$$

where F_* is any free resolution of M.

We will use Tor and Ext to compute homology with coefficients from homology with $\mathbb{Z}$ coefficients.

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23.0.1 Homology with coefficients

Recall that if X is a topological space, we have chain complex of R-modules $S_*(X;R)$ with the nth group being the free R-module generated by $\mathrm{Sing}_n(X)$. The ∂ maps are given by the alternating sums of the d_i maps.

182 **Note**

$S_*(X;R)$ may be alternatively described as $S_*(X) \otimes_{\mathbb{Z}} R$, which also calculates the derived tensor product.

We can even make a more general definition

183 **Definition**

Suppose M is an abelian group and X is a topological space. We can define $S_*(X;M) \coloneqq S_*(X) \otimes_{\mathbb{Z}} M$ and $H_q(X;M)$ to be H_q of $S_*(X;M)$.

If R is a commutative ring and M is an R-module, then $H_q(X;M)$ acquires the structure of an R-module.

23.0.2 Cohomology

Suppose X is a topological space and M is an abelian group. We can make a chain complex, concentrated in non-positive degrees, with the $(-n)$th term $\underline{\mathrm{Hom}}_{\mathbb{Z}\text{-mod}}(S_n(X), M)$. This new chain complex calculates $\underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(S_*(X), M)$.

184 **Definition**

The $(-q)$th homology group of the above chain complex is denoted $H^q(X;M)$, and is called the qth **cohomology group** of X with coefficients in M.

Both homology coefficients and cohomology with coefficients can be determined by homology with integer coefficients. However, these may be easier to compute, so they can be useful.

23.0.3 Universal coefficient theorems

185 **Theorem** (For cohomology)

Let X be a topological space, and M be an abelian group. For any integer q, there is an isomorphism

$$H^q(X;M) \cong \underline{\mathrm{Hom}}_{\mathrm{Ab}}(H_q(X), M) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{q-1}(X), M).$$

Note that on the right side, we have expressions that only deal with homology with integer coefficients, and on the left we have homology with coefficients in an arbitrary abelian group M.

186 **Theorem** (For homology)
Let C_* be a chain complex of free $\mathbb{Z}$-modules (e.g. $C_* = S_*(X)$ or $C_* = C_*^{\text{cell}}(X)$) and M is an abelian group. Then there is a natural short exact sequence

$$0 \longrightarrow H_q(C_*) \otimes_{\mathbb{Z}} M \longrightarrow H_q(C_* \otimes_{\mathbb{Z}} M) \longrightarrow \mathrm{Tor}_1(H_{q-1}(C_*), M) \longrightarrow 0$$

and furthermore an isomorphism

$$H_q(C_* \otimes_{\mathbb{Z}} M) \cong H_q(C_*) \otimes_{\mathbb{Z}} M \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{q-1}(C_*), M).$$

Note that similarly to the previous theorem, the left is calculating homology with coefficients but the right side is about homology with integer coefficients.

187 **Warning**
This isomorphism in theorem 186 is not natural! While the short exact sequence is natural, the direct sum decomposition of the middle term is not natural.

Let's do some computations before we prove this.

188 **Example**
Consider $\mathbb{RP}^2$. Recall that

$$C_*^{\text{cell}}(\mathbb{RP}^2) = C_*^{\text{cell}}(\mathbb{RP}^2; \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

and we have

$$H_q(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}/2\mathbb{Z} & q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We can then as what $H_q(\mathbb{RP}^2; \mathbb{F}_2)$ are.
One way we can compute this is to calculate $C_*^{\text{cell}}(\mathbb{RP}^2) \otimes_{\mathbb{Z}} \mathbb{F}_2$ directly.

$$\mathbb{F}_2 \xrightarrow{2} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2$$

This gives

$$H_q(\mathbb{RP}^2; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & q = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

The other way we can compute this is to use the universal coefficients theorem. In particular, we have

$$\begin{aligned} H_2(\mathbb{RP}^2; \mathbb{F}_2) &\cong H_2(\mathbb{RP}^2) \otimes_{\mathbb{Z}} \mathbb{F}_2 \oplus_{\mathbb{Z}} \mathrm{Tor}_1(H_1(\mathbb{RP}^2), \mathbb{F}_2) \\ &\cong 0 \otimes_{\mathbb{Z}} F_2 \oplus \mathrm{Tor}_1(\mathbb{F}_2, \mathbb{F}_2) \\ &\cong 0 \oplus H_1(\mathbb{F}_2 \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_2). \end{aligned}$$

We know

$$\begin{aligned} H_1(\mathbb{F}_2 \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_2) &= H_1\left(\left(\mathbb{Z} \xrightarrow{2} \mathbb{Z} \right) \otimes_{\mathbb{Z}} \mathbb{F}_2\right) \\ &= H_1\left(\mathbb{F}_2 \xrightarrow{2} \mathbb{F}_2 \right) \\ &= \mathbb{F}_2, \end{aligned}$$

so $H_2(\mathbb{RP}^2; \mathbb{F}_2) = \mathbb{F}_2$, as expected.

Similarly, we have

$$\begin{aligned} H_1(\mathbb{RP}^2;\mathbb{F}_2) &\cong H_1(\mathbb{RP}^2)\otimes_{\mathbb{Z}}\mathbb{F}_2 \oplus_{\mathbb{Z}} \mathrm{Tor}_1(H_0(\mathbb{RP}^2),\mathbb{F}_2) \\ &\cong \mathbb{F}_2 \otimes_{\mathbb{Z}} F_2 \oplus \mathrm{Tor}_1(\mathbb{Z},\mathbb{F}_2) \\ &\cong \mathbb{F}_2 \oplus H_1(\mathbb{Z}\otimes^{\mathbb{L}}\mathbb{F}_2) \\ &\cong \mathbb{F}_2 \oplus H_1\Big(\cdots \longrightarrow 0 \longrightarrow \mathbb{F}_2 \longrightarrow 0 \longrightarrow \cdots\Big) \\ &\cong \mathbb{F}_2 \oplus 0 = \mathbb{F}_2, \end{aligned}$$

as desired.

189 **Example**

Let's compute $H_2(S^2;\mathbb{F}_3)$. We have

$$\begin{aligned} H_2(S^2;\mathbb{F}_3) &\cong H_2(S^2)\otimes_{\mathbb{Z}}\mathbb{F}_3 \oplus_{\mathbb{Z}} \mathrm{Tor}_1(H_1(S^2),\mathbb{F}_3) \\ &\cong \mathbb{Z}\otimes_{\mathbb{Z}} F_3 \oplus \mathrm{Tor}_1(0,\mathbb{F}_3) \\ &\cong \mathbb{F}_3 \oplus H_1(0\otimes^{\mathbb{L}}\mathbb{F}_2) \\ &\cong \mathbb{F}_2 \oplus H_1(\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots) \\ &\cong \mathbb{F}_3 \oplus 0 = \mathbb{F}_3, \end{aligned}$$

Proof (theorem 186). Suppose C_* is a chain complex of free $\mathbb{Z}$-modules and M is an abelian group. We want to calculate the homology groups of $C_*\otimes_{\mathbb{Z}} M$. Consider a free resolution

$$\cdots \longrightarrow 0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0 \longrightarrow \cdots$$

of M. Since $\mathbb{Z}$ is a PID, we can find this two-term free resolution. This gives us a short exact sequence of chain complexes

$$0 \longrightarrow C_*\otimes_{\mathbb{Z}} F_1 \longrightarrow C_*\otimes_{\mathbb{Z}} F_0 \longrightarrow C_*\otimes_{\mathbb{Z}} M \longrightarrow 0$$

which gives us a long exact sequence in homology

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H_q(C_*\otimes F_1) & \longrightarrow & H_q(C_*\otimes F_0) & \longrightarrow & H_q(C_*\otimes M) & \xrightarrow{\partial} & H_{q-1}(C_*\otimes F_1) & \longrightarrow & H_{q-1}(C_*\otimes F_0) & \longrightarrow \cdots \\ & \cong & & \cong & & \cong & & \cong & & \cong & \\ \cdots \longrightarrow & H_q(C_*)\otimes F_1 & \longrightarrow & H_q(C_*)\otimes F_0 & \longrightarrow & H_q(C_*\otimes M) & \longrightarrow & H_{q-1}(C_*)\otimes F_1 & \longrightarrow & H_{q-1}(C_*)\otimes F_0 & \longrightarrow \cdots \end{array}$$

This long exact sequence implies the short exact sequence

$$0 \longrightarrow H_q(C_*)\otimes F_0/F_1 \longrightarrow H_q(C_*\otimes M) \longrightarrow \ker[H_{q-1}(C_*)\otimes F_1 \to H_{q-1}(C_*)\otimes F_0] \longrightarrow 0$$

Since $F_1 \to F_0 \to M$ is a free resolution, we have $F_0/F_1 \cong M$. Also, the third term of the short exact sequence is $\mathrm{Tor}(H_{q-1}(C_*),M)$. This gives the short exact sequence

$$0 \longrightarrow H_q(C_*)\otimes M \longrightarrow H_q(C_*\otimes M) \longrightarrow \mathrm{Tor}_1(H_{q-1}(C_*),M) \longrightarrow 0$$ □

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190 **Question**

What is the homology of a tensor product of chain complexes?

191 **Theorem** (Künneth)

Suppose that C_* and D_* are chain complexes of $\mathbb{Z}$-modules and suppose C_* is a complex of free $\mathbb{Z}$-modules. Then

$$H_n(C_* \otimes_{\mathbb{Z}} D_*) = \left(\bigoplus_{p+q=n} H_p(C_*) \otimes_{\mathbb{Z}} H_q(D_*)\right) \oplus \left(\bigoplus_{p+q=n-1} \operatorname{Tor}_1^{\mathbb{Z}}(H_p(C_*), H_q(D_*))\right).$$

Note the similarity between this theorem and the universal coefficient theorem. We have a guess for the tensor product in the first term, and then correction terms described by the Tor functor. In fact, the universal coefficient theorem is a special case of this theorem, where D_* is concentrated in degree 0.

Proof idea 1. Note that $C_* \otimes D_*$ calculates $C_* \otimes^{\mathbb{L}} D_*$ in $\mathrm{D}(\mathbb{Z})$. Its homology will not change if we replace D_* with any chain complex isomorphic to D_* in $\mathrm{D}(\mathbb{Z})$. In particular, replace D_* with the chain complex D'_* such that $(D'_*)_q = H_q(D_*)$ and all of the maps are 0 maps. Note that D'_* is a direct sum of chain complexes concentrated in single degrees, so we can reduce the theorem to the universal coefficient theorem. □

However, we may want to give a proof that does not use the derived category, since we have not talked about those rigorously. This proof will still however, have the same spirit, since we will reduce the proof to the universal coefficient theorem.

Proof idea 2. Consider the new chain complex $Z(D_*)$ where

$$Z(D_*)_n = Z_n(D_*) = \ker(\partial \colon D_n \to D_{n-1})$$

and all ∂ maps in $Z(D_*)$ are 0. Note that the is an inclusion of chain complexes $Z(D_*) \to D_*$. $Z(D_*)$ is a direct sum of chain complexes concentrated in single degrees.

Note that the quotient chain complex $D_*/Z(D_*)$ also has boundary maps that are the zero maps. Therefore, $D_*/Z(D_*)$ is also a direct sum of chain complexes concentrated in single degrees.

Consider the short exact sequence of chain complexes

$$0 \longrightarrow C_* \otimes_{\mathbb{Z}} Z(D_*) \longrightarrow C_* \otimes_{\mathbb{Z}} D_* \longrightarrow C_* \otimes_{\mathbb{Z}} D_*/Z(D_*) \longrightarrow 0$$

This gives the long exact sequence in homology from the snake lemma to write $H_n(C_* \otimes_{\mathbb{Z}} D_*)$ in terms of $H_n(C_* \otimes_{\mathbb{Z}} Z(D_*))$ and $H_{n-1}(C_* \otimes_{\mathbb{Z}} D_*/Z(D_*))$. This reduces to the universal coefficient theorem. □

192 **Remark**

This extends to all PIDs, not just $\mathbb{Z}$. Suppose that R is a PID and C_* and D_* are chain complexes of R-modules such that C_* is a chain complex of free R-modules. Then

$$H_n(C_* \otimes_R D_*) \cong \left(\bigoplus_{p+q=n} H_p(C_*) \otimes_R H_q(D_*)\right) \oplus \left(\bigoplus_{p+q=n-1} \operatorname{Tor}_1^{R}(H_p(C_*), H_q(D_*))\right).$$

The hypothesis that R has to be a PID comes from how we needed a 2-term free resolution of M in the proof of the universal coefficient theorem.

In particular, if R is a field, then

$$H_n(C_* \otimes_R D_*) \cong \bigoplus_{p+q=n} H_p(C_*) \otimes_R H_q(D_*).$$

This is because all free resolutions over a field are not interesting, so all of the Tor functors vanish.

193 **Theorem** (Eilenberg-Zilber)

Suppose X and Y are topological spaces and R is a commutative ring. Then $S_*(X \times Y; R)$ is quasi-isomorphic to $S_*(X; R) \otimes_R S_*(Y; R)$.

Proof idea 1. If X and Y are CW complexes, then we can examine the CW structure on $X \times Y$. Rather than developing the theory of product CW structures, we will directly prove the theorem instead, since not all topological spaces have a natural CW structure. □

Before we prove the theorem, let's do some examples.

194 **Example**

Let's compute the groups $H_q(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2)$. Recall that we have

$$H_q(\mathbb{RP}^2; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & q = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbb{F}_2$ is a field, this means

$$\begin{aligned}
H_0(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2) &\cong \bigoplus_{p+q=0} H_p(\mathbb{RP}^2; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H_q(\mathbb{RP}^2; \mathbb{F}_2) \\
&\cong H_0(\mathbb{RP}^2; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H_0(\mathbb{RP}^2; \mathbb{F}_2) \\
&\cong \mathbb{F}_2 \otimes_{\mathbb{F}_2} \mathbb{F}_2 \\
&\cong \mathbb{F}_2 \\
H_1(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2) &\cong \bigoplus_{p+q=1} H_p(\mathbb{RP}^2; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H_q(\mathbb{RP}^2; \mathbb{F}_2) \\
&\cong H_0(\mathbb{RP}^2; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H_1(\mathbb{RP}^2; \mathbb{F}_2) \oplus H_1(\mathbb{RP}^2; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H_0(\mathbb{RP}^2; \mathbb{F}_2) \\
&\cong \mathbb{F}_2 \otimes_{\mathbb{F}_2} \mathbb{F}_2 \oplus \mathbb{F}_2 \otimes_{\mathbb{F}_2} \mathbb{F}_2 \\
H_2(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2) &\cong (H_0 \otimes H_2) \oplus (H_1 \otimes H_1) \oplus (H_2 \otimes H_0) \\
&\cong \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 \\
H_3(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2) &\cong (H_0 \otimes H_3) \oplus (H_1 \otimes H_2) \oplus (H_2 \otimes H_1) \oplus (H_3 \otimes H_0) \\
&\cong (\mathbb{F}_2 \otimes 0) \oplus (\mathbb{F}_2 \otimes \mathbb{F}_2) \oplus (\mathbb{F}_2 \otimes \mathbb{F}_2) \oplus (0 \otimes \mathbb{F}_2) \\
&\cong \mathbb{F}_2 \oplus \mathbb{F}_2 \\
H_4(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_4) &\cong (H_0 \otimes H_4) \oplus (H_1 \otimes H_3) \oplus (H_2 \otimes H_2) \oplus (H_3 \otimes H_1) \oplus (H_4 \otimes H_0) \\
&\cong (\mathbb{F}_2 \otimes 0) \oplus (\mathbb{F}_2 \otimes 0) \oplus (\mathbb{F}_2 \otimes \mathbb{F}_2) \oplus (0 \otimes \mathbb{F}_2) \oplus (0 \otimes \mathbb{F}_2) \\
&\cong \mathbb{F}_2.
\end{aligned}$$

This gives

$$H_q(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & q = 0 \\ \mathbb{F}_2 \oplus \mathbb{F}_2 & q = 1 \\ \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 & q = 2 \\ \mathbb{F}_2 \oplus \mathbb{F}_2 & q = 3 \\ \mathbb{F}_2 & q = 4 \\ 0 & \text{otherwise.} \end{cases}$$

One thing to note is that these groups are symmetric around $q = 2$. This is called Poincaré duality, and we will prove it later.

