# MATH414 - Sensitivities in Finance

Charles Gendreau, Eliott Van Dieren

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# Summary

# 1. Motivation for sensitivities in finance

- Sensitivities in finance (also called greeks) are indicators which quantify the change of an option price given a variation in a certain parameter.
- The $\delta$ measures how much the change in the current stock price affects the price of the option,
- The $\nu$ measures how much the change in volatility affects the option price,
- The $\gamma$ measures how much the change in the $\delta$ affects the option price.
- **Assumption:** The volatility and the interest rate remain constant. (B-S Model)

# 2. Notations, Mathematical models and definition

- The price of a stock at time $T$ can be modeled as follows

### The Black-Scholes Model

$$S_T = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma W_T\right)$$

where $S_0$ is the current stock price, $r$ the interest rate, $\sigma$ the volatility and $\{W_t\}_{t\in[0,T]}$ is a Brownian motion.

- Given a stock price $S_T$ and a map $f$ such that $f(S_T)$ is the price of a given option we define:

### Definition of $\delta$, $\nu$ and $\gamma$

$$\delta := \frac{\partial \mathbb{E}[f(S_T)]}{\partial S_0}, \quad \nu := \frac{\partial \mathbb{E}[f(S_T)]}{\partial \sigma}, \quad \gamma := \frac{\partial^2 \mathbb{E}[f(S_T)]}{\partial S_0^2}$$

- Some options prices will depend on the path of the stock price $\{S_t\}_{t\in[0,T]}$. This definition can be generalized considering $f(S_{t_1}, ..., S_{t_m})$ as for the Asian option.

## 3.1 Estimating sensitivities : Finite Difference method

- Let $I := \mathbb{E}[f(S_T)]$, we are interested in computing $\frac{\partial I}{\partial \theta}$, for $\theta = S_0, \sigma$.
- The finite difference method gives us:

$$\frac{\partial I}{\partial \theta} = \lim_{\Delta\theta \to 0} \frac{I(\theta + \Delta\theta) - I(\theta - \Delta\theta)}{2\Delta\theta}$$

  if $I$ is sufficiently differentiable

- Applications of this method will be discussed in the next section

# 3.2 Estimating sensitivities : Pathwise derivative method

- If we can interchange the derivative and the expectation, the pathwise derivative method is:

$$\frac{\partial I}{\partial \theta} = \frac{\partial}{\partial \theta}\mathbb{E}[f(S_T)] = \mathbb{E}\left[\frac{\partial}{\partial \theta} f(S_T)\right] = \mathbb{E}\left[f'(S_T)\frac{\partial S_T}{\partial \theta}\right]$$

$$\frac{\partial^2 I}{\partial \theta^2} = \frac{\partial}{\partial \theta}\mathbb{E}\left[f'(S_T)\frac{\partial S_T}{\partial \theta}\right] = \mathbb{E}\left[f''(S_T)\left(\frac{\partial S_T}{\partial \theta}\right)^2 + f'(S_T)\frac{\partial^2 S_T}{\partial \theta^2}\right]$$

- Rule of thumb: we can interchange derivative in $\theta$ and expectation if $f(s)$ is continuous w.r.t. $s$.

## 3.3 Estimating sensitivities : Likelihood Ratio method

- By definition of $f(S_T)$, one gets

$$I = \mathbb{E}[f(S_T)] = \int_{\mathbb{R}} f\left(S_0 \exp\left\{(r - \frac{1}{2}\sigma^2)T + \sigma w\right\}\right) p_{W_T}(w)dw$$

- By replacing $w$ in the expression above from the Black-Scholes model, one gets the pdf for $S_T$ :

$$p_{S_T}(s) = \frac{1}{s\sigma\sqrt{2\pi T}} \exp\left\{-\frac{1}{2}\left(\frac{\log(s/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right\}$$

.

- and if $s \mapsto f(s)$ does not depend on $\theta$, then we get the partial derivative of $I$ as follows :

$$\frac{\partial I}{\partial \theta} = \int_0^\infty f(s)\frac{\partial p_{S_T}(s)}{\partial \theta} ds = \mathbb{E}\left[f(S_T)\frac{\dot{p}_\theta(S_T)}{p_{S_T}(S_T)}\right]$$

where $\dot{p}_\theta(S_T) = \frac{\partial p_{S_T}(s)}{\partial \theta}$

- We require the integrand to be differentiable (w.r.t. $s$) with derivative w.r.t $\theta$ in $\mathcal{L}^1$.

## 4.1 Estimation of Greeks : European Call Option

- Consider $f(S_T) = e^{-rT}[S_T - K]^+$, where $K$ is the strike price.

**Theoretical Values from Black-Scholes model**

$$\delta = \Phi(d_1), \quad \nu = S_0\sqrt{T}\phi(d_1), \quad \gamma = \frac{\phi(d_1)}{\sigma S_0\sqrt{T}}, \quad \text{with } d_1 := \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

and where $\Phi(\cdot)$ and $\phi(\cdot)$ are respectively the cdf and the pdf of the standard normal distribution.

