Bochner Integration and an application in Stochastic Processes

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Table of Contents

Some Basic Terms

Recall that, given a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} := \{\mathscr{F}_t\}_{0\leq t\leq T}, \mathbb{P})$

- A *stochastic process* is a real-valued function $X : [0,T] \times \Omega \to \mathbb{R}$ that is $(\mathscr{B}_{[0,T]} \times \mathscr{F}, \mathscr{B}_{\mathbb{R}})$ measurable.
- For given fixed $t \in [0,T]$, we view $X_t := X(t,\cdot) : \Omega \to \mathbb{R}$ as a *random variable* (simply a $(\mathscr{F}, \mathscr{B}_{\mathbb{R}})$-measurable function). Equivalently, we could view X as $X : [0,T] \to L^0(\Omega)$ with $X(t) := X_t$, where $L^0(\Omega)$ is the space of all random variables, and $L^p(\Omega)$ are as usual.
- a stochastic process X is *progressively measurable* if its restriction $X|_{[0,t]} : [0,t] \times \Omega \to \mathbb{R}$ is $(\mathscr{B}_{[0,t]} \times \mathscr{F}_t, \mathscr{B}_{\mathbb{R}})$ measurable for all $t \in [0,T]$, denoted as $X \in \mathbb{L}^0(\mathbb{F})$. For $p \in [1,\infty]$, $\mathbb{L}^p(\mathbb{F}) := \mathbb{L}^0(\mathbb{F}) \cap L^p([0,T] \times \Omega)$.

Some Basic Terms

- The *distribution* of a random variable Z is a Borel measure, denoted as $\mu_Z : \mathscr{B}_{\mathbb{R}} \to [0,1]$, is defined as

$$\mu_Z(A) := \mathbb{P}(Z \in A).$$

  We say such two random variables Z_1, Z_2 have the same distribution, denoted as $Z_1 \overset{(d)}{=} Z_2$, if $\mu_{Z_1} = \mu_{Z_2}$.
- The *finite distribution* of a stochastic process X is a collection of Borel measures

$$\{\mu_{X_{t_1},\dots,X_{t_n}}\}_{\text{all } 0 \le t_1 < t_2 < \dots < t_n \le T},$$

  where

$$\mu_{X_{t_1},\dots,X_{t_n}}(A) := \mathbb{P}\Big((X_{t_1},\dots,X_{t_n}) \in A\Big)$$

  for all $A \in \mathscr{B}_{\mathbb{R}^n}$.
- Moreover, we say such two stochastic processes X, Y have the same finite distribution, denoted as $X \overset{(d)}{=} Y$, if $\mu_{X_{t_1},\dots,X_{t_n}} = \mu_{Y_{t_1},\dots,Y_{t_n}}$ for any $0 \le t_1 < t_2 < \dots < t_n \le T$.

Problem-Driven Background

Problem

Suppose $X, Y \in \mathbb{L}^0(\mathbb{F})$ *have the same finite distribution. Is it* ***True*** *that* $\int_0^T X_t dt$ *and* $\int_0^T Y_t dt$ *have the same distribution?*

Intuition

Intuitively, the integral of a stochastic process over time should give us a random varible as we could naively “think" the integration as summation. More concretely, if X and Y satisfy some integrability condition, for example $|X_t|, |Y_t| \leq B$ for any $(t, \omega) \in [0, T] \times \Omega$, we could consider X_t, Y_t as vectors in some infinite-dimensional vector space, and hence the integration should be in the sense of vector-valued function.

