Methods to Nuke Math Competition Problems

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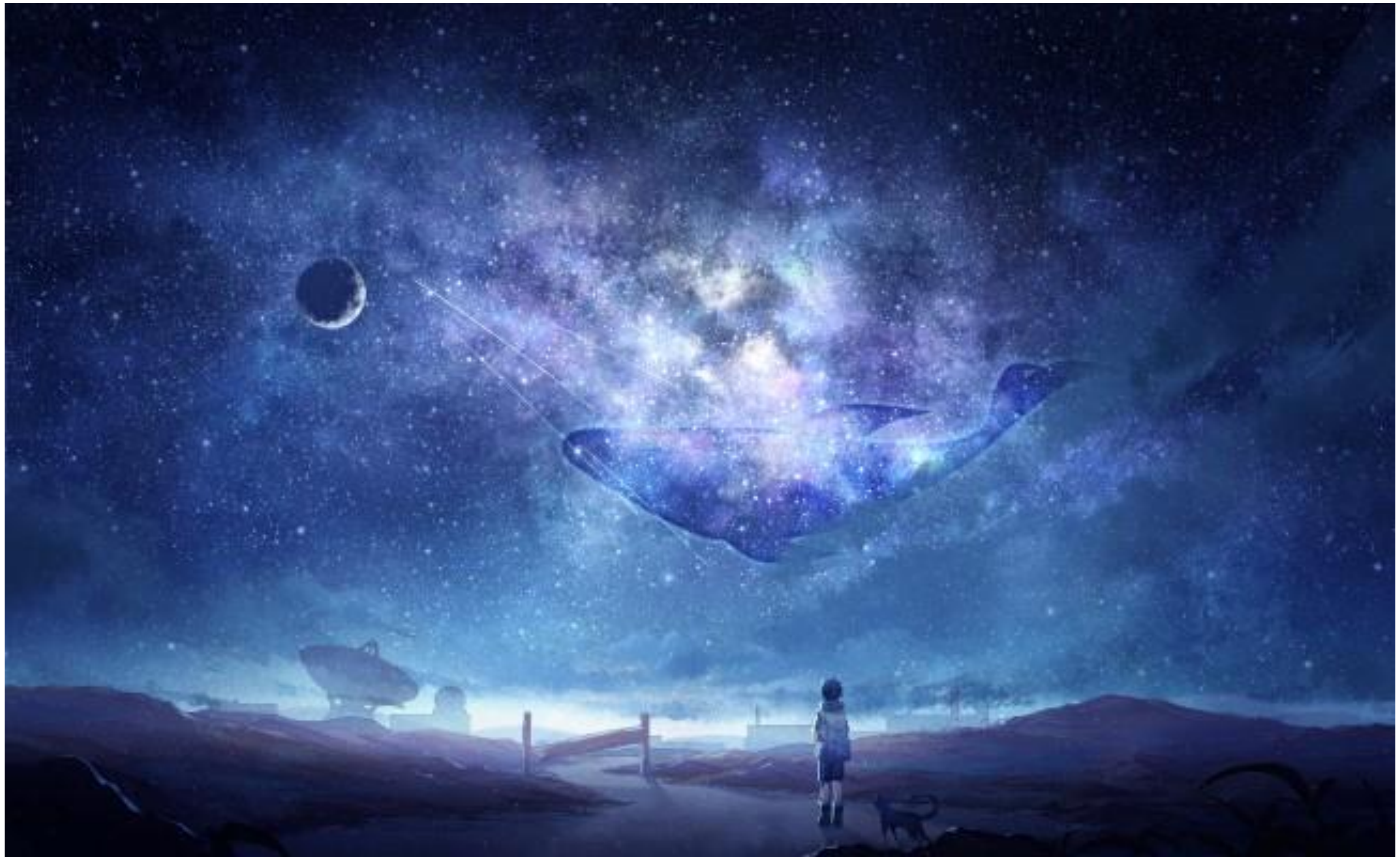
Section 1: Probabilistic Method



Disclaimer

- I am using PowerPoint which literally cannot support any form of LaTeX
- However, PowerPoint has cool Design Ideas soooo... forgive me?

Section 1 Citations

Citations for Section 1:

[1] Evan Chen's *Expected Uses of Probability*

[2] Howard Halim's *Probability States*

[3] https://artofproblemsolving.com/community/c6h23507p1943887

Note: I borrow a ton from [1]. The purpose of this presentation is to format [1] in a way that is engaging to novices while providing many practice problems from [2] and [3].