- For the simulations we set: $K = 120$, $T = 1$, $S_0 = 100$, $r = 0.05$, $\sigma = 0.25$ and $N = 10^6$

# 4.1 European Call Option : Finite Difference method

- By using the formula in in 2.1, we get :

$$\delta, \nu = \frac{\partial I}{\partial \theta} \approx \frac{I(\theta + \Delta\theta) - I(\theta - \Delta\theta)}{2\Delta\theta} = \mathbb{E}\left[\frac{f(S_T^{\theta+\Delta\theta}) - f(S_T^{\theta-\Delta\theta})}{2\Delta\theta}\right]$$

  for $\theta = S_0, \sigma$ respectively. We set $\Delta\theta = 10^{-6}$

- $(W_T)_{i \in [0,\dots,N]} \sim N(0, T)$ has to be the same for both $(S_T^{\theta \pm \Delta\theta})_{i \in [0,\dots,N]}$
- Convergence of the CMC estimator is observed as $N$ increases with order of convergence of $\mathcal{O}(N^{-1/2})$

# 4.1 European Call Option: Finite Difference - Results

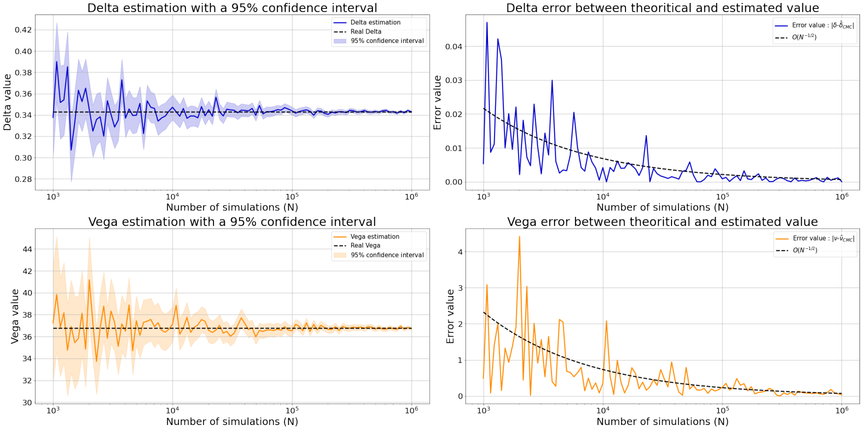


Figure: Delta and Vega estimation w.r.t. number of simulation (left column), and error bounds (right) with the finite differences method

# 4.1 European Call Option: Finite Difference method - Bias

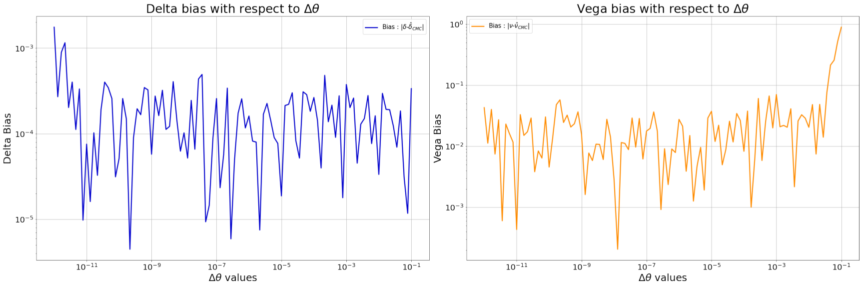


Figure: Evolution of Bias w.r.t $\Delta\theta$

- Counter intuitive results as the bias does not decrease with $\Delta\theta$.
- Expected theoretical bias growth: $\mathcal{O}(\Delta\theta)$ (via Taylor expansion)

# 4.1 European Call Option: Finite Difference - Simulation of $I(\theta \pm \Delta\theta)$

- We need to use the **same** samples of $W_T$ to simulate $I(\theta + \Delta\theta)$ and $I(\theta - \Delta\theta)$
- Using two independent samples would give ([1])

$$\mathrm{Var}(\hat{\mu}_{CMC}) = \frac{\mathrm{Var}(f(S_{T,\theta+\Delta\theta}) - f(S_{T,\theta-\Delta\theta}))}{4N(\Delta\theta)^2} \in \mathcal{O}\left(\Delta\theta^{-2}\right)$$

- The following numerical calculations also justify the use of the same samples of $W_T$:

| Delta | Indep. $W_T$ | Same $W_T$ |
|---|---|---|
| Estimated variance | $7.8 \cdot 10^7$ | $3.5 \cdot 10^{-7}$ |
| Theoretical variance | $10^6$ | $10^{-6}$ |
| **Vega** | **Indep. $W_T$** | **Same $W_T$** |
| Estimated variance | $7.7 \cdot 10^7$ | $6 \cdot 10^{-3}$ |
| Theoretical variance | $10^6$ | $10^{-6}$ |

Table: Finite difference estimator's variance w.r.t. independence of $W_T$, $\Delta\theta = 10^{-6}$, $N = 10^6$

# 4.1 European Call Option : Pathwise derivative method

- $f(S_T) = e^{-rT}[S_T - K]^+$ is continuous and piecewise differentiable w.r.t $\theta = S_0, \sigma$.

