Lecture6_QHardware_class1

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Quantum Hardware – Optical Models

Class I

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Based on the book, Quantum Computation and Quantum Information, by M. Nielsen and I. Chuang

Introduction

BASIC QUESTION: Whate are the experimental requirements for building a Quantum Computer?

- Ψ) The elementary units of the theory are the quantum bits (qubits) – They are two-level quantum systems
- Ψ) In lecture 1, we used a simple example related to the electron spin (Stern-Gerlach experiment), to justify the qubits existence in nature
- Ψ) To realize a QC, we must not only give qubits a physical representation in which they retain the quantum properties
- Ψ) It is also necessary to select a system in which they can be made to evolve as desired
- Ψ) Furthermore, we must be able to prepare qubits in some specified set of initial states and measure the final output state of the system
- Ψ) The challenge of experimental realization is that these basic requirements can often only be partially met
- Ψ) Example: A QC has to be well isolated to retain its quantum properties, but at the same time, the qubits have to be accessible

In the first lecture, we briefly discussed the four basic requirements for Quantum Computation, let us remind them here:

- Ψ) Robustly represent quantum information (QI)
- Ψ) Perform a universal family of unitary transformations
- Ψ) Prepare a fiducial initial state
- Ψ) Measure the output result

In the next sections, let us discuss a little bit each of these items. At the end of this lecture, we will show an example of a particle in a potential well.

Representation of Quantum Information

- Ψ) Quantum computation is based on the transformation of quantum states
- Ψ) Quantum bits are two-level quantum systems, and provide convenient labeling for pairs of states and their physical realization
- Ψ) An important detail is that, for computation, the crucial realization is that the set of accessible states should be finite
- Ψ) In fact, it is also desirable to have some aspect of symmetry dictating the finiteness of the state space, in order to minimize decoherence
- Ψ) Example:, a spin-$1/2$ particle lives in a Hilbert space spanned by "up" and "down" states; the spin can not be anything outside this two-dimensional space, and thus is a nearly ideal qubit when well isolated

Performance of Unitary Transformations

- Ψ) Closed quantum systems evolve unitarily by their Hamiltonians
- Ψ) To perform quantum computation in a real system, we must be able to control the Hamiltonian to effect an arbitrary selection from a universal family of unitary transformation
- Ψ) Example: Any of those transformations can be composed of single operations by the gates $X, Y, Z, H, R_i(\theta)$, etc, and CNOT gates.
- Ψ) Thus realization of those two kinds of quantum logic gates is a natural goal for experimental QC
- Ψ) The control parameters in the Hamiltonian are approximately classical controls – in reality, the controlling system is just another quantum system, and the Hamiltonian should include the back-action of the control system upon the quantum computer

- Ψ) Example: We will see during this course the Jaynes-Commings type atom-photon interaction Hamiltonian
  - After interacting with a qubit, a photon can carry away information about the state of the qubit, and this is thus a decoherence process

Initial States

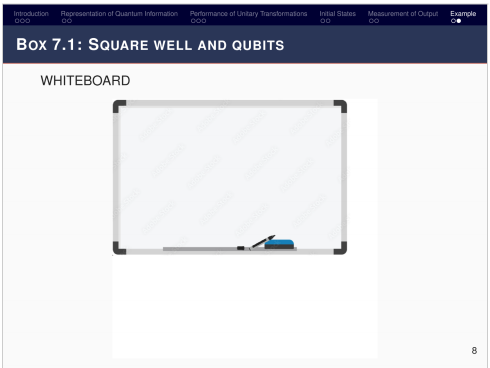
- Ψ) One of the most important requirements for being able to perform a useful computation is to be able to prepare the desired input
- Ψ) This input state preparation is a significant problem for most physical systems
- Ψ) Example 1: Ions can be prepared in good input states by cooling them into their ground state (which is a challenging thing)
- Ψ) Example 2: In nuclear magnetic resonance models, each molecule can be thought of as a single QC. Although qubits can remain in arbitrary superposition states for relatively long times. It is difficult to put all of the qubits in all of the molecules into the same state because the energy difference $\hbar w$ between the $|0\rangle$ and $|1\rangle$ states is much smaller than $K_b T$

Measurement of Output

What measurement capability is required for quantum computation?