Random Variable

- Variable which takes random values – each with different probability.
- For example, let X be random variable representing outcome of dice roll then

P(X=1)=P(X=2)=…=P(x=6)=1/6.

(We use P(X=y) to represent probability X takes on value y).

Expected Value

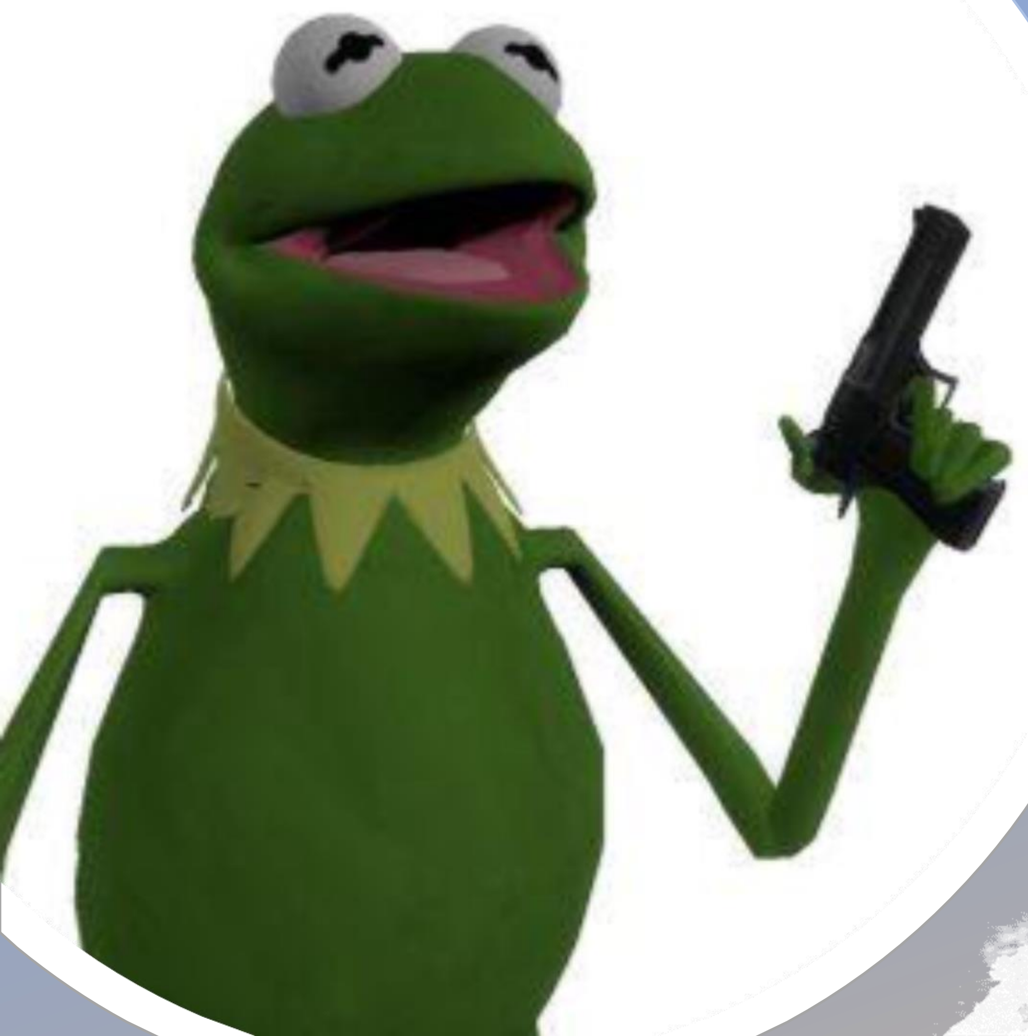
E[X] is sum of:

Values X can take on * its probability

So E[X]=3.5 (calculations left to audience).

Takeaway: think “weighted average.”

