Quick Sort

IIITS

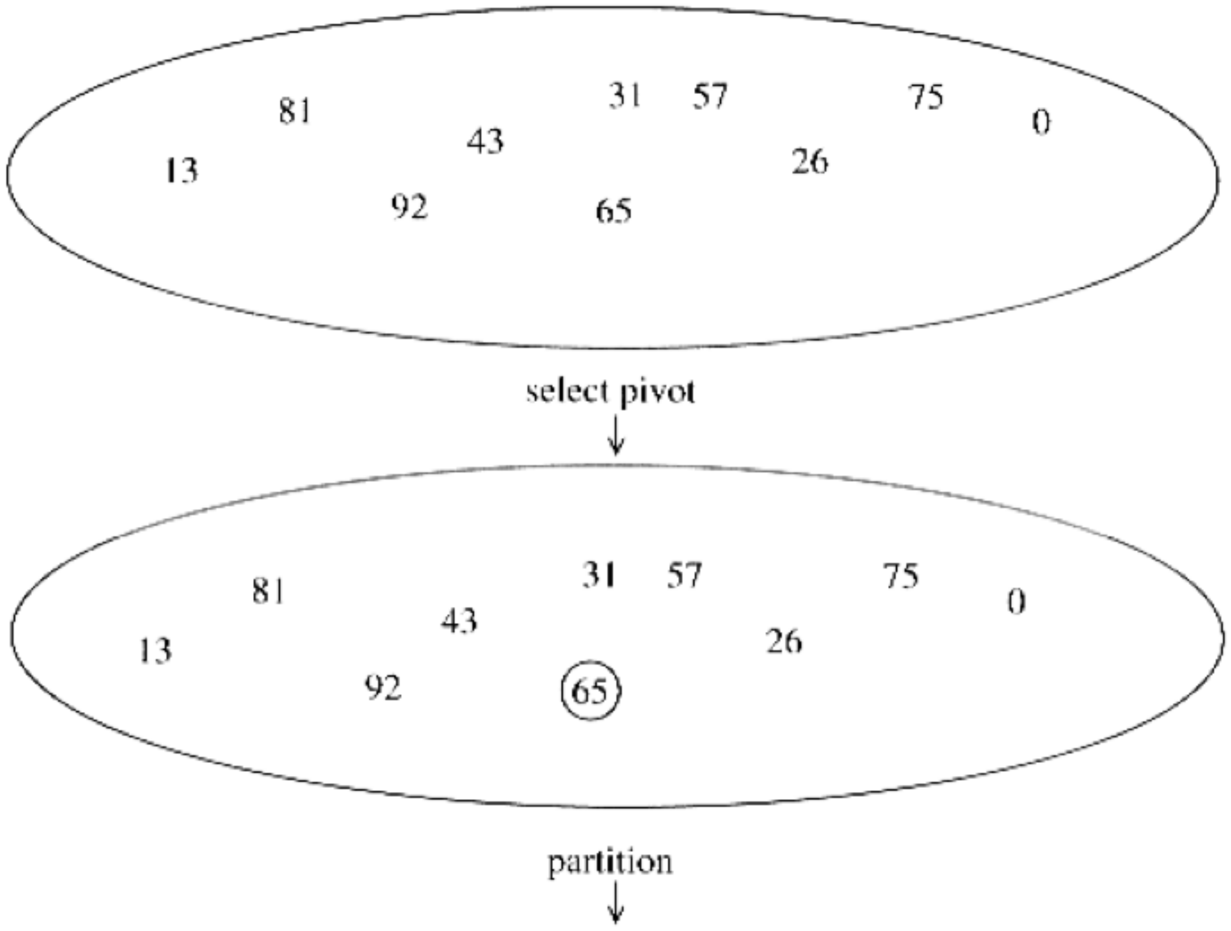
Quick Sort

- **Fastest** known sorting algorithm in practice
- Average case: O(N log N)
- Worst case: $O(N^2)$
  - But the worst case can be made exponentially unlikely.
- A divide-and-conquer recursive algorithm.

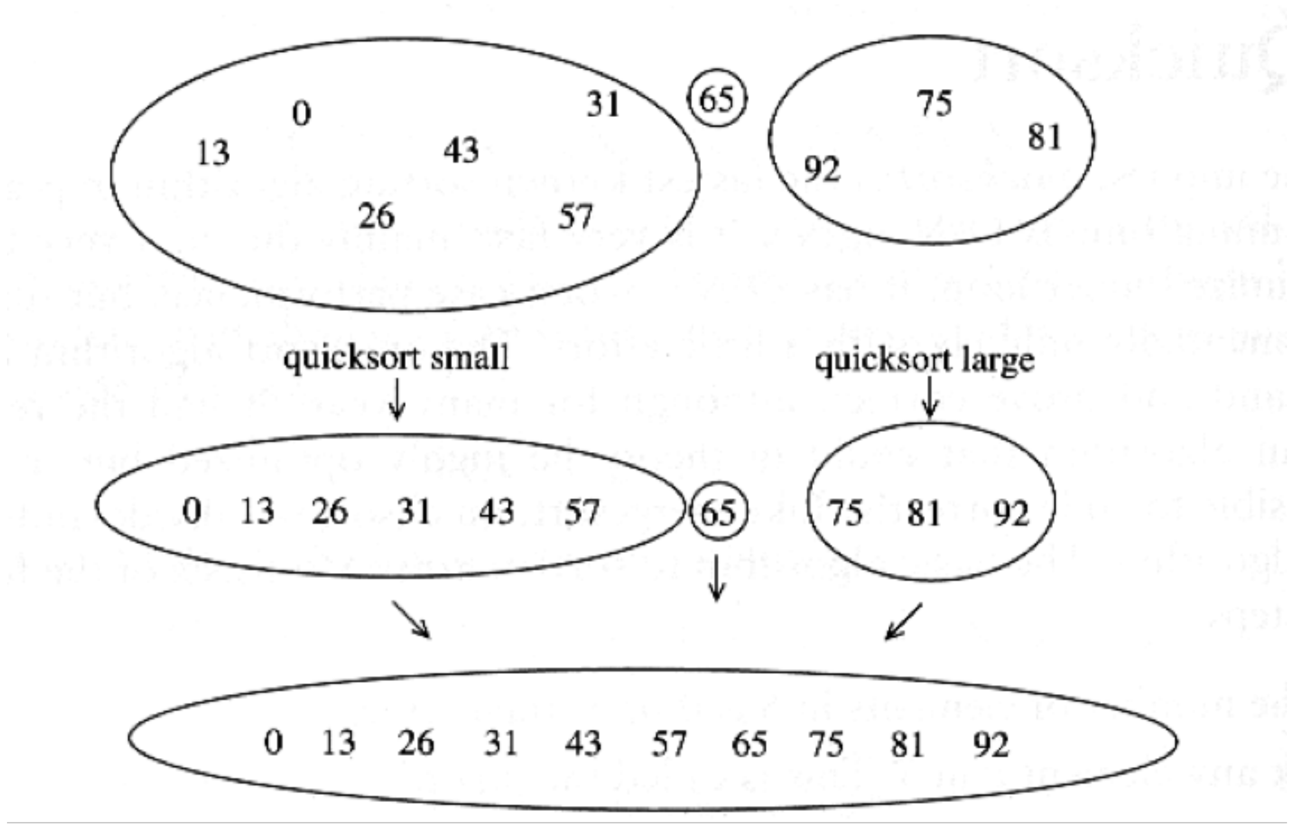
Quick Sort: Main Idea

1. If the number of elements in S is 0 or 1, then return (base case).
2. Pick any element v in S (called the pivot).
3. Partition the elements in S except v into two disjoint groups:
    1. $S_1 = \{x \in S - \{v\} \mid x \leq v\}$
    2. $S_2 = \{x \in S - \{v\} \mid x \geq v\}$
4. Return {QuickSort(S_1) + v + QuickSort(S_2)}

Quick Sort: Example



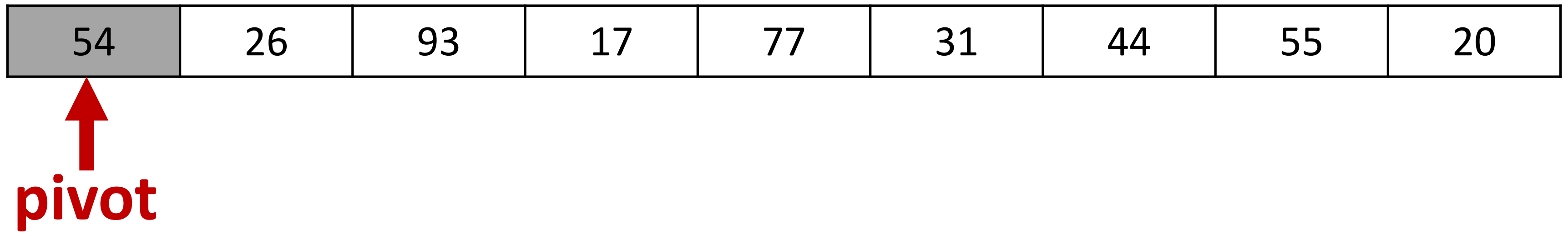
Example of Quick Sort...