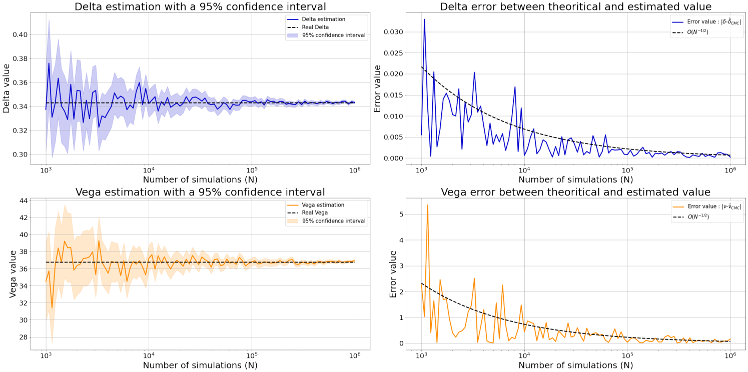


Figure: Delta and Vega estimation w.r.t. number of simulation (left column), and error bounds (right) with the pathwise derivative method

- Convergence rate: $\mathcal{O}(N^{-\frac{1}{2}})$

# 4.1 European Call Option : Likelihood ratio method

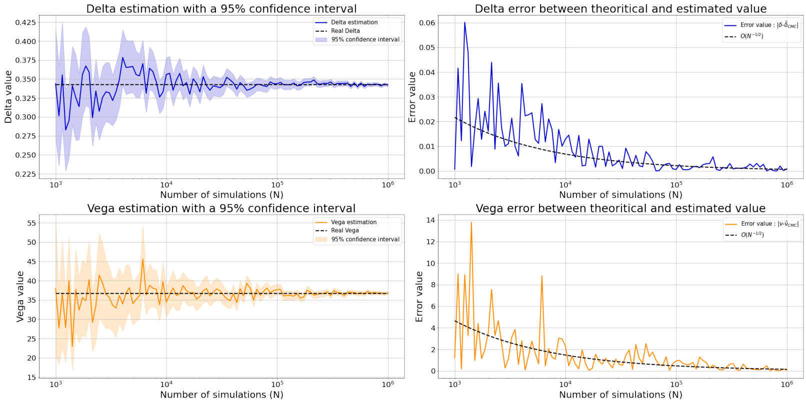


Figure: Delta and Vega estimation w.r.t. number of simulation (left column), and error bounds (right) with the likelihood ratio method

- Convergence rate: $\mathcal{O}(N^{-\frac{1}{2}})$
- Higher error for the Vega estimation than the two other methods.

# 4.1 European Call Option : Estimation of $\gamma$

- **LR - LR:** $\frac{\partial^2 I}{\partial S_0^2} = \frac{\partial}{\partial S_0}\int_0^\infty f(s)\frac{\partial p_{S_T}(s)}{\partial S_0}ds = \int_0^\infty f(s)\frac{\partial^2 p_{S_T}(s)}{\partial S_0^2}ds = \mathbb{E}\left[f(S_T)\frac{\ddot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right]$
- **LR - PD:** $\frac{\partial^2 I}{\partial S_0^2} = \mathbb{E}\left[\frac{\partial}{\partial S_0}\left(f(S_T)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right)\right] = \mathbb{E}\left[\left(\frac{f'(S_T)S_T - f(S_T)}{S_0}\right)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right]$
- **PD - LR:** $\frac{\partial^2 I}{\partial S_0^2} = \frac{\partial}{\partial S_0}\mathbb{E}[g(S_T)] = \mathbb{E}\left[g(S_T)\left(\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)} - \frac{1}{S_0}\right)\right]$
  where $g(S_T) = f'(S_T)\frac{\partial S_T}{\partial S_0}$
- **PD - PD:** $\frac{\partial^2}{\partial S_0^2}\mathbb{E}[f(S_T)] = \frac{\partial}{\partial S_0}\mathbb{E}[g(S_T)]$ where $g(s) = e^{-rT}\mathbb{1}_{\{s>K\}}\frac{s}{S_0}$ is not continuous w.r.t $s$. Hence, we cannot apply a second time the PD method.

# 4.1 European Call option: Estimation of $\gamma$

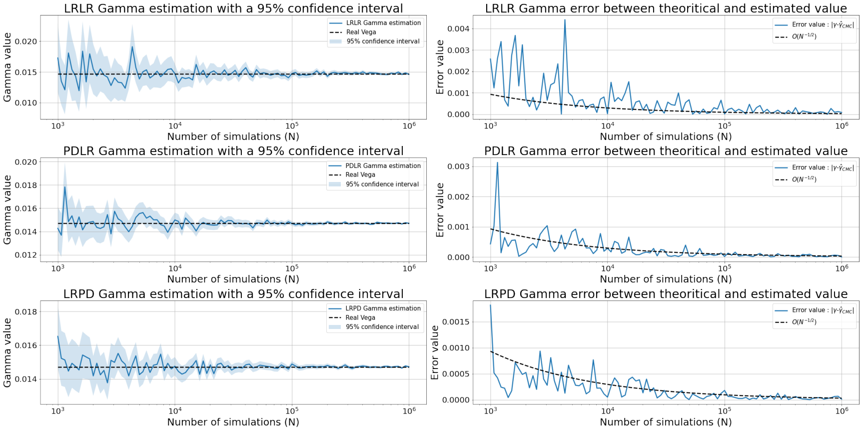


Figure: Gamma estimations w.r.t. number of simulation (left column), and error bounds (right) for different combinations of the LR and PD methods

# 4.2 Estimation of Greeks : Digital Call Option

- We now consider the payoff of a Digital call option

$$f(S_T) = e^{-rT}\mathbb{1}_{\{S_T>K\}}$$

and want to approximate $\delta$.