- Ψ) Let us think of measurement as a process of coupling one or more qubits to a classical system
- Ψ) After some time, the state of the qubits is indicated by the state of the classical system
- Ψ) Example: A qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$ represented by the ground state and excited state of a two-level atom might be measured, thus ⇒ If after the measurement the detector indicates the result $|1\rangle$ state, the ket $|\psi\rangle$ collapsed to $|1\rangle$ with probability $|b|^2$.
- Ψ) Many difficulties with measurement can be imagined like inefficient photon counters and thermal noise
- Ψ) Furthermore, during the measurement process, it is required that the coupling between the quantum and classical systems be large and switchable since measurements should not occur when not desired otherwise they can be a decoherence process

Example



- CONSIDER A PARTICLE IN A ONE-DIMENSIONAL BOX

- IT BEHAVES ACCORDING TO THE SCHRÖDINGER EQUATION

$$H = \frac{p^2}{2m} + V(x) \quad , \qquad V(x) = 0 \text{ for } 0 < x < L$$
$$V(x) = \infty \text{ , OTHERWISE}$$

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

$$H|\psi\rangle = E|\psi\rangle \Rightarrow \left[\frac{p^2}{2m} + V(x)\right]|\psi\rangle = E|\psi\rangle \quad , \quad V(x) = 0 \text{, INSIDE THE BOX}$$

$$\frac{p^2}{2m}|\psi\rangle = E|\psi\rangle$$

$$\hat{p} \rightarrow i\hbar \frac{\partial}{\partial x} \quad \text{(1-DIMENSION)}$$

- SEE APPENDIX FROM THIS PDF

- FOR NOW, IT IS MORE CONVENIENT WORK IN THE $\psi(x)$ REPRESENTATION

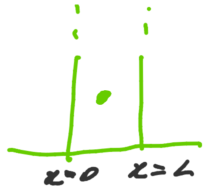
$$\frac{p^2}{2m}\underbrace{\langle x|\psi\rangle}_{\psi(x)} = E\langle x|\psi\rangle \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\,\psi(x)$$

$$\frac{d^2}{dx^2}\psi(x) = -\underbrace{\frac{2mE}{\hbar^2}}_{k^2}\psi(x) = -k^2\psi(x)$$

LET'S CALL d $\hat{p}$ ... p^2 ... $= -k^2\psi$ for $\psi \neq 0$

LET'S CALL $\frac{d}{dx} = \overset{K^2}{\hat{D}} \Rightarrow D^2\psi = -K^2\psi$, FOR $\psi \neq 0$

$$D = \pm iK$$



$$D\psi = \pm iK\psi \Rightarrow \frac{d}{dx}\psi(x) = \pm iK\psi(x) \Rightarrow \ln\psi = \pm iKx + c$$

$$\psi(x) = e^{\pm iKx + c} = A\cos(Kx) + B\sin(Kx)$$

LET'S FIND A and B: CONDITIONS $\Rightarrow \psi(0) = \psi(L) = 0$

USING THESE CONDITIONS IN THE $|\psi\rangle$ SOLUTION

- $\psi(0) = A\overset{=1}{\cos(0)} + B\overset{=0}{\sin(0)} = 0 \Rightarrow A = 0$
- $\psi(L) = B\sin(KL) = 0 \Rightarrow \sin(KL) = 0$
  $B \neq 0$

$$K_nL = n\pi, \quad n = 0, 1, 2, \dots$$

$$\boxed{K_n = \frac{n\pi}{L}}$$

REMEMBER: $K = \sqrt{\frac{2mE}{\hbar^2}} \Rightarrow \frac{n\pi}{L} = \sqrt{\frac{2mE_n}{\hbar^2}} \Rightarrow \boxed{E_n = \frac{n^2\pi^2\hbar^2}{2mL}}$