Quick Sort - Example

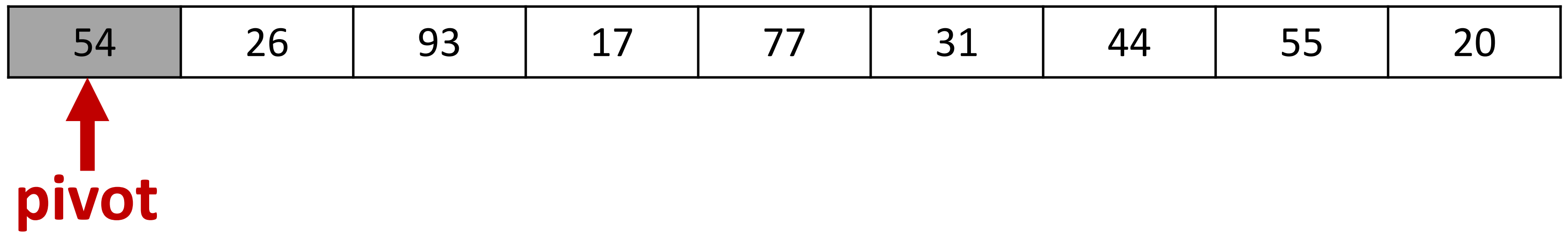
54	26	93	17	77	31	44	55	20

Quick Sort - Example



Quick Sort - Example

Partitioning begins by locating two position markers—let's call them leftmark and rightmark—at the beginning and end of the remaining items in the list.
The goal of the partition process is to move items that are on the wrong side with respect to the pivot value while also converging on the split point.



Quick Sort - Example

Partitioning begins by locating two position markers—let’s call them leftmark and rightmark—at the beginning and end of the remaining items in the list.
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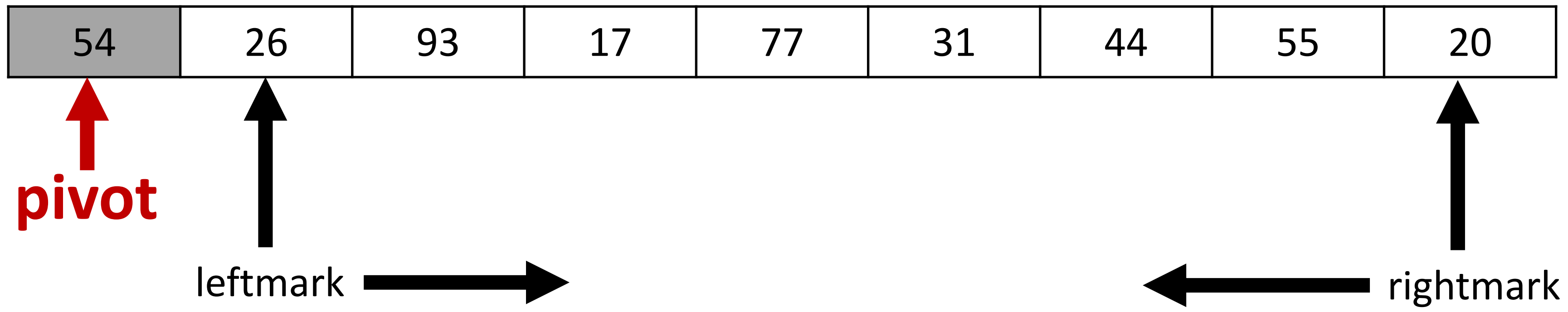
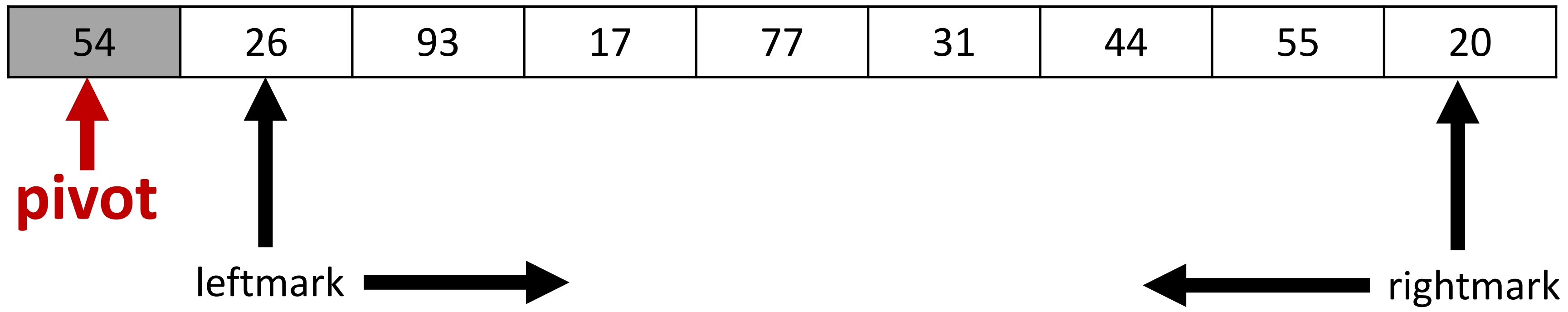


Quick Sort - Example

Partitioning begins by locating two position markers—let's call them leftmark and rightmark—at the beginning and end of the remaining items in the list.
The goal of the partition process is to move items that are on the wrong side with respect to the pivot value while also converging on the split point.



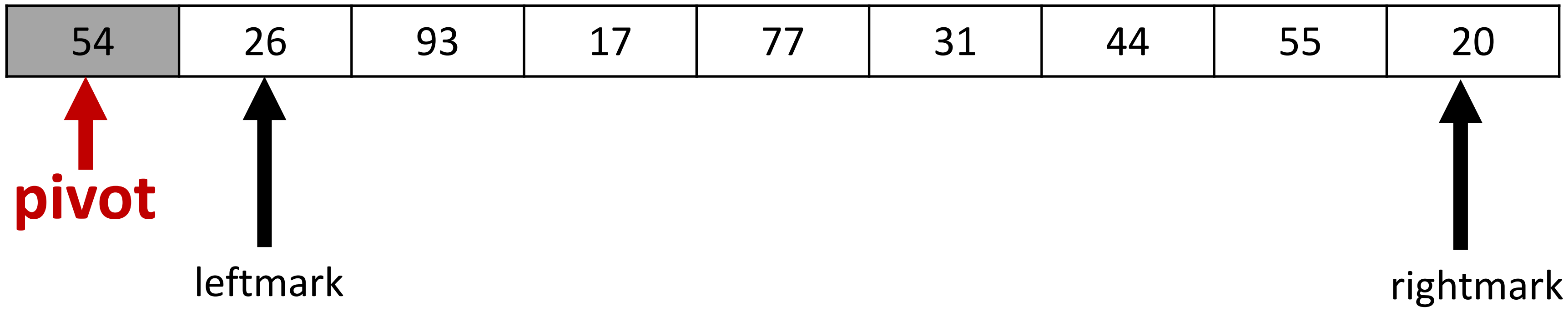
leftmark and the rightmark will converge on split point