Heuristics

Recall from [G.B. Folland, 1999], given a measure space (X, Σ, μ), after the integral of real-valued functions is defined, the integral for a $(\Sigma, \mathscr{B}_{\mathbb{C}})$ measurable function $f : X \to \mathbb{C}$ is defined as

$$\int_X f d\mu := \int_X \mathrm{Re} f d\mu + i \int_X \mathrm{Im} f d\mu,$$

whenever $\int_X |f| d\mu < \infty$. This gives us the intuition to generalize the integration to vector-valued function, as we could consider the vector space isomorphism $\mathbb{C} \cong \mathbb{R}^2$. Namely, for $(\Sigma, \mathscr{B}_{\mathbb{R}^n})$ measurable function $f : X \to \mathbb{R}^n$, we could define

$$\int_X f d\mu := \sum_{i=1}^{n} e_i \int_X \langle f, e_i \rangle d\mu,$$

whenever $\int_X |f|_2 d\mu < \infty$, where $\langle \cdot, \cdot \rangle, | \cdot |_2 := \sqrt{\langle \cdot, \cdot \rangle}$, $\{e_i\}_{i=1}^n$ is an inner product, the inner product-induced norm, and an orthonormal basis for $\mathbb{R}^n$, respectively.

Heuristics

How about f takes value in a Hilbert space, even a general Banach space? In which cases, we don't have the orthogonality or separability to follow this intuition.
But still, we could define the integration for those Banach-valued functions in the exact same manner as real-valued functions in Lebesgue integration.

Introduction

Given a measure space (X, Σ, μ) and a real Banach Space $(V, |\cdot|_V)$, we say a $(\Sigma, \mathscr{B}_V)$ measurable function $s : X \to V$ is *simple* if

$$s(x) = \sum_{i=1}^{n} \mathbb{1}_{E_i}(x) v_i,$$

where $\{E_i\} \subset \Sigma$ are disjoint, and v_i are distinct elements in V.

Definition

We say such s is *integrable*, and its integral is defined as

$$\int_X s d\mu := \sum_{i=1}^{n} \mu(E_i) v_i,$$

if $\mu(E_i)$ is finite whenever $v_i \neq 0$.

Generally,

Definition

A $(\Sigma, \mathscr{B}_V)$ measurable function $f : X \to V$ is said to be *Bochner integrable*, if there exists a sequence of integrable simple functions $\{s_n\}_{n=1}^{\infty}$ s.t.

$$\lim_{n\to\infty} \int_X |f - s_n|_V \, d\mu = 0,$$

and its integral is defined as

$$\int_X f \, d\mu := \lim_{n\to\infty} \int_X s_n \, d\mu,$$

where the first integral is in the sense of Lebesgue integral and the second convergence is in the sense of $|\cdot|_V$ norm topology.

we make several natural remarks here:

- It's easy to show that $\{\int_X s_n d\mu\}_{n=1}^{\infty}$ is a Cauchy sequence in the Banach space V, and hence by its completeness the limit exists. If f takes value in a general normed space, we may have trouble that this Cauchy sequence does not necessary to converge.
- It's also easy to show that if there is another such sequence of simple functions, the integral of f does not change. That is, the Bochner integral is well-defined (independent of choices).
- Moreover, f is Bochner integrable if and only if $\int_X |f|_V d\mu < \infty$, and we may denote $\mathcal{L}^1(X;V)$ to be the set of all such functions. It is easy to see $\mathcal{L}^1(X;V)$ is a vector space (linearity is here!) and $\int_X |\cdot|_V d\mu$ is a semi-norm on it, and hence by using the standard trick: taking the quotient space w.r.t the equivalence relation $f \sim g$ if $f = g$ μ-a.e.,

$$L^1(X;V) := \mathcal{L}^1(X;V)/\sim$$

is a Banach space equiped with the norm $|\cdot|_{L^1(X;V)} := \int_X |\cdot|_V d\mu$.

Introduction

- From our definition, simple functions are naturally dense in $L^1(X;V)$, of course w.r.t. the $|\cdot|_{L^1(X;V)}$ norm topology.
- If f is Bochner integrable, then the following usual inequlity holds:

$$|\int_E f d\mu|_V \leq \int_E |f|_V d\mu,$$

  for all $E \in \Sigma$, where $\int_E f d\mu \coloneqq \int_X f \mathbb{1}_E d\mu$ as usual.
- Typically, Suppose $T : V \to V'$ is a continous linear operator between Banach space V and V', and f is Bochner integrable. Then,
  - $Tf : X \to V'$ is also Bochner integrable;
  - the Bochner integral commutes with T:

$$\int_X T f d\mu = T \int_X f d\mu$$

- Lastly, given a Bochner integrable function f, for all $E \in \Sigma$, $\nu : E \mapsto \int_E f d\mu$ is a countably-additive V-valued measure on X, and $\nu \ll \mu$.

