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# DEVELOPING OPTIMAL SOLUTIONS FOR ORGANIZATIONAL AND BUSINESS NEEDS USING OR (OPERATIONS RESEARCH) AND AI (ARTIFICIAL INTELLIGENCE)

DAY 1: Introduction to optimization using Linear Programming, Integer Programming and Constraint Programming

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[https://github.com/chgogos/INTRO\\_OR\\_AI](https://github.com/chgogos/INTRO_OR_AI)

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# INTRODUCTION TO OPTIMIZATION

- Descriptive analytics (Statistics)
- Predictive analytics (A.I.=Artificial Intelligence)
- Prescriptive analytics (O.R.=Operations Research)

O.R. and A.I. are complementary. Predictive models can inform prescriptive models, and that combination is where the real power lies



<https://doi.org/10.1287/orms.2022.04.14>

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# MATHEMATICAL PROGRAMMING

- Problem description → Mathematical Modeling → Mathematical Programming → Solve using solver (e.g., Gurobi, IBM ILOG CPLEX, Google OR-Tools, HIGHS, ...) → Evaluate results → Reiterate
- Mathematical Programming is a branch of O.R. (Operations Research)

## **State of Mathematical Optimization Report, 2021**

<https://cdn.gurobi.com/wp-content/uploads/2022/08/Report-StateOfMathematicalOptimizationReport.pdf>

85% of Fortune 500 companies use optimization

# O.R. & ANALYTICS SUCCESS STORIES

**INFORMS** = Institute for Operations Research and the Management Sciences



**Improving Energy Efficiency**  
Maximizing power plant usage to improve reliability of energy resources



<https://www.informs.org/Impact/O.R.-Analytics-Success-Stories>



Faster and Further  
Improving train travel mile by mile

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Transforming Aviation Security  
Safeguarding aircraft, crew and passengers

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Increasing Bike-share Efficiency  
Transforming how we travel

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Enhancing Product Platforms  
Barco improves product features and shortens production time

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More Effective Television Advertising  
Turner Broadcasting uses analytics to provide targeted advertising to viewers

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Improving Organ Donation  
Pairing donors with patients in need to save lives

[Read more](#)

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# A SIMPLE EXAMPLE OF MODELING: INVESTMENT OPPORTUNITIES

An amount of 2,000,000 € will be allocated across four investment opportunities, labelled A, B, C, and D. A minimum of 200,000 € and a maximum of 1,000,000 € must be invested in each opportunity. The combined investment in A and B must not exceed 1,500,000 €. The projected returns are 9%, 11%, 5%, and 8% for opportunities A, B, C, and D, respectively.

$$\text{Maximize } Z = 0.09x_A + 0.11x_B + 0.05x_C + 0.08x_D$$

s.t.

$$x_A + x_B + x_C + x_D = 2,000,000 \quad (1)$$

$$x_A \geq 200,000 \quad (2)$$

$$x_B \geq 200,000 \quad (3)$$

$$x_C \geq 200,000 \quad (4)$$

$$x_D \geq 200,000 \quad (5)$$

$$x_A \leq 1,000,000 \quad (6)$$

$$x_B \leq 1,000,000 \quad (7)$$

$$x_C \leq 1,000,000 \quad (8)$$

$$x_D \leq 1,000,000 \quad (9)$$

$$x_A + x_B \leq 1,500,000 \quad (10)$$

---

# SOLUTION FOR THE SIMPLE INVESTMENT EXAMPLE USING OR-TOOLS

Maximize  $Z = 0.09x_A + 0.11x_B + 0.05x_C + 0.08x_D$   
s.t.  
 $x_A + x_B + x_C + x_D = 2,000,000$   
 $x_A \geq 200,000$   
 $x_B \geq 200,000$   
 $x_C \geq 200,000$   
 $x_D \geq 200,000$   
 $x_A \leq 1,000,000$   
 $x_B \leq 1,000,000$   
 $x_C \leq 1,000,000$   
 $x_D \leq 1,000,000$   
 $x_A + x_B \leq 1,500,000$

---

```
from ortools.linear_solver import pywraplp

solver = pywraplp.Solver.CreateSolver('SCIP')

# Decision variables
x_A = solver.NumVar(200_000, 1_000_000, 'x_A')
x_B = solver.NumVar(200_000, 1_000_000, 'x_B')
x_C = solver.NumVar(200_000, 1_000_000, 'x_C')
x_D = solver.NumVar(200_000, 1_000_000, 'x_D')

# Constraints
solver.Add(x_A + x_B + x_C + x_D == 2_000_000)

solver.Add(x_A + x_B <= 1_500_000)

# Objective
solver.Maximize(0.09 * x_A + 0.11 * x_B + 0.05 * x_C + 0.08 * x_D)

# Solve
status = solver.Solve()

if status == pywraplp.Solver.OPTIMAL:
    print(f'Optimal solution found:')
    print(f'x_A = {x_A.solution_value():,.2f}')
    print(f'x_B = {x_B.solution_value():,.2f}')
    print(f'x_C = {x_C.solution_value():,.2f}')
    print(f'x_D = {x_D.solution_value():,.2f}')
    print(f'Maximum Return = {solver.Objective().Value():,.2f}')
else:
    print('The problem does not have an optimal solution.')
```

```
Optimal solution found:
x_A = 500,000.00
x_B = 1,000,000.00
x_C = 200,000.00
x_D = 300,000.00
Maximum Return = 189,000.00
```

