2019-09-24 Math Club Problems

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1. Real numbers x, y, z are real numbers greater than 1 and w is a positive real number. If $\log_x w = 24, \log_y w = 40$ and $\log_{xyz} w = 12$, find $\log_z w$.

2. Let $a_1, a_2, ...$ be a sequence defined by $a_1 = 1$ and

$$a_{n+1} = \sqrt{a_n^2 - 2a_n + 3} + 1$$

for $n \geq 1$. Find a_{513}.

3. Find the minimal value of

$$\frac{9x^2 \sin^2 x + 4}{x \sin x}.$$

4. The equation

$$2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$$

has three real roots. Assume that their sum is expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y,

$$f(f(x) + y) = x + f(f(y)).$$

6. A sequence x_n of real numbers satisfy $|x_n| = |x_{n-1} + 1|$ with $x_0 = 0$ for all $n \geq 1$. Find the minimal value of the absolute value of the sum of the first 2008 terms of this sequence.

7. Define a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(k) = k^2 + k + 1$ for every integer k. Find the largest positive integer n such that

$$2015 f(1^2) f(2^2) \cdots f(n^2) \geq \Big(f(1) f(2) \cdots f(n)\Big)^2.$$

8. Let f be a function defined along the rational numbers such that $f(\frac{m}{n}) = \frac{1}{n}$ for all relatively prime positive integers m and n. Find the product of all rational numbers $0 < x < 1$ such that

$$f\left(\frac{x - f(x)}{1 - f(x)}\right) = f(x) + \frac{9}{52}.$$

9. Let a, b, and c be positive integers forming an arithmetic sequence with $a < b < c$. Let $f(x) = ax^2 + bx + c$. Two distinct real numbers r and s satisfy $f(r) = s$ and $f(s) = r$. If $rs = 2017$, determine the smallest possible value of a.

10. Karys is helping her father move basketballs from his car to the gymnasium. She carries either 3 or 4 basketballs each trip, while her father carries 6 or 7 basketballs each trip. Altogether Karys makes 15 more trips and carries 15 fewer basketballs than her father. What is the minimum and maxiumum number of basketballs she can carry?