Quick Sort - Example

Is 26 < 54

YES

Move to right

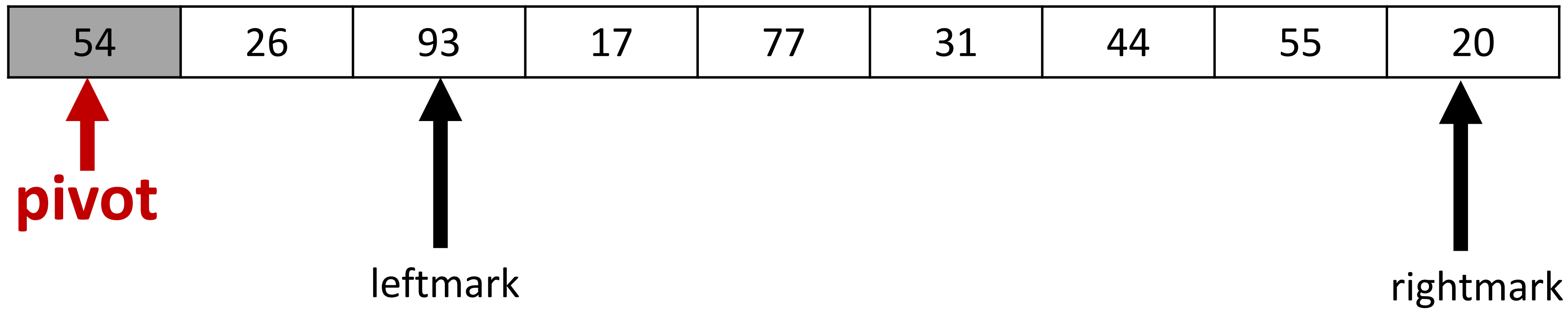


Quick Sort - Example

Is 93 < 54

NO

Stop



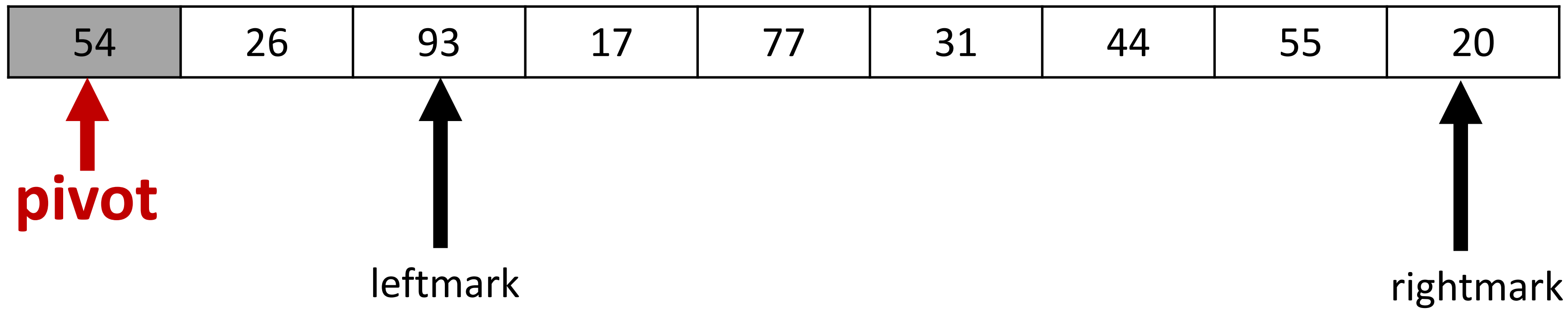
Quick Sort - Example

Now consider the rightmark

Is 20 > 54

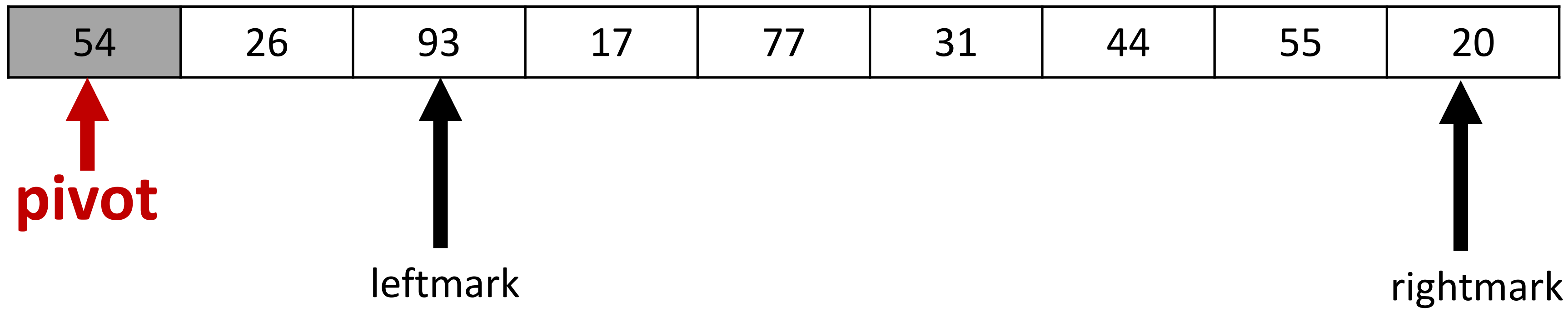
NO

Stop



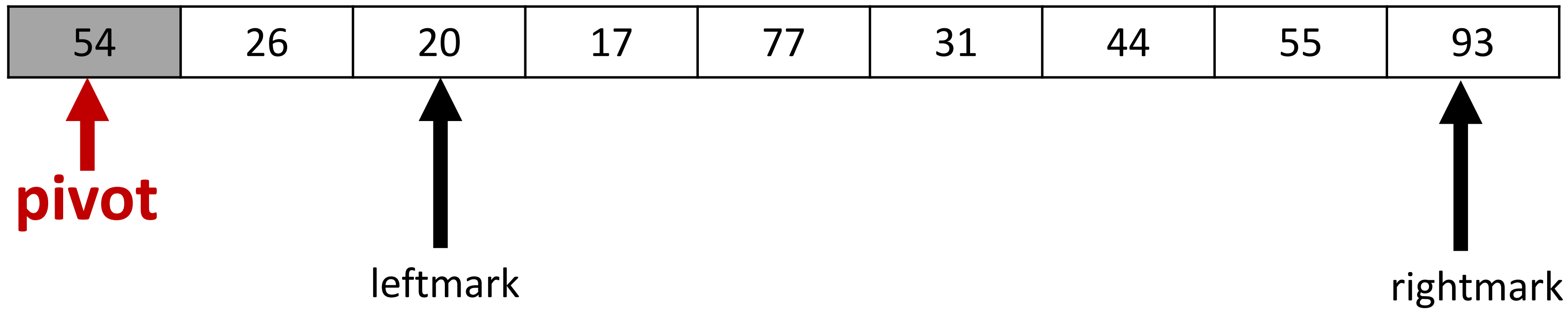
Quick Sort - Example

Exchange 20 and 93



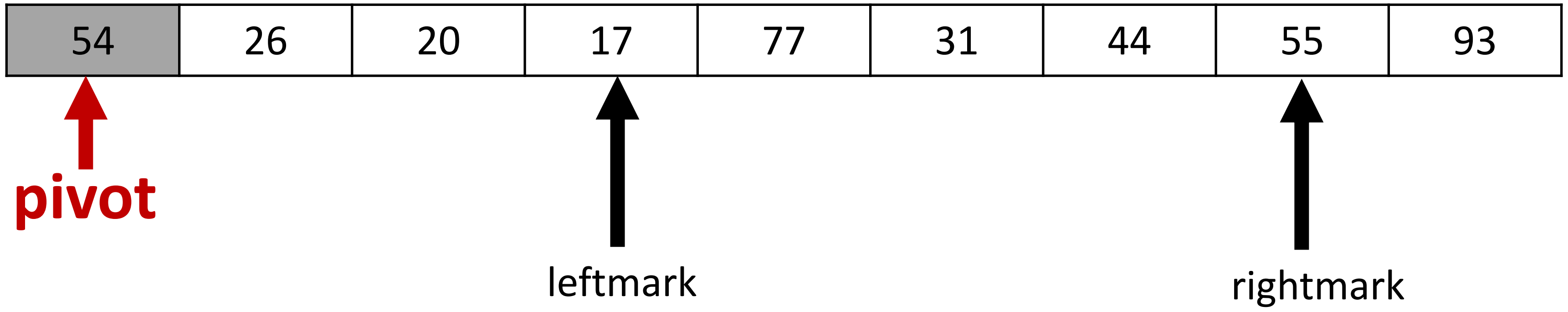
Quick Sort - Example

Exchange 20 and 93



Quick Sort - Example