---

# PYTHON BASICS AND BEYOND!

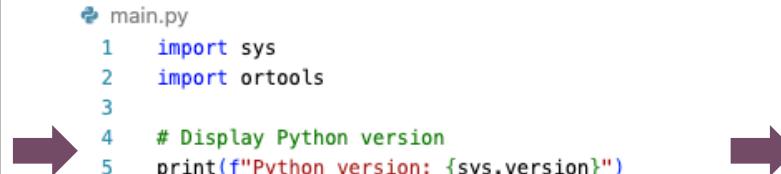
- Lists, sets, tuples, dictionaries
- Iterations over lists & dictionaries, enumerate
- Functions
- Assertions
- OOP (classes, methods)
- Comprehensions
- f-strings
- lambdas
- Read data from text files, write data to text files
- Measure execution time of code
- Debugging
- <https://learnxinyminutes.com/python/>
- Python As Fast as Possible - Learn Python in ~75 Minutes -  
<https://www.youtube.com/watch?v=VchuKL44s6E>
- <https://marko-knoebel.github.io/slides/python-all-en.html>
- <https://programming-25.mooc.fi/>

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# PACKAGE MANAGERS + VIRTUAL ENVIRONMENTS

- UV: modern high performance Python package manager (10-100x faster than pip)
- UV can act as a replacement for the traditional tools pip and virtualenv (also for conda, poetry)
- UV provides better cross-platform consistency than its peers
- UV installation instructions: <https://docs.astral.sh/uv/getting-started/installation/>

```
$ uv init prj1
Initialized project `prj1` at `/Users/a_user/dev_git_repos/prj1`
$ cd prj1
$ tree -a
.
├── main.py
├── pyproject.toml
└── README.md
uv.lock
$ uv add ortools
Using CPython 3.11.11
Creating virtual environment at: .venv
Resolved 11 packages in 682ms
Prepared 3 packages in 6.94s
Installed 10 packages in 84ms
...
+ ortools==9.12.4544
...
```



```
main.py
1 import sys
2 import ortools
3
4 # Display Python version
5 print(f"Python version: {sys.version}")
6
7 # Display OR-Tools version
8 print(f"OR-Tools version: {ortools.__version__}")
9
```

```
$ uv run main.py
Python version: 3.11.11 (main, Dec 19 2024,
14:23:18) [Clang 18.1.8 ]
OR-Tools version: 9.12.4544
```

# EXAMPLE: SEND + MORE = MONEY

```
1  from ortools.sat.python import cp_model
2
3  model = cp_model.CpModel()
4
5  s = model.new_int_var(1, 9, "S")  # cannot be 0
6  e = model.new_int_var(0, 9, "E")
7  n = model.new_int_var(0, 9, "N")
8  d = model.new_int_var(0, 9, "D")
9  m = model.new_int_var(1, 9, "M")  # cannot be 0
10 o = model.new_int_var(0, 9, "O")
11 r = model.new_int_var(0, 9, "R")
12 y = model.new_int_var(0, 9, "Y")
13
14 letters = [s, e, n, d, m, o, r, y]
15
16 model.add_all_different(letters)
17
18 # SEND + MORE = MONEY
19 send = 1000 * s + 100 * e + 10 * n + d
20 more = 1000 * m + 100 * o + 10 * r + e
21 money = 10000 * m + 1000 * o + 100 * n + 10 * e + y
22
23 model.add(send + more == money)
24
25 solver = cp_model.CpSolver()
26 status = solver.solve(model)
27
28 if status == cp_model.FEASIBLE or status == cp_model.OPTIMAL:
29     print(f"S = {solver.value(s)}")
30     print(f"E = {solver.value(e)}")
31     print(f"N = {solver.value(n)}")
32     print(f"D = {solver.value(d)}")
33     print(f"M = {solver.value(m)}")
34     print(f"O = {solver.value(o)}")
35     print(f"R = {solver.value(r)}")
36     print(f"Y = {solver.value(y)}")
37     print()
38     print(f"{solver.value(send)} + {solver.value(more)} = {solver.value(money)}")
```

```
$ uv run cryptarithmetic.py
S = 9
E = 5
N = 6
D = 7
M = 1
O = 0
R = 8
Y = 2
9567
+ 1085
= 10652
```