195 **Remark**

There is a diagonal map

$$\begin{aligned} \Delta\colon \mathbb{RP}^2 &\longrightarrow \mathbb{RP}^2 \times \mathbb{RP}^2 \\ x &\longmapsto (x, x) \end{aligned}$$

where

$$\begin{aligned} H_2(\Delta)\colon H_2(\mathbb{RP}^2; \mathbb{F}_2) &\longrightarrow H_2(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{F}_2) \\ \mathbb{F}_2 &\longrightarrow \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2. \end{aligned}$$

We would like the figure out how to describe this map. We will do this later in the course.

To prove Eilenberg-Zilber (theorem 193), we want to show $S_*(X \times Y; R)$ is quasi-isomorphic to $S_*(X; R) \times_R S_*(Y, R)$. We will show that the following map is the quasi-isomorphism.

196 **Definition**

The **Alexander-Whitney map**

$$A\colon S_*(X \times Y; R) \longrightarrow S_*(X; R) \times S_*(Y; R)$$

is a chain map defined as follows: For each integer n and $p + q = n$, let

$$A\colon S_n(X \times Y; R) \longrightarrow S_p(X; R) \otimes S_q(Y; R).$$

It suffices to define $A(\sigma)$ for $\sigma \in \mathrm{Sing}_n(X \times Y)$. Note that $\sigma\colon \Delta^n \longrightarrow X \times Y$ is determined by two maps $\alpha\colon \Delta^n \longrightarrow X$ and $\beta\colon \Delta^n \longrightarrow Y$. Let

$$A(\sigma) = (\alpha\restriction_{\Delta^p}) \otimes (\beta\restriction_{\Delta^q})$$

where the restriction to Δ^p is given by the map

$$\begin{aligned} \Delta^p &\longrightarrow \Delta^n \\ [e_0 : e_1 : \cdots : e_p] &\longmapsto [e_0 : e_1 : \cdots : e_p : 0 : \cdots : 0] \end{aligned}$$

and the restriction to Δ^q is given by the map

$$\begin{aligned} \Delta^q &\longrightarrow \Delta^n \\ [e_0 : e_1 : \cdots : e_q] &\longmapsto [0 : \cdots : 0 : e_0 : e_1 : \cdots : e_q]. \end{aligned}$$

25 October 30, 2020

197 **Theorem** (Eilenberg-Zilber)

If X and Y are topological spaces, then the Alexander-Whitney map

$$A\colon S_*(X \times Y) \to S_*(X) \otimes_{\mathbb{Z}} S_*(Y)$$

is a quasi-isomorphism.

Recall that A is a chain map that sends an n-simplex $\sigma\colon \Delta^n \to X \times Y$ to

$$A(\sigma) = \sum_{p+q=n} \alpha\restriction_{\Delta^p} \otimes \beta\restriction_{\Delta^q}$$

where $\alpha\restriction_{\Delta^p}$ is the composite

$$\Delta^p \xrightarrow{\text{first coordinates}} \Delta^n \xrightarrow{\sigma} X \times Y \xrightarrow{p_x} X$$

and $\beta\restriction_{\Delta^q}$ is the composite

$$\Delta^q \xrightarrow{\text{last coordinates}} \Delta^n \xrightarrow{\sigma} X \times Y \xrightarrow{p_y} Y.$$

The proof of this theorem is a technical elaboration of naturality and the fundamental theorem of homological algebra.

198 **Construction**

Suppose C is a category and $\mathbb{M}$ is a set of objects of C (called the set of **models**). For any $\{m_1, \ldots, m_r\} \subseteq \mathbb{M}$, we can consider the functor

$$\begin{aligned} F\colon C &\to \mathsf{Ab} \\ c &\mapsto \mathbb{Z}\Big\{\coprod_{m \in \mathbb{M}} \mathrm{Hom}(m, c)\Big\}. \end{aligned}$$

Such a functor is said to be **$\mathbb{M}$-free**.

199 **Example**

Let $C = \mathsf{Top} \times \mathsf{Top}$. Consider $\mathbb{M} = \{\Delta^n \times \Delta^n\}$. This gives the functor

$$\begin{aligned} F\colon \mathsf{Top} \times \mathsf{Ab} &\to \mathsf{Ab} \\ (X, Y) &\mapsto \mathbb{Z}\big\{\mathrm{Hom}_{\mathsf{Top}\times\mathsf{Top}}(\Delta^n \times \Delta^n, X \times Y)\big\} \\ &\mapsto \mathbb{Z}\big\{\mathrm{Hom}_{\mathsf{Top}}(\Delta^n, X) \times \mathrm{Hom}_{\mathsf{Top}}(\Delta^n, Y)\big\} \\ &\mapsto \mathbb{Z}\big\{\mathrm{Hom}_{\mathsf{Top}}(\Delta^n, X \times Y)\big\} \\ &\mapsto S_n(X \times Y). \end{aligned}$$

200 **Example**

Let $C = \mathsf{Top} \times \mathsf{Top}$ again, and consider $\{(\Delta^0, \Delta^n), (\Delta^1, \Delta^{n-1}), \ldots, (\Delta^n, \Delta^0)\}$. This gives the functor

$$F(X, Y) = \mathbb{Z}\Big\{\coprod_{p+q=n} \mathrm{Hom}_{\mathsf{Top}\times\mathsf{Top}}((\Delta^p, \Delta^q), (X, Y))\Big\}$$

$$
\begin{aligned}
&= \mathbb{Z}\left\{ \coprod_{p+q=n} \mathrm{Hom}_{\mathsf{Top}}(\Delta^p, X) \times \mathrm{Hom}_{\mathsf{Top}}(\Delta^q, Y) \right\} \\
&\cong \bigoplus_{p+q=n} \mathbb{Z}\left\{ \mathrm{Hom}_{\mathsf{Top}}(\Delta^p, X) \times \mathrm{Hom}_{\mathsf{Top}}(\Delta^q, Y) \right\} \\
&\cong \bigoplus_{p+q=n} \mathbb{Z}\left\{ S_p(X) \otimes_{\mathbb{Z}} S_q(Y) \right\} \\
&\cong (S_*(X) \otimes_{\mathbb{Z}} S_*(Y))_n.
\end{aligned}
$$

201 **Definition**

Let $\mathcal{C}$ be a category and $\mathbb{M}$ be a set of objects in $\mathcal{C}$. Suppose $F\colon \mathcal{C} \to \mathsf{Ab}$ be any functor. An **$\mathbb{M}$-free resolution** of F is a functor $F_*\colon \mathcal{C} \to \mathsf{chAb}$ with a natural transformation $\varepsilon_F\colon H_0(F_*) \to F$ satisfying:

1. Each F_n is $\mathbb{M}$-free. In particular F_* composed with taking the nth group is an $\mathbb{M}$-free functor $F_n\colon \mathcal{C} \to \mathsf{Ab}$.
2. When evaluated on an object $m \in \mathbb{M}$, $F_*(m)$ is a free resolution of $F(m)$ in the sense that

$$
H_i(F_*(m)) = \begin{cases} 0 & i \neq 0 \\ F(m) & i = 0 \end{cases}
$$

   and $\varepsilon_F\colon H_0(F_*(m)) \to F(m)$ is an isomorphism.

202 **Theorem**

Let $\Theta\colon F \to G$ be a natural transformation of functors $F, G\colon \mathcal{C} \to \mathsf{Ab}$. If F_* and G_* are $\mathbb{M}$-free resolutions of F and G, there is a natural transformation $\Theta_*\colon F_* \to G_*$ lifting Θ that is unique up to natural chain homotopy.

203 **Corollary**

The Eilenberg-Zilber theorem is true and the Alexander-Whitney map A is unique up to natural chain homotopy.

Proof. Let $\mathcal{C} = \mathsf{Top} \times \mathsf{Top}$ and

$$
\begin{aligned}
F(X, Y) &= H_0(X \times Y) \\
G(X, Y) &= H_0(S_*(X) \otimes_{\mathbb{Z}} S_*(Y)) \\
F_*(X, Y) &= S_*(X \times Y) \\
G_*(X, Y) &= S_*(X) \otimes_{\mathbb{Z}} S_*(Y).
\end{aligned}
$$

Note that when X and Y are both simplices, both F_* and G_* record chain complexes with homology only in degree zero. On the model objects, f_* and G_* are free resolutions of F and G. In general, F_* and G_* are $\mathbb{M}$-free resolutions.

It follows that if we have a map $H_0(X \times Y) \to H_0(S_*(X) \otimes_{\mathbb{Z}} S_*(Y))$, there is a unique up to natural chain homotopy lift to the Alexander-Whitney map

$$
A\colon S_*(X \times Y) \to S_*(X) \otimes_{\mathbb{Z}} S_*(Y).
$$

This means that the Alexander-Whitney map exists, and moreover that it is unique. In particular, if we want to produce an Alexander-Whitney map, it suffices to produce the map between the map between their zeroth homology and then lift the map. However, this is not too hard because zeroth homologies are just connected components.

To prove uniqueness, consider a natural isomorphism

$$
H_0(X \times Y) \to H_0(S_*(X) \otimes_{\mathbb{Z}} S_*(Y))
$$

and a natural inverse

$$H_0(S_*(X) \otimes_{\mathbb{Z}} S_*(Y)) \longrightarrow H_0(X \times Y).$$

Composing these maps gives the identity on either $H_0(X \times Y)$ or $H_0(S_*(X) \otimes_{\mathbb{Z}} S_*(Y))$, and it follows by the fundamental theorem of homological algebra, $S_*(X \times Y)$ and $S_*(X) \otimes_{\mathbb{Z}} S_*(Y)$ are chain homotopy equivalent. □

25.1 The diagonal map

Suppose $X \in \mathrm{Top}$. The diagonal map is the map $\Delta\colon X \longrightarrow X \times X$ mapping $x \mapsto (x, x)$.

If X is a space which has torsion free homology groups, i.e. the homology groups are free abelian groups. Then from Künneth,

$$H_n(X \times X) = \bigoplus_{p+q=n} H_p(X) \otimes_{\mathbb{Z}} H_q(X),$$

where all of the Tor groups are 0 because the homologies are free. Let $H_*(X) \coloneqq \bigoplus_n H_n(X)$.

204 **Example**

$H_*(T^2) = H_0(T^2) \oplus H_1(T^2) \oplus H_2(T^2) = \mathbb{Z} \oplus (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z}$.

Then we have

$$\begin{aligned} H_*(X) \otimes_{\mathbb{Z}} H_*(X) &\cong \left(\bigoplus_p H_p(X)\right) \otimes_{\mathbb{Z}} \left(\bigoplus_q H_q(X)\right) \\ &\cong \bigoplus_{p,q} H_p(X) \otimes_{\mathbb{Z}} H_q(X) \\ &\cong \bigoplus_n \bigoplus_{p+q=n} H_p(X) \otimes_{\mathbb{Z}} H_q(X) \\ &\cong \bigoplus_n H_n(X \times X) \\ &\cong H_*(X \times X). \end{aligned}$$

Therefore, $\Delta\colon X \longrightarrow X \times X$ induces a map

$$H_*(X) \longrightarrow H_*(X \times X) \cong H_*(X) \otimes_{\mathbb{Z}} H_*(X).$$

205 **Question**

In algebra, what does it mean to have an abelian group A and a map $A \longrightarrow A \otimes_{\mathbb{Z}} A$?

It is called a comultiplication, which is the dual of multiplication.

206 **Definition**

If A is an abelian group, a **multiplication** is a map of abelian groups

$$A \otimes_{\mathbb{Z}} A \longrightarrow A.$$

Note that a map $m\colon A \otimes_{\mathbb{Z}} A \longrightarrow A$ is a rule that takes a pair $a \otimes a'$ to an element $a'' = m(a \otimes a')$. Note that we have

$$\begin{aligned} m((a_1 + a_2) \otimes a') &= m(a_1 \otimes a') + m(a_2 \otimes a') \\ m(a \otimes (a_1' + a_2')) &= m(a \otimes a_1') + m(a \otimes a_2'). \end{aligned}$$

207 **Example**

A **ring** is an abelian group A with a multiplication satisfying some properties.

The fact that Δ induces a comultiplication instead of a multiplication is not connected to more familiar mathematical structures.

26 November 02, 2020

26.1 Homology with coefficients

One thing we may be wondering is why we care about homology with coefficients if we can compute them from homology with integer coefficients. Earlier, we talked about how in applied algebraic topology with huge datasets, it is provably faster to compute homology with rational coefficients than with integer coefficients.

We can also give some pure math reasons.

208 **Example** (Group homology)
If G is a group, we can form a category BG of one object, and a simplicial set $N(BG)$. We can then ask what $H_q(N(BG); M)$ is for various $q \in \mathbb{Z}$, groups G, and coefficient groups M. While we can calculate this explicitly, even in the integers, one problem is that the number of simplices grows exponentially. Therefore, while we do have an algorithm for computing them, there is in general no closed form.
In particular, if $G = \mathrm{GL}_n(\mathbb{F}_q)$, then this is known for $M = \mathbb{Q}$, but for $M = \mathbb{Z}$ it is an open problem.

209 **Example** (Configuration spaces)
Let X be a topological space and $k \in \mathbb{Z}$. Then $\mathrm{Config}_k(X)$ is the subspace of $X^{\times k}$ given by

$$\{(x_1, \ldots, x_k) \in X^{\times k} \mid x_i \neq x_j \text{ when } i \neq j\}.$$

Then the unordered configuration space is $B_k(X) = \mathrm{Config}_k(X)/S_k$, where S_k is the symmetric group on the points.

- $H_q(B_k(\mathbb{R}^n); \mathbb{Z})$ are known, but
- $H_q(B_k(R^2); \mathbb{Z})$ are unknown.

In particular, in the 1980s, people proved explicit formulas for $H_q(B_k(T^2); \mathbb{F}_2)$, and in the 2010s, people proved formulas for $H_q(B_k(T^2), \mathbb{Q})$.