- Due to continuity issues, we define

$$\mathbb{1}_{\{S_T>K\}} = f_\epsilon(S_T) + h_\epsilon(S_T) \text{ where } h_\epsilon(s) := (\mathbb{1}_{\{s>K\}} - f_\epsilon(s))$$

where $f_\epsilon(s) = \min\{1, \max\{0, \frac{s-K+\epsilon}{2\epsilon}\}\}$ is a continuous approximation of $\mathbb{1}_{\{s>K\}}$

- Therefore, we get:

$$\delta = \frac{\partial}{\partial S_0}\mathbb{E}[f(S_T)] = \delta = e^{-rT}\left(\mathbb{E}\left[f'_\epsilon(S_T)\frac{\partial S_T}{\partial S_0}\right] + \mathbb{E}\left[h_\epsilon(s)\frac{\dot{p}_{S_0}}{p_{S_T}}\right]\right)$$

where the PD method was used for $f_\epsilon$ and LR for $h_\epsilon$.

- With $\epsilon = 20$, $S_0 = K = 100$, $T = 0.25$ and $r = 0.05$, we get $\delta \approx 0.0315$

## 4.2 Digital Call Option - Estimator's variance

- To decrease the estimator's variance, one should find optimal number of samples decidaded to $f_\epsilon$ and $h_\epsilon$

### Method for optimal $m$

1. Pilot run with $m = \frac{N}{2}$ samples for the PD, and $N - m$ for the LR, which gives us the total estimator's variance

$$\begin{aligned}\text{Var}_{tot} &= \text{Var}\left(e^{-rT}\left(\hat{\mu}_{PD}(f_\epsilon) + \hat{\mu}_{LR}(h_\epsilon)\right)\right)\\ &= e^{-2rT}\left[\frac{1}{m}\text{Var}\left(f'_\epsilon(S_T)\frac{\partial S_T}{\partial S_0}\right) + \frac{1}{N-m}\text{Var}\left(h_\epsilon(S_T)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right)\right]\end{aligned}$$

2. Then, differentiate $\text{Var}_{tot}$ w.r.t. $m$ and we get

$$m^*_{1,2} = N\frac{\text{Var}\left(f'_\epsilon(S_T)\frac{\partial S_T}{\partial S_0}\right) \pm \sqrt{\text{Var}\left(f'_\epsilon(S_T)\frac{\partial S_T}{\partial S_0}\right)\text{Var}\left(h_\epsilon(S_T)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right)}}{\text{Var}\left(f'_\epsilon(S_T)\frac{\partial S_T}{\partial S_0}\right) - \text{Var}\left(h_\epsilon(S_T)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right)}$$

3. Select $m^*$ which is eligible and gives the lowest total variance.

# 4.2 Digital Call Option - Estimator's variance

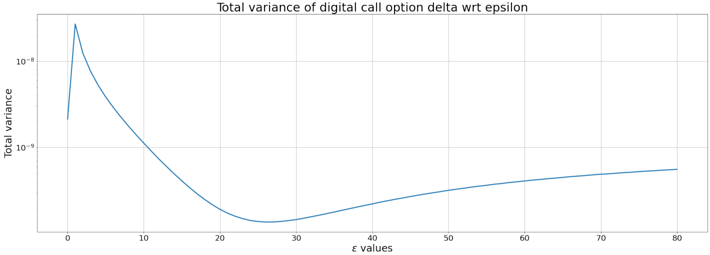


Figure: Total variance for digital option with respect to $\epsilon$

- For $\epsilon^* = 26$, $\delta = 0.0315$ and an estimator's variance in $\mathcal{O}(10^{-10})$, 30% lower than initial $\epsilon = 20$.

# 4.3 Estimation of Greeks : Path-dependant Option

- Consider $f(\mathbf{S}) = f(S_{t_1}, \ldots, S_{t_m}) = e^{-rT}[\bar{S}_T - K]^+ = \tilde{f}(\bar{S}_T)$, where $t_i = iT/m$ and $\bar{S}_T = \frac{1}{m}\sum_{i=1}^m S_{t_i}$
- **PD method:** $\frac{\partial}{\partial\theta}\mathbb{E}[\tilde{f}(\bar{S}_T)] = \mathbb{E}\left[\tilde{f}'(\bar{S}_T)\frac{\partial}{\partial\theta}\bar{S}_T\right]$, where

$$\frac{\partial \bar{S}_T}{\partial S_0} = \frac{\bar{S}_T}{S_0} \quad \text{and} \quad \frac{\partial \bar{S}_T}{\partial \sigma} = \frac{1}{m}\sum_{i=1}^m \frac{\log(S_{t_i}/S_0) - (r + \frac{1}{2}\sigma^2)t_i}{\sigma} S_{t_i}$$

### $\delta$ and $\nu$ with pathwise derivative

$$\delta = e^{-rT}\mathbb{E}\left[\mathbb{1}_{\{\bar{S}_T > K\}}\frac{\bar{S}_T}{S_0}\right], \quad \nu = e^{-rT}\mathbb{E}\left[\mathbb{1}_{\{\bar{S}_T > K\}}\frac{1}{m}\sum_{i=1}^m \frac{\log(S_{t_i}/S_0) - (r + \frac{1}{2}\sigma^2)t_i}{\sigma} S_{t_i}\right]$$