Move leftmark and rightmark

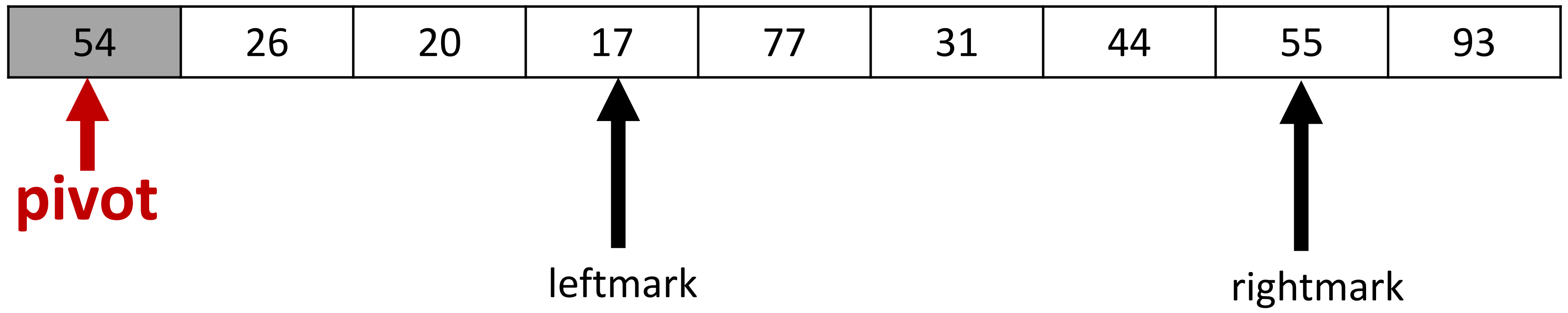


Quick Sort - Example

Is 17 < 54

YES

Move to right

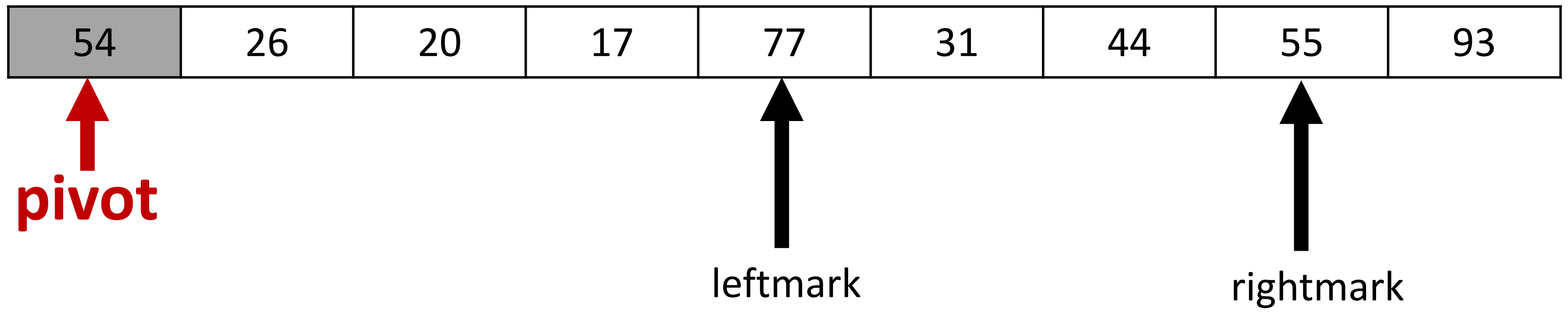


Quick Sort - Example

Is 77 < 54

NO

stop

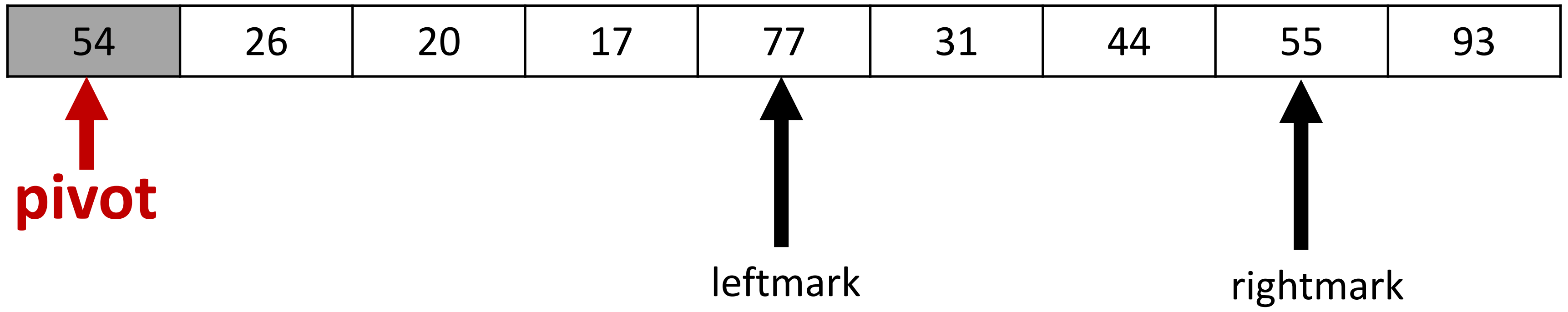


Quick Sort - Example

Is 55 > 54

YES

Move to left

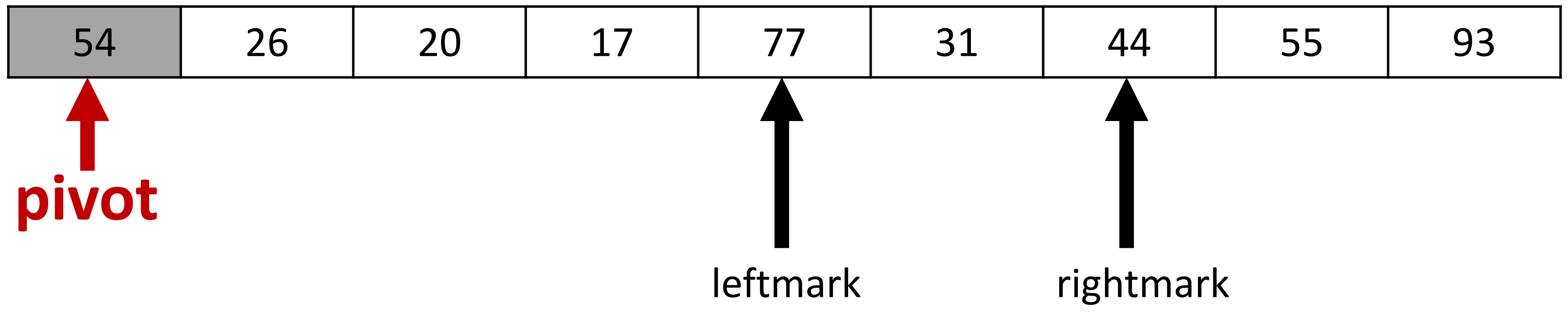


Quick Sort - Example

Is 44 > 54

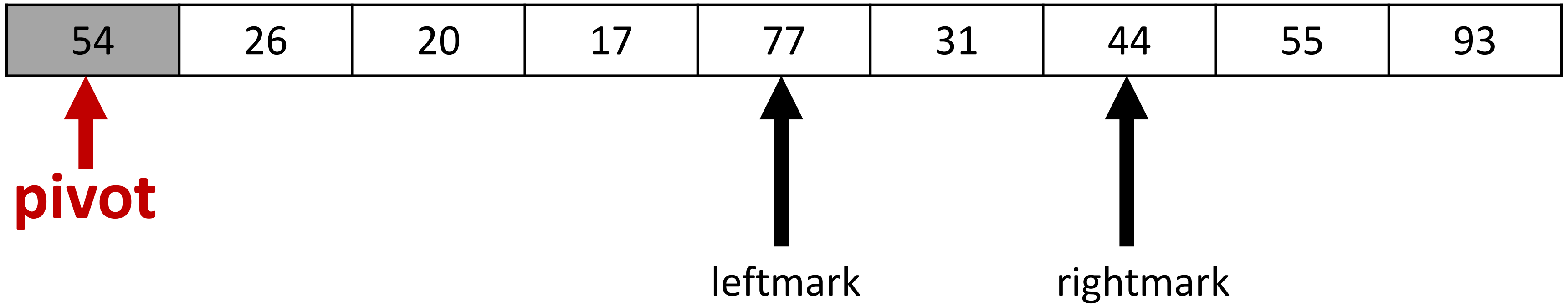
NO

stop



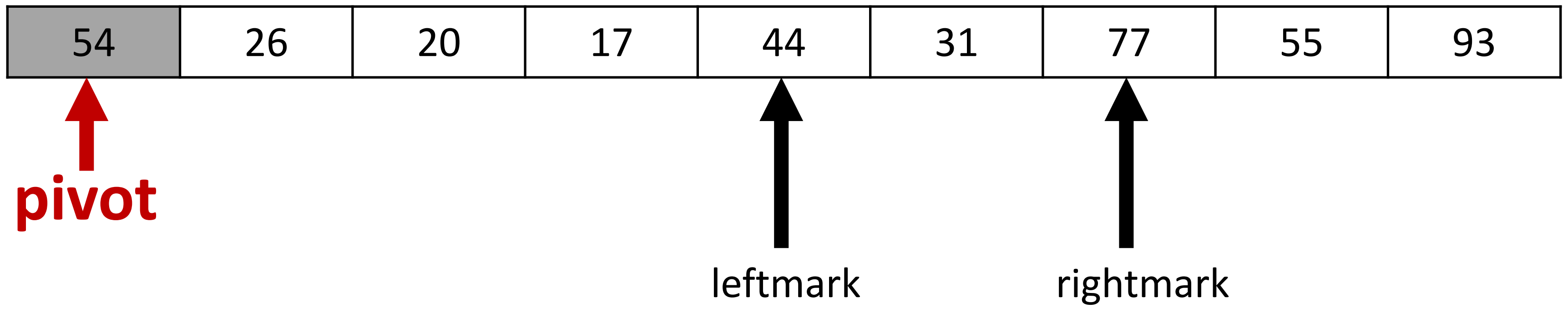
Quick Sort - Example