26.2 Cohomology

Recall that if we have a topological space$X$ and an abelian group M, then we can form $\underline{\mathrm{Hom}}_{\mathbb{Z}\text{-mod}}(S_n(X), M)$, which gives a chain $\underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(S_*(X), M)$. This is an object of $\mathrm{D}(\mathbb{Z})$, so it has a well defined homology groups. In particular, $H^q(X; M) := H_{-q}(\underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(S_*(X), M))$ is the ***q*th cohomology group** of X with coefficients in M.

210 **Definition**
Let $S^q(X; M) := \underline{\mathrm{Hom}}_{\mathrm{Ab}}(S_q(X), M)$.

Since $\underline{\mathrm{Hom}}_{\mathrm{Ab}}(-, M)\colon \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$ is functorial, the map $\partial\colon S_q(X) \to S_{q-1}(X)$ induces a map $\partial\colon S^{q-1}(X; M) \to S^q(X; M)$. From this we can form the **cochain complex**

$$S^*_{\text{Hatcher}}(X; M) = \left(\cdots \longrightarrow 0 \longrightarrow S^0(X; M) \xrightarrow{\partial} S^1(X; M) \xrightarrow{\partial} S^2(X; M) \longrightarrow \cdots \right).$$

This is a chain complex concentrated in nonpositive degrees. Then we define

$$H^q(X;M) = \frac{\ker(S^q(X;M) \to S^{q+1}(X;M))}{\operatorname{im}(S^{q-1}(X;M) \to S^q(X;M))}.$$

211 **Warning**

We can also form

$$S^*_{\text{Miller}}(X;M) = \left(\cdots \longrightarrow 0 \longrightarrow S^0(X;M) \xrightarrow{-\partial} S^1(X;M) \xrightarrow{\partial} S^2(X;M) \xrightarrow{-\partial} S^3(X;M) \longrightarrow \cdots \right)$$

where in S^*_{Miller}, the map $S^q(X;M) \to S^{q+1}(X;M)$ is $(-1)^{q+1}\partial$, where ∂ is the map in S^*_{Hatcher}.

Note that $S^*_{\text{Hatcher}}(X;M) \cong S^*_{\text{Miller}}(X;M)$ in $\mathsf{D}(\mathbb{Z})$, so they have the same cohomology groups. However, they are not exactly the same chain complex. Here, we will use $S^*(X;M)$ to denote $S^*_{\text{Miller}}(X;M)$.

212 **Remark**

$S^*(X;\mathbb{F}_2)$ is the same in both conventions.

Recall the universal coefficient theorem for cohomology:

$$H^q(X;M) \cong \operatorname{Ext}^1_{\mathbb{Z}}(H_{q-1}(X);M) \oplus \underline{\operatorname{Hom}}_{\mathrm{Ab}}(H_q(X),M).$$

This shows that cohomology with coefficients in M can be computed in terms of homology with coefficients in $\mathbb{Z}$.

213 **Example**

Suppose we want to calculate $H^2(\mathbb{RP}^2;\mathbb{F}_2)$.

The first method is to use the cellular cochain complex. First, we have the chain complex

$$C^{\text{cell}}_*(\mathbb{RP}^2,\mathbb{Z}) \cong \left(\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \right)$$

This gives the cochain complex

$$\begin{aligned} C^*_{\text{cell}}(\mathbb{RP}^2,\mathbb{Z}) &\cong \left(\cdots \longrightarrow 0 \longrightarrow \underline{\operatorname{Hom}}_{\mathrm{Ab}}(\mathbb{Z},\mathbb{F}_2) \xrightarrow{2} \underline{\operatorname{Hom}}_{\mathrm{Ab}}(\mathbb{Z},\mathbb{F}_2) \xrightarrow{0} \underline{\operatorname{Hom}}_{\mathrm{Ab}}(\mathbb{Z},\mathbb{F}_2) \longrightarrow 0 \longrightarrow \cdots \right) \\ &\cong \left(\cdots \longrightarrow 0 \longrightarrow \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \longrightarrow 0 \longrightarrow \cdots \right) \end{aligned}$$

Therefore,

$$H^q(\mathbb{RP}^2;\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & q = 0,1,2 \\ 0 & \text{otherwise.} \end{cases}$$

The second method we can use is to apply the universal coefficient theorem. We have

$$\begin{aligned} H^2(\mathbb{RP}^2;\mathbb{F}_2) &\cong \operatorname{Ext}^1_{\mathbb{Z}}(H_q(\mathbb{RP}^2),\mathbb{F}_2) \oplus \underline{\operatorname{Hom}}_{\mathrm{Ab}}(H_2(\mathbb{RP}^2),F_2) \\ &\cong \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{F}_2) \oplus \underline{\operatorname{Hom}}_{\mathrm{Ab}}(0,\mathbb{F}^2) \\ &\cong \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{F}_2,\mathbb{F}_2). \end{aligned}$$

To compute $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{F}_2,\mathbb{F}_2)$, we need to compute

$$\begin{aligned} H_{-1}\,\underline{\operatorname{Hom}}_{\mathsf{D}(\mathbb{Z})}(\mathbb{F}_2,\mathbb{F}_2) &\cong H_{-1}\,\underline{\operatorname{Hom}}_{\mathsf{D}(\mathbb{Z})}(\mathbb{F}_2,\mathbb{F}_2) \\ &\cong H_{-1}\,\underline{\operatorname{Hom}}_{\mathsf{D}(\mathbb{Z})}(\underset{\deg 1}{\mathbb{Z}} \xrightarrow{2} \underset{\deg 0}{\mathbb{Z}},\mathbb{F}_2) \\ &\cong H_{-1}(\underset{\deg 0}{\mathbb{F}} \xrightarrow{2} \underset{\deg -1}{\mathbb{F}}) \\ &\cong \mathbb{F}_2. \end{aligned}$$

214 **Definition**

Let $H^*(X;M) \cong \bigoplus_q H^q(X;M)$.

215 **Question**

How does $H^*(-;M)$ interact with diagonal maps?

If $X \longrightarrow Y$ is a map in Top and M is an abelian group, then there is a natural map $H^q(Y;M) \longrightarrow H^q(X;M)$. In particular, if X is any topological space, then there is a natural map $H^q(X \times X;M) \longrightarrow H^q(X;M)$ induced from the diagonal map. Taking the direct sum over all q, we get a map

$$H^*(X \times X;M) \longrightarrow H^*(X;M).$$

216 **Remark**

Suppose R is a ring. Then if $H^*(X;R)$ is a free R-module,

$$H^*(X \times X;R) \cong H^*(X;R) \otimes_R H^*(X;R).$$

This is similar to the result we got for regular homology.

In particular, for any spaces X and Y and any ring R, we will construct a map

$$H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \times Y;R).$$

In the case where $H_*(X;R)$ and $H_*(Y;R)$ are free, this is an isomorphism.

217 **Corollary**

If X is a topological space and R is a ring, then there is a natural multiplication

$$H^*(X;R) \otimes_R H^*(X;R) \hookrightarrow H^*(X;R)$$

which is the composite

$$H^*(X;R) \otimes_R H^*(X;R) \longrightarrow H^*(X \times X;R) \xrightarrow{H^q(\Delta;R)} H^*(X;R).$$

This means that we a structure $H^*(X;R)$ that has both an addition and multiplication operation. It turns out that multiplication is associative and commutative, so we can make this into a ring!

27 November 04, 2020

Let X be a topological space and R be a commutative ring.

We will discuss the fact that

$$H^*(X;R) = \bigoplus_{q\geq 0} H^q(X;R)$$

is a **graded commutative ring**.

The fact that it is a **graded** abelian group means that

- Any class $x \in H^q(X;R) \subseteq H^*(X;R)$ is said to be **homogeneous of degree q**.
- Every class $x \in H^*(X;R)$ is a sum of finitely many homogeneous elements.

218 **Definition**

The **cohomology cross product** is given by the following: If X and Y are topological spaces and R is a ring, there is a natural sequence of maps

$$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{f_1} H^*(S^*(X;R) \otimes_R S^*(Y;R)) \xrightarrow{f_2} H^*(\underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(S_*(X) \otimes_{\mathbb{Z}} S_*(Y), R)) \xrightarrow{f_3} H^*(X \times Y; R)$$

which defines the cross product

$$(\times)\colon H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R).$$

The cross product is an isomorphism in many cases, but not always. In particular, it is an isomorphism if $H_q(X;R)$ is a finitely generated free R-module for each $q \in \mathbb{Z}$, or if $H_q(Y;R)$ is a finitely generated free R-module for each $q \in \mathbb{Z}$. There are two separate assumptions going on. The freeness condition is equivalent to f_1 being an isomorphism, which is related to the Künneth theorem, and the finite generation is equivalent to f_2 being an isomorphism.

219 **Proposition**

We also have f_3 is always an isomorphism.

Proof. We have the definition

$$H^*(X \times Y;R) = H^*(\underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(S_*(X \times Y), R)).$$

Then by Eilenberg-Zilber,

$$S_*(X \times Y) \cong S_*(X) \otimes_{\mathbb{Z}} S_*(Y),$$

giving the result. □

Note that f_2 is induced from a chain map

$$\underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(S_*(X), R) \otimes_{\mathbb{Z}} \underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(S_*(Y), R) \to \underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(S_*(X) \otimes_{\mathbb{Z}} S_*(Y), R),$$

because by definition

$$\underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(S_*(X), R) \otimes_{\mathbb{Z}} \underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(S_*(Y), R) = S^*(X;R) \otimes_R S^*(Y;R).$$

In particular,

$$f \otimes g \mapsto \begin{cases} x \otimes y \mapsto (-1)^{pq} f(x) g(y) & \deg x = \deg f = p \text{ and } \deg y = \deg g = q \\ 0 & \text{otherwise.} \end{cases}$$

If R is a PID, then f_1 is the map from the Künneth theorem

$$0 \longrightarrow H_*(C_*) \otimes_R H_*(D_*) \xrightarrow{f_1} H_*(C_* \otimes_R D_*) \longrightarrow \text{Tor terms} \longrightarrow 0$$

If R is not a PID, f_1 still exists and is still natural, but it is just not a part of an exact sequence.

27.0.1 Remark about $H^0(X; R)$

If X is a topological space, recall that $\pi_0 X$ is the set of path components of X, and $H_0(X; R)$ is the free R-module generated by $\pi_0 X$.

The cohomology $H^0(X; R)$ is the set $\mathrm{Hom}_{\mathrm{Set}}(\pi_0 X, R)$ equipped with the natural R-module structure. If $\pi_0 X$ is finite, then $\mathrm{Hom}_{\mathrm{Set}}(\pi_0 X, R) \cong \bigoplus_{\pi_0 X} R$. However, if $\pi_0 X$ is infinite, it is not a direct sum anymore, because not every object is not necessarily a finite sum.

27.0.2 The cohomology ring

Recall that if $X \to Y$ is a map in Top, then there is a map

$$S_*(X) \to S_*(Y)$$

and hence a map

$$\underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(S_*(Y), R) \to \underline{\mathrm{Hom}}_{\mathrm{D}(\mathbb{Z})}(S_*(X), R),$$

and hence a map

$$H^*(Y; R) \to H^*(X; R).$$

Applying this to the diagonal map $\Delta \colon X \to X \times X$, this gives us a map

$$H^*(X \times X; R) \to H^*(X; R).$$

If we compose this with a direct product, we get the map

$$H^*(X; R) \times H^*(X; R) \xrightarrow{\times} H^*(X \times X; R) \xrightarrow{H^*(\Delta; R)} H^*(X; R).$$

This is called the **cup product** in cohomology with coefficients in R.

220 **Theorem**

The cup product makes $H^*(X; R)$ into an graded-commutative ring. In particular:

1. There is a unit $1 \in H^0(X; R)$ such that if $x \in H^*(X; R)$, $1x = x1 = x$.
2. If $x, y, z \in H^*(X; R)$, then $(xy)z = x(yz)$ so we can talk about xyz unambiguously.
3. (Graded-commutativity) If $x \in H^p(X; R)$ and $y \in H^q(X; R)$, then $xy = (-1)^{pq} yx \in H^{p+q}(X; R)$.

Therefore, we have (modulo the sign issue), a familiar object, namely a ring. It is natural to ask what the ring properties tell us about the space geometrically.

221 **Example**

If we consider the torus T^2 and the cohomology $H^*(T^2; \mathbb{F}_2)$, we get

$$H^0 = \mathbb{F}_2, \qquad H^1 = \mathbb{F}_2 \otimes \mathbb{F}_2, \qquad H^2 = \mathbb{F}_2, \qquad H^3 = 0, \ldots.$$

Note that these are the same as the homology groups, which is true by Poincaré duality. Let a and b be two generators of $H_1 \cong H^1$. Then, we claim that $ab \in H^2(T^2; \mathbb{F}_2) = \mathbb{F}_2$ is the element 1 because a and b intersect. In particular, even if we deform the generators, they intersect an odd number of times.

In general, the cup product is saying something about how things intersect.

28 November 06, 2020

Proof (theorem 220). We want to prove that $H^*(X;R)$ is an associative, graded-commutative, unital ring. We will first make the cup product explicit. An element of $H^p(X;R)$ is represented by a class

$$f \in S^p(X;R) = \underline{\mathrm{Hom}}_{\mathrm{Ab}}(S_p(X), R),$$

where f is a **cocycle**, and the cohomology class represented by f does not change if we add a **coboundary** to f. Here, cocycles and coboundaries correspond to cycles and boundaries in regular homology.

Explicitly, an $f \in S^p(X;R)$ is a function $S_p(X) \to R$ that respects addition. This can be thought of as a function $f\colon \mathrm{Sing}_p(X) \to R$. In particular, for every $\sigma\colon \Delta^p \to X$, there is $f(\sigma) \in R$.

Now suppose that $f \in S^p(X;R)$ and $g \in S^q(X;R)$ are elements of the cohomology ring. The cup product of f and g is the element $fg \in S^{p+q}(X;R)$ such that for each $\sigma\colon \Delta^{p+q} \to X$, we have

$$(fg)(\sigma) = (-1)^{pq} f(\sigma\restriction_{\Delta^p}) g(\sigma\restriction_{\Delta^q}),$$

where the restriction to Δ^p is to the front p-face of Δ^{p+q}, and the restriction to Δ^q is to the back q-face of Δ^{p+q}.

Associativity Suppose $f \in S^p(X;R)$, $g \in S^q(X;R)$, and $h \in S^r(X;R)$ are cocycles, and $\sigma \in \mathrm{Sing}_{p+q+r}(X)$ is an arbitrary simplex. Then

$$\begin{aligned}
((fg)h)(\sigma) &= (-1)^{(p+q)r}(fg)(\sigma\restriction_{\Delta^{p+q}})h(\sigma\restriction_{\Delta^r}) \\
&= (-1)^{(p+q)r}(-1)^{pq} f(\sigma\restriction_{\Delta^p})g(\sigma\restriction_{\Delta^q})h(\sigma\restriction_{\Delta^r}) \\
&= (-1)^{pr+pq+qr} f(\sigma\restriction_{\Delta^p})g(\sigma\restriction_{\Delta^q})h(\sigma\restriction_{\Delta^r}),
\end{aligned}$$

where the restriction to Δ^p is the first p-simplex, the restriction to Δ^q is the middle q-simplex, and the restriction to Δ^r is the middle r-simplex. We can compare this to

$$\begin{aligned}
(f(gh))(\sigma) &= (-1)^{p(q+r)} f(\sigma\restriction_{\Delta^p})(gh)(\sigma\restriction_{\Delta^{q+r}}) \\
&= (-1)^{(p+q)r} f(\sigma\restriction_{\Delta^p})(-1)^{qr} g(\sigma\restriction_{\Delta^q})h(\sigma\restriction_{\Delta^r}) \\
&= (-1)^{pr+pq+qr} f(\sigma\restriction_{\Delta^p})g(\sigma\restriction_{\Delta^q})h(\sigma\restriction_{\Delta^r}),
\end{aligned}$$

which is the same expression. Therefore, the cup product is associative.

