

BOCHNER INTEGRATION AND AN APPLICATION IN STOCHASTIC PROCESSES

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1. INTRODUCTION

In this note, we discuss the Bochner integration and an application in showing the distributions of $\int_0^T X_t dt$ and $\int_0^T Y_t dt$ coincide, given two progressively measurable stochastic processes X, Y .

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2. BOCHNER INTEGRATION

Given a measure space (X, Σ, μ) and a real Banach Space $(V, |\cdot|_V)$, we say a (Σ, \mathcal{B}_V) measurable function $s : X \rightarrow V$ is **simple** if

$$s(x) = \sum_{i=1}^n \mathbb{1}_{E_i}(x) v_i,$$

where $\{E_i\} \subseteq \Sigma$ are disjoint, and v_i are distinct elements in V .

We say such s is **integrable**, and its integral is defined as

$$\int_X s d\mu := \sum_{i=1}^n \mu(E_i) v_i,$$

if $\mu(E_i)$ is finite whenever $v_i \neq 0$.

A (Σ, \mathcal{B}_V) measurable function $f : X \rightarrow V$ is said to be **Bochner integrable**, if there exists a sequence of integrable simple functions $\{s_n\}_{n=1}^\infty$ s.t.

$$\lim_{n \rightarrow \infty} \int_X |f - s_n|_V d\mu = 0,$$

and its integral is defined as

$$\int_X f d\mu := \lim_{n \rightarrow \infty} \int_X s_n d\mu,$$

where the first integral is in the sense of Lebesgue integral and the second convergence is in the sense of $|\cdot|_V$ norm topology.

Remark 2.1. we make several natural remarks here:

- (1) It's easy to show that $\{\int_X s_n d\mu\}_{n=1}^\infty$ is a Cauchy sequence in the Banach space V , and hence by its completeness the limit exists. If f takes value in a general normed space, we may have trouble that this Cauchy sequence does not necessary to converge.
- (2) It's also easy to show that if there is another such sequence of simple functions, the integral of f does not change. That is, the Bochner integral is well-defined (independent of choices).
- (3) Moreover, f is Bochner integrable if and only if $\int_X |f|_V d\mu < \infty$, and we may denote $\mathcal{L}^1(X; V)$ to be the set of all such functions. It is easy to see $\mathcal{L}^1(X; V)$ is a vector space (linearity is here!) and $\int_X |\cdot|_V d\mu$ is a semi-norm on it, and hence by using the standard trick: taking the quotient space w.r.t the equivalence relation $f \sim g$ if $f = g$ μ -a.e.,

$$L^1(X; V) := \mathcal{L}^1(X; V) / \sim$$

is a Banach space equipped with the norm $|\cdot|_{L^1(X; V)} := \int_X |\cdot|_V d\mu$.

- (4) From our definition, simple functions are naturally dense in $L^1(X; V)$, of course w.r.t. the $|\cdot|_{L^1(X; V)}$ norm topology.

(5) If f is Bochner integrable, then the following usual inequality holds:

$$|\int_E f d\mu|_V \leq \int_E |f|_V d\mu,$$

for all $E \in \Sigma$, where $\int_E f d\mu := \int_X f \mathbb{1}_E d\mu$ as usual.

- (6) Typically, Suppose $T : V \rightarrow V'$ is a continuous linear operator between Banach space V and V' , and f is Bochner integrable. Then,
- $Tf : X \rightarrow V'$ is also Bochner integrable;
 - the Bochner integral commutes with T :

$$\int_X Tf d\mu = T \int_X f d\mu$$

- (7) Lastly, given a Bochner integrable function f , for all $E \in \Sigma$, $\nu : E \mapsto \int_E f d\mu$ is a countably-additive V -valued measure on X , and $\nu \ll \mu$.

2.1. Dominated Convergence Theorem for Banach-valued functions.

More importantly, we still have our beloved *Dominated Convergence Theorem*, perhaps after a redefinition w.r.t to a completion of the measure μ .