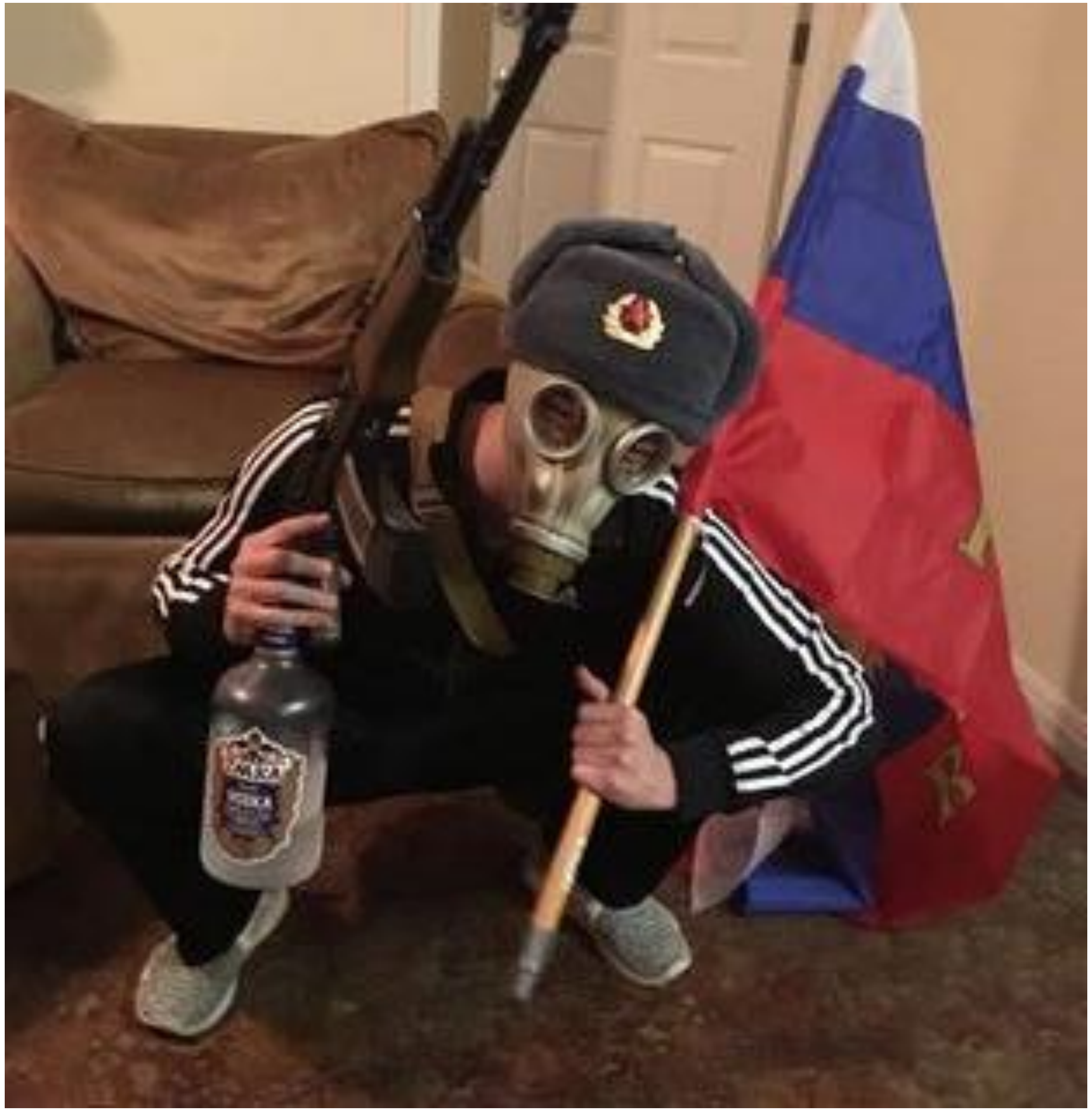
Definitions



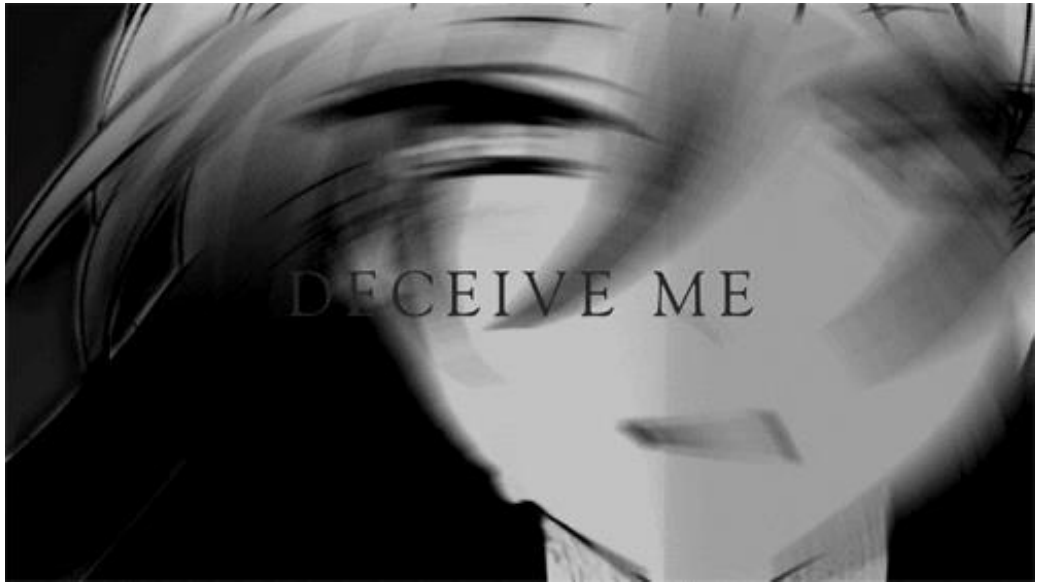
Threat Level: Noob

Nametags Problem

At a party with **n** people, the host has prepared **n** name tags (I legally don’t know how parties work). But since host drank too much vodka before the party started, he randomly hands out these nametags: 1 per person. What is expected number of people with the wrong nametag?



Approach 1



Remember definition of expected Value? We try to use it for small example n=3.

E[X] is sum of:

Value X can take on * its probability

So we can use that directly… but that’s *wayyyy* to hard to generalize.

…

Notice, however, that since the probability of each distribution of nametags is the same, namely $1/n!$.

Let y be the total number of wrong nametags. In other words, we go through every possible distribution, count the number of people with wrong nametags, and add them up to get y.

The expected value can then be thought of as $E[X] = 1/n! * y$.

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Okay, so we can count each nametag independently. Suppose a nametag had the name “Joe” on it. There are (n-1)*(n-1)! ways in which Joe’s nametag is distributed wrongly.

And this is true for all n nametags, so y=n*(n-1)*(n-1)!. As we established earlier,

$$E[X]=1/n!*y=1/n!*n*(n-1)*(n-1)!=n-1.$$

We are done… but this method is a little too noob isn’t it?

Linearity of Expectation

Given any random variables X1, X2, …, XN, we have

$$E[X1+X2+\ldots+XN]=E[X1]+E[X2]+\ldots+E[XN].$$

(Regardless if X1, X2, …, XN are dependant on each other – think what we did in the last problem by counting each nametag separately but generalized).

---USING LINEARITY OF EXPECTATION ON NAMETAGS PROBLEM---