Unitality We claim that the unit $1 \in H^0(X;R)$ can be represented by the function $1\colon \mathrm{Sing}_0(X) \to R$, mapping everything to $1 \in R$. For any cocycle $f \in S^p(X;R)$, we have

$$\begin{aligned}
(f \cdot 1)(\sigma) &= (-1)^{p\cdot 0} f(\sigma\restriction_{\Delta^p})1(\sigma\restriction_{\Delta^0}) \\
&= f(\sigma)1 = f(\sigma).
\end{aligned}$$

Graded-commutativity This property is a bit more involved. Suppose $f \in S^p(X;R)$ and $g \in S^q(X;R)$ are cocycles and let $\sigma\colon \Delta^{p+q} \to X$. Then

$$\begin{aligned}
(fg)(\sigma) &= (-1)^{pq} f(\sigma\restriction_{\Delta^p})g(\sigma\restriction_{\Delta^q}) \\
(gf)(\sigma) &= (-1)^{pq} g(\sigma\restriction_{\Delta^q})f(\sigma\restriction_{\Delta^p}).
\end{aligned}$$

However in fg, the restriction to Δ^p is the front p-face and the restriction to Δ^q is the back q-face, while in gf, the restriction to Δ^p is the back p-face and the restriction to Δ^q is the front q-face. These are not obviously related. In particular, we want to prove that $(fg) - (-1)^{pq}(gf)$ is a coboundary.

The key point is that there is an interesting chain map

$$S_*(X) \to S_*(X)$$

homotopic to the identity where in degree p it maps a simplex $\sigma\colon \Delta^p \to X$ to $(-1)^{p(p=1)/2}\tilde{\sigma}$ where $\tilde{\sigma}$ is the composite

$$\begin{array}{ccccc} \Delta^p & \longrightarrow & \Delta^p & \xrightarrow{\sigma} & X \\ [v_0 : v_1 : \cdots : v_p] & \longmapsto & [v_p : v_{p-1} : \cdots : v_0] & & \end{array}$$

Note that the number of coordinate transpositions has the same parity as $p(p+1)/2$. This is related to the sign that we introduce, which is related to other signs where when we do a reflection, we multiply by -1.

Theorem 3.11 in Hatcher uses this fact to prove that $(fg) - (-1)^{pq}(gf)$ is a coboundary. □

Note that if X is a topological space and R is a ring, then $H^*(X;R)$ is not just a graded ring, but a graded R-algebra. This means that $H^*(X;R)$ is an R-module and the multiplication is R-linear, i.e. the cup product is a map

$$H^*(X;R) \otimes_R H^*(X;R) \to H^*(X;R).$$

28.0.1 Cohomology ring of a product space

Suppose A^* and B^* are two graded R-algebras. Consider $A^* \otimes_R B^*$ as the tensor product of R-modules. We claim that this has a tensor product graded R-algebra structure. For homogeneous a, b, a', b', we define

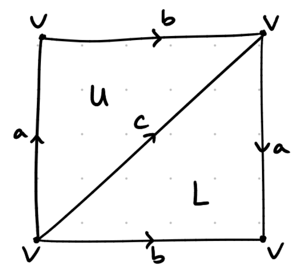
$$(a' \otimes b')(a \otimes b) = (-1)^{|b'||a|}(a'a \otimes b'b).$$

29 November 09, 2020

Today is the take-home midterm, so we will do some examples for more relaxing content.

29.1 Calculating cup products by brute force

Let's calculate $H^*(K;\mathbb{F}_2)$, where K is the Klein bottle, along with the ring structure. We have the following semisimplicial structure:



The chain complex $S_*(K)$ is

$$\cdots \longrightarrow \mathbb{Z}\{U,L\} \longrightarrow \mathbb{Z}\{a,b,c\} \longrightarrow \mathbb{Z}\{v\} \longrightarrow 0 \longrightarrow \cdots$$

To compute the cochain complex $S^*(K;\mathbb{F}_2)$, we have that $S^0(K;\mathbb{F}_2) = \underline{\mathrm{Hom}}(\mathbb{Z}\{v\},\mathbb{F}_2)$. In particular, a function in $S^0(K;\mathbb{F}_2)$ is determined by where it maps each generator in $S_0(K)$, so $S^0(K;\mathbb{F}_2) = \mathbb{F}_2\{\delta_v\}$, where $\delta_v\colon \mathbb{Z}\{v\} \to \mathbb{F}_2$ is the element corresponding to the generator in the dual vector space. Similarly, $S^1(K;\mathbb{F}_2) = \mathbb{F}_2\{\delta_a,\delta_b,\delta_c\}$, and $S^2(K;\mathbb{F}_2) = \mathbb{F}_2\{\delta_U,\delta_L\}$. This gives the entire cochain complex $S^*(K;\mathbb{F}_2)$:

$$\cdots \longrightarrow \mathbb{F}_2\{\delta_v\} \xrightarrow{\partial} \mathbb{F}_2\{\delta_a,\delta_b,\delta_c\} \xrightarrow{\partial} \mathbb{F}_2\{\delta_U,\delta_L\} \longrightarrow 0 \longrightarrow \cdots$$

Let's determine what the coboundary morphisms do. We can use the universal coefficient theorem, but we will do it manually here. To calculate $\partial(\delta_v)$, note that it is the map $\mathbb{Z}\{a,b,c\} \to \mathbb{F}_2$ obtained by composing δ_v with the boundary map ∂ in the regular chain complex. However, note that ∂ maps $a,b,c \mapsto 0$, so $\partial(\delta_v)(a) = \partial(\delta_v)(b) = \partial(\delta_v)(c) = 0$, and $\partial(\delta_v) = 0$.

Now let's calculate $\partial(\delta_a)$, $\partial(\delta_b)$, and. In the regular chain complex, we have $U \mapsto a+b-c$ and $L \mapsto c+a-b$. This gives

$$\begin{aligned}
\partial(\delta_a)(U) &= \delta_a(a+b-c) = 1 \\
\partial(\delta_a)(L) &= \delta_a(c+a-b) = 1 \\
\partial(\delta_b)(U) &= \delta_b(a+b-c) = 1 \\
\partial(\delta_b)(L) &= \delta_b(c+a-b) = -1 = 1 \\
\partial(\delta_c)(U) &= \delta_c(a+b-c) = -1 = 1 \\
\partial(\delta_c)(L) &= \delta_c(c+a-b) = 1.
\end{aligned}$$

Therefore, ∂ maps $\delta_a, \delta_b, \delta_c \mapsto \delta_U, \delta_L$. This gives the cochain complex

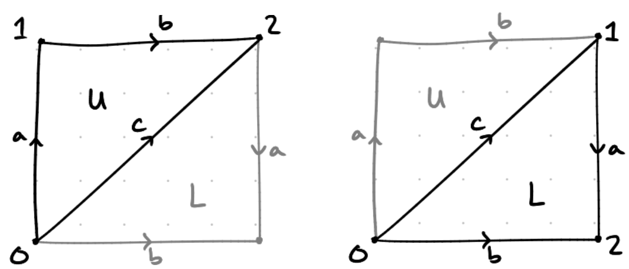
$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathbb{F}_2\{\delta_v\} & \xrightarrow{\partial=0} & \mathbb{F}_2\{\delta_a,\delta_b,\delta_c\} & \xrightarrow{\partial} & \mathbb{F}_2\{\delta_U,\delta_L\} & \longrightarrow 0 \longrightarrow \cdots \\ & & & \delta_a & \longmapsto & \delta_U+\delta_L & \\ & & & \delta_b & \longmapsto & \delta_U+\delta_L & \\ & & & \delta_c & \longmapsto & \delta_U+\delta_L & \end{array}$$

We can then calculate the cohomology:

$$\begin{aligned} H^0(K;\mathbb{F}_2) &\cong \mathbb{F}_2\{\delta_v\} \\ H^1(K;\mathbb{F}_2) &\cong \ker\partial \cong \mathbb{F}_2\{\delta_a+\delta_b, \delta_b+\delta_c\} \\ H^2(K;\mathbb{F}_2) &= \frac{\mathbb{F}_2\{\delta_U,\delta_L\}}{\delta_U+\delta_L} \cong \mathbb{F}_2\{k\}. \end{aligned}$$

Let $\alpha := \delta_a+\delta_b$, $\beta := \delta_b+\delta_c$, and $k := \delta_U$.

We can also compute the cup product structure. For instance, we can calculate $\alpha^2 = (\delta_a+\delta_b)(\delta_a+\delta_b) \in H^2(K;\mathbb{F}_2)$. To do this, we determine how it acts on U and L.



The front 1-face of U is a and the back 1-face of U is b. Therefore,

$$\big((\delta_a+\delta_b)(\delta_a+\delta_b)\big)(U) = \big((\delta_a+\delta_b)(a)\big)\big((\delta_a+\delta_b)(b)\big) = (1+0)(0+1) = 1.$$

Similarly, the front 1-face of U is a and the back 1-face of U is b, so

$$\big((\delta_a+\delta_b)(\delta_a+\delta_b)\big)(L) = \big((\delta_a+\delta_b)(c)\big)\big((\delta_a+\delta_b)(a)\big) = (0+0)(1+0) = 0.$$

Therefore, $\alpha^2 = (\delta_a+\delta_b)(\delta_a+\delta_b) = \delta_U = k$.

Similarly, we can calculate

$$\begin{aligned} \big((\delta_b+\delta_c)(\delta_b+\delta_c)\big)(U) &= \big((\delta_b+\delta_c)(a)\big)\big((\delta_b+\delta_c)(b)\big) = (0+0)(1+0) = 0 \\ \big((\delta_b+\delta_c)(\delta_b+\delta_c)\big)(L) &= \big((\delta_b+\delta_c)(c)\big)\big((\delta_b+\delta_c)(a)\big) = (0+1)(0+0) = 0 \\ \big((\delta_a+\delta_b)(\delta_b+\delta_c)\big)(U) &= \big((\delta_a+\delta_b)(a)\big)\big((\delta_b+\delta_c)(b)\big) = (1+0)(1+0) = 1 \\ \big((\delta_a+\delta_b)(\delta_b+\delta_c)\big)(L) &= \big((\delta_a+\delta_b)(c)\big)\big((\delta_b+\delta_c)(a)\big) = (0+0)(0+0) = 0, \end{aligned}$$

which means $\beta^2 = 0$ and $\alpha\beta = k$.

Therefore, if we consider

$$H^*(K;\mathbb{F}_2) \cong \underbrace{\mathbb{F}_2\{1\}}_{\text{deg }0} \oplus \underbrace{\mathbb{F}_2\{\alpha\} \oplus \mathbb{F}_2\{\beta\}}_{\text{deg }1} \oplus \underbrace{\mathbb{F}_2\{k\}}_{\text{deg }2},$$

we have $\alpha^2 = \alpha\beta = k$, $\beta^2 = 0$, and $k\alpha = k\beta = k^2 = 0$.

30 November 13, 2020

For some spaces, cohomology rings are easy to compute.

222 **Example** (S^2)

Let's consider $H^*(S^2;\mathbb{Z})$. This is

$$\begin{aligned} H^*(S^2;\mathbb{Z}) &= \bigoplus_p H^p(S^2;\mathbb{Z}) \\ &= \bigoplus_p \underline{\mathrm{Hom}}_{\mathrm{Ab}}(H_p(S^2),\mathbb{Z}) \oplus \mathrm{Ext}(\,\cdot\,). \end{aligned}$$

Since all of the homologies are free, all of the Ext terms are zero, so

$$H^0(S^2;\mathbb{Z}) \cong \mathbb{Z}$$
$$H^1(S^2;\mathbb{Z}) \cong 0$$
$$H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}.$$

Therefore, $H^*(S^2;\mathbb{Z}) = \underbrace{\mathbb{Z}\{1\}}_{\deg 0} \oplus \underbrace{\mathbb{Z}\{x\}}_{\deg 2}$ where $1\cdot x = x \cdot 1 = x$ and $x^2 = 0$ for degree reasons. Therefore, $H^*(S^2;\mathbb{Z}) \cong \mathbb{Z}[x]/x^2$. We write $|x| = 2$ to mean that x is homogeneous of degree 2.

In general, for any ring R and any integer $n \geq 1$,

$$H^*(S^n;R) \cong R[x]/x^2$$

where $|x| = n$. As a module, this is isomorphic to $R \oplus R$.

223 **Example** (Klein bottle)

Last time, we saw that

$$H^*(K;\mathbb{F}_2) \cong \underbrace{\mathbb{F}_2\{1\}}_{\deg 0} \oplus \underbrace{\mathbb{F}_2\{\alpha\} \oplus \mathbb{F}_2\{\beta\}}_{\deg 1} \oplus \underbrace{\mathbb{F}_2\{k\}}_{\deg 2},$$

where $\alpha^2 = k$, $\alpha\beta = \beta\alpha = k$, $\beta^2 = 0$, and all other nontrivial products are zero. Therefore,

$$H^*(K;\mathbb{F}_2) \cong \mathbb{F}_2[\alpha,\beta,k]/(\alpha^2 - k, \beta^2, \alpha\beta - k, \beta\alpha - k, \alpha k, \beta k, k^2).$$

224 **Question**

Is there a continuous map $S^2 \xrightarrow{f} K$ such that

$$H^2(f;\mathbb{F}_2)\colon H^2(K;\mathbb{F}_2) \to H^2(S^2;\mathbb{F}_2)?$$

Note that if $f\colon S^2 \to J$ exists, it also induces a map of rings

$$\begin{array}{ccc} H^*(K;\mathbb{F}_2) & \xrightarrow{H^*(f)} & H^*(S^2;\mathbb{F}_2) \\ \mathbb{F}_2[\alpha,\beta,k]/(\alpha\beta - k,\ldots) & \longrightarrow & \mathbb{F}_2[x]/x^2. \end{array}$$

Since this is a map of graded rings, it must send α and β to 0, because there are no degree 1 elements in $H^*(S^2;\mathbb{F}_2)$. Therefore, we must also send $\alpha^2 = k$ to zero, so there is no continuous map $S^2 \to K$.

225 **Construction**

Suppose R is a ring and A^*, B^* are two graded R-algebras. Then $A^* \otimes_R B^*$ has a canonical R-algebra structure:

- $A^* \otimes_R B^*$ is (obviously) an R-module.
- If $a \in A^*$ is homogeneous of degree p and $b \in B^*$ is homogeneous of degree q, then $|a \otimes b| = p + q$.
- The multiplication is given by

$$(a' \otimes b')(a \otimes b) = (-1)^{|b'||a|}(a'a \otimes b'b).$$

226 **Theorem**

If $X, Y \in \mathrm{Top}$, then the cross product

$$\times\colon H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$$

is a homomorphism of graded R-algebras.