## 4.3 Estimation of Greeks : Path-dependant Option - Likelihood Ratio

- $\frac{\partial}{\partial\theta}\mathbb{E}[f(\mathbf{S})] = \frac{\partial}{\partial\theta}\int_{\mathbb{R}^m} f(\mathbf{s})p_S(\mathbf{s})d\mathbf{s} = \int_{\mathbb{R}^m} f(\mathbf{s})\frac{\partial}{\partial\theta}p_S(\mathbf{s})d\mathbf{s} = \mathbb{E}\left[f(\mathbf{S})\frac{\partial}{\partial\theta}\log p_S(\mathbf{S})\right]$
- $\{S_t\}_{t\in[0,T]}$ is a Markov process $\implies p_S(\mathbf{s}) = \prod_{k=1}^{m} p_{S_k|S_{k-1}}(s_k|s_{k-1})$
- Adapt the expression of $p_{S_T} = p_{S_T|S_0}$ to get one for $p_{S_k|S_{k-1}}(s_k|s_{k-1})$
- Case $\theta = S_0$: $\frac{\partial}{\partial\theta}\log p_S(\mathbf{s}) = \frac{\partial}{\partial\theta}p_{S_1|S_0}(s_1|s_0)$ as $p_{S_k|S_{k-1}}(s_k|s_{k-1})$ is independent of $S_0$ for $k = 2, ..., m$
- Case $\theta = \sigma$: We can derive a formula for $\frac{\partial}{\partial\sigma}p_{S_k|S_{k-1}}(s_k|s_{k-1})$ as we did for $\frac{\partial}{\partial\sigma}p_{S_T}(s)$. This gives a closed form formula of $\frac{\partial}{\partial\sigma}\log p_S(\mathbf{s})$.

# 4.3 Estimation of Greeks : Path-dependant Option - Likelihood Ratio

- We can estimate the following by generating independent samples of $(W_{t_1}, W_{t_2}, ..., W_{t_m})$.

### Delta - Likelihood ratio

$$\delta = \mathbb{E}\left[e^{-rT}[\bar{S}_T - K]^+ \frac{1}{S_0\sigma\sqrt{T/m}}\left(\frac{\log(S_1/S_0) - (r - \frac{1}{2}\sigma^2)\frac{T}{m}}{\sigma\sqrt{T/m}}\right)\right]$$

### Vega - Likelihood ratio

$$\nu = \mathbb{E}\left[e^{-rT}[\bar{S}_T - K]^+ u(\mathbf{S})\right]$$

where

$$u(\mathbf{S}) = \sum_{k=1}^{m} \frac{-1}{\sigma} - \left(\frac{\log(S_k/S_{k-1}) - (r - \frac{1}{2}\sigma^2)\frac{T}{m})}{\sigma\sqrt{T/m}}\right)\left(\frac{r\frac{T}{m} - \log(S_k/S_{k-1})}{\sigma^2\sqrt{T/m}} + \frac{\sqrt{T/m}}{2}\right)$$

# Conclusion

- Several methods to compute financial sensitivities.
- Overall, good results for estimation, except bias.
- Further steps : Using variance reduction methods (Control variate, importance sampling) and further numerical tests on estimators.
- Apply the methods to other types of securities and other Greeks ($\rho$, *Vanna*,...) if applicable.

# References

Paul Glasserman.
Estimating Sensitivities.
In Paul Glasserman, editor, *Monte Carlo Methods in Financial Engineering*, Stochastic Modelling and Applied Probability, pages 377–420. Springer, New York, NY, 2003.

# Backup Slides - Formulas

## Expression of $\frac{\partial S_T}{\partial \theta}$

$$\frac{\partial S_T}{\partial S_0} = e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T} = \frac{S_T}{S_0}$$

$$\frac{\partial S_T}{\partial \sigma} = (W_T - \sigma T)S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T} = (W_T - \sigma T)S_T.$$

## LR calculations

$$\frac{\dot{p_\theta}(s)}{p_{S_T}(s)} = \frac{\partial}{\partial \theta}(\log p_{S_T}(s)) = \frac{\partial}{\partial \theta}(-\log(s\sigma\sqrt{2\pi T})) - \frac{1}{2}\frac{\partial}{\partial \theta}\left(\frac{\log(s/S_0) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2$$

$$\frac{\dot{p_{S_0}}(s)}{p_{S_T}(s)} = \frac{1}{S_0\sigma\sqrt{T}}\left(\frac{\log(s/S_0) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$\frac{\dot{p_\sigma}(s)}{p_{S_T}(s)} = \frac{-1}{\sigma} - \left(\frac{\log(s/S_0) - (r-\frac{1}{2}\sigma^2)T)}{\sigma\sqrt{T}}\right)\left(\frac{rT - \log(s/S_0)}{\sigma^2\sqrt{T}} + \frac{\sqrt{T}}{2}\right).$$

# Rule of thumb - 1

- When is it allowed to interchange the derivative w.r.t. $\theta$ and the expectation ?
- We are allowed to interchange the derivative and the expectation (usually) only if $\theta \mapsto f(S_T)$ is continuous and piecewise differentiable.
- "Proof" : One could approximate $f$ by a increasing sequence of smooth functions $\{f_n\}_{n\geq 1}$. Applying the monotone convergence theorem, we would get that $\lim_{n\to\infty} \mathbb{E}[f_n(S_T)] = \mathbb{E}[f(S_T)]$. If $f$ is continuous, we could choose a sequence of $f_n$ that converges uniformly to $f$ allowing us to interchange the limit in $n$ and the derivative in $\theta$. Note that we cannot do this if $f$ is not continuous.

