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Quantum Hardware – Optical Models

Class II

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Based on the book, Quantum Computation and Quantum Information, by M. Nielsen and I. Chuang

Harmonic Oscillator - Part I

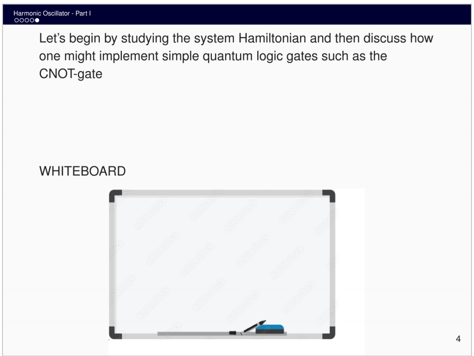
In this lecture, we will Quantize the Simple Harmonic Oscillator (SHO). This is a very elementary system in physics and the formalism we will learn in this example will serve as a basis for studying other physical systems

- Ψ) Actually, a large number of systems are governed (at least approximately) by the SHO equations
- Ψ) An example of a SHO is a particle in a parabolic potential well $V(x) = mw^2x^2/2$
- Ψ) In the classical world this could be a mass on a spring, which oscillates back and forth as energy is transferred between the potential energy of the spring and the kinetic energy of the mass
- Ψ) It could also be a resonant electrical circuit, where the energy sloshes back and forth between the inductor and the capacitor
- Ψ) In these examples the energy of the system is a continuous parameter
- Ψ) In the quantum domain, which is reached when coupling to the external world becomes very small, the total energy of the system can only take a discrete set of values

- Ψ) We know that in a cavity, there exists an infinite number of possible stationary waves (normal modes of the cavity)
- Ψ) The electromagnetic (EM) field can be expanded in terms of these modes and each of the coefficients of this expansion obeys a differential equation that is identical to that of a SHO with angular frequency associated with each mode
- Ψ) In other words, the EM field is formally equivalent to a set of independent harmonic oscillators
- Ψ) The quantization of the field is obtained by quantizing these oscillators associated with the various normal modes of the cavity.
- Ψ) Recall. that it was the study of the behavior of these oscillators at thermal equilibrium (blackbody radiation) that led Planck to introduce the constant h

Example: A single mode of electromagnetic radiation trapped in a cavity

- Ψ) The total amount of energy (up to a fixed offset) can only be integer multiples of $\hbar w$
- Ψ) An energy scale which is determined by the fundamental constant $\hbar$ and frequency of trapped radiation w
- Ψ) The set of discrete energy eigenstates of a SHO can be labeled as $|n\rangle$, when $n = 0, 1, ..., \infty$
- Ψ) The relationship to quantum information comes by taking a finite subset of these states to represent qubits
- Ψ) Moreover, unitary transforms can be applied by simply allowing the system to involve the time. However, there are problems with this scheme as will become clear soon



BOX 7.2 THE QUANTUM HARMONIC OSCILLATOR

– CONSIDER A PARTICLE IN A ONE-DIMENSIONAL PARABOLIC POTENTIAL. THE HAMILTONIAN IS

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$p \to$ PARTICLE MOMENTUM OPERATOR, $x \to$ POSITION OPERATOR
$m \to$ MASS, $\omega \to$ FREQUENCY

– TO UNDERSTAND THIS SYSTEM LET'S SOLVE THE SCHRÖDINGER EQUATION (TIME-INDEPENDENT)

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + \frac{1}{2} m \omega^2 x^2 \psi(x) = E \psi(x)$$

– WE COULD SIMPLY SOLVE THIS DIFFERENTIAL EQUATION AND FIND A SOLUTION FOR $\psi(x)$

– BUT INSTEAD WE WILL USE A NEW AND USEFUL METHOD

– SIMPLE REWRITING H WITH NEW OPERATORS

– SINCE

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

BECAUSE p and x ARE BOTH OPERATORS, LET'S DEFINE

$$a = \frac{1}{\sqrt{2m\omega\hbar}} (m\omega x + ip), \quad a^{\dagger} = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega x - ip)$$

$$a = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x + ip) \quad , \quad a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x - ip)$$

– LET'S SEE SOME PROPERTIES OF a AND $a^\dagger$

$$[a, a^\dagger] = \frac{1}{2m\hbar\omega}\Big[(m\omega x + ip), (m\omega x - ip)\Big]$$

$$= \frac{1}{2m\hbar\omega}\Big\{m^2\omega^2\underbrace{[x,x]}_{=0} + im\omega\underbrace{[p,x]}_{=-i\hbar} - im\omega\underbrace{[x,p]}_{=i\hbar} + \underbrace{[p,p]}_{=0}\Big\}$$