Theorem 2.2 (Dominated Convergence Theorem for Banach-valued functions). *Suppose $\{f_n\}_{n=1}^\infty \subseteq L^1(X; V)$, $f_n \rightarrow f$ μ -a.e., and there $\exists g \in L^1(X, \Sigma, \mu)$ s.t. $|f_n(x)|_V \leq g(x)$ for μ -a.e. $x \in X$. Then, $f \in L^1(X; V)$,*

$$f_n \rightarrow f \text{ in } L^1(X; V),$$

and hence

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu,$$

where the second convergence is in the sense of $|\cdot|_V$ norm topology.

The proof is clear by applying the usual Dominated Convergence Theorem [Folland Theorem 2.24] to $|f_n - f|_V \leq 2g$ and observe

$$|\int_X f d\mu - \int_X f_n d\mu|_V \leq \int_X |f_n - f|_V d\mu \rightarrow 0.$$

2.2. Luzin's Theorem for Banach-valued functions.

Now we have sufficient ingredients, it is time to present the main result: *Luzin's Theorem* for Banach-valued functions.

Theorem 2.3 (Luzin's Theorem for Banach-valued functions). *If $f : [a, b] \rightarrow V$ is $(\mathcal{L}_{[a,b]}, \mathcal{B}_V)$ -measurable ($\mathcal{L}_{[a,b]}$ is the class of all Lebesgue measurable sets included in $[a, b]$) and $\varepsilon > 0$, then there is a compact set $K_\varepsilon \subseteq [a, b]$ s.t. the Lebesgue measure $m(K_\varepsilon^c) < \varepsilon$ and $f|_{K_\varepsilon}$ is continuous.*

To prove this version of Luzin's, we give several natural lemmas¹.

Lemma 2.4. *If $f_n \rightarrow f$ in $L^1(X; V)$, there is a subsequence $\{f_{n_j}\}$ s.t. $f_{n_j} \rightarrow f$ a.e.*

Lemma 2.5 (Egorov's for Banach-valued functions). *Suppose $\mu(X) < \infty$, and $\{f_n\}, f$ are (Σ, \mathcal{B}_V) -measurable functions on X s.t. $f_n \rightarrow f$ a.e. Then, $\forall \varepsilon > 0, \exists E_\varepsilon \subseteq X$ s.t. $\mu(E) < \varepsilon$ and $f_n \rightarrow f$ uniformly on E^c .*

Lemma 2.6. *If μ is a Lebesgue-Stieltjes measure on \mathbb{R} , then compactly supported continuous functions $f : \mathbb{R} \rightarrow V$ are dense in $L^1(\mathbb{R}; V)$, with respect to the $\|\cdot\|_{L^1(\mathbb{R}; V)}$ norm topology.*

Now we could prove the Luzin's in the usual way

Proof. We first assume f is bounded, i.e. $\sup_{x \in [a, b]} |f(x)|_V < \infty$, and then we combine lemma 2.4 and lemma 2.6 that there is a sequence of compactly supported continuous functions $\{f_n\}$ s.t. $f_n \rightarrow f$ a.e. Then, as $m([a, b]) = b - a < \infty$, we could apply Egorov's (lemma 2.5), for a given $\varepsilon > 0, \exists E_\varepsilon \subseteq [a, b]$ s.t. $m(E_\varepsilon) < \varepsilon$ and $f_n \rightarrow f$ uniformly on E_ε^c . That is, as the uniform limit of continuous functions on E_ε^c , f is continuous on E_ε^c . Finally, recall Lebesgue measurable sets could be approached from inside by compact sets. That is, as $m(E_\varepsilon^c) > b - a - \varepsilon$, $b - a - \varepsilon$ is not the sup and hence \exists compact $K_\varepsilon \subseteq E_\varepsilon^c$ s.t. $m(K_\varepsilon) > b - a - \varepsilon$. K_ε is what we are looking for.