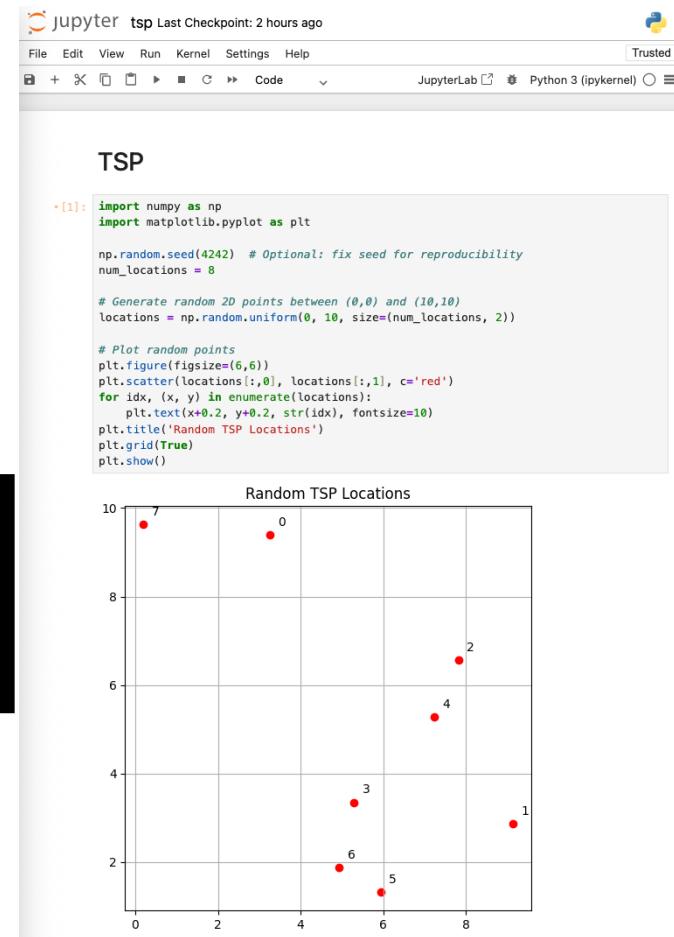
How about:  
**TWO + TWO = FOUR**

?

# JUPYTER NOTEBOOK

- Jupyter notebook is a popular tool for interactive computing, data analysis, and visualization

```
$ uv init prj2
Initialized project `prj2` at `/Users/a_user/dev_git_repos/prj2`
$ cd prj2
$ uv add jupyter ortools matplotlib numpy
...
$ uv run jupyter notebook
[I 2025-04-28 11:50:20.242 ServerApp] http://localhost:8888/tree?token=c96782dbcf41a97121a3bb5382da454b1a6d2375da26e0e7
[I 2025-04-28 11:50:20.242 ServerApp] http://127.0.0.1:8888/tree?token=c96782dbcf41a97121a3bb5382da454b1a6d2375da26e0e7
[I 2025-04-28 11:50:20.242 ServerApp] Use Control-C to stop this server and shut down all kernels (twice to skip confirmation).
...
```



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# VSCODE + PYTHON

- <https://marko-knoebl.github.io/slides/python-beginner-collection-en.html> - VS Code for Python



# GOOGLE COLAB

- Google's Colab is a cloud based Jupyter Notebook service that requires no setup to use and provides **free access to computing resources**
- Google colab notebooks can be saved into Google Drive



[https://colab.research.google.com/github/google/or-tools/blob/stable/examples/notebook/sat/solve\\_with\\_time\\_limit\\_sample.ipynb](https://colab.research.google.com/github/google/or-tools/blob/stable/examples/notebook/sat/solve_with_time_limit_sample.ipynb)

The screenshot shows a Google Colab notebook titled "solve\_with\_time\_limit\_sample.ipynb". The notebook interface includes a toolbar with File, Edit, View, Insert, Runtime, Tools, and Help; a search bar; and tabs for Commands, Code, and Text. The main content area displays the Apache License 2.0 notice, the file structure, and the code for solving a CP-SAT problem. The code uses the ortools library to create a model, set a time limit, and solve it. The output section shows the results: x = 1, y = 0, z = 0.

```
File Edit View Insert Runtime Tools Help
Q, Commands + Code + Text
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solve_with_time_limit_sample_sat

Run in Google Colab View source on GitHub

First, you must install ortools package in this colab.

%pip install ortools
Show hidden output

Solves a problem with a time limit.

from ortools.sat.python import cp_model

def solve_with_time_limit_sample():
    """Minimal CP-SAT example to showcase calling the solver."""
    # Creates the model.
    model = cp_model.CpModel()
    # Creates the variables.
    num_vals = 3
    x = model.NewIntVar(0, num_vals - 1, "x")
    y = model.NewIntVar(0, num_vals - 1, "y")
    z = model.NewIntVar(0, num_vals - 1, "z")
    # Adds an all-different constraint.
    model.Add(x != y)

    # Creates a solver and solves the model.
    solver = cp_model.CpSolver()

    # Sets a time limit of 10 seconds.
    solver.parameters.max_time_in_seconds = 10.0

    status = solver.Solve(model)

    if status == cp_model.OPTIMAL:
        print(f"x = {solver.Value(x)}")
        print(f"y = {solver.Value(y)}")
        print(f"z = {solver.Value(z)}")

solve_with_time_limit_sample()

x = 1
y = 0
z = 0
```

# GOOGLE OR-TOOLS - RESOURCES

- <https://developers.google.com/optimization>
- [https://github.com/or-tools/awesome\\_or-tools](https://github.com/or-tools/awesome_or-tools)
- <https://d-krupke.github.io/cpsat-primer/>
- <https://kunlei.github.io/python-ortools-notes/>

The screenshot shows the official Google OR-Tools website. At the top, there's a navigation bar with links for 'Google OR-Tools' (with a logo), 'OR-Tools', 'OR API', 'Search' (with a search icon), 'English' (with a dropdown arrow), and 'Sign in'. Below the navigation is a blue header section with the text 'Route. Schedule. Plan. Assign. Pack. Solve.' and a subtext 'OR-Tools is fast and portable software for combinatorial optimization.' In the center, there are three main sections: 'Get started with OR-Tools' featuring a large question mark icon, 'Install OR-Tools' featuring a blue cube icon, and a sidebar note about winning gold in international constraint programming competitions. Below these are sections for 'About OR-Tools' (describing it as an open source suite for optimization) and 'Discussion forum' (linking to Visit our forum). To the right, there's a 'Google AI' logo and social media links for 'Discussion forum', 'Discord', 'GitHub', and 'Stack Overflow'.