SO, ENERGY IS QUANTIZED

— $\psi_n(x) = B\sin(K_n x)$

USING THE NORMALIZATION CONDITION: $\langle\psi|\psi\rangle = 1$

$$\langle\psi_n|\psi_n\rangle = \int dx \, \langle\psi_n|x\rangle\langle x|\psi_n\rangle = 1$$

HOMEWORK: PAGE 121 FROM COHEN-TANNOUDJI BOOK, $\int dx \, |x\rangle\langle x| = I$

$$\int_0^L dx \, \psi_n^*(x)\,\psi_n(x) = 1 \Rightarrow \int_0^L dx \, |B|^2 \sin^2(K_n x) = 1$$

$$|B|^2\left[\frac{\cancel{K_n}L - \cos(K_nL)\overset{=0}{\sin(K_nL)}}{2\cancel{K_n}}\right] = 1 \Rightarrow B = \sqrt{\frac{2}{L}}$$

$$|B|^2\left[\frac{K_n L - \cos(K_n L)\sin(K_n L)}{2\cancel{K}_n}\right] = 1 \Rightarrow B = \sqrt{\frac{2}{L}}$$

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{L}}\,\sin(K_n x)}$$

- THE COMPLETE SOLUTION REQUIRES A TIME-DEPENT CONTRIBUTION
- FOR A TIME JUST TO SOLVE

$$i\hbar \frac{d\psi_n(t)}{dt} = H\psi(t) = E_n \psi_n(t)$$

$$\psi_n(t) = e^{-iE_n t/\hbar}$$

– THUS, THE SOLUTION

$$\Psi_n(x,t) = \psi_n(t)\,\psi_n(x) = e^{-iE_n t/\hbar}\,\psi_n(x)$$

– IN DIRAC, NOTATION

$$\langle x|\psi_n(t)\rangle = e^{-iE_n t/\hbar}\langle x|\psi_n\rangle$$

$$|\psi_n(t)\rangle = e^{-iE_n t/\hbar}\,|\psi_n\rangle$$

– NOW, THE COMPLETE SOLUTION:

$$|\psi(t)\rangle = \sum_n a_n |\psi_n(t)\rangle \quad , \quad n = 1, 2, 3, \ldots$$

– THESE STATES HAVE A DISCRETE SPECTRUM

– IN PARTICULAR, SUPPOSE THAT WE ARRANGE MATTERS, SUCH THAT ONLY THE <u>TWO LOWEST ENERGY LEVELS</u> NEED TO BE CONSIDERED

– LET'S DEFINE AN ARBITRARY WAVE FUNCTION OF THE TYPE:

$$|\psi(t)\rangle = a_1|\psi_1(t)\rangle + a_2|\psi_2(t)\rangle$$

– LET'S USE THE NIELSEN–CHUANG NOTATION: $\begin{cases} \alpha_1 = a \\ \alpha_2 = b \end{cases}$

$$|\Psi(t)\rangle = a|\Psi_1^{(t)}\rangle + b|\Psi_2^{(t)}\rangle = a e^{-iE_1 t/\hbar}|\Psi_1\rangle + b e^{-iE_2 t/\hbar}|\Psi_2\rangle$$

LETS SIMPLIFY THIS EXPRESSION PUTTING $e^{-i(E_1+E_2)t/2\hbar}$

$$|\Psi(t)\rangle = e^{-i(E_1+E_2)t/2\hbar}\left[a e^{-i\omega t/2}|\Psi_1\rangle + b e^{i\omega t/2}|\Psi_2\rangle\right]$$

WHERE, $\omega = \dfrac{E_1 - E_2}{\hbar}$, WE COLD ALSO DO: $|\Psi_1\rangle = |0\rangle$

$|\Psi_2\rangle = |1\rangle$

$$|\Psi(t)\rangle = e^{-i(E_1+E_2)t/2\hbar}\left[a e^{-i\omega t/2}|0\rangle + b e^{+i\omega t/2}|1\rangle\right]$$

$$|\Psi(t)\rangle = e^{-i(E_1+E_2)t/2\hbar}\begin{pmatrix} a e^{-i\omega t/2} \\ b e^{i\omega t/2} \end{pmatrix} = \begin{pmatrix} \bar{A} \\ \bar{B} \end{pmatrix}$$

– THIS IS A TWO-LEVEL SYSTEM THAT REPRESENTS A QUBIT!