If f is unbounded, we can consider the sets $E_n = [a, b] \cap \{x : |f(x)|_V \leq n\}$ for $n \in \mathbb{N}$, which is measurable and have finite measure since $[a, b]$ has finite measure. Then, we can apply the previous arguments to each $f|_{E_n}$, which is a bounded function, to obtain a sequence of compact $K_n \subseteq E_n$ s.t.

$$m(K_n^c) < \frac{\varepsilon}{2^n},$$

and $f|_{K_n}$ is continuous. Then, typically, $f|_{\cap_{n \in \mathbb{N}} K_n}$ is continuous and $\cap_{n \in \mathbb{N}} K_n$ is still compact, with

$$m(\cup_{n \in \mathbb{N}} K_n^c) \leq \sum_{n \in \mathbb{N}} m(K_n^c) < \varepsilon.$$

□

3. AN APPLICATION

3.1. Stochastic Analysis Preliminaries.

Recall that, given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$

- A *stochastic process* is a real-valued² function $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ that is $(\mathcal{B}_{[0, T]} \times \mathcal{F}, \mathcal{B}_{\mathbb{R}})$ measurable.

¹We attached the proofs in the appendix.

²We use the real-valued functions for simplicity, but similar notion could be developed for complex-valued functions.

- For given fixed $t \in [0, T]$, we view $X_t := X(t, \cdot) : \Omega \rightarrow \mathbb{R}$ as a *random variable* (simply a $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ -measurable function). Equivalently, we could view X as $X : [0, T] \rightarrow L^0(\Omega)$ with $X(t) := X_t$, where $L^0(\Omega)$ is the space of all random variables $Z : \Omega \rightarrow \mathbb{R}$. For $p \in [0, \infty]$, $L^p(\Omega)$ are as usual.
- A stochastic process X is *progressively measurable* if its restriction $X|_{[0,t]} : [0, t] \times \Omega \rightarrow \mathbb{R}$ is $(\mathcal{B}_{[0,t]} \times \mathcal{F}_t, \mathcal{B}_{\mathbb{R}})$ measurable for all $t \in [0, T]$, denoted as $X \in \mathbb{L}^0(\mathbb{F})$. For $p \in [1, \infty]$, $\mathbb{L}^p(\mathbb{F}) := \mathbb{L}^0(\mathbb{F}) \cap L^p([0, T] \times \Omega)$ are as usual.
- The *distribution* of a random variable Z is a Borel measure, denoted as $\mu_Z : \mathcal{B}_{\mathbb{R}} \rightarrow [0, 1]$, is defined as

$$\mu_Z(A) := \mathbb{P}(Z \in A).$$

We say such two random variables Z_1, Z_2 have the same distribution, denoted as $Z_1 \stackrel{(d)}{=} Z_2$, if $\mu_{Z_1} = \mu_{Z_2}$.

- The *finite distribution* of a stochastic process X is a collection of Borel measures

$$\{\mu_{X_{t_1}, \dots, X_{t_n}}\}_{\text{all } 0 \leq t_1 < t_2 < \dots < t_n \leq T},$$

where

$$\mu_{X_{t_1}, \dots, X_{t_n}}(A) := \mathbb{P}((X_{t_1}, \dots, X_{t_n}) \in A)$$

for all $A \in \mathcal{B}_{\mathbb{R}^n}$.

- Moreover, we say such two stochastic processes X, Y have the same finite distribution, denoted as $X \stackrel{(d)}{=} Y$, if $\mu_{X_{t_1}, \dots, X_{t_n}} = \mu_{Y_{t_1}, \dots, Y_{t_n}}$, for any $0 \leq t_1 < t_2 < \dots < t_n \leq T$.

3.2. Main Proposition.

We want to show that

Proposition 3.1. Given $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ and $X, Y \in \mathbb{L}^0(\mathbb{F})$, if $X \stackrel{(d)}{=} Y$, then $\int_0^T X_t dt \stackrel{(d)}{=} \int_0^T Y_t dt$.

We first give out a lemma.

Lemma 3.2. Suppose $X, Y \in \mathbb{L}^0(\mathbb{F})$ and $X \stackrel{(d)}{=} Y$. If X is continuous in t for a.e. ω , then Y is also continuous in t a.e. ω .