Exchange 77 and 44



Quick Sort - Example

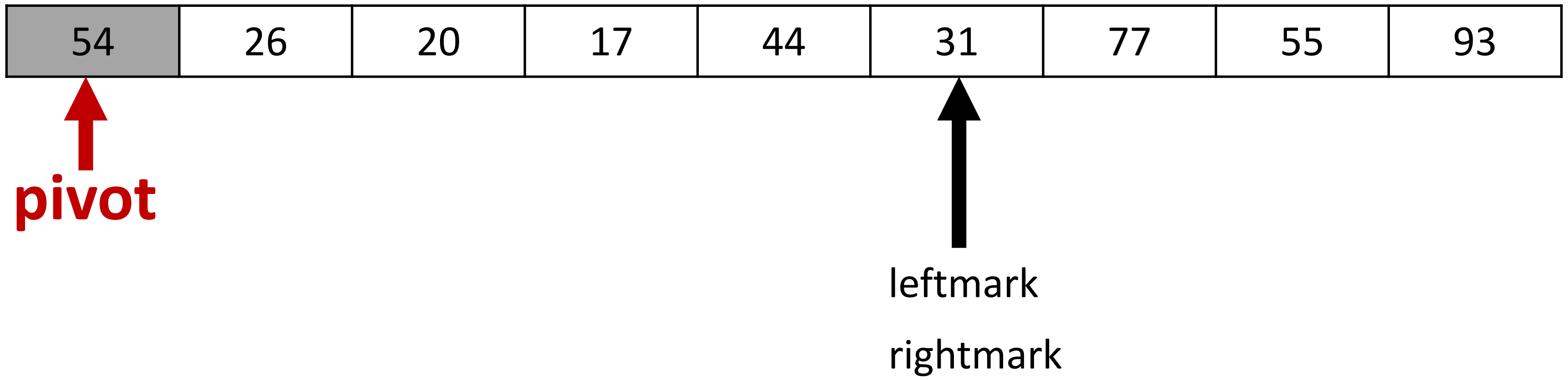
Exchange 77 and 44



Quick Sort - Example

Change the leftmark

Change the rightmark

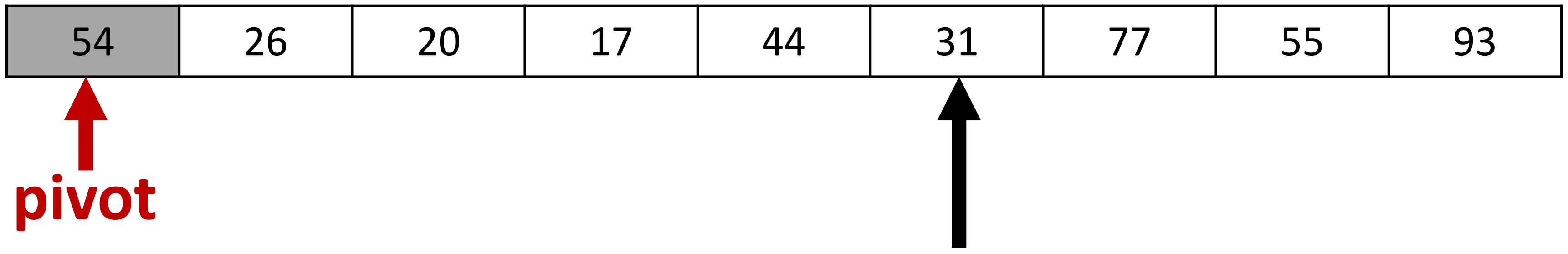


Quick Sort - Example

If 31 < 54

YES

Move to right



leftmark

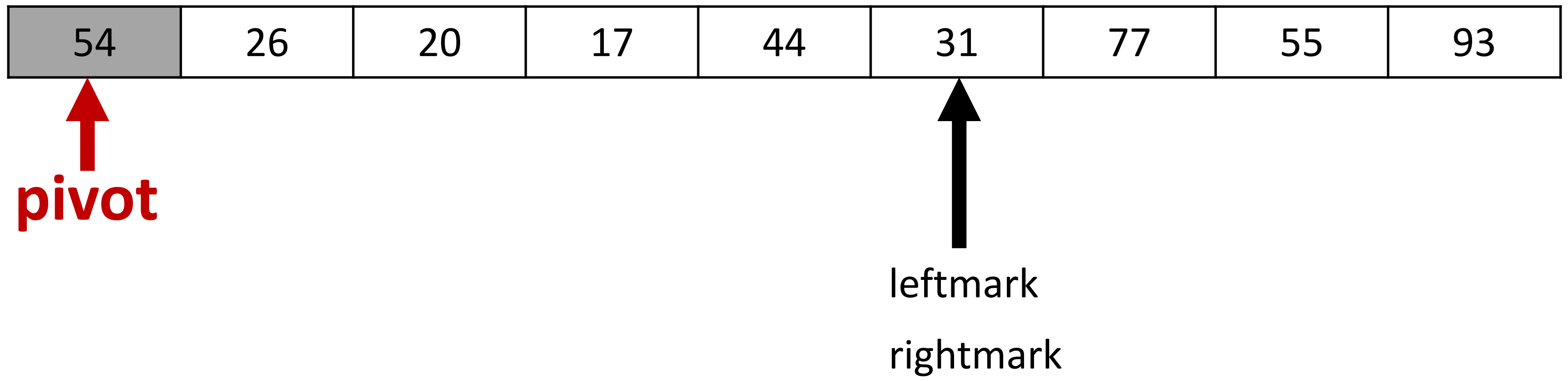
rightmark

When leftmark == rightmark – split point found

exchange 54 and 31

Quick Sort - Example

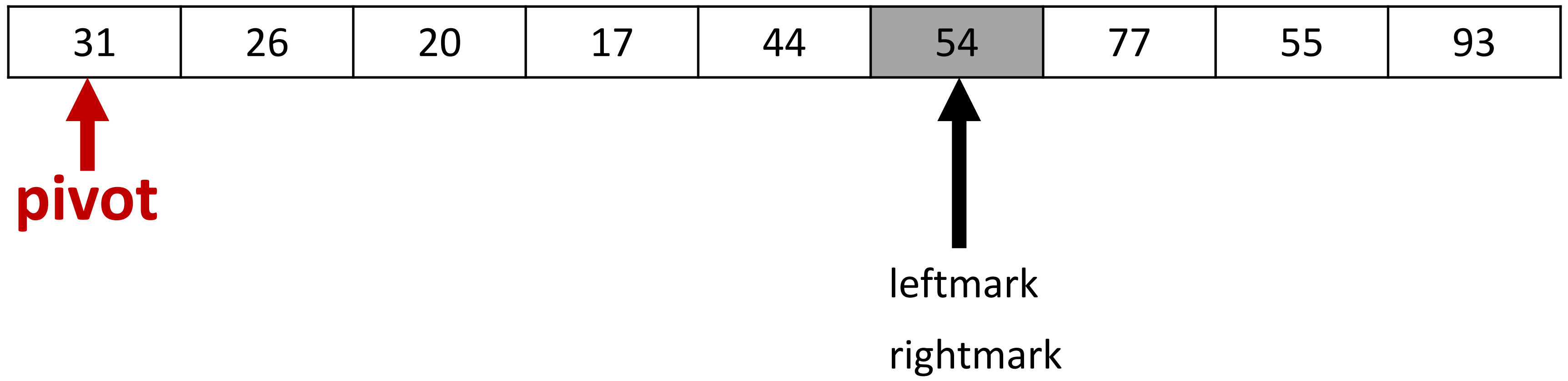
leftmark and rightmark crossing each other