Let Y1 be random variable defined as 1 if nametag 1 is distributed wrongly, 0 if correct. Likewise let Y2, Y3, … be defined the same way.

Then, X=Y1+Y2+…+YN so

$$E[X]=E[Y1]+E[Y2]+\ldots+E[YN]=(n-1)/n+(n-1)/n+\ldots+(n-1)/n=n-1.$$

Threat Level: Towed Artillery

Two Standard Examples

2006 Gangsters stand in a circle each armed with noob gun. Suddenly, every gangster either shoots the person to his left or right (person shot dies of blood loss). What is the expected number of people dead?

Coin is flipped 2 times in a row. What is the expected number of 2 heads flipped in a row? Note that HHH is considered two occurrences of 2 heads in a row: **HH**H and H**HH**.

Practice

Problems 2.4 and 2.5 of [1].

Harder Problem

Minecraft Hypixel Anvil Spleef

One day, a bishop and a knight were on squares in the same row of an infinite chessboard, when a huge anvil shower occurred, placing an anvil in each square on the chessboard independently and randomly with probability p. Neither the bishop nor the knight were hit, but their movement may have been obstructed by the anvils. For what value of p is the expected number of valid squares that the bishop can move to (in one move) equal to the expected number of squares that the knight can move to (in one move)?

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The knight can move to 8 squares. Linearity of Expectation tells us that 8p of these squares are expected to have an anvil on it so expected that the knight can move to 8-8p=8(1-p) squares.

The bishop can move infinitely in 4 directions. Consider moving along each direction until hitting an anvil…

Then, the expected combined distance is 4((1-p)+(1-p)^2+…). We can then solve 8(1-p)=4((1-p)+(1-p)^2+…) using geometric sequences which we will leave for the audience to do.