$$\frac{\partial}{\partial\theta}\mathbb{E}[f(S_T)] = \frac{\partial}{\partial\theta}\lim_{n\to\infty}\mathbb{E}[f_n(S_T)] = \lim_{n\to\infty}\frac{\partial}{\partial\theta}\mathbb{E}[f_n(S_T)]$$

Now as $f_n$ is smooth enough, we can interchange the limit of the derivative with the expectation and then again the limit and the derivative to get

$$\frac{\partial}{\partial\theta}\mathbb{E}[f(S_T)] = \lim_{n\to\infty}\frac{\partial}{\partial\theta}\mathbb{E}[f_n(S_T)] = \mathbb{E}\left[\lim_{n\to\infty}\frac{\partial}{\partial\theta}f_n(S_T)\right] = \mathbb{E}\left[\frac{\partial}{\partial\theta}f(S_T)\right] .$$

# Rule of thumb - 2

Sufficient conditions to apply the pathwise derivative method are described in [1] (chapter 7.2.2). Using only the Black-Scholes model which is very smooth w.r.t to $\theta = S_0, \sigma$, these sufficient conditions summarize as:

**Theorem 1**

Let $f : \Omega \to \mathbb{R}$, ($\Omega \subseteq \mathbb{R}$ or $\mathbb{R}^m$ a domain) and suppose that the following conditions hold:

1. $\mathbb{P}(S_T \in D_f) = 1$, where $D_f := \{s : s \mapsto f(s) \text{ is differentiable}\}$.
2. $s \mapsto f(s)$ is Lipschitz.

Then, $\frac{\partial}{\partial\theta}\mathbb{E}[f(S_T)] = \mathbb{E}\left[\frac{\partial}{\partial\theta}f(S_T)\right]$ (in the multivariate case, $S_T$ would be a random vector as seen in section 3.3).

**Rigorous conditions:** $f$ and $S_T$ almost surely differentiable and Lipschitz respectively w.r.t. $s$ and $\theta$.

# Likelihood ratio condition

The following theorem is useful to prove that we can interchange the derivative and the integral.

## Theorem 2

Let $\Omega \subset \mathbb{R}^n$ open and $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ a function such that

Theorem 2For every $\theta \in [\theta_1, \theta_2]$, the function $s \mapsto \varphi(s, \theta)$ is integrable;for a.e. $s$, the function $s \mapsto \varphi(s, \theta)$ is differentiable on $(\theta_1, \theta_2)$;there exists $\psi \in \mathcal{L}^1(\Omega, \mathbb{R})$ such that $|\frac{\partial \varphi(s,\theta)}{\partial \theta}| \leq \psi(s)$.

Then $\theta \mapsto \int_\Omega \varphi(s, \theta) ds$ is differentiable on $(\theta_1, \theta_2)$ and

$$\frac{\partial}{\partial \theta} \int_\Omega \varphi(s, \theta) ds = \int_\Omega \frac{\partial}{\partial \theta} \varphi(s, \theta) ds \ .$$

We use it by applying $\varphi(s, \theta) = f(s) p_{S_T}(s, \theta)$. Observe that thanks to the smoothness w.r.t. $\theta = S_0, \sigma$ of $p_{S_T}$

# Variance - Finite differences

Once can write the variance of the estimator $\hat{\mu}_{CMC}$ as

$$\text{Var}(\hat{\mu}_{CMC}) = \frac{\text{Var}(f(S_{T,\theta+\Delta\theta}) - f(S_{T,\theta-\Delta\theta}))}{4N(\Delta\theta)^2}$$

with $S_{T,\theta}$ being the simulated underlying price at time $T$ with a model using $\theta$ as one of its parameter, with all its other parameters fixed.
Three cases emerge from this variance:

1. The first one is assuming that $S_{T,\theta+\Delta\theta} \perp\!\!\!\perp S_{T,\theta-\Delta\theta}$, i.e. that two independent samples $W_T$ are drawn for each finite difference estimation. In this case, $\text{Var}(f(S_{T,\theta+\Delta\theta}) - f(S_{T,\theta-\Delta\theta}))$ is in $\mathcal{O}(1)$ with respect to $\Delta\theta$. This means that the estimator's variance $\text{Var}(\hat{\mu}_{CMC}) \in \mathcal{O}\left(\Delta\theta^{-2}\right)$, which might grow quite quickly when $\Delta\theta \ll 1$.
2. The second case is taking the same iid sample $W_T$, which lands that $\text{Var}(f(S_{T,\theta+\Delta\theta}) - f(S_{T,\theta-\Delta\theta}))$ is in $\mathcal{O}(\Delta\theta)$. This gives us $\text{Var}(\hat{\mu}_{CMC}) \in \mathcal{O}(\Delta\theta)$.
3. The third case is $f(S_{T,\theta})$ to be continuous a.s. in $\theta$. If this is the case, then we get that $\text{Var}(f(S_{T,\theta+\Delta\theta}) - f(S_{T,\theta-\Delta\theta}))$ is in $\mathcal{O}(\Delta\theta^2)$, and therefore that $\text{Var}(\hat{\mu}_{CMC}) \in \mathcal{O}(1)$.

