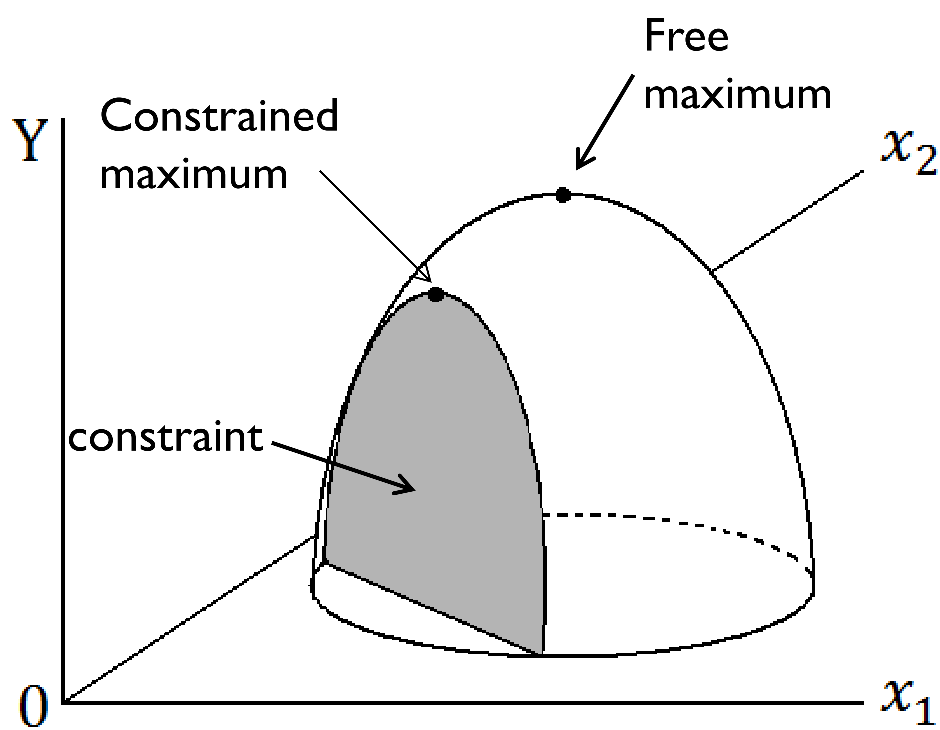
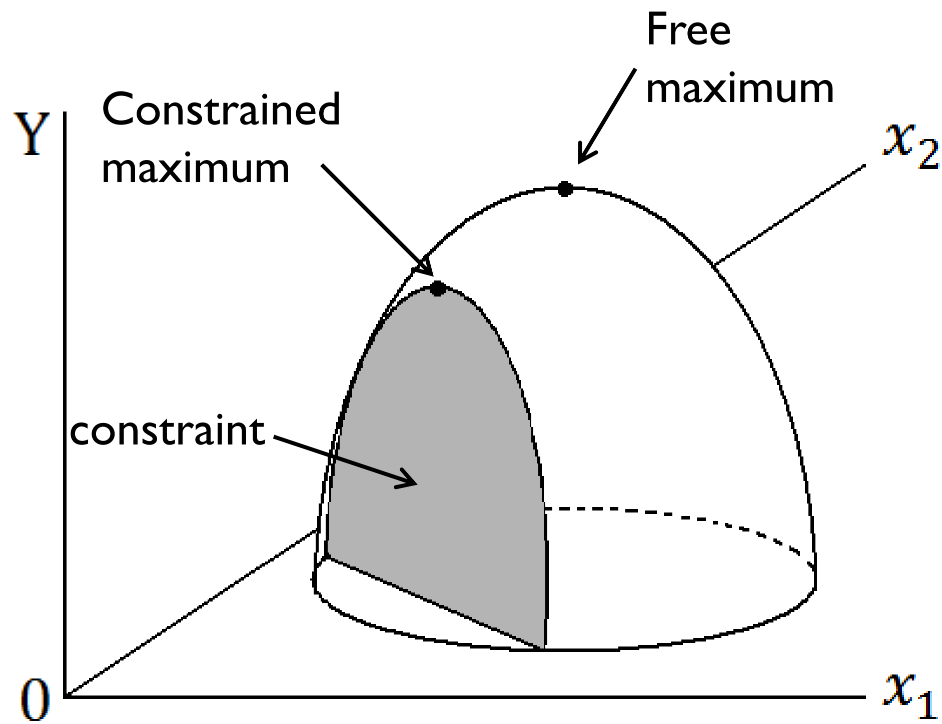
# Constrained Optimization

We want to minimize (or maximize) the objective function, at the same time the solution should obey certain constraints.