Key Idea: Recursion

We can also use recursion to solve expectation problems.

Consider an 8×8 grid of squares. A rook is placed in the lower left corner, and every minute it moves to a square in the same row or column with equal probability (the rook must move; i.e. it cannot stay in the same square). What is the expected number of minutes until the rook reaches the upper right corner?

And Another

Even Harder Problem

Richard and Coins

Richard has a four infinitely large piles of coins: a pile of pennies, a pile of nickels, a pile of dimes, and a pile of quarters. He chooses one pile at random and takes one coin from that pile. Richard then repeats this process until the sum of the values of the coins he has taken is an integer number of dollars. What is the expected value of this final sum of money, in cents?

Practice

1. Tom has a scientific calculator. Unfortunately, all keys are broken except for one row: 1, 2, 3, + and -. Tom presses a sequence of 5 random keystrokes; at each stroke, each key is equally likely to be pressed. The calculator then evaluates the entire expression, yielding a result of E. Find the expected value of E. (Note: Negative numbers are permitted, so 13-22 gives E = −9. Any excess operators are parsed as signs, so -2-+3 gives E = −5 and -+-31 gives E = 31. Trailing operators are discarded, so 2++-+ gives E = 2. A string consisting only of operators, such as -++-+, gives E = 0.) [HINT: CONSIDER SYMMETRY].
2. Question 2 but 1, 2, 3, 4, 5, 6, 7, 8, 9 instead of 1, 2, 3.

The Probabilistic Method

Main idea: if the average of 10 integers is 9.5, then one number must be greater than 9.

Example: Perfect Matching in Bipartite Graph

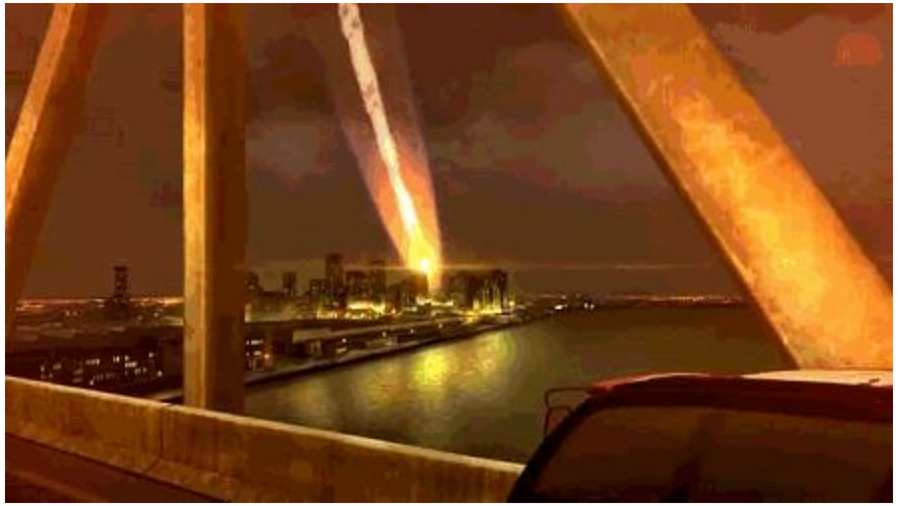
Prove that any subgraph of $K(n,n)$ with at least n^2-n+1 edges has a perfect matching.

Note by pigeonhole principle, all nodes must be present in subgraph. Then, consider randomly pairing off nodes of one set to noes of the other set. Call the score of a random pairing the number of pairs where an edge exists: we want to prove that there exist some random pairing with score of n.

Let score be $S = S1+\ldots+Sn$ where Sk is 1 if edge exists in pair k, 0 otherwise. Then $E[Sk]=(n^2-1)C(n^2-n)/(n^2)C(n^2-n+1)=(n^2-n+1)/n^2$. So, $E[S]=nE[Sk]=n-1+1/n > n-1$ so there must exist pairing with score of n since scores are integer values.

:oh: Add Some Calculus…

Threat level: ICBM



A Bound on Binomial Coefficients

Best taught through a problem…

Let n and k be integers with $n \leq 2^{(k/2)}$ and $k \geq 3$. Then prove it is possible to color the edges of the complete graph on n vertices with the following property: one cannot find k vertices for which the kC2 edges among them are monochromatic. There are two colors overall.