Google OR-Tools

OR-Tools OR API

Search / English Sign in

Route. Schedule. Plan. Assign. Pack. Solve.

OR-Tools is fast and portable software for combinatorial optimization.

Get started with OR-Tools

Learn how to solve optimization problems from C++, Python, C#, or Java.

Get started

Install OR-Tools

See the [Release Notes](#) for the latest updates.

Install OR-Tools

About OR-Tools

OR-Tools is an open source software suite for optimization, tuned for tackling the world's toughest problems in vehicle routing, flows, integer and linear programming, and constraint programming.

After modeling your problem in the programming language of your choice, you can use any of a half dozen solvers to solve it: commercial solvers such as Gurobi or CPLEX, or open-source solvers such as SCIP, GLPK, or Google's GLOP and award-winning CP-SAT.

Discussion forum

Visit our forum

Discord

Join Google OR-Tools Discord server

GitHub

Download our code

Stack Overflow

Ask questions using the or-tools or cp-sat tags

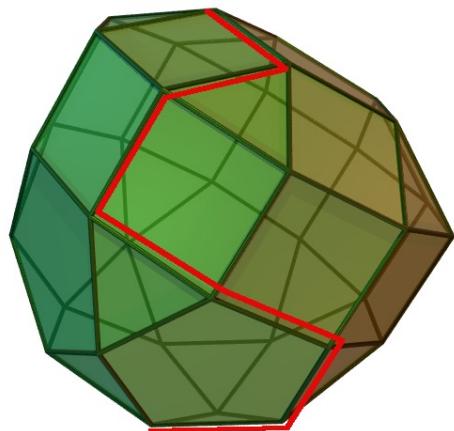
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# INTRODUCTION TO OPTIMIZATION USING LINEAR PROGRAMMING, INTEGER PROGRAMMING AND CONSTRAINT PROGRAMMING

Model and solve decision support problems using Google's OR-Tools and Python

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# LINEAR PROGRAMMING



- Linear Programming (LP) is used a lot in practice
- LP entails formulation (modeling) and then solving of linear programs
- Basic elements of a Linear Program:
  - **Decision variables** (assume continuous values)
  - **Constraints** (functions should be linear over decision variables)
  - **Objective function** (either maximize or minimize of a linear function over the decision variables)
- The general formulation of a mathematical program:

$$\begin{aligned} \max \quad & f(x_1, x_2, \dots, x_n) \\ \text{s.t.} \quad & g_i(x_1, x_2, \dots, x_n) \leq b_i \quad \forall i = 1, \dots, m \\ & x_j \in \mathbb{R} \quad \forall j = 1, \dots, n \end{aligned}$$

- Concrete example of a problem instance:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 6 \\ & 2x_1 + x_2 \leq 6 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

---

# LINEAR PROGRAMMING – COMPACT LP FORMULATIONS

- Most problem instances, in practice, have big (or huge!) numbers of variables and constraints
- Compact formulations enhance readability by use of **indices** (e.g.,  $i, j, k$ ), the **summation symbol** ( $\Sigma$ ) and the **for all symbol** ( $\forall$ )
- Example of compact formulation:

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j x_j \\ \text{s.t.} \quad & \sum_{j \in J} a_{ij} x_j \geq b_i, \quad \forall i \in I \\ & x_j \geq 0, \quad \forall j \in J \end{aligned}$$

where  $x_j$  are the decision variables for each  $j \in J$ ,  
 $c_j$  is the cost coefficient for each  $j \in J$ ,  
 $a_{ij}$  is the coefficient of variable  $j$  in constraint  $i$ ,  
 $b_i$  is the right-hand side value for constraint  $i$ ,  
 $I, J$  are finite index sets.

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# LINEAR PROGRAMMING – ALGORITHMS & SOLVERS

- Effective algorithms exist for solving LPs:
    - Simplex method (Dantzig, 1947) and its variants, i.e., Revised Simplex, Dual Simplex, Primal-Dual Simplex
    - Interior point methods (Karmarkar, 1984), i.e., Barrier
    - ~~Ellipsoid method (Khachiyan, 1979)~~
    - First-order methods (for large LPs), i.e., Primal-Dual Hybrid Gradient
  - Commercial and **open-source** software provide SoTA implementations of LP solvers:
    - Gurobi
    - IBM ILOG CPLEX
    - FICO XPRESS
    - MOSEK
    - **OR-Tools (GLOP, PDLP, and 3<sup>rd</sup> party LP solvers)**
    - **HIGHS (LP solvers)**
    - **SCIP Optimization Suite (SoPlex)**
    - **COIN-OR (CLP)**
    - **GLPK**
- The relative performance of algorithms and solvers can vary by orders of magnitude on a single instance.
  - No single algorithm and solver is uniformly better than others.
  - Crossover (enabled by default) increases solve time, sometimes substantially.
  - PDLP can solve to low precision sometimes 10 times faster than to high precision.
- [https://developers.google.com/optimization/lp/lp\\_advanced](https://developers.google.com/optimization/lp/lp_advanced)

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# LP EXAMPLE – DIET PROBLEM (PROBLEM STATEMENT)

You are tasked with planning a daily diet using 5 available food items. The goal is to minimize the total cost while ensuring that the diet provides at least the required daily amounts of 3 essential nutrients: **Calories**, **Protein**, and **Vitamins**. Each food item has a known cost per unit and provides specific amounts of these nutrients. The **nutritional requirements** are at least 1,200 calories per day, at least 50 grams of protein per day and at least 5 grams of vitamins per day.