Proof of the Proposition 3.1.

Assume X, Y bounded (i.e. $X, Y \in \mathbb{L}^\infty(\mathbb{F})$ and equivalently $X, Y : [0, T] \rightarrow L^\infty(\Omega)$.) and X continuous. By lemma 3.2, Y is also continuous. Then, consider partition of $\pi_n : \{t_i := \frac{i}{2^n}T\}$, $i = 0, \dots, 2^n$ and

$$X_t^{\pi_n} := \sum_{i=0}^{n-1} X_{t_i} \mathbb{1}_{\{t_i \leq t < t_{i+1}\}}$$

and

$$Y_t^{\pi_n} := \sum_{i=0}^{n-1} Y_{t_i} \mathbb{1}_{\{t_i \leq t < t_{i+1}\}}.$$

It is clear, as X, Y has same finite distribution, that

$$\int_0^T X_t^{\pi_n} dt = \sum_{i=0}^{n-1} X_{t_i} \Delta t_{i+1} \stackrel{(d)}{=} \sum_{i=0}^{n-1} Y_{t_i} \Delta t_{i+1} = \int_0^T Y_t^{\pi_n} dt.$$

Notice $X^{\pi_n} \rightarrow X$ pointwisely in t , $Y^{\pi_n} \rightarrow Y$ in t pointwisely, and boundedness of X, Y , by dominated convergence theorem (Theorem 2.2), we know

$$\int_0^T X_t dt = \lim_{n \rightarrow \infty} \int_0^T X_t^{\pi_n} dt \stackrel{(d)}{=} \lim_{n \rightarrow \infty} \int_0^T Y_t^{\pi_n} dt = \int_0^T Y_t dt.$$

To see the $\stackrel{(d)}{=}$ above, denote $Z_n := \int_0^T X_t^{\pi_n} dt$, $\tilde{Z}_n := \int_0^T Y_t^{\pi_n} dt$, $Z := \int_0^T X_t dt$ and $\tilde{Z} := \int_0^T Y_t dt$. By Folland 6.8(c), after taking a union of \mathbb{P} -null sets, we have $Z_n \rightarrow Z$, $\tilde{Z}_n \rightarrow \tilde{Z}$ uniformly on $\Omega \setminus E$ with $\mathbb{P}(E) = 0$. For a given closed $A \subseteq \mathbb{R}$, we denote a ε -perturbation of A , $A_\varepsilon := \{x \in \mathbb{R} : |x - y| < \varepsilon \text{ for some } y \in A\}$. we **claim**:

$$\bigcap_{\varepsilon > 0} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{\omega \notin E : Z_n(\omega) \in A_\varepsilon\} = \{\omega \notin E : Z(\omega) \in A\}$$

and similarly $\bigcap_{\varepsilon > 0} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{\omega \notin E : \tilde{Z}_n(\omega) \in A_\varepsilon\} = \{\omega \notin E : \tilde{Z}(\omega) \in A\}$. Then, as $Z_n \stackrel{(d)}{=} \tilde{Z}_n$,

$$\mathbb{P}(\{\omega \notin E : Z(\omega) \in A\}) = \mathbb{P}(\{\omega \notin E : \tilde{Z}(\omega) \in A\}),$$

and we are done as $\mathcal{B}_{\mathbb{R}}$ is generated by closed sets.

To see the claim, \supseteq direction is clear by the uniform convergence on E^c . For the other direction, let $\omega \in \bigcap_{\varepsilon > 0} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{\omega \notin E : Z_n(\omega) \in A_\varepsilon\}$. For a given $\varepsilon/2$, $\exists N_1$ s.t. $Z_n(\omega) \in A_{\varepsilon/2}$ for all $n \geq N_1$. Pick a y_ε s.t.

$$|Z_n(\omega) - y_\varepsilon| < \frac{\varepsilon}{2}.$$

Also, the uniform convergence on E^c allows us to choose N_2 s.t.

$$|Z(\omega) - Z_n(\omega)| < \frac{\varepsilon}{2}$$

for all $n \geq N_2$. We choose $N = N_1 \vee N_2$, and hence we have for all $\varepsilon, \exists y_\varepsilon \in A$

$$|Z(\omega) - y_\varepsilon| \leq |Z(\omega) - Z_n(\omega)| + |Z_n(\omega) - y_\varepsilon| < \varepsilon,$$

i.e. $Z(\omega)$ is a limit point of A and hence $Z(\omega) \in A$ as A is closed.