In this project, one can observe that $f(S_T)$ is continuous a.s. with respect to $S_0$ and $\sigma$, meaning that we should get a variance of magnitude in $\mathcal{O}(1/N)$ when taking the number of simulations ($N$) into account.

# Bias - Finite difference

By Taylor :

$$I(\theta + \Delta\theta) = I(\theta) + I'(\theta)\Delta\theta + \frac{I''(\theta)}{2}(\Delta\theta)^2 + \mathcal{O}((\Delta\theta)^2)$$
$$I(\theta - \Delta\theta) = I(\theta) - I'(\theta)\Delta\theta + \frac{I''(\theta)}{2}(\Delta\theta)^2 + \mathcal{O}((\Delta\theta)^2)$$

Then, we get

$$\frac{I(\theta + \Delta\theta) - I(\theta - \Delta\theta)}{2\Delta\theta} - I'(\theta) \in \mathcal{O}(\Delta\theta)$$

# Estimation of $\gamma$ - LRLR method

As $f(s) = e^{-rT}[s-K]^+$ does not depend on $S_0$, and $\frac{\dot{p}_\theta}{p_{S_T}}$ is smooth enough, one can check that we can apply Theorem 2 and get

$$\frac{\partial^2 I}{\partial S_0^2} = \int_0^\infty f(s) \underbrace{\frac{\partial^2 p_{S_T}(s)}{\partial S_0^2}}_{:=\ddot{p}_{S_0}(s)} ds = \mathbb{E}\left[f(S_T)\frac{\ddot{p}_{S_0}(s)}{p_{S_T}(s)}\right]$$

where

$$\frac{\ddot{p}_{S_0}(s)}{p_{S_T}(s)} = -\frac{1}{S_0^2\sigma^2 T}\left(1 + \log(\frac{s}{S_0}) - (r - \frac{1}{2}\sigma^2)T\right) + \left(\frac{\log(s/S_0) - (r - \frac{1}{2}\sigma^2)T}{S_0\sigma^2 T}\right)^2$$

We can therefore estimate $\gamma$ using the likelihood ratio method of order 1 with the new score function $\frac{\ddot{p}_{S_0}(s)}{p_{S_T}(s)}$.

# Estimation of $\gamma$ - LRPD method

By applying the LR method:

$$\frac{\partial^2 I}{\partial S_0^2} = \frac{\partial}{\partial S_0}\mathbb{E}\left[f(S_T)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right]$$

Observing that the function $S_0 \mapsto f(S_T)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}$ is continuous and piece-wise differentiable, one can apply the rule of thumb to interchange the derivative and the expectation to get

$$\frac{\partial^2 I}{\partial S_0^2} = \mathbb{E}\left[\frac{\partial}{\partial S_0}\left(f(S_T)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right)\right] = \mathbb{E}\left[f'(S_T)\frac{\partial S_T}{\partial S_0}\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)} + f(S_T)\frac{\partial}{\partial S_0}\left(\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right)\right].$$

With some basics computations one finds

$$\frac{\partial}{\partial S_0}\left(\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right) = \frac{-1}{S_0}\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}$$

which gives us a formula for $\gamma$ that we can easily estimate:

$$\frac{\partial^2 I}{\partial S_0^2} = \mathbb{E}\left[\left(\frac{f'(S_T)S_T - f(S_T)}{S_0}\right)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right].$$

# Estimation of $\gamma$ - PDLR method

If we first apply the the PD method, we have

$$\frac{\partial^2}{\partial S_0^2}\mathbb{E}[f(S_T)] = \frac{\partial}{\partial S_0}\mathbb{E}\left[f'(S_T)\frac{S_T}{S_0}\right] = \frac{\partial}{\partial S_0}\mathbb{E}[g(S_T)]$$

where $g(S_T) = f'(S_T)\frac{\partial S_T}{\partial S_0} = e^{-rT}1_{\{S_T \geq K\}}\frac{S_T}{S_0}$ satisfies the conditions of theorem 2. Hence, we apply the likelihood ratio to $g$ instead of $f$:

$$\begin{aligned}\frac{\partial}{\partial S_0}\mathbb{E}[g(S_T)] &= \int_0^\infty se^{-rT}1_{\{s\geq K\}}\frac{\partial}{\partial S_0}\frac{p_{S_T}(s)}{S_0}ds = \int_0^\infty se^{-rT}1_{\{s\geq K\}}\frac{\dot{p}_{S_0}(s)S_0 - p_{S_T}(s)}{S_0^2}ds \\ &= \int_0^\infty g(s)\dot{p}_{S_0}(s)ds - \frac{1}{S_0}\int_0^\infty g(s)p_{S_T}(s)ds = \mathbb{E}\left[g(S_T)\left(\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)} - \frac{1}{S_0}\right)\right]\end{aligned}$$

which we can easily simulate with the CMC estimator.