Food	Cost per Unit (€)	Calories (per unit)	Protein (grams per unit)	Vitamins (grams per unit)
Food 1	2	300	5	1
Food 2	1.5	200	10	2
Food 3	3	400	8	3
Food 4	2	250	6	2
Food 5	4	500	12	4

Nutritional Information for Each Food Item

---

# DIET PROBLEM (FORMULATION X 2 )

Food	Cost per Unit (€)	Calories (per unit)	Protein (grams per unit)	Vitamins (grams per unit)
Food 1	2	300	5	1
Food 2	1.5	200	10	2
Food 3	3	400	8	3
Food 4	2	250	6	2
Food 5	4	500	12	4

$$\min \quad Z = 2x_1 + 1.5x_2 + 3x_3 + 2x_4 + 4x_5$$

$$\min \quad Z = \mathbf{c}^T \mathbf{x}$$

$$\text{s.t.} \quad A\mathbf{x} \geq \mathbf{b}$$

$$\text{s.t.} \quad 300x_1 + 200x_2 + 400x_3 + 250x_4 + 500x_5 \geq 1200$$

$$\mathbf{x} \geq 0$$

$$5x_1 + 10x_2 + 8x_3 + 6x_4 + 12x_5 \geq 50$$

$$x_1 + 2x_2 + 3x_3 + 2x_4 + 4x_5 \geq 5$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

where

$$\mathbf{c} = \begin{pmatrix} 2 \\ 1.5 \\ 3 \\ 2 \\ 4 \end{pmatrix}, \quad A = \begin{pmatrix} 300 & 200 & 400 & 250 & 500 \\ 5 & 10 & 8 & 6 & 12 \\ 1 & 2 & 3 & 2 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1200 \\ 50 \\ 5 \end{pmatrix}$$

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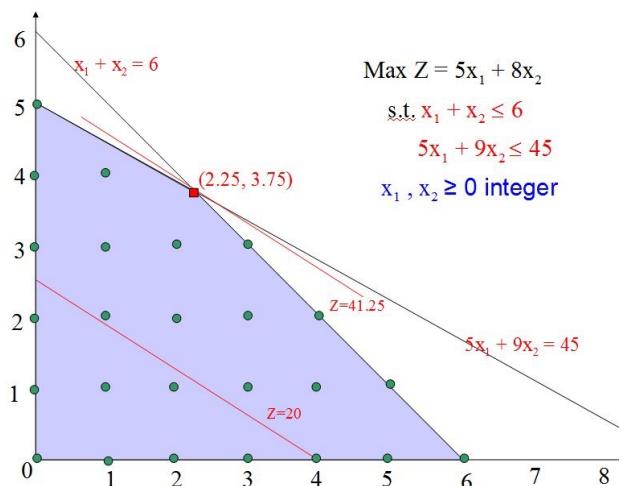
# DIET PROBLEM (SOLUTION USING OR-TOOLS GLOP)

```
1  from ortools.linear_solver import pywraplp
2
3  # Create the solver
4  solver = pywraplp.Solver.CreateSolver('GLOP')
5
6  # Variables (continuous)
7  x1 = solver.NumVar(0.0, solver.infinity(), 'x1')
8  x2 = solver.NumVar(0.0, solver.infinity(), 'x2')
9  x3 = solver.NumVar(0.0, solver.infinity(), 'x3')
10 x4 = solver.NumVar(0.0, solver.infinity(), 'x4')
11 x5 = solver.NumVar(0.0, solver.infinity(), 'x5')
12
13 # Objective function
14 solver.Minimize(2 * x1 + 1.5 * x2 + 3 * x3 + 2 * x4 + 4 * x5)
15
16 # Constraints
17 solver.Add(300 * x1 + 200 * x2 + 400 * x3 + 250 * x4 + 500 * x5 >= 1200) # Calories
18 solver.Add(5 * x1 + 10 * x2 + 8 * x3 + 6 * x4 + 12 * x5 >= 50) # Protein
19 solver.Add(x1 + 2 * x2 + 3 * x3 + 2 * x4 + 4 * x5 >= 5) # Vitamins
20
21 # Solve the problem
22 status = solver.Solve()
23
24 # Results
25 solution = {}
26 if status == pywraplp.Solver.OPTIMAL:
27     solution = {
28         'x1': x1.solution_value(),
29         'x2': x2.solution_value(),
30         'x3': x3.solution_value(),
31         'x4': x4.solution_value(),
32         'x5': x5.solution_value(),
33         'objective': solver.Objective().Value()
34     }
35
36 print(solution)

1  from ortools.linear_solver import pywraplp
2
3  # Create the solver
4  solver = pywraplp.Solver.CreateSolver('GLOP')
5
6  # Data (costs, nutritional values)
7  c = [2, 1.5, 3, 2, 4] # Cost per unit for each food
8  A = [
9      [300, 200, 400, 250, 500], # Calories
10     [5, 10, 8, 6, 12], # Protein
11     [1, 2, 3, 2, 4] # Vitamins
12 ]
13 b = [1200, 50, 5] # Minimum nutritional requirements
14
15 # Variables (continuous)
16 x = [solver.NumVar(0.0, solver.infinity(), f'x{i+1}') for i in range(5)]
17
18 # Objective function: Minimize c^T * x
19 objective = solver.Objective()
20 for i in range(5):
21     objective.SetCoefficient(x[i], c[i])
22 objective.SetMinimization()
23
24 # Constraints: A * x >= b
25 for i in range(3): # 3 constraints
26     constraint = solver.Constraint(b[i], solver.infinity())
27     for j in range(5): # 5 foods
28         constraint.SetCoefficient(x[j], A[i][j])
29
30 # Solve the problem
31 status = solver.Solve()
32
33 # Results
34 solution = {}
35 if status == pywraplp.Solver.OPTIMAL:
36     solution = {
37         'x': [x[i].solution_value() for i in range(5)],
38         'objective': solver.Objective().Value()
39     }
40
41 print(solution)
```

