Homework 4

ECE 269

Due : 11:59pm PT on Feb 28

1. Given a matrix A with eigenvalues $\lambda_1, ..., \lambda_n$, calculate eigenvalues of the following matrices (show the procedure).

(1) $(2I + A^2)^{-1}$ (5 points)

Answer:

Given

$$A = XDX^{-1}, D = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{bmatrix}$$

We can calculate

$$A^2 = XD^2X^{-1}, D^2 = \begin{bmatrix} (\lambda_1)^2 & 0 & ... & 0 \\ 0 & (\lambda_2)^2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & (\lambda_n)^2 \end{bmatrix}$$

$$2I + A^2 = X(2I + D^2)X^{-1}, 2I + D^2 = \begin{bmatrix} 2 + (\lambda_1)^2 & 0 & ... & 0 \\ 0 & 2 + (\lambda_2)^2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 2 + (\lambda_n)^2 \end{bmatrix}$$

$$(2I + A^2)^{-1} = X(2I + D^2)^{-1}X^{-1}, (2I + D^2)^{-1} = \begin{bmatrix} \frac{1}{2+(\lambda_1)^2} & 0 & ... & 0 \\ 0 & \frac{1}{2+(\lambda_2)^2} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \frac{1}{2+(\lambda_n)^2} \end{bmatrix}$$

So the eigenvalues of $(2I + A^2)^{-1}$ are

$$\frac{1}{2 + (\lambda_1)^2}, \frac{1}{2 + (\lambda_2)^2}, ..., \frac{1}{2 + (\lambda_n)^2}$$

(2) $(3A + A^{-1})^2$ (5 points)

Answer:

Given

$$A = XDX^{-1}, D = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{bmatrix}$$

We can calculate

$$3A = X(3D)X^{-1}, 3D = \begin{bmatrix} 3\lambda_1 & 0 & ... & 0 \\ 0 & 3\lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 3\lambda_n \end{bmatrix}$$

$$A^{-1} = XD^{-1}X^{-1}, D^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & ... & 0 \\ 0 & \frac{1}{\lambda_2} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \frac{1}{\lambda_n} \end{bmatrix}$$

$$3A + A^{-1} = X(3D + D^{-1})X^{-1}, 3D + D^{-1} = \begin{bmatrix} 3\lambda_1 + \frac{1}{\lambda_1} & 0 & ... & 0 \\ 0 & 3\lambda_2 + \frac{1}{\lambda_2} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 3\lambda_n + \frac{1}{\lambda_n} \end{bmatrix}$$

$$(3A + A^{-1})^2 = X(3D + D^{-1})^2X^{-1}, (3D + D^{-1})^2 = \begin{bmatrix} (3\lambda_1 + \frac{1}{\lambda_1})^2 & 0 & ... & 0 \\ 0 & (3\lambda_2 + \frac{1}{\lambda_2})^2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & (3\lambda_n + \frac{1}{\lambda_n})^2 \end{bmatrix}$$

So the eigenvalues of $(3A + A^{-1})^2$ are

$$(3\lambda_1 + \frac{1}{\lambda_1})^2, (3\lambda_2 + \frac{1}{\lambda_2})^2, ..., (3\lambda_n + \frac{1}{\lambda_n})^2$$

(3) $((3A + A^{-1})^2 + 5A^{-1})^3$ (5 points)

Answer:

From (1) and (2) we have

$$(3A + A^{-1})^2 = X(3D + D^{-1})^2X^{-1}, (3D + D^{-1})^2 = \begin{bmatrix} (3\lambda_1 + \frac{1}{\lambda_1})^2 & 0 & ... & 0 \\ 0 & (3\lambda_2 + \frac{1}{\lambda_2})^2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & (3\lambda_n + \frac{1}{\lambda_n})^2 \end{bmatrix}$$

$$5A^{-1} = X(5D^{-1})X^{-1}, 5D^{-1} = \begin{bmatrix} \frac{5}{\lambda_1} & 0 & ... & 0 \\ 0 & \frac{5}{\lambda_2} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \frac{5}{\lambda_n} \end{bmatrix}$$

$$((3A + A^{-1})^2 + 5A^{-1})^3 = X((3D + D^{-1})^2 + 5D^{-1})^3X^{-1}$$

$$((3D + D^{-1})^2 + 5D^{-1})^3 = \begin{bmatrix} ((3\lambda_1 + \frac{1}{\lambda_1})^2 + \frac{5}{\lambda_1})^3 & 0 & ... & 0 \\ 0 & ((3\lambda_2 + \frac{1}{\lambda_2})^2 + \frac{5}{\lambda_2})^3 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & ((3\lambda_n + \frac{1}{\lambda_n})^2 + \frac{5}{\lambda_n})^3 \end{bmatrix}$$