# Constrained Optimization

Graphically, the difference between the free optima and the constrained optima can be shown as:






- The free optima occurs at the peak of the surface.

- If we specify a specific relationship between variables $x_1$ and $x_2$ (a constraint) then the search for an optimum is restricted to <u>a slice of the surface.</u> The constrained maximum occurs at the peak of the slice.

# Constrained Optimization

- Since economists deal with the allocation of scarce resources among alternative uses, the concept of constraints or restrictions is important.

- There are two approaches to solving constrained optima problems:
  - (i) substitution method
  - (ii) Lagrange multipliers

# Substitution Method

- Consider a firm producing commodity $y$ with the following production function:

$$y = 5x_1x_2$$

- Without any constraints, the firm can produce an unlimited quantity by utilizing an unlimited amount of $x_1$ and $x_2$.

# Substitution Method

- But suppose the firm has a budget constraint:

  Let $p_{x_1} = \$2/unit$

  $p_{x_2} = \$1/unit$

- For simplicity, assume that the maximum amount the firm can spend on these two inputs is \$100.

- So we have the following constraint:

$$2x_1 + x_2 = 100$$

# Substitution Method

- Suppose the economic question facing this firm is maximizing production subject to this budget constraint.

- The solution via the substitution method is to substitute:

  - First, write the constraint in terms of $x_2$:

$$2x_1 + x_2 = 100$$

$$\therefore x_2 = 100 - 2x_1$$

# Substitution Method

$$x_2 = 100 - 2x_1$$

- Then substitute this value into the production function, such that:

$$y = 5x_1x_2$$

$$y = 5x_1(100 - 2x_1)$$

$$= 500x_1 - 10x_1^2$$

- With this substitution, the constrained maxima problem is reduced to a free maxima problem with one independent variable.

# Substitution Method

- Now apply the usual optimization procedure:

$$\frac{dy}{dx_1} = 500 - 20x_1 = 0$$

$$-20x_1 = -500$$

(critical value) $$\therefore x_1 = \frac{-500}{-20} = 25$$

# Substitution Method

$$\frac{dy}{dx_1} = 500 - 20x_1 = 0$$

$$\frac{d^2y}{dx_1^2} = -20 < 0 \qquad \therefore \text{relative max}$$

$$\therefore \text{if } x_1 = 25 \text{ then } 100 = 2(25) + x_2$$

$$\therefore 100 - 50 = 50 = x_2$$

- The method of substitution is one way to solve constrained optima problems. This is manageable in some cases. In others, the constraint may be very complicated and substitution becomes complex.

# Lagrange Multipliers

- The constrained optima problem can be stated as finding the extreme value of $y = f(x_1, x_2)$ subject to $g(x_1, x_2) = 0$.

- So Lagrange (a mathematician) formed the <u>augmented function</u>.

$$L = f(x_1, x_2) + \alpha\left(g(x_1, x_2)\right)$$

denotes the augmented function called the Lagrangian, will behave like the function if the constraint is followed.

# Lagrange Multipliers

- Given the augmented function, the first order condition for optimization (where the independent variables are $x_1$, $x_2$ and λ) is as follows:

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= f_1 + \alpha g_1 = 0 \\ \frac{\partial L}{\partial x_2} &= f_2 + \alpha g_2 = 0 \\ \frac{\partial L}{\partial \alpha} &= g(x_1, x_2) = 0 \end{aligned}\right\} \text{Solve simultaneously for critical values}$$

# Lagrange Multipliers

- Using the previous example:

$$L = 5x_1x_2 + \alpha(100 - 2x_1 - x_2)$$

note: $100 = 2x_1 + x_2$

$\therefore 100 - 2x_1 - x_2 = 0$ to be on the budget line

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= 5x_2 - 2\alpha = 0 \\ \frac{\partial L}{\partial x_2} &= 5x_1 - \alpha = 0 \\ \frac{\partial L}{\partial \alpha} &= 100 - 2x_1 - x_2 = 0 \end{aligned}\right\}$$