---

# INTEGER PROGRAMMING



- In some problems, variables must only assume **integer values**
- **Integer Programming (IP)** is the process of formulating and solving problems with integer variables
- An IP is actually an ILP (Integer Linear Program)
- **MIP** (Mixed Integer Programming) and **BIP** (Binary Integer Programming) are special cases of IP
- Non-Linear Programs (e.g., using the quadratic function over a decision variable in the objective or the constraints) is another class of problems (NLPs)

---

# INTEGER PROGRAMMING – ALGORITHMS & SOLVERS

- Effective\* algorithms exist for solving IPs:
  - Branch and Bound (B&B) – solves linear relaxations and systematically partitions the problem space (branching) while using bounds (bounding) to prune subproblems
  - Branch and Cut (B&C) – an extension to B&B that adds cutting planes (valid inequalities) during the search to tighten the LP relaxation and speed up convergence
- Commercial and [open-source](#) software provide SoTA implementations of IP solvers:
  - Gurobi
  - IBM ILOG CPLEX
  - FICO XPRESS
  - [OR-Tools](#) (uses 3<sup>rd</sup> party IP solvers such as [SCIP](#) and [CBC](#))
  - [HIGHs](#) (MIP solver)
  - [SCIP Optimization Suite \(SCIP\)](#)
  - [COIN-OR \(CBC\)](#)
  - [GLPK](#) (supports both LP and IP)

\*although, less effective than their LP counterparts

---

# IP EXAMPLE: THE 0/1 KNAKSACK PROBLEM

The 0/1 Knapsack Problem is a classic combinatorial optimization problem where, given a set of items each with a weight and a value, the goal is to determine the most valuable subset of items to include in a knapsack without exceeding its weight capacity



$$\begin{aligned} \max \quad & \sum_{i=1}^n v_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^n w_i x_i \leq W \\ & x_i \in \{0, 1\}, \quad i = 1, 2, \dots, n \end{aligned}$$

where:

- $v_i$  = value (or profit) of item  $i$ ,
- $w_i$  = weight of item  $i$ ,
- $W$  = capacity of the knapsack.

---

There are special purpose algorithms for the knapsack problem that can solve it efficiently (e.g., Dynamic Programming)

---

# THE 0/1 KNAPSACK PROBLEM – FORMULATION



Item	1	2	3	4	5
Value (\$)	4	2	1	2	10
Weight (kg)	12	1	1	2	4

$$\begin{aligned} \text{max } & 4x_1 + 2x_2 + x_3 + 2x_4 + 10x_5 \\ \text{s.t. } & 12x_1 + x_2 + x_3 + 2x_4 + 4x_5 \leq 15 \\ & x_i \in \{0, 1\}, \quad \forall i = 1, \dots, 5 \end{aligned}$$

```
1  from ortools.linear_solver import pywraplp
2
3  solver = pywraplp.Solver.CreateSolver("SCIP")
4
5  # Binary Decision Variables
6  x = []
7  for i in range(5):
8      x.append(solver.IntVar(0, 1, f"x{i+1}"))
9
10 # Objective function
11 solver.Maximize(4 * x[0] + 2 * x[1] + x[2] + 2 * x[3] + 10 * x[4])
12
13 # Constraint
14 solver.Add(12 * x[0] + x[1] + x[2] + 2 * x[3] + 4 * x[4] <= 15)
15
16 status = solver.Solve()
17
18 if status == pywraplp.Solver.OPTIMAL:
19     print("Solution:")
20     for i in range(5):
21         print(f"x{i+1} = {int(x[i].solution_value())}")
22         print(f"Maximum objective value = {solver.Objective().Value()}")
23 else:
24     print("No optimal solution found.")
```

# P VS NP PROBLEM

 Clay Mathematics Institute

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Unsolved

## P vs NP

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

<https://www.claymath.org/millennium/p-vs-np/>



"I can't find an efficient algorithm, but neither can all these famous people."

An example of an NP problem is the **Subset Sum Problem (SSP)**: Given a set of integers and a target sum  $T$ , is there a subset whose sum equals  $T$ ?