This is often an isomorphic, in particular when $H_q(X;R)$ is a finitely generated free R-module for all $q \in \mathbb{Z}$.

Proof. Suppose $\alpha_1, \alpha_2 \in H^*(X;R)$ and $\beta_1, \beta_2 \in H^*(Y;R)$ are homogeneous. We need to prove

$$(\alpha_1 \times \beta_1)(\alpha_2 \times \beta_2) = (-1)^{|\alpha_2||\beta_1|}((\alpha_1\alpha_2) \times (\beta_1\beta_2)).$$

Consider

$$\begin{tikzcd} X \times Y \arrow[rr, "\Delta_{X\times Y}"] \arrow[dr, "\Delta_X \times \Delta_Y"'] & & X \times Y \times X \times Y \\ & X \times X \times Y \times Y \arrow[ur, "1\times\mathrm{swap}\times 1"'] & \end{tikzcd}$$

We have

$$\begin{aligned}
(\alpha_1 \times \beta_1)(\alpha_2 \times \beta_2) &= H^*(\Delta_{X\times Y})(\alpha_1 \times \beta_1 \times \alpha_2 \times \beta_2) \\
&= \big(H^*(\Delta_X \times \Delta_Y) \circ H^*(1 \times \mathrm{swap} \times 1)\big)(\alpha_1 \times \beta_1 \times \alpha_2 \times \beta_2) \\
&= H^*(\Delta_X \times \Delta_Y)\big((-1)^{|\beta_1||\alpha_2|}(\alpha_1 \times \alpha_2 \times \beta_1 \times \beta_2)\big) \\
&= (-1)^{|\beta_1||\alpha_2|}((\alpha_1\alpha_2) \times (\beta_1\beta_2)).
\end{aligned}$$

□

227 **Example**

Suppose we want to calculate $H^*(T^2;\mathbb{Z})$. Note that we have $T^2 = S^1 \times S^1$. First, note that

$$\times\colon H^*(S^1;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^1;\mathbb{Z}) \to H^*(S^1 \times S^1;\mathbb{Z})$$

is an isomorphism. If we let the two copies of $H^*(S^1;\mathbb{Z})$ be $\mathbb{Z}\{1\} \oplus \mathbb{Z}\{a\}$, $|a| = 1$, $a^2 = 0$ and $\mathbb{Z}\{1\} \oplus \mathbb{Z}\{b\}$, $|b| = 1$, $b^2 = 0$, then

$$H^*(S^1;Z) \otimes_{\mathbb{Z}} H^*(S^1;Z) \cong \underbrace{\mathbb{Z}\{1 \otimes 1\}}_{\deg 0} \oplus \underbrace{\mathbb{Z}\{a \otimes 1\} \oplus \mathbb{Z}\{1 \otimes b\}}_{\deg 1} \oplus \underbrace{\mathbb{Z}\{a \otimes b\}}_{\deg 2}.$$

The only interesting tensor products are the ones between degree 1 elements. We have

$$\begin{aligned}
(a \otimes 1)(1 \otimes b) &= (-1)^{0\cdot 0}(a1 \otimes 1b) = a \otimes b \\
(1 \otimes b)(a \otimes 1) &= (-1)^{1\cdot 1}(1a \otimes b1) = -(a \otimes b) \\
(a \otimes 1)(a \otimes 1) &= (-1)^{0\cdot 1}(aa \otimes 11) = (0 \otimes 1) = 0
\end{aligned}$$

$$(1 \otimes b)(1 \otimes b) = (-1)^{1 \cdot 0}(11 \otimes bb) = (1 \otimes 0) = 0.$$

If we let $x = a \otimes 1$, $y = 1 \otimes b$, and $z = a \otimes b$, we get

$$H^*(T^2; \mathbb{Z}) \cong \mathbb{Z}[x, y, z]/(xy - z, x^2, y^2, xz, yz, z^2).$$

One trick we could have used to compute x^2 is

$$x^2 = x \cdot x = (-1)^1 x \cdot x \implies 2x^2 = 0 \implies x^2 = 0.$$

31 November 16, 2020

Recall that if A^* and B^* are graded R-algebras, then $A^* \oplus B^*$ is also a graded algebra with multiplication given by $(a,b)(a',b') = (aa',bb')$, with unit $(1,1)$.

228 **Remark**

This direct sum is the product of A^* and B^* in the category of graded R-modules.

229 **Proposition**

Suppose X and Y are topological spaces. Then $H^*(X \amalg Y; R) \cong H^*(X;R) \oplus H^*(Y;R)$ as R-algebras.

Proof. The inclusions $X \hookrightarrow X \amalg Y$ and $X \hookrightarrow X \amalg Y$ induce maps of graded R-algebras:

$$H^*(X \amalg Y; R) \to H^*(X;R)$$
$$H^*(X \amalg Y; R) \to H^*(Y;R).$$

This induces a map, by the universal property of a product in a category

$$H^*(X \amalg Y; R) \to H^*(X;R) \oplus H(Y;R).$$

This is automatically a map of graded R-algebras. To prove it is an isomorphism, we need to just check that it is an isomorphism of abelian groups. $\square$

31.0.1 Cohomology of wedge products

Suppose X is a topological space with $x \in X$, and Y is a topological space with $y \in Y$. Then recall that $X \vee Y = X \amalg Y/(x \sim y)$. Note that there is a quotient map

$$r \colon X \amalg Y \to X \vee Y.$$

Using the long exact sequence of a pair, we see that $H_q(X \amalg Y) \to H_q(X \vee Y)$ is an isomorphism for $q > 0$. By the universal coefficient theorem,

$$H^q(X \vee Y; R) \to H^q(X \amalg Y; R)$$

is also an isomorphism.

230 **Corollary**

The quotient map

$$X \amalg Y \xrightarrow{r} X \vee Y$$

induces a map of graded R-algebras

$$H^*(X \vee Y) \to H^*(X \amalg Y)$$

which is an isomorphism in positive degrees.

In degree 0, it is an injection.

231 **Example**

Consider $S^2 \vee S^1 \vee S^1$. The cohomology ring $H^*(S^2 \vee S^1 \vee S^1; \mathbb{Z})$ injects into

$$H^*(S^2;\mathbb{Z}) \oplus H^*(S^1;\mathbb{Z}) \oplus H^*(S^1;\mathbb{Z})$$

that is an isomorphism in positive degrees, and an injection in degree 0. In particular, the cohomology ring injects into

$$\underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}_{\deg 0} \oplus \underbrace{\mathbb{Z} \oplus \mathbb{Z}}_{\deg 1} \oplus \underbrace{\mathbb{Z} \oplus}_{\deg 2},$$

where the degree 0 component is generated by $(1_{S^2}, 0, 0)$, $(0, 1_{S^1}, 0)$, and $(0, 0, 1_{S^1})$; the degree 1 component is generated by $(0, x, 0)$ and $(0, 0, x)$; and the degree 2 component is generated by $(y, 0, 0)$, where $x \in H^1(S^1;\mathbb{Z})$ and $y \in H^2(S^2;\mathbb{Z})$ are generators.

Then $H^*(S^2 \vee S^1 \vee S^1;\mathbb{Z})$ is the subring generated by

$$\underbrace{(1_{S^2}, 1_{S^2}, 1_{S^2})}_{\deg 0}, \underbrace{(0, x, 0), (0, 0, x)}_{\deg 1}, \underbrace{(y, 0, 0)}_{\deg 2},$$

i.e.

$$H^*(S^2 \vee S^1 \vee S^1;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

All products of positive degree classes are trivial.

232 **Question**

Is $S^2 \vee S^1 \vee S^1$ homotopy equivalent to the torus T^2? Note that $H_q(S^2 \vee S^1 \vee S^1;\mathbb{Z}) \cong H_q(T;\mathbb{Z})$ for all q.

However, the cohomology ring of $S^2 \vee S^1 \vee S^1$ has trivial positive degree products, but the cohomology ring of T^2 is $\mathbb{Z}[x,y,z]/(x^2, y^2, xy - z, z^2, xz, yz)$. In particular, $xy = z$, even though x and y are both in positive degrees.

Therefore, these spaces are not homotopy equivalent, or in other words

$$S^2 \vee S^1 \vee S^1 \ncong T^2$$

in Ho(Top). Another way to think about this is that the diagonal maps

$$(S^2 \vee S^1 \vee S^1) \to (S^2 \vee S^1 \vee S^1) \times (S^2 \vee S^1 \vee S^1)$$
$$T^2 \to T^2 \times T^2$$

induce different maps in homology.

31.1 Poincaré duality

Suppose A is an abelian group and R is a ring. Then there is a map

$$\underline{\mathrm{Hom}}_{\mathsf{Ab}}(A, R) \otimes_{\mathbb{Z}} A \xrightarrow{f} R$$

adjoint, under the currying isomorphism, to the identity

$$\underline{\mathrm{Hom}}_{\mathsf{Ab}}(A, R) \xrightarrow{g} \underline{\mathrm{Hom}}_{\mathsf{Ab}}(A, R).$$

In particular, this is the evaluation map $\varphi \otimes m \mapsto \varphi(m)$.

Similarly, if M is an R-module, there is a map

$$\underline{\mathrm{Hom}}_{R\text{-mod}}(M, R) \otimes_R M \xrightarrow{f} R$$

adjoint to the identity

$$\underline{\mathrm{Hom}}_{R\text{-mod}}(M, R) \to \underline{\mathrm{Hom}}_{R\text{-mod}}(M, R).$$

31.1.1 Kronecker pairing

Suppose X is a topological space and R is a ring. Then there is a pairing map

$$\langle -, - \rangle \colon H^q(X;R) \otimes_R H_q(X;R) \to R.$$

To see this, note that a class in $H^q(X;R)$ is represented by a cocycle in $S^q(X;R)$ modulo coboundaries, i.e. functions $\mathrm{Sing}_q(X) \to R$. A class in $H^q(X;R)$ is represented by a formal R-linear combination of classes in $\mathrm{Sing}_q(X)$, which are cycles modulo boundaries.

We can take any function $\mathrm{Sing}_q(X) \to R$ and evaluate it on any formal R-linear combination of classes in $\mathrm{Sing}_q(X)$. In other words, we have a map

$$S^q(X;R) \otimes_R S_q(X;R) \to R,$$

which extends to a chain map

$$S^*(X;R) \otimes_R S_*(X;R) \to R,$$

or a map

$$H^*(X;R) \otimes_R H_*(X;R) \to R.$$

32 November 18, 2020

233 **Proposition** (Problem 1 of PSET)

If X is a finite type CW complex, then the map, adjoint to the Kronecker pairing

$$H^q(X;\mathbb{F}_2) \longrightarrow \underline{\mathrm{Hom}}_{\mathbb{F}_2\text{-mod}}(H_q(X;\mathbb{F}_2),\mathbb{F}_2)$$

is an isomorphism.

This is an example of a perfect pairing.

234 **Definition**

A **perfect pairing** of two finitely generated free R-modules V and W is an R-linear map

$$V \otimes_R W \longrightarrow R$$

such that the adjoint map

$$V \longrightarrow \underline{\mathrm{Hom}}_R(W,R)$$

is an isomorphism of R modules.

235 **Definition**

An n-dimensional **manifold** M is a Hausdorff topological space such that every point has an open neighborhood homeomorphic to $\mathbb{R}^n$.

236 **Example**

A 2-dimensional manifold is called a **surface**. Examples include S^2, T^2, the Klein bottle K, $\mathbb{RP}^2$, and $\mathbb{R}^2$.

Non-examples include $S^2 \vee S^2$, because at the wedge point, there is no open neighborhood homeomorphic to $\mathbb{R}^2$.

Examples of 3D manifolds include $\mathbb{R}^3$, S^3, $S^1 \times S^1 \times S^1$, $\mathbb{RP}^3$, etc.

Other manifolds include

- smooth algebraic varieties over $\mathbb{R}$ or $\mathbb{C}$,
- configuration spaces in physics.

237 **Fact**

Any compact manifold is homotopy equivalent to a finite type CW complex.

238 **Theorem** (Poincaré duality)

Let M be a compact n-dimensional manifold. Then there exists a unique class $[M] \in H_n(M;\mathbb{F}_2)$ called the **fundamental class** of the manifold, such that for all $p,q \in \mathbb{Z}$ such that $p+q=n$, the map

$$H^p(M;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^q(M;\mathbb{F}_2) \xrightarrow{\cup} H^n(M;\mathbb{F}_2) \xrightarrow{\langle -,[M]\rangle} \mathbb{F}_2$$

is a perfect pairing.

In other words, $H^p(M;\mathbb{F}_2)$ is canonically isomorphic to the F_2-linear dual of $H^q(M;\mathbb{F}_2)$.

239 **Example**

Suppose M is a compact 3D manifold with $H^0(M;\mathbb{F}_2) = \mathbb{F}_2 \oplus \mathbb{F}_2$ and $H^1(M;\mathbb{F}_2) = \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$. Then we can determine *all* of the homology and cohomology groups of M with $\mathbb{F}_2$ coefficients. In particular,

$$\begin{aligned} H^2(M;\mathbb{F}_2) &\cong \underline{\mathrm{Hom}}(H^1(M;\mathbb{F}_2),\mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 \\ H^3(M;\mathbb{F}_2) &\cong \underline{\mathrm{Hom}}(H^0(M;\mathbb{F}_2),\mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2, \end{aligned}$$

and all other cohomology groups vanish. All other cohomology groups vanish, because e.g.

$$H^4(M;\mathbb{F}_2) \cong H^{-1}(M;\mathbb{F}_2)^\vee \cong 0^\vee \cong 0.$$

From proposition 233, we have

$$\begin{aligned} H_0(M;\mathbb{F}_2) &\cong \mathbb{F}_2 \oplus \mathbb{F}_2 \\ H_1(M;\mathbb{F}_2) &\cong \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 \\ H_2(M;\mathbb{F}_2) &\cong \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 \\ H_3(M;\mathbb{F}_2) &\cong \mathbb{F}_2 \oplus \mathbb{F}_2, \end{aligned}$$

and $H_q(M;\mathbb{F}_2) \cong 0$ for $q > 3$.