Now for general bounded X, Y , we apply the Luzin's (Theorem 2.3), and, after taking a intersection of two compact sets, we may assume that $\forall \delta > 0$, $\exists K_\delta \subseteq [0, T]$ s.t. $m(K_\delta^c) < \delta$, and $X|_{K_\delta}, Y|_{K_\delta}$ are continuous. Apply the same discretization and perturbation technique, we have

$$\int_{K_\delta} X_t dt = \lim_{n \rightarrow \infty} \int_{K_\delta} X_t^{\pi_n} dt \stackrel{(d)}{=} \lim_{n \rightarrow \infty} \int_{K_\delta} Y_t^{\pi_n} dt = \int_{K_\delta} Y_t dt.$$

Then, we choose $\delta_m := \frac{1}{m|X|_\infty} \wedge \frac{1}{m|Y|_\infty}$, denote $K_m := K_{\delta_m}$, and hence we have

$$\int_{K_m^c} |X_t|_\infty dt \leq |X|_\infty m(K_m^c) < |X|_\infty \delta_m \leq 1/m,$$

and observe that

$$\left| \int_0^T X_t dt - \int_{K_m} X_t dt \right|_\infty = \left| \int_{K_m^c} X_t dt \right|_\infty \leq \int_{K_m^c} |X_t|_\infty dt < \frac{1}{m},$$

i.e. $\int_{K_m} X_t dt \rightarrow \int_0^T X_t dt$ uniformly except on a \mathbb{P} -null set E . Similarly, after taking a union of \mathbb{P} -null sets, we may say $\int_{K_m} Y_t dt \rightarrow \int_0^T Y_t dt$ uniformly except on the same \mathbb{P} -null set E . Then, we further apply the perturbation technique on $\int_{K_m} X_t dt, \int_0^T X_t dt$ and we may conclude $\int_0^T X_t dt \stackrel{(d)}{=} \int_0^T Y_t dt$.

Finally, for general X, Y , we follow the standard space truncation $X^{(n)} := X \wedge n$. It gives an a.e. convergence and hence allows us to follow the same perturbation procedure. \square

Remark 3.3. In fact this is a trivial application, as all we need to apply the perturbation technique is the a.e. convergence, which could be achieved by using \mathbb{P} -a.e. defined $(\int_0^T X_t dt)(\omega) := \int_0^T X(t, \omega) dt$ from the Fubini's. Anyway, it is good to realize the integration for vector-valued functions.

4. REFERENCE

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5. APPENDIX

To prove Lemma 2.4, we add the following generalized notion of convergence in measure and lemma.

We say that a sequence of (Σ, \mathcal{B}_V) -measurable functions $\{f_n\}$ on (X, Σ, μ)

- **converges in measure** to f if, for every $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} \mu(\{x : |f_n(x) - f(x)|_V \geq \varepsilon\}) = 0.$$

- **Cauchy in measure** if, for every $\varepsilon > 0$, we have

$$\mu(\{x : |f_n(x) - f_m(x)|_V \geq \varepsilon\}) \rightarrow 0$$

as $n, m \rightarrow \infty$.

Lemma 5.1. Suppose $\{f_n\}$ is Cauchy in measure, then there is (Σ, \mathcal{B}_V) -measurable function f s.t. $f_n \rightarrow f$ in measure, and there is a subsequence f_{n_j} that $f_{n_j} \rightarrow f$ a.e. Moreover, if $f_n \rightarrow g$ in measure, then $f = g$ a.e.