When leftmark == rightmark – split point found

exchange 54 and 31

Quick Sort - Example

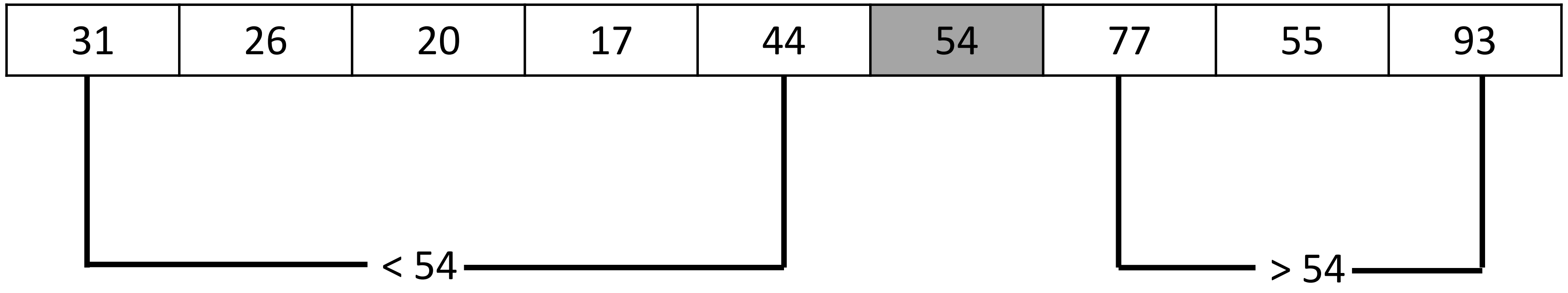


When leftmark == rightmark – split point found

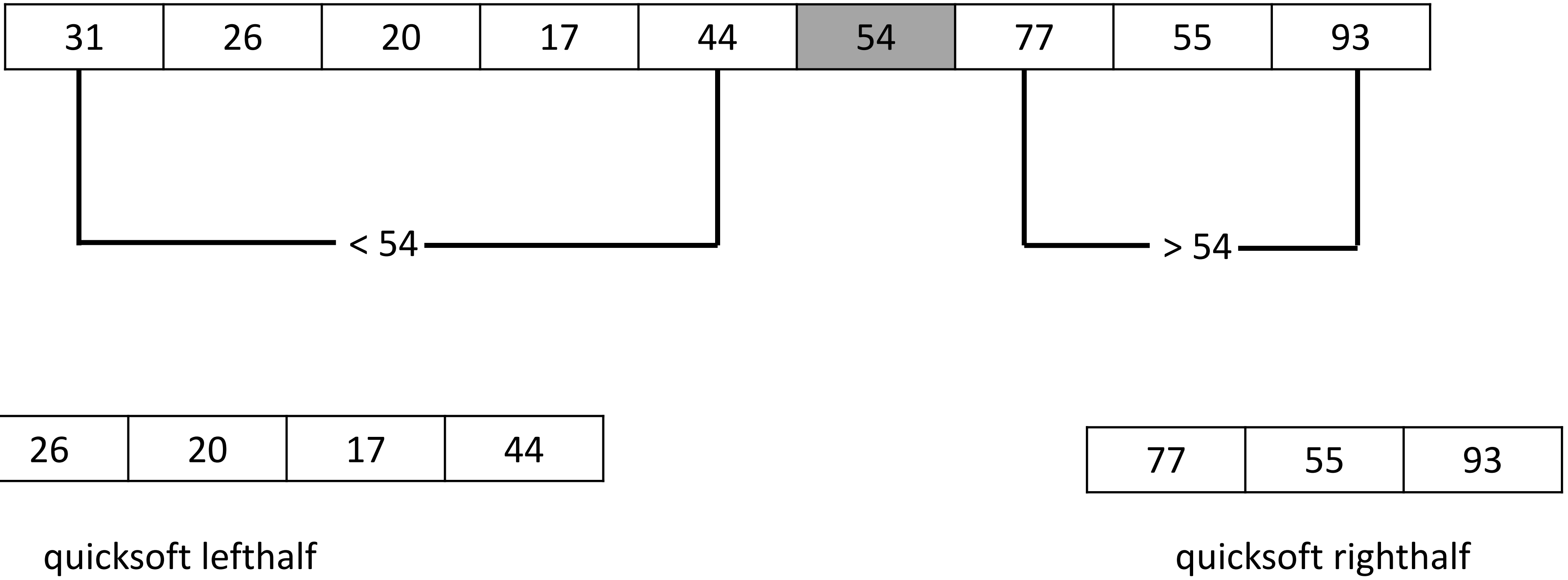
exchange 54 and 31

Quick Sort - Example

54 is in place



Quick Sort - Example

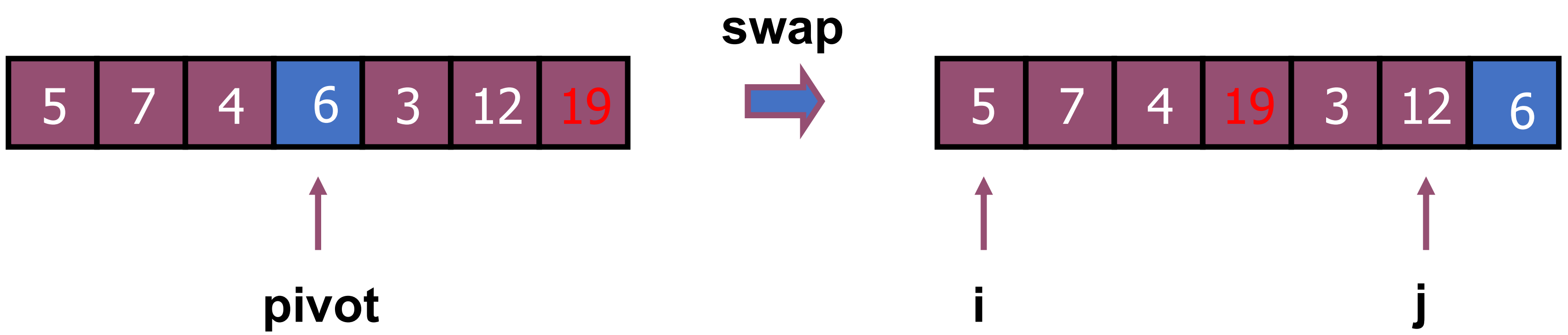


Issues To Consider

- How to partition?
  - Several methods exist.
  - The one we consider is known to give good results and to be easy and efficient.
  - We discuss the partition strategy first.
- How to pick the pivot?
  - Many methods

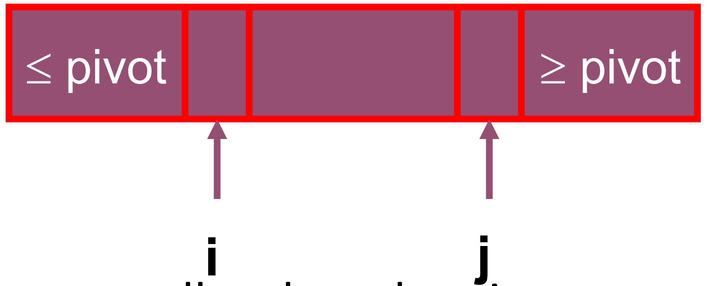
Partitioning Strategy

- For now, assume that pivot = A[(left+right)/2].
- We want to partition array A[left .. right].
- First, get the pivot element out of the way by swapping it with the last element (swap pivot and A[right]).
- Let i start at the first element and j start at the next-to-last element (i = left, j = right – 1)

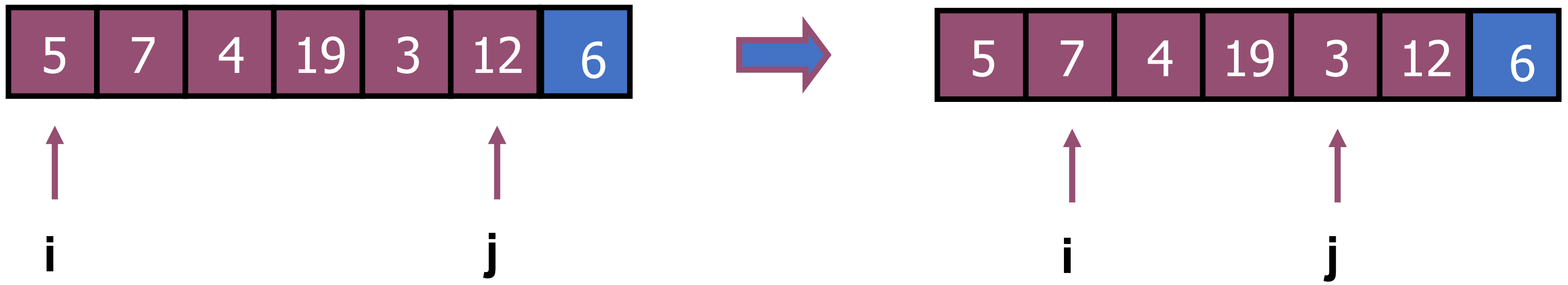


Partitioning Strategy

- Want to have
  - A[k] ≤ pivot, for k < i
  - A[k] ≥ pivot, for k > j

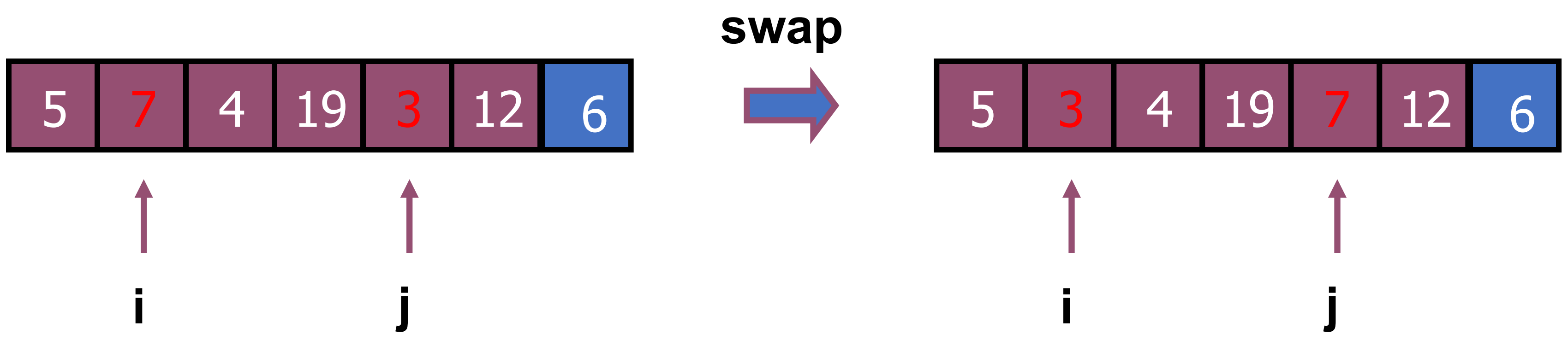


- When i < j
  - Move i right, skipping over elements smaller than the pivot
  - Move j left, skipping over elements greater than the pivot
  - When both i and j have stopped
    - A[i] ≥ pivot
    - A[j] ≤ pivot ⇒ A[i] and A[j] should now be swapped