240 **Example**

Recall

$$\begin{aligned} H^*(T^2;\mathbb{F}_2) &= \mathbb{F}_2[x,y,z]/(xy - z, x^2, y^2, xz, yz, z^2) \\ &\cong \underbrace{\mathbb{F}_2\{1\}}_{\deg 0} \oplus \underbrace{\mathbb{F}_2\{x,y\}}_{\deg 1} \oplus \underbrace{\mathbb{F}_2\{z\}}_{\deg 2}. \end{aligned}$$

The fundamental class of the torus is an element of $H_2(T^2;\mathbb{F}_2) \cong \mathbb{F}_2\{\delta_z\}$. The fundamental class is δ_z.
By Poincaré duality, there is a perfect pairing

$$P\colon H^1(T^2;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^1(T^2;\mathbb{F}_2) \longrightarrow \mathbb{F}_2$$

where

$$\begin{aligned} P(x \otimes y) &= \langle xy, \delta_z\rangle = \langle z, \delta_z\rangle = \delta_z(z) = 1 \\ P(x \otimes x) &= \langle xx, \delta_z\rangle = \langle 0, \delta_z\rangle = \delta_z(0) = 0, \end{aligned}$$

etc.
There is another pairing

$$P\colon H^0(T^2;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^2(T^2;\mathbb{F}_2) \longrightarrow \mathbb{F}_2$$

where

$$P(1 \otimes z) = \langle 1z, \delta_z\rangle = \langle z, \delta_z\rangle = \delta_z(z) = 1.$$

241 **Example**

What is $H^*(\mathbb{RP}^2;\mathbb{F}_2)$? Note that $\mathbb{RP}^2$ is a 2D compact manifold. We can compute

$$H^0(\mathbb{RP}^2;\mathbb{F}_2) \cong \mathbb{F}_2$$
$$H^1(\mathbb{RP}^2;\mathbb{F}_2) \cong \mathbb{F}_2$$

because $\mathbb{RP}^2$ has one path component. For H^1, we have to manually calculate it. However, for H^2, we can use Poincaré duality, which gives

$$H^2(\mathbb{RP}^2;\mathbb{F}_2) \cong H^0(\mathbb{RP}^2;\mathbb{F}_2) \cong \mathbb{F}_2.$$

Therefore, we can write

$$H^*(\mathbb{RP}^2;\mathbb{F}_2) \cong \underbrace{\mathbb{F}_2\{1\}}_{\deg 0} \oplus \underbrace{\mathbb{F}_2\{a\}}_{\deg 1} \oplus \underbrace{\mathbb{F}_2\{b\}}_{\deg 2}.$$

We know that $b^2 = ab = 0$ for degree reasons, so the only question is if $a^2 = b$ or $a^2 = 0$.

There is a perfect pairing

$$P\colon H^1(\mathbb{RP}^2;\mathbb{F}_2) \otimes_{\mathbb{F}_2} H^1(\mathbb{RP}^2;\mathbb{F}_2) \to \mathbb{F}_2.$$

This map must be nontrivial in order for its adjoint to be an isomorphism. Therefore, $P(a \otimes a) = \langle a^2, [\mathbb{RP}^2]\rangle \neq 0$, so $a^2 \neq 0$, and furthermore $a^2 = b$.

Therefore,

$$H^*(\mathbb{RP}^2;\mathbb{F}_2) \cong \mathbb{F}_2[a,b]/(a^2 - b, ab, b^2),$$

where $|a| = 1$ and $|b| = 2$.

242 **Remark**

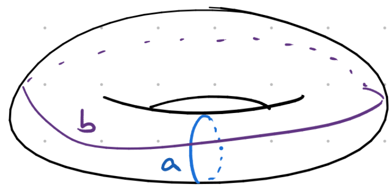
Note that $H^*(S^2 \vee S^1;\mathbb{F}_2) \cong \mathbb{F}_2[a,b]/(a^2, ab, b^2)$ where $|a| = 1$ and $|b| = 2$. So the cohomology groups are the same as $\mathbb{RP}^2$, but the ring structure is different.

In particular, $S^2 \vee S^1$ is not a manifold, so we cannot apply Poincaré duality.

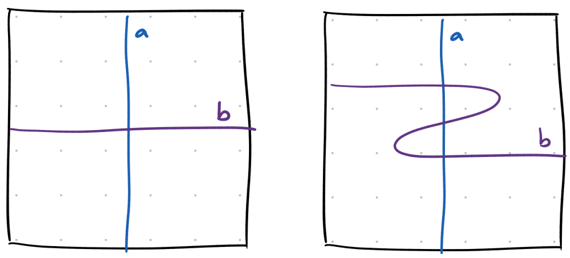
33 November 20, 2020

33.0.1 Geometric intuition of Poincaré duality

Consider the torus T^2 with the two standard loops:

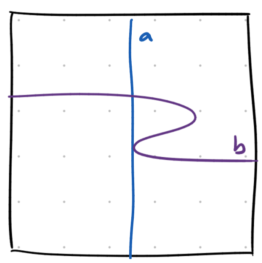


The loops a and b are geometric pictures of cycles representing generators of $H_q(T;\mathbb{F}_2) \cong \mathbb{F}_2\{a, b\}$.



How many times do a and b intersect modulo 2? No matter how we deform a and b, they will intersect an odd number of times.

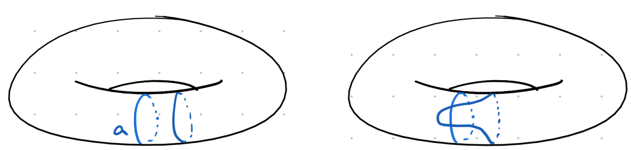
In very special configurations, they intersect an even number of times.



However, this situation is unstable, i.e. deforming a or b slightly will return to an odd number of intersections.

In a class on smooth manifolds, one could prove that the first two pictures involve **transverse** interactions, while the last one involves not transverse interactions. In order to do this, we would need to go into smooth geometry, where we can talk about tangent lines. However, we can use the cup product to talk about these intersections. In particular, this is a purely topological structure that does not care about geometry.

We can ask stranger things such as: How many times does a intersect a? We should think about this by thinking about a as a generic representative for its homology.



We can see that generically, a will intersect a an even number of times.

We can define an $\mathbb{F}_2$-module homomorphism

$$f_a\colon H_1(R;\mathbb{F}_2) \cong \mathbb{F}_2\{a,b\} \longrightarrow \mathbb{F}_2$$

which sends $a \mapsto 0$ and $b \mapsto 1$, i.e. it counts the number of intersections with a modulo 2.

Using the isomorphism

$$H^1(T;\mathbb{F}_2) \cong \underline{\mathrm{Hom}}_{\mathbb{F}_2\text{-mod}}(H_1(T,\mathbb{F}_2),\mathbb{F}_2),$$

we get that f_a represents a class in $H^1(T;\mathbb{F}_2)$, which is the **Poincaré dual** of a.

In general, suppose M is a compact n-dimensional manifold where p and q are integers such that $p+q=n$. If we fix a cycle $a \in H_p(M;\mathbb{F}_2)$, this represents a p-dimensional submanifold of M. For any $b \in H_q(M;\mathbb{F}_2)$, a and b will intersect at some number of points, since the dimensions are complementary. It turns out that the number of points modulo 2 is independent of the generic geometric representatives we chose for a and b.

Overall, if we fix $a \in H_p(M;\mathbb{F}_2)$ and vary $b \in H_q(M;\mathbb{F}_2)$, we get a function

$$f_a\colon H_q(M;\mathbb{F}_2) \longrightarrow \mathbb{F}_2,$$

which can be viewed as a class in $H^q(M;\mathbb{F}_2)$. The Poincaré duality theorem says that this defines an isomorphism

$$\begin{aligned} H_p(M;\mathbb{F}_2) &\cong H^q(M;\mathbb{F}_2) \\ a &\mapsto f_a. \end{aligned}$$

In particular, this states a cycle, modulo boundaries, is determined by the number of times it intersects all other cycles in complementary dimension.

Last time, we had the perfect pairing

$$\begin{aligned} P\colon H^q(X;\mathbb{F}_2) \otimes H^p(X;\mathbb{F}_2) &\longrightarrow \mathbb{F}_2 \\ u \otimes v &\mapsto \langle uv, [M]\rangle \end{aligned}$$

which induces an isomorphism

$$\begin{aligned} H^q(M;\mathbb{F}_2) &\cong \underline{\mathrm{Hom}}_{\mathbb{F}_2\text{-mod}}(H^p(X;\mathbb{F}_2),\mathbb{F}_2) \\ &\cong H_p(X;\mathbb{F}_2). \end{aligned}$$

243 **Claim**

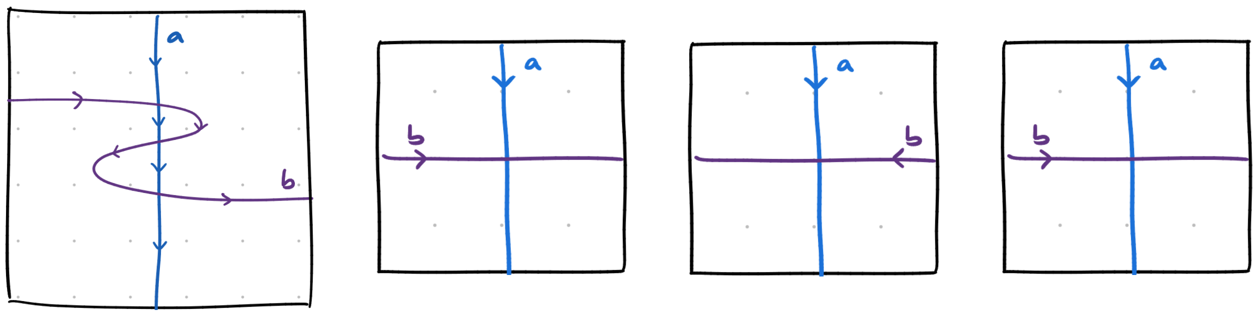
These two isomorphisms are equivalent.

244 **Remark**

The fundamental class $[M] \in H_n(M;\mathbb{F}_2)$ is Poincaré dual to $1 \in H^0(M;\mathbb{F}_2) \subseteq H^*(M;\mathbb{F}_2)$, the function that takes every path component to 1. Geometrically, this is the class that covers the whole manifold if we think about intersections.

There is also a version of Poincaré duality with integer coefficients, but it's more complicated and only works for oriented manifold. The idea is as follows:

Consider the torus T^2, which will be oriented, and the loops a and b from before, where they intersect three times and we give them directions. In particular, consider the diagram with three intersections, and zoom into the three intersections.



The idea is that the last two diagrams differ by a reflection, and when we count intersections, we do a signed count, where a reflection introduces a minus sign. Therefore, the last two diagrams cancel each other out, and there is only one signed intersection.

However, this only works for oriented manifolds because for nonoriented manifolds, going around in a loop might cause the diagrams to be inconsistent, i.e. they will become reflected.

33.1 More applications of Poincaré duality

Last time, we computed $H^*(\mathbb{RP}^2;\mathbb{F}_2) \cong \mathbb{F}_2[a]/a^3$. But more generally, we can compute $H^*(\mathbb{RP}^n;\mathbb{F}_2)$.

245 **Example** ($H^*(\mathbb{RP}^3;\mathbb{F}_2)$)

As a graded vector space, we have

$$H^*(\mathbb{RP}^3;\mathbb{F}_2) \cong \underbrace{\mathbb{F}_2\{1\}}_{\deg 0} \oplus \underbrace{\mathbb{F}_2\{a\}}_{\deg 1} \oplus \underbrace{\mathbb{F}_2\{b\}}_{\deg 2} \oplus \underbrace{\mathbb{F}_2\{c\}}_{\deg 3}.$$

In particular, we have the perfect pairing

$$P\colon H^1(\mathbb{RP}^2) \otimes_{\mathbb{F}_2} H^2(\mathbb{RP}^2) \cong \mathbb{F}_2\{a \otimes b\} \to \mathbb{F}_2$$

where $P(a \otimes b) = \langle ab, [\mathbb{RP}^3]\rangle$. In particular, this is a nontrivial map $\mathbb{F}_2 \to \mathbb{F}_2$, so $ab \neq 0$, and furthermore $ab = c$.

We can completely determine $H^*(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}_2[a]/a^{n+1}$ where $|a| = 1$.

246 **Theorem** (Borsuk-Ulam)

Let $f\colon S^n \to \mathbb{R}^n$ be a continuous function. There exists $x \in S^n$ such that $f(x) = f(-x)$.

247 **Example**

At any moment in time, some antipodal pair of points on Earth share the same temperature and air pressure.

Proof. Suppose no such x exists. Consider

$$\begin{aligned} g\colon S^n &\to S^{n-1} \\ x &\mapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}. \end{aligned}$$

This is well defined because the denominator is never zero. Note that $g(-x) = -g(x)$. Hence, there is a continuous map $\bar{g}\colon \mathbb{RP}^n \to \mathbb{RP}^{n-1}$ induced by g.

We claim that the map

$$H_1(\bar{g};\mathbb{F}_2)\colon H_1(\mathbb{RP}^n;\mathbb{F}_2) \cong \mathbb{F}_2 \to H_1(\mathbb{RP}^{n-1};\mathbb{F}_2) \cong \mathbb{F}_2$$

is nontrivial. Consider the path in S^n going from the north pole to the south pole. Upon projection to $\mathbb{RP}^n$, this becomes a cycle that generates $H_1(\mathbb{RP}^n;\mathbb{F}_2)$. Mapping under $\bar{g}$ gives a nontrivial element of $H_1(\mathbb{RP}^{n-1};\mathbb{F}_2)$.

We now proceed with the remainder of the proof. By Poincaré duality, we have that $H^1(\bar{g}; \mathbb{F}_2)$ is nontrivial. In particular, the following map must satisfy

$$\begin{aligned} H^*(\bar{g}; \mathbb{F}_2) \colon & H^*(\mathbb{RP}^{n-1}; \mathbb{F}_2) \to H^*(\mathbb{RP}^n; \mathbb{F}_2) \\ \colon & \mathbb{F}_2[a]/a^n \to \mathbb{F}_2[a]a^{n+!} \\ & a \mapsto a, \end{aligned}$$

but there is no such ring map, which is a contradiction. □

34 December 02, 2020

34.1 Proof of Poincaré duality

34.1.1 Constructions

So far, we have the constructions relating homology and cohomology:

Cross products

$$\times\colon H^p(X;R) \otimes_R H^q(Y;R) \to H^{p+q}(X \times Y;R)$$

Cup products

$$\times\colon H^p(X;R) \otimes_R H^q(X;R) \to H^{p+q}(X \times X;R),$$

which is the cross product combined with the diagonal map.

Kronecker pairing

$$H^p(X;R) \otimes_R H_p(X;R) \to R,$$

which is taking some function on homology (i.e. and element of cohomology) and evaluating it.

248 **Definition**
The **cap product** is the operation

$$\cap\colon H^p(X;R) \otimes_R H_n(X;R) \to H_{n-p}(X;R).$$

is defined by applying homology to the chain map

$$S^p(X;R) \otimes_R S_n(X;R) \xrightarrow{\text{AW}} S^p(X;R) \otimes_R S_p(X;R) \otimes_R S_{n-p}(X;R) \xrightarrow{\langle -,- \rangle} R \otimes_R S_{n-p}(X;R) \cong S_{n-p}(X;R),$$

where AW is the Alexander-Whitney map.

This is a variation of the Kronecker pairing that doesn't require the elements to come from a homology and cohomology of the same degree.

249 **Note**
All of these constructions come from

- the cross product,
- the diagonal map on a space, and
- the Kronecker pairing.

Here are some properties of the cap product:

1. $(a \cup b) \cap x = a \cap (b \cap x)$ and $1 \cap x = x$. In other words, $H_*(X;R)$ is a module for $H^*(X;R)$.
2. $\langle a \cup b, x \rangle = \langle a, b \cap x \rangle$.