$[x,p] = i\hbar$, $[p,x] = -i\hbar$, THESE ARE THE COMUTATIONS RELATIONS VALID AT EVERY MOMENT OF TIME

HOMEWORK

NIELSEN–CHUANG : BOX 2.4 (PAGE 89)

: EXERCISES 7.1, 7.2, 7.3, 7.4, 7.5 (PAGE 284)

$$[a, a^\dagger] = \frac{1}{2m\hbar\omega}\Big\{im\omega(-i\hbar) - im\omega(i\hbar)\Big\} = 1$$

– NOW WE WILL DEFINE A NEW OPERATOR

$$N = a^\dagger a$$

NOTE THAT N IS HERMITIAN : $N^\dagger = (a^\dagger a)^\dagger = a^\dagger (a^\dagger)^\dagger = a^\dagger a = N$

– LET'S FIND A RELATION BETWEEN N AND H

– FIRST NOTE THAT

$$N = a^\dagger a = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega x - ip)\frac{1}{\sqrt{2m\hbar\omega}}(m\omega x + ip)$$

$$= \frac{1}{2m\hbar\omega}\Big(m^2\omega^2 x^2 + im\omega\underbrace{[x,p]}_{=i\hbar} + p^2\Big)$$

$$= \frac{1}{2m\hbar\omega}\left(m^2\omega^2x^2 + im\omega\overset{=i\hbar}{[x,p]} + p^2\right)$$

$$= \frac{1}{2m\hbar\omega}\left(m^2\omega^2x^2 - m\omega\hbar + p^2\right) = \frac{1}{\hbar\omega}\underbrace{\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2\right)}_{H} - \frac{1}{2}$$

$$N = \frac{H}{\hbar\omega} - \frac{1}{2} \Rightarrow H = \hbar\omega(N + 1/2)$$

— LET'S DENOTE THE EIGENKET OF N BY ITS EIGENVALUE n

$$N|n\rangle = n|n\rangle$$

— WE WILL LATER SHOW THAT n MUST BE A NON-NEGATIVE INTEGER

— BUT FOR NOW, USE THE HAMILTONIAN ON $|n\rangle$

$$H|n\rangle = E|n\rangle$$

$$\hbar\omega(N+1/2)|n\rangle = E|n\rangle \Rightarrow \hbar\omega(n+1/2)|n\rangle = E|n\rangle$$

$$E \rightarrow E_n = \hbar\omega(n+1/2)$$

— NOW TO APPRECIATE THE PHYSICAL SIGNIFICANCE OF $a, a^\dagger$, LET'S FIRST NOTE THAT

- $[N,a] = [a^\dagger a, a] = a^\dagger\overset{=0}{[a,a]} + \overset{=-1}{[a^\dagger,a]}a = -a$

SEE COHEN-TANNOUDJI PAGE 168

— THUS, WE CAN DERIVE

- $[N,a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger$

— AS A RESULT, WE HAVE

- $N a^\dagger|n\rangle = \left([N,a^\dagger] + a^\dagger N\right)|n\rangle = \left(a^\dagger + a^\dagger N\right)|n\rangle$

- $N a^{+}|n\rangle = \left([N, a^{+}] + a^{+}N\right)|n\rangle = \left(a^{+} + a^{+}N\right)|n\rangle$

$$= a^{+}|n\rangle + a^{+}N|n\rangle = a^{+} + n a^{+}|n\rangle$$

$$= (1+n)\, a^{+}|n\rangle \qquad (*)$$

- $N a|n\rangle = (-1+n)\, a|n\rangle \qquad (**) \qquad$ (HOMEWORK)

— THESE RELATIONS IMPLIES THAT $a^{+}|n\rangle$ AND $a|n\rangle$ IS ALSO AN EIGENKET OF N WITH INCREASED (DECREASED) BY ONE

— BECAUSE THE INCREASE (DECREASE) OF n BY ONE AMOUNTS TO THE CREATION (ANNIHILATION) OF ONE QUANTUM UNIT OF ENERGY $\hbar\omega$