SSP is directly related to the 0/1 knapsack problem

# CONSTRAINT PROGRAMMING

*“Constraint programming represents one of the closest approaches computer science has yet made to the Holy Grail of programming: the user states the problem, the computer solves it.”*

*Eugene Freuder, 1996*

- **Constraint Programming (CP)** is a paradigm for solving combinatorial problems by stating constraints that must be satisfied
- Core idea: The problem is modelled using variables that have finite domains (e.g.,  $x \in \{1, 2, 3, 4\}$ ,  $y \in \{1 \dots 10\}$ ,  $z = \{2, 4, 6, 8, 10\}$ ) and constraints (e.g.,  $3x \neq y$ ,  $x * y = z$ )
- The solving process:
  - Constraint propagation (reduce domains by enforcing constraints)
  - Search (explore assignments using backtracking if necessary)
- CP can be used for feasibility problems and for optimization problems
- CP enables a more flexible modelling framework than LP and IP, as it supports general combinatorial and logical constraints beyond linear forms

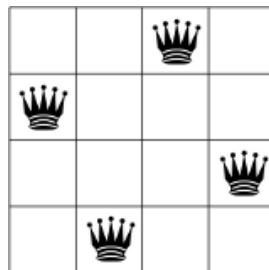
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# CONSTRAINT PROGRAMMING – ALGORITHMS & SOLVERS

- Several algorithms have been developed for solving Constraint Programming problems:
  - Arc consistency (e.g., AC-3, AC-4, AC-2001)
  - Global constraint propagators (e.g., all-different, if-then, cumulative)
  - Backtracking search and Backjumping
  - Branch and Bound (B&B)
  - Heuristics (e.g., MRV, LCV)
  - Lazy Clause Generation (LCG)
  - Large Neighborhood Search (LNS)
- Commercial and [open-source](#) software provide SoTA implementations of CP solvers:
  - IBM ILOG CPLEX (CP-Optimizer)
  - FICO XPRESS (XPRESS-CP)
  - [OR-Tools \(CP, CP-SAT\)](#)
  - [Choco Solver](#)
  - [Gecode](#)
  - [Chuffed](#)

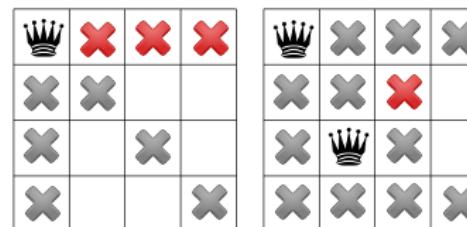
# CP EXAMPLE – THE N-QUEENS PROBLEM

How can N queens be placed on an NxN chessboard so that no two of them attack each other?

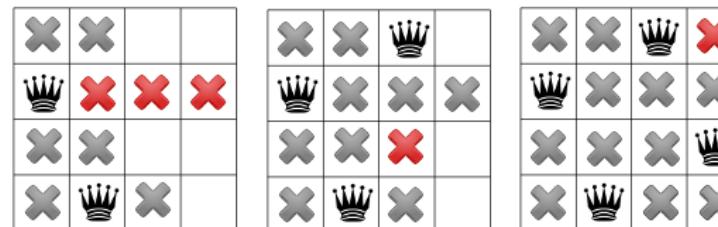


<https://developers.google.com/optimization/cp/queens>

- N-Queens is not an optimization problem
- Propagation + backtracking



backtrack occurs here!



---

# THE N-QUEENS PROBLEM – FORMULATION

```
1  from ortools.constraint_solver import pywrapcp
2
3  def main(board_size):
4      solver = pywrapcp.Solver("n-queens")
5
6      # Variables: The array index is the column, and the value is the row.
7      queens = [solver.IntVar(0, board_size - 1, f"x{i}") for i in range(board_size)]
8
9      # All rows must be different.
10     solver.Add(solver.AllDifferent(queens))
11
12     # No two queens can be on the same diagonal.
13     solver.Add(solver.AllDifferent([queens[i] + i for i in range(board_size)]))
14     solver.Add(solver.AllDifferent([queens[i] - i for i in range(board_size)]))
15
16     db = solver.Phase(queens, solver.CHOOSE_FIRST_UNBOUND, solver.ASSIGN_MIN_VALUE)
17
18     solver.NewSearch(db)
19     while solver.NextSolution():
20         for i in range(board_size):
21             for j in range(board_size):
22                 if queens[j].Value() == i:
23                     print("Q", end=" ")
24                 else:
25                     print("_", end=" ")
26             print()
27         print()
28     solver.EndSearch()
29
30 main(8)
```

1 of the 92 solutions for the 8 x 8 board

queens[0]=0	Q	-	-	-	-	-	-	-
queens[1]=4	-	-	-	-	-	-	-	Q
...	-	-	-	-	-	-	Q	-
queens[7]=3	-	-	-	-	-	Q	-	-
	-	-	-	-	Q	-	-	-
	-	-	-	-	-	Q	-	-
	-	-	-	-	-	-	Q	-
	-	-	-	-	-	-	-	Q
	-	-	-	-	-	-	-	-

A different formulation, that solves the problem using IP, can be found at:  
<https://kunlei.github.io/python-ortools-notes/ip-nqueens.html>

---

# THE N-QUEENS PROBLEM – IMPLEMENTATION USING CP-SAT

```
1  from ortools.sat.python import cp_model
2
3
4  def solve(board_size):
5      # Create the solver
6      model = cp_model.CpModel()
7
8      # Variables: The array index is the column, and the value is the row.
9      queens = [model.new_int_var(0, board_size - 1, f"x{i}") for i in range(board_size)]
10
11     # All rows must be different.
12     model.add_all_different(queens)
13
14     # No two queens can be on the same diagonal.
15     model.add_all_different([queens[i] + i for i in range(board_size)])
16     model.add_all_different([queens[i] - i for i in range(board_size)])
17
18     # Solve the model.
19     solver = cp_model.CpSolver()
20     solver.solve(model)
21
22     # Display the first solution found and terminate.
23     all_queens = range(board_size)
24     for i in all_queens:
25         for j in all_queens:
26             if solver.value(queens[j]) == i:
27                 # There is a queen in column j, row i.
28                 print("Q", end=" ")
29             else:
30                 print("_", end=" ")
31     print()
32
33
34  if __name__ == "__main__":
35      solve(8)
```