3. Let $\varepsilon\colon H_0(X;R) \to R$ be the element adjoint to $1 \in H^0(X;R)$. Then if $b \in H^p(X;R)$ and $x \in H_p(X;R)$, then
$$\varepsilon(b \cap x) = \langle b, x\rangle.$$
4. (Projection formula) Given a map $f\colon X \to Y$, $b \in H^p(Y)$, and $x \in H_n(X)$,
$$H_*(f)(H^*(f)(b) \cap x) = b \cap (H_*(f)(x)).$$

250 **Theorem** (Poincaré duality)
Let M be a compact n-dimensional manifold equipped with an R-orientation for a PID R. Then there is a unique $[M] \in H_n(M;R)$ that restricts to the local orientation in $H_n(M; M - \{x\}) \otimes_{\mathbb{Z}} R$ for each point $x \in M$, such that
$$\begin{aligned} H^p(M;R) &\to H_{n-p}(M;R) \\ a &\mapsto a \cap [M] \end{aligned}$$
is an isomorphism for all $p \in \mathbb{Z}$.

To understand this geometrically, consider the reverse map. It takes a homology class and records the function on pth homology that counts intersections with sign depending on the R-orientation around each intersection.

We will prove this with $R = \mathbb{Z}$ for the notation simplicity. The more general proof is not any harder. We will prove Poincaré duality by formulating a more general statement that works for compact subsets of arbitrary manifolds.

34.1.2 Relative cap products

Suppose $A \subseteq X$ is a subspace of X. Then there is a **relative cap product**
$$H^p(X) \otimes H_n(X, A) \to H_{n-p}(X, A)$$
making $H_*(X, A)$ a module over $H^*(X)$. This is constructed via the following diagram.
$$\begin{tikzcd} 0 \arrow[d] & & 0 \arrow[d] \\ S^p(X) \otimes S_n(A) \arrow[r] \arrow[d] & S^p(A) \otimes S_n(A) \arrow[r, "\cap"] & S_{n-p}(A) \arrow[d] \\ S^p(X) \otimes S_n(X) \arrow[rr, "\cap"] \arrow[d] & & S_{n-p}(X) \arrow[d] \\ S^p(X) \otimes S_n(X, A) \arrow[rr, dashed] \arrow[d] & & S_{n-p}(X, A) \arrow[d] \\ 0 & & 0 \end{tikzcd}$$
We have a short exact sequence of chain complexes on the left from the definition of $S_n(X, A) \cong S_n(X/A)$. We then have the map $S^p(X) \otimes S_n(A) \to S^p(A) \otimes S_n(A)$ from the inclusion $A \subseteq X$. Then, the rectangle commutes by the projection formula. Then, the dashed arrow exists by the universal property of a quotient. Taking the homology of the dashed arrow gives the relative cap product.

34.1.3 Čech cohomology

Suppose $K \subseteq X$ is a closed subset of a topological space X. By excision, we have
$$H_n(X, X - K) \cong H_n(U, U - K)$$

for any open U containing K. Note that we have a relative cap product

$$H^p(U) \otimes H_n(U, U-K) \to H_{n-k}(U, U-K).$$

Thus, $H_*(X, X-K)$ is a module over $H^*(U)$ for every open U containing K.

251 **Definition**

The pth **Čech cohomology** of K is denoted $\check{H}^p(K)$. An element $x \in \check{H}^p(K)$ is an element of $H^p(U)$ for some open U containing K. If $K \subseteq U \subseteq V$, we let $x \in H^p(V) \subseteq \check{H}^p(X)$ be equal to

$$H^*(i)(x) \in H^p(U) \subseteq H^p(X).$$

Informally, this tries to capture the cohomology rings of arbitrarily small open subsets of K.

252 **Remark**

$\check{H}^p(K)$ depends on how K sits in X. For "reasonable" $K \subseteq X$, we have $\check{H}^*(K) \cong H^*(K)$, but this might fail for things like the Cantor set.

An instance of "reasonable" is when K is a deformation retract of an open set.

Note that $\check{H}^*(K)$ is a ring. Furthermore, $H_n(X, X-K)$ is a module over $\check{H}^p(K)$.

We now have

$$\cap\colon \check{H}^p(K) \otimes H_n(X, X-K) \to H_{n-p}(X, X-K).$$

We will want to show that this is an isomorphism or perfect pairing under various conditions on $X \subseteq X$. We will prove this by breaking X and K into smaller pieces that we understand and reassembling the desired statement from Mayor-Vietoris and the five lemma.

253 **Example** (Topologist's sine curve)

Consider the graph of $\sin(2\pi/x)$ for $0 < x < 1$. Let $K \subseteq \mathbb{R}^2$ be the union of this curve, $\{0\} \times [-1, 1]$, and γ defined to be a curve connecting $(0, -1)$ and $(1, 0)$ such that it doesn't intersect the rest of the curves.

This looks like a circle, but it gets really complicated near the y-axis.

We have

$$H^*(K) \cong \begin{cases} \mathbb{Z} & \deg = 0 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\check{H}^*(K) \cong H^*(S^1) \cong \begin{cases} \mathbb{Z} & \deg = 0 \text{ and } \deg = 1 \\ 0 & \text{otherwise.} \end{cases}$$

35 December 04, 2020

Recall that if X is a topological space, we defined the Čech cohomology of a closed subset $K \subseteq X$ to be

$$\check{H}^*(K) = \coprod_{K \subseteq U} H^*(U)/\sim,$$

where U ranges over the set of open neighborhoods of K, where we consider two classes to be the same if one is an inclusion of the other.

If $L \subseteq K$ is a pair of closed subsets of X, then we can form

$$\check{H}^*(K, L) := \coprod_{\substack{L \subseteq V \\ \cap \quad \cap \\ K \subseteq U}} H^*(U, V)/\sim$$

where (U, V) ranges over all pairs $Y \subseteq U$ such that V is an open neighborhood of L and U is an open neighborhood of K.

The theorems that we expect to hold from standard cohomology indeed hold.

254 **Theorem** (LES)

Let (K, L) be a closed pair in X. Then there is a natural long exact sequence

$$\cdots \longrightarrow \check{H}^p(K, L) \longrightarrow \check{H}^p(K) \longrightarrow \check{H}^p(L) \xrightarrow{\delta} \check{H}^{p+q}(K, L) \longrightarrow \cdots$$

255 **Theorem** (Excision)

Suppose A and B are compact subsets of a Hausdorff space X. Then the inclusion

$$(B, A \cap B) \subseteq (A \cup B, A)$$

induces an isomorphism

$$\check{H}^p(A \cup B, A) \cong \check{H}^p(B, A \cap B)$$

for all p.

Before we had a cap produce for a closed subspace $K \subseteq X$

$$\cap \colon \check{H}^p(K) \otimes_{\mathbb{Z}} H_n(X, X - K) \to H_{n-p}(X, X - K).$$

We can extend this to a **fully relative cap product**.

256 **Definition**

Let $L \subseteq K$ be a pair of closed subspaces of X. Then there is a map

$$\cap \colon \check{H}^p(K, L) \otimes_{\mathbb{Z}} H_n(X, X - K) \to H_{n-p}(X, X - K).$$

Then, this fully relative cap produce commutes with Mayer-Vietoris sequence.

257 **Theorem**

Let A and B be compact subsets of a Hausdorff space X. Let $x_{A\cup B} \in H_n(X, X - A\cup B)$ be a homology class. This gives us canonical restrictions

$$x_A \in H_n(X, X-A), x_B \in H_n(X, X-B), \text{ and } x_{A\cap B} \in H_n(X, X-A\cap B).$$

Then there is a map of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & \check{H}^p(A\cup B) \arrow[r] \arrow[d, "-\cap x_{A\cup B}"] & \check{H}^p(A) \oplus H^p(B) \arrow[r] \arrow[d, "(-\cap x_A)\oplus(-\cap x_B)"] & \check{H}^p(A\cap B) \arrow[r, "\delta"] \arrow[d, "-\cap x_{A\cap B}"] & \check{H}^{p+1}(A\cup B) \arrow[r] \arrow[d, "-\cap x_{A\cup B}"] & \cdots \\ \cdots \arrow[r] & H_{n-p}(X, X-A\cup B) \arrow[r] & H_{n-p}(X, X-A) \oplus H_{n-p}(X, X-B) \arrow[r] & H_{n-p}(X, X-A\cap B) \arrow[r, "\delta"] & H_{n-p-1}(X, X-A\cup B) \arrow[r] & \cdots \end{tikzcd}$$

35.1 Poincaré duality, again

Let M be an n-dimensional manifold and let K be a compact subset. Recall that

$$H_n(M, M-K) \to \Gamma(K; o_M) = \{f\colon K \to o_M \mid K \xrightarrow{f} o_M \xrightarrow{\pi} K = \mathrm{id}\}.$$

A $\mathbb{Z}$-orientation along a closed subset K is a section of o_M over K (i.e. an element of $\Gamma(K; o_M)$) that restricts to a generator of $H_n(M, M-\{x\})$ for every $x \in K$. The corresponding class in $H_n(M, M-K)$ is called the **fundamental class along K**, denoted $[M]_K$.

If $L \subseteq K$ is an inclusion of compact subsets of M, then the map

$$H_n(M, M-K) \to H_n(M, M-L)$$

sends $[M]_K$ to a fundamental class $[M]_L$. In particular, an inclusion of subsets gives a restriction of the orientation.

Also, we have a cap product

$$\cap\colon \check{H}^p(K,L) \otimes_{\mathbb{Z}} H_n(M, M-K) \to H_{n-p}(M-L, M-K).$$

258 **Theorem** (Poincaré duality)

Let M be an n-dimensional manifold and let $L \subseteq K$ be a pair of compact subspaces. Assume we have a $\mathbb{Z}$-orientation along K with fundamental class $[M]_K$. Then the map

$$-\cap [M]_K \colon \check{H}^p(K,L) \to H_{n-p}(M-L, M-K)$$

is an isomorphism.

Sketch. Read Miller's notes for more details.

1. Prove the theorem for $M = \mathbb{R}^n$, where K and L are compact convex subsets.
2. Prove for $M = \mathbb{R}^n$ where K and L are a finite union of compact convex subsets of $\mathbb{R}^n$.
3. Prove for $M = \mathbb{R}^n$ where K and L are any compact subsets of $\mathbb{R}^n$.
4. Prove for arbitrary M where K and L are finite unions of compact Euclidean subspace of M.
5. Prove for arbitrary M where K and L are arbitrary compact subspaces. ◻

35.1.1 Applications

259 **Theorem**

Let M be an n-dimensional manifold and K be a compact subset. A $\mathbb{Z}$-orientation along K determines $[M]_K \in H_n(M, M-K)$ and capping with it gives an isomorphism

$$\check{H}^{n-p}(K) \longrightarrow H_p(M, M-K).$$

260 **Corollary** (Alexander duality)

Suppose K is a compact subset of $\mathbb{R}^n$. The composite

$$\check{H}^{n-p}(K) \xrightarrow{\cong} H_p(\mathbb{R}^n, \mathbb{R}^n - K) \xrightarrow{\partial} \tilde{H}_{p-1}(\mathbb{R}^n - K)$$

is an isomorphism.

261 **Example** (Jordan curve theorem)

Suppose $n = 2$. Let K be a closed loop in $\mathbb{R}^2$ with $\check{H}^1(K) \cong \mathbb{Z}$. (This will almost always be the case except for pathological embeddings of the circle into $\mathbb{R}^2$.)

Then $\tilde{H}_0(\mathbb{R}^2 - K) \cong \check{H}^1(K) \cong \mathbb{Z}$, so $H_0(\mathbb{R}^2 - K) \cong \mathbb{Z} \oplus \mathbb{Z}$.

In other words, $\mathbb{R}^2 - K$ has two path components.

262 **Warning**

The analog of this in $\mathbb{R}^3$ is false. In particular, there are maps $f : S^2 \to \mathbb{R}^3$ where $\mathbb{R}^3 - \operatorname{im} f$ does not give two path components. The counterexample is called the Alexander horned sphere.

36 December 07, 2020

The next two days will be about the future, i.e. what exists in algebraic topology outside of this class. In this course, we focused on four topics:

- category theory
- homological algebra (chAb, $D(\mathbb{Z})$, Tor, Ext)
- homotopy theory (study of Ho(Top))
- algebraic topology (study of Top, and in particular manifolds)

36.1 Category theory

We studied products and pushouts, which are special cases of **limits** and **colimits**. We also studied curring isomorphisms, which are special cases of **adjunctions**. Since the early 2000s, people also like to study **higher categories** or **∞-categories**, where there are

1. objects,
2. morphisms,
3. 2-morphisms, or morphisms between morphisms,

and these can keep going on.

263 **Example**

$H\mathbb{Z}$-modules are the ∞-category of chain complexes, where

- objects are chain complex of abelian groups,
- morphisms are chain maps between complexes,
- 2-morphisms are chain homotopies between chain maps.

264 **Example**

$\mathcal{S}$ is the ∞-category of homotopy theory, where

- objects are topological spaces,
- morphisms are continuous maps, and
- 2-morphisms are homotopies.

We can make an equivalent ∞-category with a combinatorial definition with simplicial sets.

265 **Example**

Cat_2 is the ∞-category of categories, where

- objects are categories,
- morphisms are functors,
- 2-morphisms are natural transformations.

To learn more, Google `Kerodon`.

36.2 Homological algebra

Homological algebra a developed much more in algebraic geometry (i.e. 18.726).

In particular, there are things called sheaf Hom, sheaf Ext, sheaf cohomology, etc.

266 **Idea**

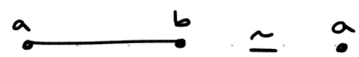
$\{x^2 + y^2 - 1 = 0\} \subseteq \mathbb{R}^2$ is an algebraic way of describing the circle. Can one come up with a purely algebraic algorithm for calculating, starting with a set of polynomials with real coefficients, the homology of the set of those solutions?

What happens if one applies these algorithms to polynomials with coefficients in a number field?

36.3 Homotopy theory

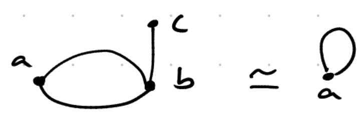
This is "the study of the equals sign", or better yet, "the study of the isomorphism sign".

Suppose there is a CW-complex with two objects and a path. This can be thought of as two object, and the path between them tells us that $a = b$. In particular, we can identify the two objects, and think of this as a single object.



On the other hand, if we have two objects with two paths between them, i.e. they are equal in two different ways. In particular, there are two different equalities between them. We can use one path to identify them, but then we have the other equality left. This is the same as a single object with a nontrivial automorphism.

Similarly, if we have an extra object c that is uniquely identified with b, then we identify b and c, and then it reduces to the previous case.



Note that these are just homotopy equivalences of graphs. In the spirit of higher category theory, we can also think about equalities between equalities.

This is equivalent to saying $D^2 \simeq *$.

In general, homotopy theory is about objects, equalities between objects, 2-equalities between equalities, etc.

36.4 Algebraic topology

One example of a question we ask in algebraic topology is: "Can we classify all n-dimensional compact manifolds up to homomorphism? What about smooth manifolds up to diffeomorphism?"

For example, surfaces (2-dimensional compact manifolds) are classified. For example, $\mathbb{Z}$-oriented surfaces are classified by their genus. In particular, it is determined by its first homology group.

However, when we increase the dimension, the manifolds get very complicated, so we can ask the question about manifolds with particularly simple homology.

267 **Theorem** (Kervaire-Milnor, 1961)

Classification of all compact simply connect n-dimensional manifolds M with

$$H^q(M) \cong \begin{cases} \mathbb{Z} & \text{if } q = 0, n \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 5$. For lower dimensions, we have known $n = 2$ since around 1900, we know $n = 3$ due to Perelman in 2003, and $n = 4$ is still unknown.