DOES OUR TWO-LEVEL SYSTEM TRANSFORM LIKE A QUBIT?

– TO PERFORM OPERATIONS TO THIS QUBIT, WE WILL PERTURB THE HAMILTONIAN

– CONSIDER THE EFFECT OF ADDING THE TERM:

$$\delta V(x) = -V_0(t)\,\frac{9\pi^2}{L}\left(\frac{x}{2} - \frac{1}{2}\right)$$

– IN THE BASIS OF OUR TWO-LEVEL SYSTEM, THIS CAN BE WRITTEN BY TAKING THE FOLLOWING MATRIX ELEMENTS:

$$V_{nm} = \langle\Psi_n|\delta V(x)|\Psi_m\rangle =$$
$$= \iint dx\,dx'\,\langle\Psi_n|x'\rangle\langle x'|\delta V(x)|x\rangle\langle x|\Psi\rangle$$

$$= \int\int dx\,dx'\,\delta V(x)\,\langle \psi_n | x'\rangle \underbrace{\langle x' | x\rangle}_{\delta(x'-x)} \langle x | \psi\rangle$$

$\delta(x-x') \rightarrow$ DELTA OF DIRAC, SEE PAGE 121, COHEN

PROPERTIE: $\psi(r) = \int dr\, \psi(r')\, \delta(r-r')$

$$V_{nm} = \int dx\, \delta V(x)\, \langle \psi_n | x\rangle \langle x | \psi_m\rangle = \int dx\, \delta V(x)\, \psi_n^*(x)\, \psi_m(x)$$

SO, FOR $n=m$

- $$V_{nn} = \int_0^L dx \left[\overbrace{-V_0(t)\frac{9\pi^2}{16L}}^{c(t)} \left(\frac{x}{L} - \frac{1}{2}\right) \sin^2(k_n x)\right.$$

WE HAVE TWO TYPES OF INTEGRALS:
- $\int_0^L dx\, x \sin^2(k_n x)$
- $\int_0^L dx\, \sin^2(k_n x)$

$$V_{nn} = c(t)\left[\frac{L}{4} - \frac{L}{4}\right] = 0$$

AND, FOR $n \neq m$

$$\left.\begin{array}{l} V_{12} = \langle \psi_1 | \delta V(x) | \psi_2\rangle \\ V_{21} = \langle \psi_2 | \delta V(x) | \psi_1\rangle \end{array}\right\rangle \int_0^L dx \sqrt{\frac{2}{L}} \sin(k_1 x)\, \delta V(x) \cdot \sqrt{\frac{2}{L}} \sin(k_2 x)$$

$$V_{12} = -V_0 \frac{9\pi^2}{16} \frac{2}{L} \int_0^L dx \left(\frac{x}{L} - \frac{1}{2}\right) \text{sen}(k_1 x)\, \text{sen}(k_2 x)$$

$$V_{12} = V_0(t) = V_{21}$$

SUCH THAT, TO LOWEST ORDER IN V_0, THE PERTURBATION OF H IS

$$H_1 = V_0(t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = V_0(t)\, X$$

WHERE X IS THE PAULI X-GATE

— SO, THIS PERTURBATION GENERATES ROTATIONS AROUND $\hat{x}$-AXIS

— SIMILAR TECNIQUES CAN BE USED TO PERFORM SINGLE QUBIT OPERATIONS BY MANIPULATING THE POTENTIAL

— WE JUST SHOWED HOW A SINGLE QUBIT CAN BE REPRESENTED BY THE TWO LOWEST LEVELS IN A SQUARE WELL POTENTIAL

— AND HOW PERTURBATIONS OF THE POTENTIAL CAN EFFECT COMPUTATIONAL OPERATION ON THE QUBIT

— HOWEVER, PERTURBATIONS ALSO INTRODUCE HIGHER ORDER EFFECTS SO OTHER LEVELS COULD ENTER IN THIS PICTURE, AND OUR TWO-LEVEL SYSTEM BEGINS TO FAIL