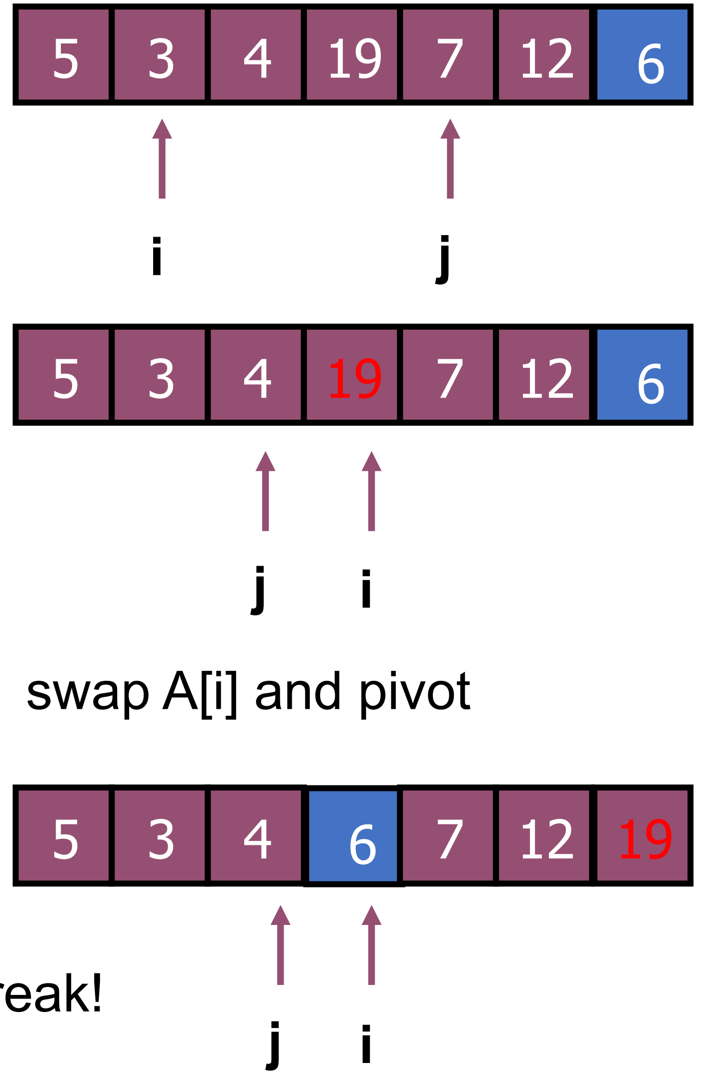
Partitioning Strategy (2)

- When i and j have stopped and i is to the left of j (thus legal)
  - Swap A[i] and A[j]
    - The large element is pushed to the right and the small element is pushed to the left
  - After swapping
    - A[i] $\leq$ pivot
    - A[j] $\geq$ pivot
  - Repeat the process until i and j cross



Partitioning Strategy (3)

- When i and j have crossed
  - swap A[i] and pivot
- Result:
  - A[k] ≤ pivot, for k < i
  - A[k] ≥ pivot, for k > i



Picking the Pivot

- There are several ways to pick a pivot.

- Objective: Choose a pivot so that we will get 2 partitions of (almost) equal size.

Picking the Pivot (2)

- Use the first element as pivot
  - if the input is random, ok.
  - if the input is presorted (or in reverse order)
    - all the elements go into S_2 (or S_1).
    - this happens consistently throughout the recursive calls.
    - results in $O(N^2)$ behavior (we analyze this case later).
- Choose the pivot randomly
  - generally safe,
  - but random number generation can be expensive and does not reduce the running time of the algorithm.

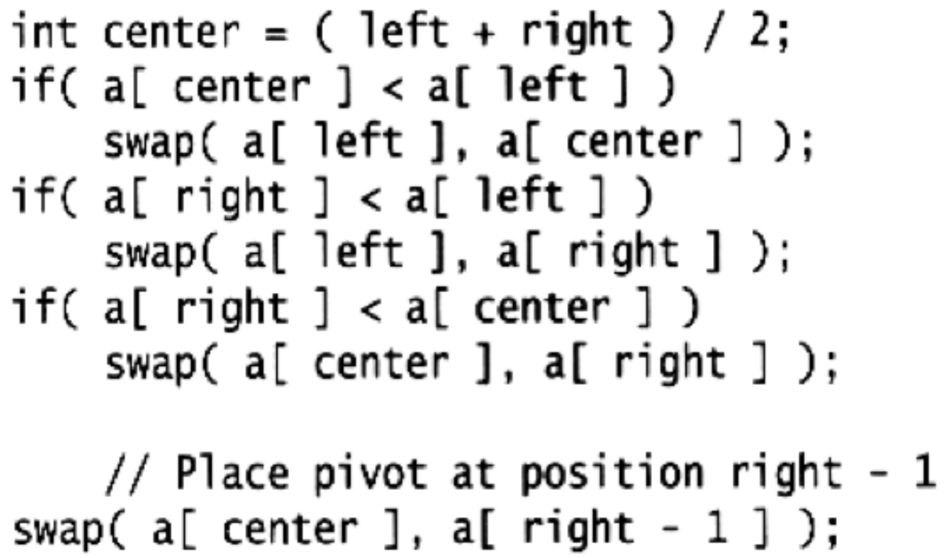
Picking the Pivot (3)

- Use the median of the array (ideal pivot)
    - The $\lceil N/2 \rceil$ *th* largest element
    - Partitioning always cuts the array into roughly half
    - An optimal quick sort (O(N log N))
    - However, hard to find the exact median

- Median-of-three partitioning
    - eliminates the bad case for sorted input.
    - reduces the number of comparisons by 14%.

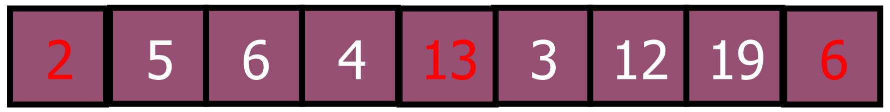
Median of Three Method

- Compare just three elements: the leftmost, rightmost and center
  - Swap these elements if necessary so that
    - A[left] = Smallest
    - A[right] = Largest
    - A[center] = Median of three
  - Pick A[center] as the pivot.
  - Swap A[center] and A[right – 1] so that the pivot is at the second last position (why?)

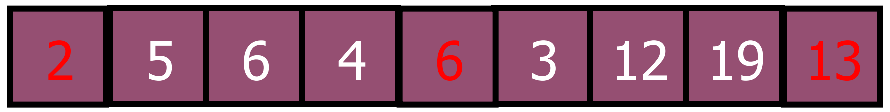


```
int center = ( left + right ) / 2;
if( a[ center ] < a[ left ] )
    swap( a[ left ], a[ center ] );
if( a[ right ] < a[ left ] )
    swap( a[ left ], a[ right ] );
if( a[ right ] < a[ center ] )
    swap( a[ center ], a[ right ] );

    // Place pivot at position right - 1
swap( a[ center ], a[ right - 1 ] );
```

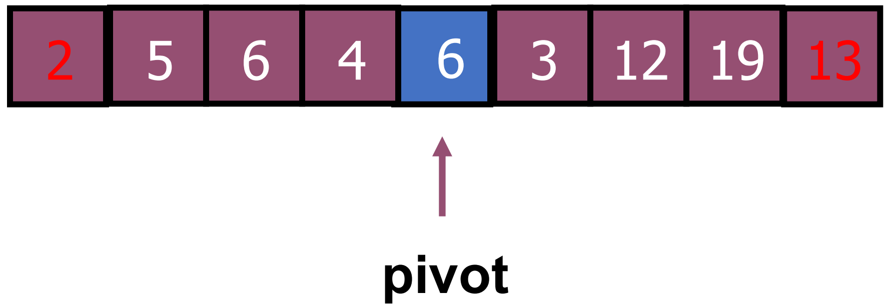
Median of Three: Example



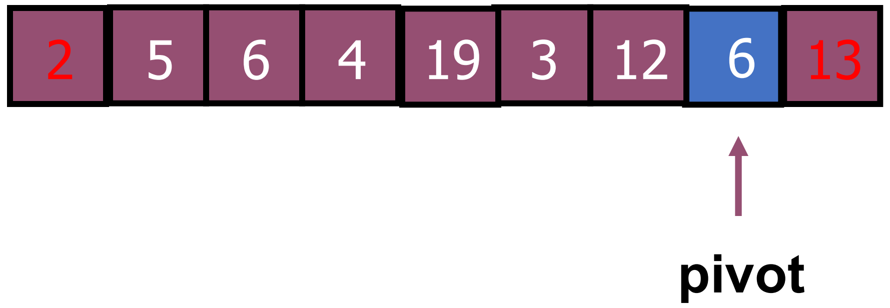
A[left] = 2, A[center] = 13,
A[right] = 6



Swap A[center] and A[right]