# Estimation of $\gamma$ - PDPD method

One could also think about applying the Pathwise derivative method twice. However, we cannot interchange the derivative and the expectation two times for the European Call option. Indeed, one has

$$\frac{\partial^2}{\partial S_0^2}\mathbb{E}[f(S_T)] = \frac{\partial}{\partial S_0}\mathbb{E}\left[f'(S_T)\frac{S_T}{S_0}\right] = \frac{\partial}{\partial S_0}\mathbb{E}[g(S_T)]$$

where $g(s) = f'(s)\frac{s}{S_0} = e^{-rT}\mathbb{1}_{\{s>K\}}\frac{s}{S_0}$ is defined for $s \neq K$. As $g$ is not continuous w.r.t. $s$, the rule of thumb states that one cannot interchange the derivative in $S_0$ and the expectation. One can also see that $\frac{\partial}{\partial S_0}\mathbb{E}[g(S_T)] \neq \mathbb{E}[\frac{\partial}{\partial S_0}g(S_T)]$. Indeed, by estimating an approximation of the left-hand term with the finite differences method we see that it is non zero, while $\mathbb{E}\left[\frac{\partial}{\partial S_0}g(S_T)\right] = \mathbb{E}\left[f''(S_T)(\frac{\partial S_T}{\partial S_0})^2 + f'(S_T)\frac{\partial^2 S_T}{\partial S_0^2}\right] = 0$.

# Digital option - Computations

$$f_\epsilon(s) = \begin{cases} 0 & \text{if } s \leq K - \epsilon \\ \frac{s-K+\epsilon}{2\epsilon} & \text{if } K-\epsilon \leq s \leq K+\epsilon \\ 1 & \text{if } s > K+\epsilon \end{cases} \qquad f'_\epsilon(s) = \begin{cases} 0 & \text{if } s \leq K - \epsilon \\ \frac{1}{2\epsilon} & \text{if } K-\epsilon \leq s \leq K+\epsilon \\ 0 & \text{if } s > K+\epsilon \end{cases}$$

We can now express the Delta as

$$\delta = \frac{\partial}{\partial S_0}\mathbb{E}[f(S_T)] = e^{-rT}\left(\frac{\partial}{\partial S_0}\mathbb{E}[f_\epsilon(S_T))] + \frac{\partial}{\partial S_0}\mathbb{E}[h_\epsilon(S_T))]\right).$$

As $f_e$ is continuous and piece-wise differentiable w.r.t. $S_0$, from the rule of thumb we can apply the pathwise derivative method to compute $\frac{\partial}{\partial S_0}\mathbb{E}[f_\epsilon(S_T)]$:

$$\frac{\partial}{\partial S_0}\mathbb{E}[f_\epsilon(S_T)] = \mathbb{E}\left[f'_\epsilon(S_T)\frac{\partial S_T}{\partial S_0}\right]$$

and as in the previous section, Theorem 2 applies and the LR method can be used to compute $\frac{\partial}{\partial S_0}\mathbb{E}[h_\epsilon(S_T)]$:

$$\frac{\partial}{\partial S_0}\mathbb{E}[h_\epsilon(S_T)] = \int_0^\infty h_\epsilon(s)\frac{\partial p_{S_T}(s)}{\partial S_0}ds = \mathbb{E}\left[h_\epsilon(S_T)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right].$$

# Complete method for optimal sample size

1. Given $N$ random samples from $S_T$, one initially starts a pilot run with $m = \frac{N}{2}$ samples for the PD method, and $N - m$ for the LR method (note that this implies that the two estimators are independent), which gives us the total estimator's variance

$$\begin{aligned}\mathrm{Var}_{tot} &= \mathrm{Var}\left(e^{-rT}\left(\hat{\mu}_{PD}(f_\epsilon) + \hat{\mu}_{LR}(h_\epsilon)\right)\right)\\ &= e^{-2rT}\left[\frac{1}{m}\mathrm{Var}\left(f'_\epsilon(S_T)\frac{\partial S_T}{\partial S_0}\right) + \frac{1}{N-m}\mathrm{Var}\left(h_\epsilon(S_T)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right)\right].\end{aligned}$$

2. By differentiating $\mathrm{Var}_{tot}$ w.r.t. $m$, one can get the following optimal value for $m$:

$$m^*_{1,2} = N\frac{\mathrm{Var}\left(f'_\epsilon(S_T)\frac{\partial S_T}{\partial S_0}\right) \pm \sqrt{\mathrm{Var}\left(f'_\epsilon(S_T)\frac{\partial S_T}{\partial S_0}\right)\mathrm{Var}\left(h_\epsilon(S_T)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right)}}{\mathrm{Var}\left(f'_\epsilon(S_T)\frac{\partial S_T}{\partial S_0}\right) - \mathrm{Var}\left(h_\epsilon(S_T)\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}\right)}$$

then one can select optimal $m$ which is eligible i.e. $m \in (0, N)$ and if both are eligible, one selects $m$ such that it minimizes the original $\mathrm{Var}_{tot}$ function. Please note that $m^*_{1,2}$ will be minima of the function as $\mathrm{Var}_{tot}$ is convex $\forall m \in (0, N)$ as $\frac{\partial^2}{\partial m^2}\mathrm{Var}_{tot} = \frac{2}{m^3}\mathrm{Var}(f'_\epsilon(S_T)\frac{\partial S_T}{\partial S_0}) + \frac{2}{(N-m)^3}\mathrm{Var}(h_\epsilon(S_T))\frac{\dot{p}_{S_0}(S_T)}{p_{S_T}(S_T)}) \geq 0$ for all $m \in (0, N)$.