An Algebraic Identity

"Just use the identity" – Idris Tarwala.

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Factor $a^3 + b^3 + c^3 - 3abc$

As the slide title suggests, we begin by trying to factor $a^3 + b^3 + c^3 - 3abc$. Suppose that a, b, c were the roots of a polynomial $P(x) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$. This means that

$$P(a) = a^3 - (a+b+c)a^2 + (ab+bc+ca)a - abc = 0,$$

$$P(b) = b^3 - (a+b+c)b^2 + (ab+bc+ca)b - abc = 0,$$

$$P(c) = c^3 - (a+b+c)c^2 + (ab+bc+ca)c - abc = 0.$$

Adding them up gives

$$a^3+b^3+c^3-(a+b+c)(a^2+b^2+c^2)+(a+b+c)(ab+bc+ca)-3abc = 0,$$

and so

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

A result of the identity

Essentially, this means that if $a + b + c = 0$, then $a^3 + b^3 + c^3 - 3abc = 0$. Also notice how we can rewrite the identity as

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2)$$

This is useful when dealing with inequalities–as long as we prove $a + b + c \geq 0$, then we can conclude the LHS is non-negative. In fact, try proving the three variable case of AM-GM with this form.

Problems

Example 1

Factor $(x-y)^3+(y-z)^3+(z-x)^3$.

Solution

Notice that $(x-y)+(y-z)+(z-x)=0$. Therefore, we know that $(x-y)^3+(y-z)^3+(z-x)^3 = 3(x-y)(y-z)(z-x)$.

Example 2

Prove that $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$ is a rational number.

Solution

Suppose $x = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$. Then, $x - \sqrt[3]{2+\sqrt{5}} - \sqrt[3]{2-\sqrt{5}} = 0$ which by our result gives $x^3 - (2+\sqrt{5}) - (2-\sqrt{5}) = 3(x)(\sqrt[3]{2+\sqrt{5}})(\sqrt[3]{2-\sqrt{5}}) = -3x$. Thus, we get $x^3 + 3x - 4 = 0$ and $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$ is one such solution of the polynomial. It follows that $x = 1$ is a solution, and so $x^3 + 3x - 4 = (x-1)(x^2 + x + 4)$. Since $x^2 + x + 4$ has not real solutions, we have $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} = 1$ which is rational.

Practice

1. Let x, y, z be integers such that
$$(x-y)^2+(y-z)^2+(z-x)^2=xyz.$$
Prove that $x^3+y^3+z^3$ is divisible by $x+y+z+6$.
2. Let a, b, c be distinct real numbers. Prove that the following equality cannot hold:
$$\sqrt[3]{a-b}+\sqrt[3]{b-c}+\sqrt[3]{c-a}=0.$$
3. Let a, b, c be rational numbers such that
$$a+b\sqrt[3]{2}+c\sqrt[3]{4}=0.$$
Prove that $a=b=c=0$.

Problems

4 Let r be a real number such that

$$\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}} = 3.$$

Determine the value of

$$r^3 + \frac{1}{r^3}.$$

5 Find the locus of points (x, y) such that

$$x^3 + y^3 + 3xy = 1.$$

Problems

6 Let a, b, c be distinct positive integers and let k be a positive integer such that

$$ab + bc + ca \geq 3k^2 - 1.$$

Prove that

$$\frac{a^3 + b^3 + c^3}{3} - abc \geq 3k.$$

Last Problem

It is also true that

$$a^3 + b^3 + c^3 = (a + b + c)(a + wb + w^2c)(a + w^2b + wc)$$

where $w = e^{i \cdot \frac{2\pi}{3}}$, the third root of unity. Now try the following problem: let S be the set of all integers x such that $x = a^3 + b^3 + c^3 - 3abc$ for some integers a, b, c. Prove that if two elements are in S, then so is their product.

Solution

Let $P(X) = a + bX + cX^2$. Notice that $x \in S$ if and only if there exists some polynomial $P(X)$ of degree at most 2 and integer coefficients such that $x = P(1)P(w)P(w^2)$. Let x, y be two elements in S such that

$$x = P(1)P(w)P(w^2), y = Q(1)Q(w)Q(w^2).$$

Then, $xy = P(1)Q(1)P(w)Q(w)P(w^2)Q(w^2)$. Let $F(X) = P(X)Q(X) = (X^3 - 1)G(X) + R(X)$ (polynomial division). Then, since $1, w, w^2$ are roots of $X^3 - 1 = 0$, we have $P(1)Q(1) = G(1), P(w)Q(w) = G(w), P(w^2)Q(w^2) = G(w^2)$ so $xy = G(1)G(w)G(w^2) \in S$ (we can obviously guarentee integer coefficients).

Source

I took problems from Mathematical Olympiad Treasures by Titu Andreescu.