DCT

More importantly, we still have our beloved *Dominated Convergence Theorem*, perhaps after a redefinition w.r.t to a completion of the measure μ.

Theorem (Dominated Convergence Theorem for Banach-valued functions)

Suppose $\{f_n\}_{n=1}^{\infty} \subset L^1(X;V)$, $f_n \to f$ *μ-a.e., and there* $\exists g \in L^1(X, \Sigma, \mu)$ *s.t.* $|f_n(x)|_V \leq g(x)$ *for μ-a.e.* $x \in X$. *Then,* $f \in L^1(X;V)$,

$$f_n \to f \text{ in } L^1(X;V),$$

and hence

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu,$$

where the second convergence is in the sense of $|\cdot|_V$ *norm topology.*

Luzin's

Now we have sufficient ingredients, it is time to present the result mainly for the application: *Luzin's Theorem for Banach-valued functions.*

Theorem (Luzin's Theorem for Banach-valued functions)

If $f : [a, b] \to V$ *is* $(\mathcal{L}_{[a,b]}, \mathscr{B}_V)$*-measurable (*$\mathcal{L}_{[a,b]}$ *is the class of all Lebesgue measurable sets included in* $[a, b]$*) and* $\epsilon > 0$*, then there is a compact set* $K_\epsilon \subset [a, b]$ *s.t. the Lebesgue measure* $m(K_\epsilon^c) < \epsilon$ *and* $f|_K$ *is continuous.*

An Trivial Application in Stochastics Processes

Now we could go back to our original interest to see how this version of Luzin's is applied. Recall we want to show that

Proposition

Given $(\Omega, \mathscr{F}, \mathbb{F} := \{\mathscr{F}_t\}_{0 \le t \le T}, \mathbb{P})$ *and* $X, Y \in \mathbb{L}^0(\mathbb{F})$, *if* $X \overset{(d)}{=} Y$, *then* $\int_0^T X_t dt \overset{(d)}{=} \int_0^T Y_t dt$.

Before we prove this proposition, we first give out a lemma.

Lemma

Suppose $X, Y \in \mathbb{L}^0(\mathbb{F})$ *and* $X \overset{(d)}{=} Y$. *If* X *is continuous in* t *for a.e.* ω, *then* Y *is also continuous in* t *a.e.* ω.

Outline of the Proof

We now can outline the proof.
Firstly, we assume X, Y are bounded (i.e. $X, Y \in \mathbb{L}^\infty(\mathbb{F})$ and equivalently $X, Y : [0, T] \to L^\infty(\Omega)$.) and X is continuous

- By the lemma, we know Y is continuous.
- We then consider discretization of both X and Y: $X_t^{\pi_n} := \sum_{i=0}^{n-1} X_{t_i} \mathbb{1}_{\{t_i \leq t < t_{i+1}\}}$, and $Y_t^{\pi_n} := \sum_{i=0}^{n-1} Y_{t_i} \mathbb{1}_{\{t_i \leq t < t_{i+1}\}}$, where $\pi_n : \{t_i := \frac{i}{2^n} T\}$, $i = 0, ..., 2^n$ is a partition of $[0, T]$.
- We establish $\int_0^T X_t^{\pi_n} dt = \sum_{i=0}^{n-1} X_{t_i} \Delta t_{i+1} \overset{(d)}{=} \sum_{i=0}^{n-1} Y_{t_i} \Delta t_{i+1} = \int_0^T Y_t^{\pi_n} dt.$
- Continuity of X, Y gives pointwise convergences in t: $X^{\pi_n} \to X$ pointwise in t and $Y^{\pi_n} \to Y$, and, together with the boundedness of X, Y, by Dominated convergence theorem for V-valued functions, $\int_0^T X_t^{\pi_n} dt \to \int_0^T X_t dt$ uniformly except on a $\mathbb{P}$-null set E. Similarly, after taking a union of $\mathbb{P}$-null sets, we may say $\int_0^T Y_t^{\pi_n} dt \to \int_0^T Y_t dt$ uniformly except on the same $\mathbb{P}$-null set E.