So the eigenvalues of $(3A + A^{-1})^2$ are

$$((3\lambda_1 + \frac{1}{\lambda_1})^2 + \frac{5}{\lambda_1})^3, ((3\lambda_2 + \frac{1}{\lambda_2})^2 + \frac{5}{\lambda_2})^3, ..., ((3\lambda_n + \frac{1}{\lambda_n})^2 + \frac{5}{\lambda_n})^3$$

Calculate the values of the following (show the procedure):
(a) $det(A^3 + A^{-5} + 3I)$ (5 points)
Answer:
Eigenvalues of $A^3 + A^{-5} + 3I$ are $\{\lambda_i^3 + \lambda_i^{-5} + 3 | i = 1, ..., n\}$ so

$$det(A^3 + A^{-5} + 3I) = \prod_{i=1}^{n} (\lambda_i^3 + \lambda_i^{-5} + 3)$$

(b) $tr((\frac{1}{3}A^{-1} + 5I)^{-1})$(5 points)
Answer:
Eigenvalues of $\frac{1}{3}A^{-1} + 5I)^{-1}$ are $\{(\frac{1}{3\lambda_i} + 5)^{-1} | i = 1, ..., n\}$ so

$$tr((\frac{1}{3}A^{-1} + 5I)^{-1} = \sum_{i=1}^{n} (\frac{1}{3\lambda_i} + 5)^{-1} = \sum_{i=1}^{n} (\frac{3\lambda_i}{1 + 15\lambda_i})$$

2. Given two $m \times m$ matrix X and Y, where $XY = YX$.
(1) Let u be an eigenvector of X. Show that either Yu is an eigenvector of X or Yu is a zero vector. (10 points)
Answer:
If Yu is not a zero vector, given $Xu = \lambda u$

$$X(Yu) = (XY)u = (YX)u = Y(Xu) = Y(\lambda u) = \lambda(Yu)$$

this means Yu is an eigenvector of X with same eigenvalue λ. So either Yu is an eigenvector of X or Yu is a zero vector.
(2) Suppose the eigenvalues of X are mutually different from each other. Show that Y and X have the same eigenvectors. (15 points)
Answer:
Claim 1 : Eigenvectors with different eigenvalues are linearly independent.
Proof : We prove by induction on number of eigenvectors k. When $k = 1$, there is only one eigenvector, so the statement is true. When the statement holds for $k < n$, we prove that it is also true for $k = n$. Suppose that there are scalars $c_1, c_2, ..., c_n$

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0 \tag{1}$$

Multiply by A at left we have

$$A c_1 v_1 + A c_2 v_2 + ... + A c_n v_n = 0 \tag{2}$$

Which is

$$\lambda_1 c_1 v_1 + \lambda_2 c_2 v_2 + ... + \lambda_n c_n v_n = 0 \tag{3}$$

Multiply (1) by λ_1 we have

$$\lambda_1 c_1 v_1 + \lambda_1 c_2 v_2 + ... + \lambda_1 c_n v_n = 0 \tag{4}$$

Substract 3 by 4 we have

$$(\lambda_2 - \lambda_1) c_2 v_2 + ... + (\lambda_n - \lambda_1) c_n v_n = 0 \tag{5}$$

This reduces to the $k = n - 1$ case, since eigenvalues are distinct we can assert that $(\lambda_2 - \lambda_1), (\lambda_3 - \lambda_1), ..., (\lambda_n - \lambda_1)$ are nonzero. By the assumption, $(\lambda_2 - \lambda_1)c_2, (\lambda_3 - \lambda_1)c_3, ..., (\lambda_n - \lambda_1)c_n$ should be zero. This leads to the conclusion that $c_2, ..., c_n$ are all zero. Plugging back into (1) we have $c_1 v_1 = 0$ which means c_1 is also zero. So we conclude $c_1, c_2, ..., c_n$ should be zero, which means $v_1, v_2, ..., v_n$ is linearly independent. The proof is completed by induction.

Back to the problem (2): The eigenvalues of X are mutually different to each other. Say, they are $\lambda_1, \lambda_2, ...\lambda_n$. The eigenvectors of an eigenvalue form a vector space (called the eigenspace). Let B_1 be a basis for eigenspace of λ_1, B_2 a basis for eigenspace of λ_2, ..., and B_n a basis for eigenspace of λ_m. Each of $B_1, ..., B_m$ has at least 1 vector.

Claim 2 : $B_1 \cup B_2 ... \cup B_m$ is linearly independent.

Proof : Assume there are scalars not all zero such that the sum of scalar multiply vectors in $B_1 \cup B_2 ... \cup B_m$ equal to zero, we will have a $v_1 + ... + v_m = 0$, where $v_1, ..., v_m$ not all zero and add up to zero. This is a contradiction to Claim 1, so the assumption fails. This completes the proof.