Corollary 5.2. *If $f_n \rightarrow f$ in measure, then there is a subsequence $\{f_{n_j}\}$ s.t. $f_{n_j} \rightarrow f$ a.e.*

Lemma 5.3. *if $f_n \rightarrow f$ in $L^1(X; V)$, then $f_n \rightarrow f$ in measure.*

Together, lemma 2.4 is proved.

Proof of Lemma 5.1. Since $\forall \varepsilon > 0$, $\lim_{n,m \rightarrow \infty} \mu(\{x : |f_n(x) - f_m(x)|_V \geq \varepsilon\}) = 0$, we choose n_1 large enough to be the first index s.t. $\mu(\{x : |f_n(x) - f_{n+1}(x)|_V \geq 2^{-1}\}) < 2^{-1}$. Similarly, We could choose $\{g_j\} = \{f_{n_j}\}$ s.t. if $E_j := \{x : |g_j(x) - g_{j+1}(x)|_V \geq 2^{-j}\}$, then $\mu(E_j) \leq 2^{-j}$. Denote $F_k := \bigcup_{j \geq k} E_j$, and hence $\mu(F_k) \leq \sum_{j=k}^{\infty} 2^{-j} = 2^{1-k}$, and if $x \notin F_k$, i.e. $x \in \bigcap_{j \geq k} E_j^c$ for $i \geq j \geq k$, we have

$$|g_i(x) - g_j(x)|_V \leq \sum_{l=j}^{i-1} |g_l(x) - g_{l+1}(x)|_V \leq \sum_{l=j}^{i-1} 2^{-l} < 2^{1-j}.$$

Consider $x \in \bigcup_{k=1}^{\infty} F_k^c$, then $x \in F_k^c$ for some k , and in here we have $|g_i(x) - g_j(x)|_V < 2^{1-j}$ for all $i \geq j \geq k$. Pick $K \geq k$ s.t. $2^{1-K} < \varepsilon$, and then, whenever $i \geq j \geq K$

$$|g_i(x) - g_j(x)|_V < 2^{1-j} \leq 2^{1-K} < \varepsilon,$$

i.e. $\{g_j\}$ pointwise Cauchy in $\bigcup_{k=1}^{\infty} F_k^c$. Observe $\mu(\left(\bigcup_{k=1}^{\infty} F_k^c\right)^c) = \mu(\bigcap_{k=1}^{\infty} F_k) = \lim_{k \rightarrow \infty} \mu(F_k) = 0$, and hence we know $\{g_j\}$ pointwise Cauchy almost everywhere. Define

$$f(x) := \begin{cases} \lim_{j \rightarrow \infty} g_j(x) & \text{if } x \in \bigcup_{k=1}^{\infty} F_k^c \\ \mathbf{0} & \text{otherwise} \end{cases}.$$

It is easy to see f is (Σ, \mathcal{B}_V) -measurable and hence $f_{n_j} = g_j \rightarrow f$ a.e.

We left to show $f_n \rightarrow f$ in measure. To see this, observe by triangle inequality

$$\{x : |f_n(x) - f(x)|_V \geq \varepsilon\} \subseteq \{x : |f_n(x) - f_{n_j}(x)|_V \geq \frac{\varepsilon}{2}\} \cup \{x : |f_{n_j}(x) - f(x)|_V \geq \frac{\varepsilon}{2}\},$$

where the later two sets have small measure when $n, j \rightarrow \infty$, one given by convergence in measure and the other given by the a.e. pointwise convergence.

If $f_n \rightarrow g$ in measure, similarly, observe that

$$\{x : |f(x) - g(x)|_V \geq \varepsilon\} \subseteq \{x : |f(x) - f_n(x)|_V \geq \frac{\varepsilon}{2}\} \cup \{x : |f_n(x) - g(x)|_V \geq \frac{\varepsilon}{2}\},$$

i.e. $\mu(\{x : |f(x) - g(x)|_V \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$. Note

$$\{x : |f(x) - g(x)|_V > 0\} = \bigcup_{m \in \mathbb{N}} \{x : |f(x) - g(x)|_V \geq \frac{1}{m}\},$$

we conclude $f = g$ a.e. by the continuity from below. □

Proof of Lemma 5.2.

