A Bound on Binomial Coefficients

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1 Introduction

I have a PowerPoint presentation for my school's math club which borrows heavily from Evan Chen's Probabilistic Method handout. The purpose of this document is to outline a more detailed solution to example 3.3 of his handout, especially the latter part which has been left to the reader.

2 A Bound on Binomial Coefficients

We begin with the following claim.

Theorem 1. *For any positive integers n and k, we have $\binom{n}{k} < \frac{1}{e}\left(\frac{en}{k}\right)^k$.*

Solution. Notice that

$$\begin{aligned} \binom{n}{k} < \frac{n^k}{k!} &\overset{?}{<} \frac{1}{e}\left(\frac{en}{k}\right)^k, \\ \implies 0 < \frac{k^k}{e^{k-1}} &\overset{?}{<} k!, \\ \implies k\ln k - k + 1 &\overset{?}{<} \ln(1) + \ln(2) + ... + \ln(k). \end{aligned}$$

The presence of $k \ln k$ on the left hand side and the sum of the natural logarithms on the right hand side gives us an incentive to integrate $\ln k$. Furthermore, since $f(k) = \ln k$ is an increasing function, we have

$$\int_1^k \ln(k)dk < \ln 2 + \ln 3 + ... + \ln k$$

(think of integrating using rectangles of width 1 on an increasing function with the starting index we have above). We can evaluate our integral with integration by parts:

$$\begin{aligned} \int_1^k \ln(k)dk &= k\ln(k) - \int_1^k kd\ln(k) \\ &= k\ln(k) - \int_1^k k\frac{d\ln(k)}{dk}dk \\ &= k\ln(k) - \int_1^k 1dk \\ &= k\ln(k) - k + 1. \end{aligned}$$

We can thus conclude that indeed $k\ln k - k + 1 < \ln 1 + \ln 2 + ... + \ln k$. All the steps with a question mark on the $<$ are reversible and we are thus done. Note that in some of the above expressions, we may have omitted $\ln 1 = 0$ in the sum for clarity. ∎

3 Problem

Problem. Let n and k be integers with $n \leq 2^{\frac{k}{2}} andk \geq 3$. Prove it is possible to color the edges (two colors) of a complete graph on n vertices with the following property: one cannot find k vertices such that the $\binom{k}{2}$ vertices among them are not monochromatic.

Solution. Let us randomly color the edges of the graph, assigning a score X to each coloring. Call a collection of k vertices bad if they are monochromatic. The score is simply the total number of bad collections in a random coloring. If we prove $E[X] < 1$, then there must exist a coloring with score 0 which will finish the problem. Since there are $\binom{n}{k}$ ways to choose k vertices, and probability $\left(\frac{1}{2}\right)^{\binom{k}{2}-1}$ of the $\binom{k}{2}$ edges of being a single color,

$$E[X] = \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}-1}.$$

To prove $E[X]$ is less than 1, we will use theorem 1.

$$\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}-1} < \frac{1}{e}\left(\frac{en}{k}\right)^k\left(\frac{1}{2}\right)^{\frac{k^2-k-2}{2}} \overset{?}{<} 1.$$

Using the condition $n \leq 2^{\frac{k}{2}}$ and rearranging, we arrive at

$$\left(\frac{\sqrt{2}e}{k}\right)^k \overset{?}{<} 2e$$

for $k \geq 3$. When $k = 3$, we can manually check that the inequality holds. We will prove that $\left(\frac{\sqrt{2}e}{k}\right)^k$ is decreasing after $k = 3$. Let $y = \left(\frac{\sqrt{2}e}{k}\right)^k$. We get

$$\begin{aligned}
\ln y &= k(\ln(\sqrt{2}e) - \ln k),\\
\frac{1}{y}y' &= (\ln(\sqrt{2}e) - \ln k) + k(-\frac{1}{k}).
\end{aligned}$$

$$\implies y' = y(\ln\sqrt{2} - \ln k).$$

This is negative when $k \geq 3$ since $y > 0$ and $\ln\sqrt{2} < \ln k$ with this condition. We can conclude that the function is decreasing with $k > 3$ and so $E[X] < 1$. We are done. ■