Homework 3

ECE 269

Due : 11:59pm PT on Feb 15

1. Given two matrices X and Y which are $d \times d$ and invertible, judge whether the following matrices are invertible and give proofs.
(1) (5 points)

$$XYX$$

Answer:
Yes, its inverse is $X^{-1}YX^{-1}$.
Proof:

$$X^{-1}Y^{-1}X^{-1}XYX = X^{-1}Y^{-1}(X^{-1}X)YX = X^{-1}Y^{-1}IYX =$$
$$X^{-1}Y^{-1}YX = X^{-1}IX = X^{-1}IX = X^{-1}X = I$$

$$XYXX^{-1}Y^{-1}X^{-1} = XY(XX^{-1})Y^{-1}X^{-1} = XYY^{-1}X^{-1} =$$
$$X(YY^{-1})X^{-1} = X(I)X^{-1} = XX^{-1} = I$$

(2) (5 points)

$$X + Y$$

Answer:
No. A counter example :
$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has inverse $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has inverse $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $X + Y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is non-invertible.
(3) (10 points)

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$$

Answer:
Yes, its inverse is $\begin{bmatrix} X^{-1} & 0 \\ 0 & Y^{-1} \end{bmatrix}$.

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ 0 & Y^{-1} \end{bmatrix} = \begin{bmatrix} XX^{-1} + 00 & X0 + 0Y^{-1} \\ 0X^{-1} + Y0 & 00 + YY^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

(4) (10 points)

$$\begin{bmatrix} X & X + Y \\ 0 & Y \end{bmatrix}$$

Answer: Yes, its inverse is $\begin{bmatrix} X^{-1} & -X^{-1}(X+Y)Y^{-1} \\ 0 & Y^{-1} \end{bmatrix}$.

$$\begin{bmatrix} X & (X+Y) \\ 0 & Y \end{bmatrix} \begin{bmatrix} X^{-1} & -X^{-1}(X+Y)Y^{-1} \\ 0 & Y^{-1} \end{bmatrix} =$$

$$\begin{bmatrix} XX^{-1}+(X+Y)0 & -XX^{-1}(X+Y)Y^{-1}+(X+Y)Y^{-1} \\ 0X^{-1}+Y0 & -0X^{-1}(X+Y)Y^{-1}+YY^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} X^{-1} & -X^{-1}(X+Y)Y^{-1} \\ 0 & Y^{-1} \end{bmatrix} \begin{bmatrix} X & (X+Y) \\ 0 & Y \end{bmatrix} =$$

$$\begin{bmatrix} X^{-1}X+00 & X^{-1}(X+Y)-X^{-1}(X+Y)Y^{-1}Y \\ 0X+Y^{-1}0 & 0(X+Y)+Y^{-1}Y \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

2. Given a $m \times n$ matrix X and a $n \times m$ matrix Y , suppose $I+XY$ is invertible.
(1) Show that $I+YX$ is also invertible. (25 points)

$$det(I+XY) = det(\begin{bmatrix} I+XY & 0 \\ Y & I \end{bmatrix}) = det(\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\begin{bmatrix} I & -X \\ Y & I \end{bmatrix}) = det(\begin{bmatrix} I & X \\ 0 & I \end{bmatrix})det(\begin{bmatrix} I & -X \\ Y & I \end{bmatrix})$$

$$= det(\begin{bmatrix} I & -X \\ Y & I \end{bmatrix})det(\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}) = det(\begin{bmatrix} I & -X \\ Y & I \end{bmatrix}\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}) = det(\begin{bmatrix} I & 0 \\ Y & I+YX \end{bmatrix}) = det(I+YX)$$

So $det(I+XY) = det(I+YX)$, this mean $I+XY$ is invertible iff $I+YX$ is invertible. Since $I+XY$ is invertible given by the problem, $I+YX$ is invertible.
(2) Show that $Y(I+XY)^{-1} = (I+YX)^{-1}Y$. (20 points)

$$(I+YX)Y = Y+YXY = Y(I+XY)$$

and since $I+YX$ and $I+XY$ are invertible, multiply $(I+YX)^{-1}$ on the left and $(I+XY)^{-1}$ on the right we have

$$(I+YX)^{-1}(I+YX)Y(I+XY)^{-1} = (I+YX)^{-1}(Y+YXY)Y(I+XY)^{-1} = (I+YX)^{-1}Y(I+XY)(I+XY)^{-1}$$

which means

$$Y(I+XY)^{-1} = (I+YX)^{-1}Y$$

3. (25 points) Given $n \times n$ matrices A, B, C, D where A is invertible and $AC = CA$, show that

$$det(\begin{bmatrix} A & B \\ C & D \end{bmatrix}) = det(AD-CB)$$

Claim 1

$$det(\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}) = det(D), \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = det(A), det(\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}) = 1$$

Proof : $\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = det(A)$ and $det(\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}) = det(D)$ can be easily seen by cofactor expansion.

$$C_{ij} = (-1)^{(i+j)}det(A_{ij})$$

$$det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}$$

when the size of the I matrix increase, the value determinant remain the same in the cofactor expansion formula.

$$det(\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}) = 1$$

can also be seen by the cofactor expansion. It is also uppper trangular whose determinant is product of diagonal elements.

Claim 2

$$det(\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}) = det(A)det(D)$$

Proof :

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$

so using Claim1

$$det(\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}) = det(\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}) = det(\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix})det(\begin{bmatrix} I & B \\ 0 & I \end{bmatrix})det(\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}) =$$

$$det(D)det(1)det(A) = det(A)det(D)$$

Claim 3

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

so

$$det(\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}) = det(\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix})$$

which means

$$det(\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix})det(\begin{bmatrix} A & B \\ C & D \end{bmatrix})det(\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix}) = det(\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix})$$

so

$$det(\begin{bmatrix} A & B \\ C & D \end{bmatrix}) = det(\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}) = det(A)det(D - CA^{-1}B)$$