— THUS, THE TERM CREATION (ANNIHILATION) OPERATOR FOR a^{+} (a) IS APPROPRIATED

— NOTE THAT, FROM EQUATION $(**)$

$a|n\rangle$ AND $|n-1\rangle$ ARE THE SAME UP TO A MULTIPLICATIVE CONSTANT, BECAUSE IF

$$N|n\rangle = n|n\rangle$$

$$N|n-1\rangle = (n-1)|n-1\rangle, \quad \text{SO FROM}$$

$$N a|n\rangle = (n-1)\, a|n\rangle$$

COMPARING THE LAST TWO EXPRESSIONS,

$$a|n\rangle = c\,|n-1\rangle$$

WHERE c IS A NUMERICAL CONSTANT TO BE DETERMINED BY THE REQUIREMENT THAT BOTH $|n\rangle$ AND $|n-1\rangle$ ARE NORMALIZED

— SO, NOTE THAT

$$\langle n|a^{+}a|n\rangle = \langle n-1|c^{*}c|n-1\rangle = |c|^2\langle n-1|n-1\rangle = |c|^2$$

— SO, NOTE THAT

$$\langle n|\underbrace{a^\dagger a}_{N}|n\rangle = \langle n-1|c^* c|n-1\rangle = |c|^2 \langle n-1|n-1\rangle = |c|^2$$

$$\langle n|N|n\rangle = |c|^2 \Rightarrow \langle n|n|n\rangle = |c|^2 \Rightarrow n\langle n|n\rangle = |c|^2$$

$$c = \sqrt{n}$$

TAKING c TO BE POSITIVE (BY CONVENTION). SO WE OBTAIN

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

— SIMILARLY, IT IS POSSIBLE TO SHOW THAT

$$N a^\dagger|n\rangle = (1+n)a^\dagger|n\rangle$$

— SO, $a^\dagger|n\rangle$ AND $|n+1\rangle$ ARE THE SAME UP TO A MULTIPLICA THE CONSTANT

$$a^\dagger|n\rangle = c\,|n+1\rangle \text{ , THUS}$$

$$\langle n|a a^\dagger|n\rangle = |c|^2\langle n+1|n+1\rangle$$

AND USING THE EXPRESSION $[a, a^\dagger] = 1 \Rightarrow a a^\dagger - \underbrace{a^\dagger a}_{N} = 1 \Rightarrow a a^\dagger = 1+N$

$$\langle n|a a^\dagger|n\rangle = \langle n|(1+N)|n\rangle = |c|^2$$

$$1+n = |c|^2 \Rightarrow c = \sqrt{1+n}$$

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

— SUPPOSE THAT WE KEEP ON APPLYING THE OPERATOR a TO BOTH SIDES OF THE EQUATION

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

- $a^2|n\rangle = \sqrt{n}\,a|n-1\rangle = \sqrt{n}\sqrt{n-1}\,|n-2\rangle = \sqrt{n(n-1)}\,|n-2\rangle$

- $a^3|n\rangle = \sqrt{n(n-1)}\,a|n-2\rangle = \sqrt{n(n-1)(n-2)}\,|n-3\rangle$

⋮

- WE CAN OBTAIN NUMERICAL OPERATOR EIGENKETS WITH SMALLER AND SMALLER n UNTIL THE SEQUENCE TERMINATES WHICH IS BOUND TO HAPPEN WHENEVER WE START WITH A POSITIVE INTEGER n

- ONE MAY ARGUE THAT IF WE START WITH A NON-INTEGER n THE SEQUENCE WILL NOT FINISH, LEADING TO EIGENKETS WITH NEGATIVE VALUE OF n

- REMEMBER THAT WE ALSO HAVE THE POSITIVITY REQUIREMENT FOR THE NORM OF $a|n\rangle$

NORM OF $a|n\rangle$: $(\langle n|a^{+})(a|n\rangle) \geq 0$

$$\langle n|a^{+}a|n\rangle \geq 0 \Rightarrow \langle n|N|n\rangle \geq 0$$

$$n \geq 0$$

WHICH IMPLIES THAT n CAN NEVER BE NEGATIVE

- SO, WE CONCLUDE THAT THE SEQUENCE MUST STOP WITH $n=0$

- AND THE ALLOWED VALUES OF n ARE NON-NEGATIVE INTEGERS

- BECAUSE, THE SMALLEST VALUE OF n IS ZERO, THE GROUND STATE OF ENERGY OF THE SHO IS

$$E_n = \hbar\omega(n+1/2)$$

$$E_0 = \frac{\hbar\omega}{2} \quad \text{(GROUND STATE)}$$