Introduction to Functional Equations Part II

"Plug in 0" – Idris Tarwala.

Freeman Cheng

Erindale Secondary School

Overview

Last week, we went over problems only involving substitution. This week, we'll look at some more techniques.

Isolation

Example 1

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2)$$

for all real x and y (WOOT 2019).

Solution

Setting $x = y$ gives us $-2xf(0) = 0$ so $f(0) = 0$. Because of the abundance of $x + y$ and $x - y$, let $a = x + y$ and $b = x - y$. Our condition becomes

$$bf(a) - af(b) = (a^2 - b^2)ab = a^3b + ab^3.$$

Assuming both variables are not 0 and rearranging becomes

$$\frac{b^3 - f(b)}{b} = \frac{a^3 - f(a)}{a}.$$

Since the left side is independent from the right side, both sides must equal to some constant k. Therefore,

$$\frac{a^3 - f(a)}{a} = k \implies f(a) = a^3 + ka$$

(this also satisfies $f(0) = 0$). We can check this solution works by substituting it back into our initial condition.

Iterated Functions

Define a function f as cyclic with order n if composing f with itself n times ie. $(f \circ f \circ f \circ ... \circ f)(x) = f^n(x) = x$. An example of a cyclic function with order 2 is $f(x) = 1 - x$ since $f(f(x)) = x$. The main thing we can do when encountering a cyclic function is to keep substituting!

Example 2

Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation

$$2f(x) + 3f(\frac{2x+29}{x-2}) = 100x + 80$$

for all $x \neq 2$. Find $f(3)$ (WOOT 2019).

Solution

We notice that $g(x) = \frac{2x+29}{x-2}$ is cyclic with order 2 (try to always check if cyclic, we leave the computation to the reader).
Therefore, we let $x = 3$ to get

$$2f(3) + 3f(35) = 380.$$

Notice how if we let $x = 35$, $\frac{2x+29}{x-2}$ will equal 3. In other words, letting $x = 35$, we have

$$2f(35) + 3f(3) = 3580.$$

We can thus solve for $f(3) = 1996$ (again, we leave the computation to the reader).

Another Iterated Functions Problem

Here's one for practice:

$$2f(x) + f(1/x) = \frac{2x^2 + 3x + 1}{x},$$

find $f(x)$ (original).

Iterated Functions Continued

Notice that if $f^n(x) = g(x)$, then we have two ways to evaluate $f^{n+1}(x)$ namely $f^{n+1}(x) = f(g(x)) = g(f(x))$ (we can use this information).

Example 3

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all integers m,

$$f(f(m)) = m + 1.$$

Solution

Note that $f(f(f(m))) = f(m+1) = f(m)+1$. By fixing $f(0) = c$, we can get $f(k) = c + k$ (induction). However, if we plug this into our original condition, we arrive at $f(f(m)) = m + 2c = m + 1$ so $c = \frac{1}{2}$ which is not an integer, so there are no solutions!

Problems

Happy solving!

1. Find all $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)f(y) - f(xy) = x + y$ for all real x and y.
2. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(a)f(b) = f(a+b)$$

(Math Olympiad Discord)

3. (HARD) Find all functions f, defined on the nonnegative real numbers and taking nonnegative real values, such that: (i) $f(xf(y))f(y) = f(x+y)$ for all $x, y \geq 0$, (ii) $f(2) = 0$, (iii) $f(x) \neq 0$ for $0 \leq x < 2$ (WOOT 2019).