Binomial Theorem

Freeman Cheng

1 Factorials

Factorial. The factorial of a non-negative integer n is written as "$n!$," and is read as "n factorial." By definition,

$$n! = (n) \cdot (n-1) \cdot (n-2) \cdot ... \cdot 3 \cdot 2 \cdot 1$$

if $n > 0$. The value of $0!$ is defined as 1.

1. (a) Compute the values of 4!, 5!, and 6!.

   (b) Find the units digit of the sum
   $$\sum_{n=1}^{2019} (n!)^2.$$

2. (a) How many ways can a person arrange n distinct items on a line?

   (b) Find an argument as to why defining $0! = 1$ is a valid choice.

3. (a) Given the sequence of letters "AABCDDD," how many permutations of this sequence are there?

   (b) The same sequence of letters is to be put on a circle. How many ways is there to do this? Two placements are considered the same if one can be rotated to be the other.

   (c) Generalize.

4. A group of 8 children are divided into three groups of sizes 2, 3, and 3 respectively. How many ways are there to do this?

5. *Prove Legendre's formula stated below.

Legendre's formula. The largest prime power p which divides $n!$ can be calculated as
$$\sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor.$$
(The notation $\lfloor x \rfloor$ is read as the "floor of x." The greatest integer less than or equal to x is denoted as $\lfloor x \rfloor$).

2 Binomial Coefficient

Binomial coefficient. A binomial coefficient is two numbers A and B written in the form $\binom{A}{B}$. This is read as "A choose B." Numerically,

$$\binom{A}{B} = \frac{A!}{B! \cdot (A-B)!}.$$

Suppose you have 6 distinct balls in a bag. You want to simultaneously choose 3 balls from the bag. How many different outcomes are there?

We assign 3 imaginary slots for each ball we will take out. There are 6 choices for which ball to put in the first slot, 5 choices for the second slot, and 4 choices for the third and final slot. The number of different outcomes for choosing 3 of 6 balls is

$$6 \cdot 5 \cdot 4 = 120$$

if there is order of choice. However, in this problem, the balls are chosen simultaneously, so no such order exist. We have over-counted our answer since having ball A, B, C is the same as having ball C, A, B in the first, second, and third slots respectively. Thus, we have counted each outcome a total of $3! = 6$ times (the number of ways to permute the 3 balls). Thus, our answer is actually

$$\frac{6 \cdot 5 \cdot 4}{3!} = 20.$$

(A note that in this lesson, all binomials $\binom{A}{B}$ will refer to ones where $A \geq B \geq 0$. Negative binomials do exist, but we don't need to worry about them at the moment).

1. It turns out that the number of ways to choose a group of B items from A distinct items is $\binom{A}{B} = \frac{A!}{B! \cdot (A-B)!}$. Prove why this is the case.
2. One may notice that
$$\binom{A}{B} = \binom{A}{A-B}.$$
    (a) Prove why this is the case with the formula.
    (b) Use a counting argument to justify this.
3. How many permutations are there for this sequence of 7 letters: "AAABBBB?"
    (a) Solve this with factorials.
    (b) It turns out that the answer is exactly $\binom{7}{3}$:
        i. Explain with the binomial coefficient formula.
        ii. Use a counting argument to justify this.
    (c) Generalize for a sequence containing a A's and b B's with a and b being non-negative integers.

4. A robot is moving on the Cartesian plane, starting at point $(0, 0)$. For each move, the robot can either increase either his x or y coordinate by 1.
   (a) The robot wants to reach the point $(3, 3)$. In how many ways can he do this?
   (b) The robot wants to reach the point (a, b) where a and b are positive integers. How many ways can he do this?
   (c) Generalize this for 3 dimensions. The robot starts at point $(0, 0, 0)$ and wants to arrive at (a, b, c) where a, b, and c are positive integers. As always, the robot can either increment the x, y, or z coordinate value by 1 for each move.
5. *Prove why the following are true.

Stars and bars. There number of ways to assign positive integers to k values $a_1, a_2, a_3..., a_{k-2}, a_{k-1}, a_k$ such that the sum

$$a_1 + a_2 + a_3 + ... + a_{k-2} + a_{k-1} + a_k = n$$

is

$$\binom{n-1}{k-1}.$$

Stars and bars (modification). The number of ways to assign non-negative integers to the aforementioned k values is

$$\binom{n+k-1}{k-1}.$$

(Note that determining the number of ways to divide n children into k groups is a equivalent problem).

6. **Four vertices of a square is labelled 1, 2, 3, and 4. A robot can draw a straight line connecting any two previously unconnected vertices. How many ways can he connect enough vertices so that there is no group of vertices who are unconnected from another group. (Hint: calculate the opposite).

3 Binomial Theorem

Binomial theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

The coefficients of the term $x^{n-k}y^k$ the binomial expansion $(x+y)^n$ represents how many times $x^{n-k}y^k$ shows up if you purely expand

$$(x+y)(x+y)(x+y)...(x+y)(x+y)(x+y).$$

Since the degree of x in $x^{n-k}y^k$ is $n-k$, the coefficient of $x^{n-k}y^k$ is the number of ways to choose $n-k$ x's in the n groups of "$(x+y)$." There are

$$\binom{n}{n-k} = \binom{n}{k}$$

such ways, and thus that is the coefficient of the term $x^{n-k}y^k$ in the expansion.

The following exercises explores the relationship between Pascal's triangle and the coefficients.

1. The robot returns. This time, he starts at the top of the Pascal's triangle and walks in accordance with the following rules: the robot can only travel downwards either to the left or right. Prove that the number of routes the robot can take to arrive at a certain number on the Pascal's triangle is that number itself.

$$\begin{array}{l ccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\cline{2-14}
 & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & & 6
\end{array}$$

2. *Determine the relationship between the numbers on the pascal triangle and binomial coefficients.

3. Practice the following binomial theorem questions.

   (a) Expand $(x+2y)^4$.

   (b) Expand $(2x-y)^5$.

   (c) Find the constant term in the expansion of $(x - \frac{2}{x^3})^9$.

4. Prove Pascal's identity by using algebra, association with Pascal's triangle, and by creating a counting argument.

Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

5. Using the binomial theorem, prove why

Useful identity.

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

6. Have a second look at the Pascal's triangle. It would follow that the sum of the numbers in each row is a power of 2. Use a counting argument to justify why this is the case.
7. Let S be a set of size n. Why are there 2^n total subsets of S?
   (a) Explain with simple counting argument.
   (b) Use binomial coefficients.
8. Prove why

Another useful identity.

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = 2^{n-1}.$$

In other words, if n is even,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + ... + \binom{n}{n-4} + \binom{n}{n-2} + \binom{n}{n} = 2^{n-1}.$$

If n is odd,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + ... + \binom{n}{n-5} + \binom{n}{n-3} + \binom{n}{n-1} = 2^{n-1}.$$