3 unknowns: $x_1, x_2, \alpha$

3 equations

# Lagrange Multipliers

$$\left.\begin{aligned}\frac{\partial L}{\partial x_1} &= 5x_2 - 2\alpha = 0\\ \frac{\partial L}{\partial x_2} &= 5x_1 - \alpha = 0\\ \frac{\partial L}{\partial \alpha} &= 100 - 2x_1 - x_2 = 0\end{aligned}\right\} \begin{array}{c}\text{3 unknowns:}\\ x_1, x_2, \alpha\\ \text{3 equations}\end{array}$$

- Solving these 3 equations simultaneously:

$$5x_2 - 2\alpha = 0$$

$$5x_2 = 2\alpha$$

$$\therefore x_2 = \frac{2\alpha}{5}$$

# Lagrange Multipliers

$$\left.\begin{aligned}\frac{\partial L}{\partial x_1} &= 5x_2 - 2\alpha = 0\\ \frac{\partial L}{\partial x_2} &= 5x_1 - \alpha = 0\\ \frac{\partial L}{\partial \alpha} &= 100 - 2x_1 - x_2 = 0\end{aligned}\right\}\begin{array}{c}\text{3 unknowns:}\\ x_1, x_2, \alpha\\ \text{3 equations}\end{array}$$

- Solving these 3 equations simultaneously (cont'd):

$$x_1 = \frac{\alpha}{5}$$

# Lagrange Multipliers

$$x_1 = \frac{\alpha}{5}$$

$$x_2 = \frac{2\alpha}{5}$$

- Solving these 3 equations simultaneously (cont'd):

$$\therefore 100 - 2\left(\frac{\alpha}{5}\right) - \left(\frac{2\alpha}{5}\right) = 0$$

$$500 = 4\alpha$$

$$100 = \frac{4\alpha}{5}$$

$$\alpha = 125$$

# Lagrange Multipliers

- This solution yields the same answer as the substitution method, i.e.,

$$x_1 = 25 \qquad x_2 = 50$$

# Lagrange Multipliers

- Economists prefer using the Lagrange technique over the substitution method, because:

  (i) easier to handle for most cases and

  (ii) provides additional information.

- The Lagrange multiplier gives how much sensitive the constraint is

# KKT Conditions

- In the presence of inequality constraints, one can use KKT conditions which are necessary conditions for the optimality.
  - These are sufficient also, provided the objective is convex and constraints are linear. (This is what exactly happens in the case of SVMs).

# Constrained Optimization Problem

- Minimize $f(v)$
  Subject to the constraints $g_j(v) \leq 0, \;\; 1 \leq j \leq n$.
- Lagrangian,

$$\mathcal{L} = f(v) + \sum_{j=1}^{n} \alpha_j \, g_j(v)$$

  where $v$ is called *primary* variables and $\alpha_j$ are the Lagrangian multipliers which are also called *dual* variables.
- $\mathcal{L}$ has to be minimized with respect to primal varibles and maximized with respect to dual variables.

# K.K.T Conditions

$$(i)\ \nabla_v L = 0$$

$$\left.\begin{array}{l} (ii)\ \alpha_j \geq 0 \\ (iii)\ \alpha_j g_j(v) = 0 \\ (iv)\ g_j(v) \leq 0 \end{array}\right\} \text{for all } j = 1 \text{ to } n$$

•K.K.T. Coditions, in general are necessary, i.e., at optimal point these are satisfied. So, if these are not satisfied we know that the point we are concerned is not optimal.

•However, when the objective is a convex function and constraints are all linear functions, then K.K.T conditions are sufficient also.

• The (iv) th one is within the problem statement. So normally first 3 conditions are called the KKT conditions.

# Example.

Minimize $f(x) = (x-4)^2 + 5$, such that $x \geq 6$.

$L = (x-4)^2 + 5 + \alpha(-x+6)$

KKT conditions:

(1) $\frac{\partial L}{\partial x} = 0$, So $x = \frac{1}{2}(\alpha + 8)$

(2) $\alpha(-x+6) = 0$

(3) $\alpha \geq 0$

(2) and (3) along with the problem constraint, gives $x = 6$.

# Example

Minimize $f(v_1, v_2) = v_1 + v_2$, such that $v_1^2 + v_2^2 \leq 1$.

Solution: $L = (v_1 + v_2) + \alpha(v_1^2 + v_2^2 - 1)$.