— IN REAL PHYSICAL SYSTEMS BOXES ARE NOT INFINITY, THUS TRANSITIONS FROM THE BOUND STATES TO THE CONTINUUM OF UNBOUND STATES WOULD BE POSSIBLE

— ALL THESE ISSUES LEAD TO DÉCOHERENCE, SINCE IT COULD DESTROY THE SUPERPOSITION STATES

APPENDIX 1

FOURIER TRANSFORMS IN QUANTUM MECHANICS

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \, e^{-ikx} f(x)$$

THIS IS THE FOURIER TRANSFORM OF $f(x)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dK \, e^{ikx} \tilde{f}(K)$$

IS THE FOURIER TRANSFORM OF $\tilde{f}(K)$

- IN QUANTUM MECHANICS, IT IS USED A SLIGHTLY DIFFERENT CONVENTION

- IF $\psi(x)$ IS A (ONE-DIMENSIONAL) WAVE FUNCTION

- IT'S FOURIER TRANSFORM $\overline{\Psi}(p)$ IS DEFINED IN TERMS OF $p = \hbar K \Rightarrow \frac{dp}{\hbar}\ dK$ (p 1-DIMENSION OF MOMENTUM). THUS,

$$\tilde{f}\left(\frac{p}{\hbar}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \, e^{-ipx/\hbar} f(x), \text{ AND}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{dp}{\hbar} \, e^{ipx/\hbar} \tilde{f}\left(\frac{p}{\hbar}\right)$$

BY CONVENTION: • $\overline{\Psi}(p) = \frac{1}{\sqrt{\hbar}} \tilde{f}\left(\frac{p}{\hbar}\right)$

• $\psi(x) = f(x)$

SO, WE FIND

$$\overline{\Psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx \, e^{-ipx/\hbar} \psi(x) \qquad (*)$$

AND

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \, e^{ipx/\hbar} \overline{\Psi}(p)$$

OKAY! LET'S CONSIDER NOW THE MOMENTUM OPERATOR $\hat{p}$

$$\hat{p}\,|p\rangle = p\,|p\rangle \qquad (1-0)$$

SO, IT FOLLOWS FROM THE CLOSURE RELATION THAT

$$\langle\psi|\hat{p}|\psi\rangle = \int dp\ \langle\psi|\hat{p}|p\rangle\langle p|\psi\rangle = \int dp\ p\,\langle\psi|p\rangle\langle p|\psi\rangle$$

$$= \int dp\ p\ \bar{\Psi}^*(p)\,\bar{\Psi}(p)$$

USING EQ. (*)

$$\langle\psi|\hat{p}|\psi\rangle = \int dp\ \bar{\Psi}^*(p)\underbrace{\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} dx\ e^{-ipx/\hbar}\,\psi(x)}_{\bar{\Psi}(p)}\ p$$

$$\langle\psi|\hat{p}|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} dp\ \bar{\Psi}^*(p)\ \underbrace{\int_{-\infty}^{+\infty} dx\ e^{-ipx/\hbar}\,p\,\psi(x)}_{INT}$$

$$INT = \int_{-\infty}^{+\infty} dx\ p\ e^{-ipx/\hbar}\,\psi(x) = -\int_{-\infty}^{+\infty} dx\ \frac{\hbar}{i}\frac{\partial}{\partial x}\left[e^{-ipx/\hbar}\right]\psi(x)$$

$$= -\int_{-\infty}^{+\infty} dx\ \frac{\hbar}{i}\left\{\frac{\partial}{\partial x}\left[e^{-ipx/\hbar}\,\psi(x)\right] - e^{-ipx/\hbar}\frac{\partial}{\partial x}\psi(x)\right\}$$

$$= \underbrace{-\frac{\hbar}{i}\,e^{-ipx/\hbar}\,\psi(x)\Big|_{-\infty}^{+\infty}}_{=0} + \int_{-\infty}^{+\infty} dx\ e^{-ipx/\hbar}\,\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)$$