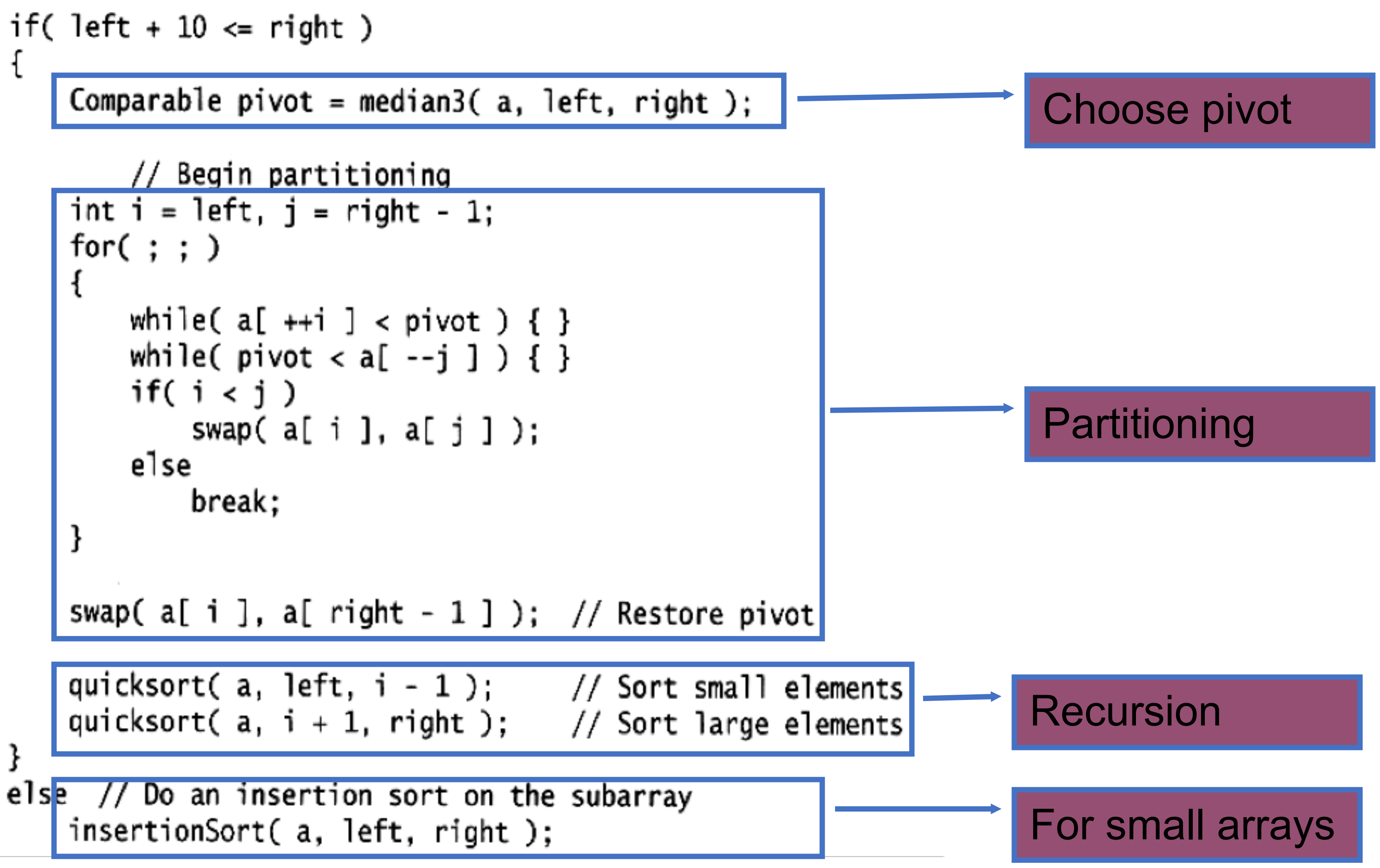
Choose A[center] as pivot



Swap pivot and A[right – 1]

We only need to partition A[left + 1, …, right – 2].

Quick Sort: Pseudo-code

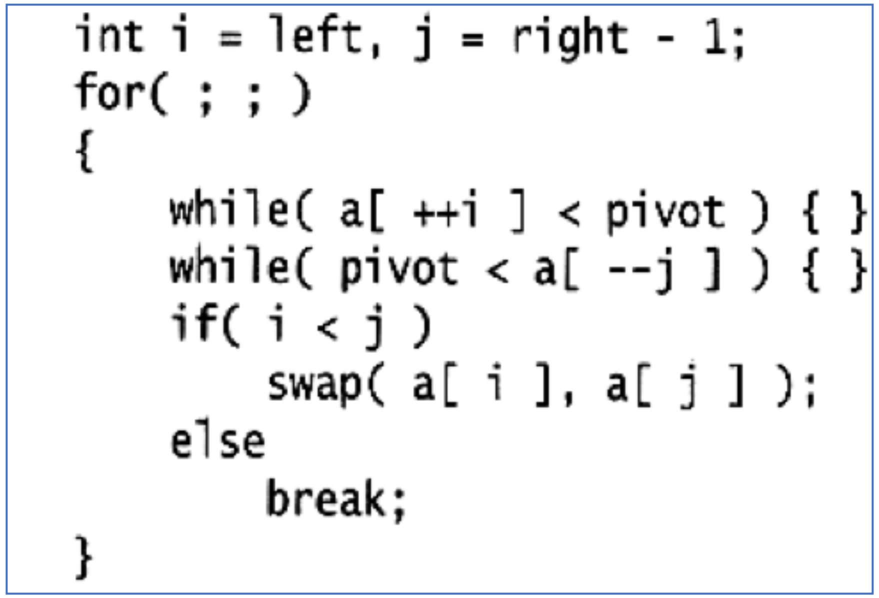


```
if( left + 10 <= right )
{
    Comparable pivot = median3( a, left, right );

        // Begin partitioning
    int i = left, j = right - 1;
    for( ; ; )
    {
        while( a[ ++i ] < pivot ) { }
        while( pivot < a[ --j ] ) { }
        if( i < j )
            swap( a[ i ], a[ j ] );
        else
            break;
    }

    swap( a[ i ], a[ right - 1 ] );   // Restore pivot

    quicksort( a, left, i - 1 );      // Sort small elements
    quicksort( a, i + 1, right );     // Sort large elements
}
else  // Do an insertion sort on the subarray
    insertionSort( a, left, right );
```

Choose pivot
Partitioning
Recursion
For small arrays

Partitioning Part

- The partitioning code we just saw works only if pivot is picked as median-of-three.
  - A[left] ≤ pivot and A[right] ≥ pivot
  - Need to partition only A[left + 1, …, right – 2]
- j will not run past the beginning
  - because A[left] ≤ pivot
- i will not run past the end
  - because A[right-1] = pivot



```
int i = left, j = right - 1;
for( ; ; )
{
    while( a[ ++i ] < pivot ) { }
    while( pivot < a[ --j ] ) { }
    if( i < j )
        swap( a[ i ], a[ j ] );
    else
        break;
}
```

```
/* The main function that implements QuickSort
 arr[] --> Array to be sorted,
  low  --> Starting index,
  high  --> Ending index */
void quickSort(int arr[], int low, int high)
{
    if (low < high)
    {
        /* pi is partitioning index, arr[p] is now
           at right place */
        int pi = partition(arr, low, high);

        // Separately sort elements before
        // partition and after partition
        quickSort(arr, low, pi - 1);
        quickSort(arr, pi + 1, high);
    }
}
```

```c
/* This function takes last element as pivot, places
   the pivot element at its correct position in sorted
    array, and places all smaller (smaller than pivot)
   to left of pivot and all greater elements to right
   of pivot */
int partition (int arr[], int low, int high)
{
    int pivot = arr[high];    // pivot
    int i = (low - 1);  // Index of smaller element

    for (int j = low; j <= high- 1; j++)
    {
        // If current element is smaller than or
        // equal to pivot
        if (arr[j] <= pivot)
        {
            i++;    // increment index of smaller element
            swap(&arr[i], &arr[j]);
        }
    }
    swap(&arr[i + 1], &arr[high]);
    return (i + 1);
}
```

Small Arrays - Nuances

- For very small arrays, quick sort does not perform as well as insertion sort
- Do not use quick sort recursively for small arrays
  - Use a sorting algorithm that is efficient for small arrays, such as insertion sort.
- When using quick sort recursively, switch to insertion sort when the sub-arrays have between 5 to 20 elements (10 is usually good).
  - saves about 15% in the running time.
  - avoids taking the median of three when the sub-array has only 1 or 2 elements.

Assignment

- Assume the pivot is chosen as the middle element of an array: pivot = a[(left+right)/2] - Median of Three . Rewrite the partitioning code and the whole quick sort algorithm.

Analysis

Assumptions:

- A random pivot (no median-of-three partitioning)
- No cutoff for small arrays (to make it simple)

1. If the number of elements in S is 0 or 1, then return (base case).
2. Pick an element v in S (called the pivot).
3. Partition the elements in S except v into two disjoint groups:
    1. $S_1 = \{x \in S - \{v\} \mid x \leq v\}$
    2. $S_2 = \{x \in S - \{v\} \mid x \geq v\}$
4. Return {QuickSort(S_1) + v + QuickSort(S_2)}

Worst-Case Scenario

- What will be the worst case?
  - The pivot is the smallest element, all the time
  - Partition is always unbalanced

$$\begin{aligned}
T(N) &= T(N-1) + cN \\
T(N-1) &= T(N-2) + c(N-1) \\
T(N-2) &= T(N-3) + c(N-2) \\
&\vdots \\
T(2) &= T(1) + c(2) \\
T(N) &= T(1) + c\sum_{i=2}^{N} i = O(N^2)
\end{aligned}$$

Best-Case Scenario

- What will be the best case?
  - Partition is perfectly balanced.
  - Pivot is always in the middle (median of the array).

- T(N) = T(N/2) + T(N/2) + cN = 2T(N/2) + cN

- This recurrence is similar to the merge sort recurrence.
- The result is O(NlogN).

Average-Case Analysis

- Assume that each of the sizes for S_1 is equally likely $\Rightarrow$ has probability 1/N.
- This assumption is valid for the pivoting and partitioning strategy just discussed (but may not be for some others),
- On average, the running time is O(N log N).
- Proof: pp 272–273, Data Structures and Algorithm Analysis by M. A. Weiss, 2nd edition