KKT Conditions

(1) $\frac{\partial L}{\partial v_1} = 1 + 2\alpha v_1 = 0$, $\frac{\partial L}{\partial v_2} = 1 + 2\alpha v_2 = 0$.

$So,\ v_1 = v_2 = -\frac{1}{2\alpha}$. So $\alpha \neq 0$.

(2) $\alpha \geq 0$

(3) $\alpha(v_1^2 + v_2^2 - 1) = 0$.

From (2) and (3) since $\alpha > 0$, we have $v_1^2 + v_2^2 - 1 = 0$.

This gives $\alpha = \frac{1}{\sqrt{2}}$.

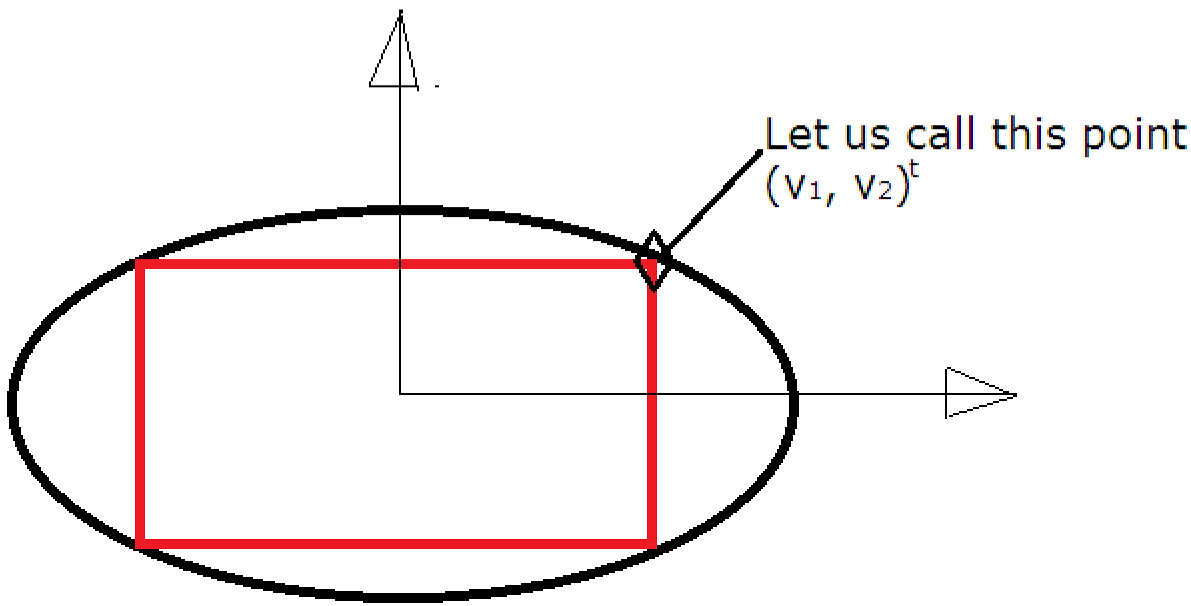
So, we get $v_1 = v_2 = -\frac{1}{\sqrt{2}}$

# With both equality and inequality constraints

- We see an example.

# An Example.

- Find maximum perimeter rectangle that is inscribed in the ellipse $x^2 + 4y^2 = 4$.



- Maximize the perimeter= $4(v_1 + v_2)$, subject to $v_1^2 + 4v_2^2 - 4 = 0$. Also, note we have constraints $v_1 \geq 0$, and $v_2 \geq 0$.

- Lagrangian, $L(v_1, v_2, \alpha_1, \alpha_2, \alpha_3) = -4(v_1 + v_2) + \alpha_1(v_1^2 + 4v_2^2 - 4) + \alpha_2(-v_1) + \alpha_3(-v_2)$.

# KKT Conditions

- $\frac{\partial L}{\partial v_1} = 0$
- $\frac{\partial L}{\partial v_2} = 0$
- $\frac{\partial L}{\partial \alpha_1} = 0$
- $v_1 \alpha_2 = 0, \qquad v_2 \alpha_3 = 0$
- $\alpha_1 \geq 0, \qquad \alpha_2 \geq 0, \alpha_3 \geq 0$

- We get $\alpha_1 = \frac{\sqrt{5}}{2}, \alpha_2 = \alpha_3 = 0.$
- We get the solution, …