Back to the problem (2): Consider the set $B_1 \cup B_2 ... \cup B_m$, it is also linearly independent (According to Claim 2). It has at least m elements, and they can have at most m elements since the dimension of the vector space of m-dimension vector is m (Any linearly independent set can have at most m elements, which is the dimension of the vector space). So $B_1 \cup B_2 ... \cup B_m$ has exactly m elements, which means each B_i has exactly 1 element. This means the dimension for each eigenspace of λ_i is 1.

From (1) we have Yu is an eigenvector of X with eigenvalue λ given u is an eigenvector of X with eigenvalue λ. Since the dimension of each eigenspace is 1, we can conclude that have $Yu = cu$ for some scalar c. This means that u is also an eigenvector of Y. So we conclude that any eigenvector of X is an eigenvector of Y. Recall that each eigenspace for X has dimension 1. Take a vector for each eigenspace, this forms a basis b (Since they are linearly independent and there are m of them) for the vector space of m dimensional vector. The vectors in b are also eigenvectors of Y.

If there exist a vector v such that it is an eigenvector of Y but it is not a eigenvector of X, this means that v is not in $span(B)$ and adding v to B forms a linearly independent set. This cannot be true since $|B + v| = m + 1$ which is larger than the dimension of the vector space of m-dimensional vector (The cardinality of any linearly independent set is smaller or equal to the dimension of the vector space). So such v does not exist. This means that Y has exactly the same set of eigenvectors as X.

3. Let p, m and n be positive integers and F a field. Let V be the space of $m \times n$ matrices over F and W the space of $p \times n$ matrices over F. Let B be a fixed $p \times m$ matrix and let T be the linear transformation from V into W defined by $T(A) = BA$. Prove that T is invertible if and only if $p = m$ and B is and invertible $m \times m$ matrix. (25 points)

Answer:

If :

If m=p and B is an invertible $m \times m$ matrix, then

(1) T is $1 - 1$:

If $T(v_1) = T(v_2)$ this means $Bv_1 = Bv_2$ and $B(v_1 - v_2) = 0$. Since B is invertible we have $B^{-1}B(v_1 - v_2) = v_1 - v_2 = 0$ so $v_1 = v_2$, this means T is $1 - 1$.

(2) T is onto :

For any u in W we have $B^{-1}u$ is in V ($B^{-1}u$ has dimension $m \times n$ so it is in V) and $T(B^{-1}u) = BB^{-1}u = u$ so T is onto.

This means T is invertible.

Only If :

If T is invertible, this means T is $1-1$ and onto. T is $1-1$ means $dim(V) = dim(T(V)) <= dim(W)$. T is onto means $T(V) = W$ so $dim(T(V)) = dim(W)$. Combining both we get $dim(V) = dim(T(V)) = dim(W)$ so $m = p$ and B is $m \times m$.

Since T is $1-1$, $N(T)$ must be $\{0\}$ (Otherwise suppose there is $x \in N(T)$ which is nonzero, and we will have $T(v) = T(v+x) = T(v) + T(x) = T(v) + 0 = T(v)$, which contradicts to the fact that T is $1-1$). If B is not invertible, this means there exist m dimensional vector $v \neq 0$ such that $Bv = 0$. Consider the $m \times n$ dimensional matrix which each column is v, T maps this matrix to zero, which is a contradictions to the fact that $N(T) = \{0\}$. So B should be invertible.

4. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $rank(T^2) = rank(T)$. Prove that the range and null space of T are disjoint, i.e., have only the zero vector in common. (25 points)

Answer:

Since T is a linear operator, we have $T(V) \subseteq V$, so $T(T(V)) \subseteq T(V)$. T and T^2 are both linear operators on V so by the rank-nullity theorem $dimV = rank(T) + nullity(T) = rank(T^2) + nullity(T^2)$. Since $rank(T^2) = rank(T)$ we have $nullity(T^2) = nullity(T)$. We also have $N(T) \subseteq N(T^2)$ (since $T(x) = 0$ yields $T(T(x)) = T(0) = 0$, linear transforms always map zero to zero), combining $nullity(T^2) = nullity(T)$ we have $N(T) = N(T^2)$.

Suppose that $x \in R(T) \cap N(T)$. $x \in R(T)$ means that there exist y in V such that $T(y) = x$. $x \in N(T)$ means that $T(x) = 0$. Combining both we get $T(T(y)) = T(x) = 0$, so $y \in N(T^2)$. Since $N(T^2) = N(T)$ we can conclude that $N(T) = 0$ and thus $T(y) = 0$, which means $x = 0$. So $R(T) \cap N(T) = \{0\}$.