Fibonacci Sequence With Generating Functions

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2019

1 Introduction

> "A generating function is a clothesline on which we hang up a sequence of numbers for display" (Herbert S. Wilf).

We begin this article with an appropriate quote which cleverly depicts what a generating function is.

Defintion 1.1. *A generating function of the infinite sequence* $\{a_n\}_{n=0}^{\infty}$ *is*

$$G(x) = \sum_{n=0}^{\infty} a_n x^n.$$

In other words, the generating function of a sequence with index origin 0 is a power series with the value of index n as the coefficient of x^n. We will refer to the coefficient of x^n in $G(x)$ as $a_n = [x^n]G(x)$. As long as x does not take on a value, we do not need to worry about convergence in our context.

2 Recursive Sequences

2.0.1 Introduction

We may use partial functions and the geometric series formula to find closed forms of recursive sequences. For example, consider the sequence $a_n = \frac{a_{n-1}+1}{2}$ for $a_0 = 2$. We want to find a closed form for a_n. The generating function for our sequence is

$$G(x) = \sum_{n=0}^{\infty} a_n x^n.$$

We use the recursive definition of a_n to write an equation in $G(x)$. Notice $xG(x)$ shifts the index of the sequence by 1 as $a_{n-1} = [x^n]xG(x)$. Thus,

$$G(x) = a_0 + \frac{xG(x)}{2} + \left(\frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{2} + ...\right), \tag{1}$$

$$G(x) = 2 + \frac{xG(x)}{2} + \frac{\frac{x}{2}}{1-x}, \tag{2}$$

$$G(x) = \frac{4-3x}{(1-x)(2-x)}. \tag{3}$$

Partial fractions on (3) yields

$$G(x) = \frac{1}{1-x} + \frac{1}{1-\frac{x}{2}}.$$

Expansion via geometric series formula makes

$$G(x) = (1 + x + x^2 + ...) + (1 + \frac{x}{2} + \frac{x^2}{2^2} + ...),$$

thus

$$a_n = [x^n]G(x) = \boxed{1 + \frac{1}{2^n}}.$$

2.1 Asymptotic Formula

Generating functions is useful in attaining the asymptotic formula for a recursive sequence.

Defintion 2.1. *$A(x)$ is asymptotically equivalent to $B(x)$, or that $A(x) \sim B(x)$ if and only if*

$$\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1.$$

Consider the Fibonacci sequence, defined as $F_n = F_{n=1} + F_{n-2}$ for $F_0 = 0, F_1 = 1$. It is very difficult to figure out how fast the sequence grows with the recursive formula, thus we hope to find a closed form with generating functions. As always, let

$$G(x) = \sum_{n=0}^{\infty} F_n x^n.$$

Again, we hope to write an equation in terms of $G(x)$ with the help of our recursive definition. It follows that

$$G(x) = x + xG(x) + x^2 G(x)$$

and therefore

$$G(x) = \frac{x}{1 - x - x^2}.$$

Finding the solutions to $1 - x - x^2 = 0$, we get $x = \frac{-1 \pm \sqrt{5}}{2}$, and partial fractions gives

$$G(x) = \frac{-x}{(x + \frac{1+\sqrt{5}}{2})(x + \frac{1-\sqrt{5}}{2})}, \tag{4}$$

$$G(x) = \frac{1}{\sqrt{5}} \cdot \frac{\frac{1-\sqrt{5}}{2}}{x + \frac{1-\sqrt{5}}{2}} - \frac{1}{\sqrt{5}} \cdot \frac{\frac{1+\sqrt{5}}{2}}{x + \frac{1+\sqrt{5}}{2}}, \tag{5}$$

$$G(x) = \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - (\frac{1+\sqrt{5}}{2})x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - (\frac{1-\sqrt{5}}{2})x}, \tag{6}$$

$$G(x) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \left((\frac{1+\sqrt{5}}{2})^n x^n - (\frac{1-\sqrt{5}}{2})^n x^n \right). \tag{7}$$

Finally,

$$F_n = [x^n]G(x) = \boxed{\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n}.$$

This result is called Binet's Formula. Because

$$0 < |\frac{1-\sqrt{5}}{2}| < 1,$$

we have

$$\lim_{n \to \infty} \left(\frac{1-\sqrt{5}}{2} \right)^n = 0,$$

and therefore we achieve

$$\boxed{F_n \sim \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n}.$$

The Fibonacci sequence grows exponentially.