THIS IS THE CASE IF $\int |\psi|^2 dx = 1$, I.E. IF THE PROBABILITY TO FIND A PARTICLE SOMEWHERE IS TO BE UNITY. THUS,

$$INT = \int_{-\infty}^{+\infty} dx\ e^{-ipx/\hbar}\,\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)$$

— RETURNING TO $\langle\psi|\hat{p}|\psi\rangle$,

$$\langle\psi|\hat{p}|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} dp\,\bar{\Psi}^*(p)\int_{-\infty}^{+\infty} dx\ e^{-ipx/\hbar}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi(x)$$

$$= \int_{-\infty}^{+\infty} dx \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi(x) \underbrace{\left[\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp\, \overline{\Psi}^{*}(p)\, e^{-ipx/\hbar}\right]}_{\psi^{*}(x)}$$

so,

$$\langle\psi|\hat{p}|\psi\rangle = \int_{-\infty}^{+\infty} dx\, \psi^{*}(x) \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi(x)$$

$$\int_{-\infty}^{+\infty} dx\, \langle\psi|x\rangle\hat{p}\langle x|\psi\rangle = \int_{-\infty}^{+\infty} dx\, \psi^{*}(x)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi(x)$$

$$\int_{-\infty}^{+\infty} dx\, \psi^{*}(x)\,\hat{p}\,\psi(x) = \int_{-\infty}^{+\infty} dx\, \psi^{*}(x)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi(x)$$

$$\hat{p} \to \frac{\hbar}{i}\frac{\partial}{\partial x} = -i\hbar\frac{\partial}{\partial x}$$

APPENDIX 2

– A PLANE WAVE FOR ELECTROMAGNETIC WAVE EQUATION CAN BE WRITTEN IN THE FORM $\frac{\partial^2\psi(x,t)}{\partial t^2} = c^2\frac{\partial^2\phi(x,t)}{\partial x^2}$ (1–DIMENTION)

THE SOLUTION IS OF THE TYPE: $\psi(x,t) = A\, e^{i(kx-\omega t)}$

– WE CAN USE THE PHOTON CONCEPT BY IDENTIFYING THE PHOTON ENERGY WITH ITS FREQUENCY: $E = h\nu = \hbar\omega$ (EINSTEIN–PLANCK RELATION)

AND ITS MOMENTUM WITH ITS WAVELENGTH: $p = \frac{h}{\lambda} = \hbar k$ (dE–BROGLIE RELATION)

– THE PLANE WAVE SOLUTION TAKE THE FORM

$$\psi(x,t) = A\, e^{i(px - Et)/\hbar}$$

THIS SATISFY THE ORIGINAL EQUATION BY

$$\frac{(-iE)^2}{\hbar^2}\psi(x,t) = \left(\frac{ip}{\hbar}\right)^2 c^2 \psi(x,t) \Rightarrow E^2\psi(x,t) = (pc)^2\psi(x,t) \Rightarrow E = pc$$

WHICH IS THE PHOTON ENERGY

— COMPARING WITH $\frac{\partial^2}{\partial t^2}\psi(x,t) = c^2 \frac{\partial^2 \psi(x,t)}{\partial x^2}$, WE CAN IDENTIFY

$$E \to \hat{E} = i\hbar \frac{\partial}{\partial t}, \quad p \to \hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

— FOR MASSIVE AND NON-RELATIVISTIC PARTICLES

$$E = \frac{p^2}{2m} \Rightarrow \hat{E}\psi(x,t) = \frac{\hat{p}^2}{2m}\psi(x,t)$$

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x}$$

WHICH IS THE TIME DEPENDENT SCHRÖDINGER EQUATION