Let $E_{n,\varepsilon} := \{x : |f_n(x) - f(x)|_V \geq \varepsilon\}$, then

$$\varepsilon \mu(E_{n,\varepsilon}) \leq \int_{E_{n,\varepsilon}} |f_n - f| d\mu \leq \int_X |f_n - f| d\mu \rightarrow 0$$

as $n \rightarrow \infty$. □

Proof of Lemma 2.5.

Let $f_n \rightarrow f$ pointwise on K with $\mu(K^c) = 0$. Note

$$K = \bigcap_{k=1}^{\infty} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{x : |f_n(x) - f(x)|_V < \frac{1}{k}\}.$$

That is, for any fixed k , $K(k) := \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{x : |f_n(x) - f(x)|_V < \frac{1}{k}\}$ has full measure. For $N \in \mathbb{N}$, denote

$$E_N(k) := \bigcup_{n \geq N} \{x : |f_n(x) - f(x)|_V \geq \frac{1}{k}\}.$$

Observe $E_N(k)$ decreases as N increases, and

$$K(k) = \bigcup_{N \in \mathbb{N}} E_N(k)^c = \left(\bigcap_{N \in \mathbb{N}} E_N(k) \right)^c.$$

Thus, $F(k) := \bigcap_{N \in \mathbb{N}} E_N(k)$ is a null set. By continuity from above, we may choose N_k s.t. $\mu(E_{N_k}(k)) < \varepsilon 2^{-k}$. Denote $E := \bigcup_{k=1}^{\infty} E_{N_k}(k)$, so we have $\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_{N_k}(k)) < \varepsilon$.

On E^c , observe $E^c \subseteq E_{N_k}(k)^c$ for any k . Therefore, For any $\varepsilon' > 0$, there is a k s.t. $\frac{1}{k} < \varepsilon$, and for $\forall x \in E^c \subseteq E_{N_k}(k)^c$, $|f_n(x) - f(x)|_V < \frac{1}{k} < \varepsilon'$ whenever $n \geq N_k$, i.e. $f_n \rightarrow f$ uniformly on E^c . □

Proof of Lemma 2.6.

By linearity and denseness of simple functions in $L^1(\mathbb{R}; V)$, it suffice to show any simple function of form $s(x) := \mathbb{1}_E(x) \mathbf{1}$ (we abuse the term to say it is a *indicator*) could approximated arbitrarily close in $\|\cdot\|_{L^1(X; V)}$ by a compactly supported continuous function, where $E \in \mathcal{L}_{\mathbb{R}}$ Lebesgue-Stieltjes measurable set, and $\mathbf{1}$ is a unit vector in V .

Recall that E could be approximated by a finite union of open intervals $\{I_i\}_{i=1}^n$ s.t. $\mu(E \Delta \bigcup_{i=1}^n I_i) < \varepsilon$. Note

$$(1) \quad \int_X |\mathbb{1}_E \mathbf{1} - \mathbb{1}_{\bigcup_{i=1}^n I_i} \mathbf{1}|_V d\mu = \int_X |\mathbb{1}_E - \mathbb{1}_{\bigcup_{i=1}^n I_i}| d\mu = \mu(E \Delta \bigcup_{i=1}^n I_i) < \varepsilon.$$

So we know the indicator of any E could be approximated arbitrarily by a indicator of finite union of open intervals.

Meanwhile, for the indicator of an open interval $\mathbb{1}_{(a,b)}\mathbf{1}$, we could approximate by constructing

$$f_\varepsilon(x) := \begin{cases} \mathbf{1} & \text{if } x \in [a + \frac{\varepsilon}{2}, b - \frac{\varepsilon}{2}] \\ \text{Linear} & \text{if } x \in [a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2}] \cup [b - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}] . \\ \mathbf{0} & \text{otherwise} \end{cases}$$

□