Outline of the Proof

- Establish
$$\lim_{n\to\infty}\int_0^T X_t^{\pi_n}dt \overset{(d)}{=} \lim_{n\to\infty}\int_0^T Y_t^{\pi_n}dt$$
thru standard perturbation technique: given a closed set $A \subset \mathbb{R}$: denote $A_\epsilon := \{x \in \mathbb{R} : |x-y| < \epsilon \text{ for some } y \in A\}$, and observe
$$\begin{aligned}&\bigcap_{\epsilon>0}\bigcup_{N\in\mathbb{N}}\bigcap_{n\geq N}\{\omega \notin E : [\int_0^T X_t^{\pi_n}dt](\omega) \in A_\epsilon\}\\ =&\{\omega \notin E : [\int_0^T X_t dt](\omega) \in A\},\end{aligned} \tag{1}$$
and use $\int_0^T X_t^{\pi_n}dt \overset{(d)}{=} \int_0^T Y_t^{\pi_n}dt$.

We are done for this case.

Outline of the Proof

Now for general bounded X, Y, we

- apply the V-valued version of Luzin's, and, after taking a intersection, we may assume that $\forall \delta > 0$, $\exists K_\delta \subset [0, T]$ s.t. $m(K_\delta^c) < \delta$, and $X|_{K_\delta}$, $Y|_{K_\delta}$ are continuous.
- then we apply the discretization to K_δ and the perturbation, similarly we have

$$\int_{K_\delta} X_t dt = \lim_{n\to\infty} \int_{K_\delta} X_t^{\pi_n} dt \overset{(d)}{=} \lim_{n\to\infty} \int_{K_\delta} Y_t^{\pi_n} dt = \int_{K_\delta} Y_t dt.$$

- We choose $\delta_m := \frac{1}{m|X|_\infty} \wedge \frac{1}{m|Y|_\infty}$ s.t. $K_m := K_{\delta_m}$ and $\int_{K_m^c} |X_t|_\infty dt < 1/m$,
- Observe $|\int_0^T X_t dt - \int_{K_m} X_t dt|_\infty \leq \int_{K_m^c} |X_t|_\infty dt < \frac{1}{m}$, i.e. $\int_{K_m} X_t dt \to \int_0^T X_t dt$ uniformly except on a $\mathbb{P}$-null set E. Similarly, after taking a union of $\mathbb{P}$-null sets, we may say $\int_{K_m} Y_t dt \to \int_0^T Y_t dt$ uniformly except on the same $\mathbb{P}$-null set E.

Outline of the Proof

- We can conclude $\int_0^T X_t dt \overset{(d)}{=} \int_0^T Y_t dt$ by the standard perturbation control of a closed set $A \subset \mathbb{R}$ and use $\int_{K_n} X_t dt \overset{(d)}{=} \int_{K_n} Y_t dt$.

Lastly, for general X, Y, we follow the standard space truncation $X^{(n)} := X \wedge n$ and follow the same procedure.

We remark that this is a trivial application as all we need for the perturbation control is a.e. pointwise convergence, which could be established directly from the start by using Fubini's, but anyway it is good to know the integration for vector-valued functions.

References

G.B. Folland. *Real Analysis: Modern Techniques and Their Applications*, Second Edition, 1999.

K. Yosida, *Functional analysis*, Springer, Ch. 8, §1., 1980.

W. Rudin. *Functional analysis*. New York: McGraw-Hill, 1973.