This code displays only the first solution

Q	-	-	-	-	-	-	-
-	-	-	-	-	-	-	Q
-	-	-	-	-	-	Q	-
-	-	-	-	-	-	-	Q
-	-	-	-	-	-	Q	-
-	-	-	-	Q	-	-	-
-	-	-	-	-	Q	-	-
-	-	-	Q	-	-	-	-

---

# HANDS-ON ACTIVITIES



Resource Allocation



Workforce Planning



Cryptarithmetic puzzle

---

# HANDS-ON – RESOURCE ALLOCATION – THE DIET PROBLEM

1. Find what the “Stigler Diet” is by consulting either the Web or Open AI’s ChatGPT
2. Navigate through the OR-Tools web site and **locate the code** that using LP finds the optimal solution for the “Stigler diet”
3. Execute locally the code, using GLOP (i.e., the Linear Programming solver developed as part of OR-Tools) and another LP solver (e.g., CLP of COIN-OR) and compare the results
4. What would have been the cost of the optimal solution in current money value (you can use <https://data.bls.gov/cgi-bin/cpicalc.pl>) ?

## Extra

- Visit: <https://www.feedcalculator.org/>
- Read: Dantzig, George B. “The diet problem.” Interfaces 20.4 (1990): 43-47.



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# HANDS-ON – WORKFORCE PLANNING

- Assume that we operate a department store and we know the number of employees needed for each day of the week. Each employee must work for five consecutive days. How many employees should we hire for each one of the 7 shifts:
  - Monday to Friday = ?
  - Tuesday to Saturday = ?
  - Wednesday to Sunday = ?
  - Thursday to Monday = ?
  - Friday to Tuesday = ?
  - Saturday to Wednesday = ?
  - Sunday to Thursday = ?
- The goal is to minimize the number of employees hired

Mon	Tue	Wed	Thu	Fri	Sat	Sun
90	80	100	60	80	150	140

Tip: Use 7 decision variables, the first one should assume the value of the number of employees that are hired in shift Monday to Friday, the second one should assume the value of the number of employees that are hired in shift Tuesday to Saturday, etc.

---

## HANDS-ON – WORKFORCE PLANNING – FORMULATION

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\ \text{s.t.} \quad & x_1 + x_4 + x_5 + x_6 + x_7 \geq 90 \\ & x_1 + x_2 + x_5 + x_6 + x_7 \geq 80 \\ & x_1 + x_2 + x_3 + x_6 + x_7 \geq 100 \\ & x_1 + x_2 + x_3 + x_4 + x_7 \geq 60 \\ & x_1 + x_2 + x_3 + x_4 + x_5 \geq 80 \\ & x_2 + x_3 + x_4 + x_5 + x_6 \geq 150 \\ & x_3 + x_4 + x_5 + x_6 + x_7 \geq 140 \\ & x_i \geq 0 \quad \forall i = 1, \dots, 7 \end{aligned}$$

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# HANDS-ON – CRYPTARITHMETIC PUZZLES

Solve the following cryptarithmetic puzzle:

$$\mathbf{NINE - FOUR = FIVE}$$

The constraints are the following:

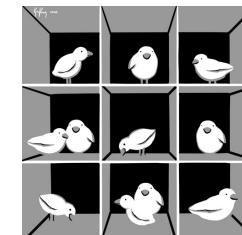
- All letters map to distinct digits
- Leading letters cannot be zero
- The subtraction must hold numerically

Use OR-Tools CP-SAT to find a single solution (if it exists)

Do you believe that the following cryptarithmetic puzzles have solutions?

$$\mathbf{BASE \times BALL = GAMES}$$

$$\mathbf{PEACE + MIND = RELAX}$$



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# MANY OTHER OPTIMIZATION APPROACHES EXIST

- Gradient Descent (GD) & Stochastic Gradient Descent (SGD)
- Dynamic Programming (DP)
- Simulated Annealing (SA)
- Tabu Search (TS)
- Genetic Algorithms (GA)
- Particle Swarm Optimization (PSO)
- Ant Colony Optimization (ACO)
- Iterated Local Search (ILS)
- Variable Neighborhood Search (VNS)
- Reinforcement Learning (RL)
- Quantum Annealing (QA)
- Quantum Unconstrained Binary Optimization (QUBO)

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# REFERENCES

- The Burrito Optimization Game, <https://doi.org/10.1287/orms.2022.04.14>
- The state of the Mathematical Optimization Report 2021, <https://cdn.gurobi.com/wp-content/uploads/2022/08/Report-StateOfMathematicalOptimizationReport.pdf>
- CP-SAT Primer, <https://d-krupke.github.io/cpsat-primer/>
- Freuder, E. C. (1997). In pursuit of the holy grail. *Constraints*, 2, 57-61.
- ACP Summer School 2023: "Intro to Constraint Programming" by Tias Guns - <https://www.youtube.com/watch?v=JpxbiEAFe80&list=PLcByDTr7vRTYJ2s6DL-3bjjGwtQif33y3>