We can also ask if we can classify simply connected and compact $2n$-dimensional manifolds M where $H_0(M) \cong \mathbb{Z}$ and

$$H_1(M) \cong H_2(M) \cong \cdots \cong H_{n-1}(M) \cong 0.$$

Every simply connected space is $\mathbb{Z}$-orientable, so it follows that

$$H_{n+1}(M) \cong H_{n+2}(M) \cong \cdots \cong H_{2n}(M) \cong \mathbb{Z}.$$

$H_n(M)$ can be anything. This is classified in all dimensions $2n$ except $2n = 4, 24, 126$.

- $n \equiv 6 \bmod 8$ was proved by Wall in 1962
- $n \equiv 5 \bmod 8$ was proved by Brown and Peterson in 1966
- $n \equiv 3 \bmod 8$ was proved by Browder in 1969
- $n \equiv 2 \bmod 8$ was proved by Schultz in 1972
- $n \equiv 1 \bmod 8$ was proved by Stolz in 1985
- $n \equiv 7 \bmod 8$ and $n \neq 63$ was done by Hill-Hopkins-Ravenel in 2009. At the time it was proved, Hopkins was a professor at MIT and Hill was a graduate student at MIT. He is now at UCLA and is the head of the LGBTQ society of mathematicians.
- $n \equiv 0 \bmod 4$ and $n \neq 12$ was done by Burkland-Senger-Hahn in 2019. Burkland and Senger are grad students at MIT, and Hahn is the lecturer of this course! Adela Zhang is thinking about extending this to when two homology groups are nonzero.

36.5 More homotopy theory

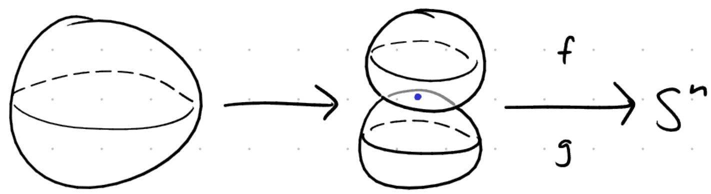
268 **Question**

If m and n are positive integers, how many maps are there from $S^m \to S^n$ up to homotopy? The set of maps $S^m \to S^n$ up to homotopy is denoted $\pi_m S^n$.

On the homework, we proved $\pi_3 S^2$ has more than one element. In 18.906, we will see that

1. If $m < n$, all maps $S^m \to S^n$ are homotopic, i.e. $\pi_m S^n$ has one element.
2. If $m = n$, then $\pi_m S^m \cong \mathbb{Z}$. Maps $S^m \to S^m$ are determined up to homotopy by their degrees.
3. If $m > n$, then unless $m = 2n - 1$, $\pi_m S^n$ is finite.

A natural question to ask is how large is this set? Moreover, note that $\pi_m S^n$ is not just a set, but a group in the following way: If we have two maps $f, g \colon S^n \to S^n$, we first ensure (by homotopy) that the south pole of f is mapped to the same point as the north pole of g. Then, $f * g$ is the map depicted in the following diagram, where we first pinch the equator, and then we map the top half by f and the bottom half by g.


f
g
S^n

37 December 09, 2020

37.1 Homotopy groups of spheres

Recall that $\pi_m S^n$ is the group of continuous maps from S^m to S^n up to homotopy.

269 **Theorem** (Freudenthal)

For $n \geq k+2$, $\pi_{n+k} S^n$ is independent of n. This group is denoted $\pi_k \mathbb{S}$.

270 **Example**

$\pi_3 S^2 \cong \mathbb{Z}$, $\pi_4 S^3 \cong \mathbb{Z}/2\mathbb{Z}$, $\pi_5 S^4 \cong \mathbb{Z}/2\mathbb{Z}$, $\ldots$, so $\pi_1 \mathbb{S} \cong \mathbb{Z}/2\mathbb{Z}$.

The first couple of stable groups are

- $\pi_1 \mathbb{S} \cong \mathbb{Z}/2\mathbb{Z}$
- $\pi_2 \mathbb{S} \cong \mathbb{Z}/2\mathbb{Z}$
- $\pi_3 \mathbb{S} \cong \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
- $\pi_4 \mathbb{S} \cong 0$
- $\pi_5 \mathbb{S} \cong 0$
- $\pi_6 \mathbb{S} \cong \mathbb{Z}/2\mathbb{Z}$
- $\pi_7 \mathbb{S} \cong \mathbb{Z}/16\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$
- $\pi_8 \mathbb{S} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

271 **Question**

How can we use homology to understand these groups?

An element of $\pi_7 \mathbb{S}^n$ is a map

$$f : S^{7+n} \to S^n$$

for $n \gg 0$. Then $H_*(f) : H_*(S^{7+n}) \to H_*(S^n)$ is going to trivial no matter what f is.

272 **Definition**

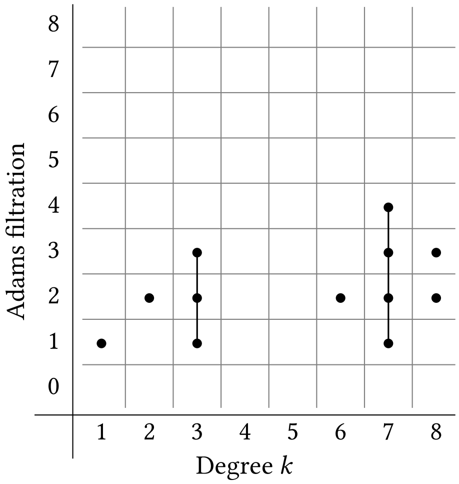
A map of spheres $S^m \to S^n$ has **$\mathbb{F}_p$-Adams filtration** at least k if it can be factored as a composite

$$S^m \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3 \xrightarrow{f_4} \cdots \xrightarrow{f_k} X_k = S^n$$

such that each $H_*(f_i; \mathbb{F}_p)$ is trivial.

Intuitively, this means that it is hard to figure out what is going on when viewed through homology, because we can factor the map but still don't understand what the individual maps are.

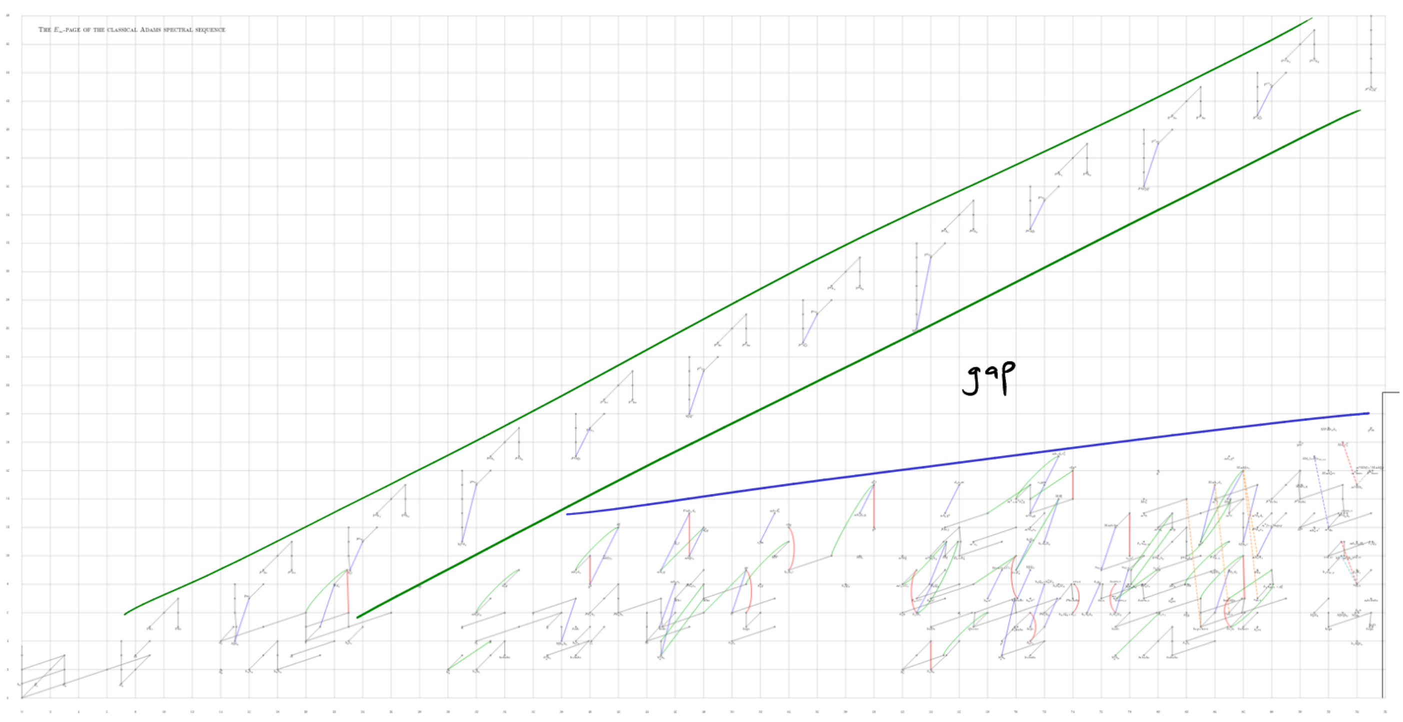
One way that we can think about it graphically is with a table like the following



- $\pi_1\mathbb{S}_{(2)} = \mathbb{Z}/2\mathbb{Z}$
- $\pi_2\mathbb{S}_{(2)} = \mathbb{Z}/2\mathbb{Z}$
- $\pi_3\mathbb{S}_{(2)} = \mathbb{Z}/8\mathbb{Z}$
- $\pi_4\mathbb{S}_{(2)} = 0$
- $\pi_5\mathbb{S}_{(2)} = 0$
- $\pi_6\mathbb{S}_{(2)} = \mathbb{Z}/2\mathbb{Z}$
- $\pi_7\mathbb{S}_{(2)} = \mathbb{Z}/16\mathbb{Z}$
- $\pi_8\mathbb{S}_{(2)} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

Here, we're localizing at the prime 2. In particular, since $\pi_3\mathbb{S} = \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, we have $\pi_3\mathbb{S}_{(2)} \cong \mathbb{Z}/8\mathbb{Z}$. To read the table, we look at each column individually. The number of dots tells us what the group is, and the connections tell us how large the powers of the primes are. The placement of the first dot tells us what the Adams filtration is. For instance, for degree 3, we can interpret the table as saying that $\frac{1}{2}$ of the elements in the homotopy groups are in Adams filtration 1, $\frac{1}{4}$ of the elements are in Adams filtration 2, and $\frac{1}{8}$ of the elements are in Adams filtration 3.

When we extend the chart, we start to become limited by computation power. Ten years ago, we were able to compute the chart up to around degree ~ 50, and now we have the chart up to degree 96. This chart is on page 8 of `Adamschart.pdf`.



When we look at the chart, we see that there is a simple collection of points with larger slope than the rest of the points, indicated with green. There is also a chaotic pattern of points below it, with a gap in between.

In particular, there are a simple group of homotopy groups that are hard to detect with homology, and then there is a messy collection of homotopy groups that are easier to detect (but still quite hard) with homology.

There is indeed a good understanding of the maps corresponding to very high Adams filtration, and this region is called the v_1-periodic homotopy groups of spheres. Then there seems to be a gap, and an open problem is whether this gap continues, and how big it is. Another natural question is whether we can understand the non-v_1-periodic stuff.

273 **Definition**

An **extraordinary (co)homology theory** E_* is a functor

$$E_*\colon \mathsf{Ho}(\mathsf{Top}) \to \text{graded Ab groups}$$

satisfying all of the Eilenberg-Steenrod axioms except the dimension axiom.

They can directly tell, sometimes, the differences between various maps $f\colon S^m \to S^n$. In particular, $E_*(f)\colon E_*(S^m) \to E_*(S^n)$ could be nontrivial, because the homologies of a sphere do not have to be concentrated in one degree.

The most important example is $E_* = KO_*$, called **topological K-theory**. This theory sees the v_1-periodic part of $\pi_*\mathbb{S}$. It has a geometric definition in terms of vector bundles, which assemble to define K-theory. It also has a more algebraic/combinatorial definition.

274 **Question**

How do we make extraordinary (co)homology theories that detect other elements in $\pi_*\mathbb{S}$?

275 **Definition** (Idea)

An **$\mathbb{E}_\infty$-ring** is a cohomology theory E^* valued in graded commutative rings.

For example, we could have $E^*(X) = H^*(X;R)$, where R is a commutative ring. We can also have $E^*(X) = KO^*(X)$.

Given an $\mathbb{E}_\infty$-ring E^*, we can extract a classical ring, which is $E^*(*)$. For example, the classical ring underlying $H^*(-;R)$ is $H^*(*;R) \cong R$ in degree 0. On the other hand, the classical ring underlying $KO^*(-)$ is $KO^*(*)$, which is 8-periodic. (Bott periodicity)

The main idea of algebraic homotopy theory is that we should develop all of commutative algebra replacing rings with $\mathbb{E}_\infty$-rings. In particular, wherever we have concepts about rings like PIDs or Nakayama's lemma, we should try to find an analogous theorem about $\mathbb{E}_\infty$-rings. This is what Waldhausen called "brave new algebra", which is today sometimes called "higher algebra", "spectral algebra", or "derived algebra". A lot of the results we get are the same, but sometimes there are difference.

276 **Example**

In classical algebra or number theory, people study elliptic curves. To imitate the theory of elliptic curves in $\mathbb{E}_\infty$-rings, we are led to the notion of **elliptic cohomology theories**.

One intersecting thing that doesn't happen in classical number theory is that there is a universal elliptic cohomology theory. In particular, there is an elliptic cohomology theory that captures the information of all other elliptic cohomology theories. This is a new phenomenon that we cannot understand with regular rings. This universal theory is called **topological modular forms** (tmf).

We can try to understand the underlying classical ring by looking at the underlying classical ring $\mathrm{tmf}^*(*) \otimes_{\mathbb{Z}} \mathbb{Q}$, and it turns out to be the classical ring of modular forms.

One can keep going into things like automorphic forms or abelian varieties and try to construct analogs in this land of brave new algebra. $\mathbb{E}_\infty$-rings are about commutative algebra, but we can also talk about associative rings, which are called $\mathbb{E}_1$- rings. When we do this, we get a lot of analogs, and oftentimes we get more objects that we can construct compared to classical ring theory, and a lot of the time they are universal. This is the topic of study in **chromatic homotopy theory**, which assembles the homotopy groups $\pi_*\mathbb{S}$ out of things detected by $\mathbb{E}_\infty$-rings. Each $\mathbb{E}_\infty$-ring has a chromatic height that roughly measures how much of the homotopy group it can see.

- Height 0 is ordinary homology,
- Height 1 is topological K-theory,
- Height 2 is TMF, and so on.

Furthermore, we have the chromatic convergence theorem, which says that each element in $\pi_* \mathbb{S}$ is detected at some chromatic height.

Another big open question in this subject is whether there is a geometric construction of TMF. For example, we can understand ordinary homology through cycles and maps of simplices, and we can understand K-theory with vector bundles. Physicists tell us that it should be related to the Dirac operator on the space of loops in X. However, nobody has been able to do this mathematically in a well defined way.