**Homework 2**

**ECE 269**

**Due : 11:59pm PT on Jan 31**

1. Let $V$be the subspace of $R[x]$ of polynomials of degree at most 3. Equip $V$ with the inner product

$$(f|g) = \int_0^1 f(t)g(t)dt$$

(a) Find the orthogonal complement of the subspace of scalar polynomials.
(b) Apply the Gram-Schmidt process to the basis $1, x, x^2, x^3$.
Answer :

(a) For nonzero scalar $c$

$$g(t) = c_0 + c_1 t + c_2 t^2$$

$$(c|g(t)) = \int_0^1 (c)(c_0 + c_1 t + c_2 t^2)dt = c\int_0^1 (c_0 + c_1 t + c_2 t^2)dt = 0$$

$$\rightarrow \int_0^1 (c_0 + c_1 t + c_2 t^2)dt = (c_0 t + c_1\frac{t^2}{2} + c_2\frac{t^3}{3})|_0^1 = c_0 + \frac{c_1}{2} + \frac{c_2}{3} = 0$$

$$\rightarrow c_2 = \frac{-6c_0 - 3c_1}{2}$$

So orthogronal complement of scalar is $\{c_0 + c_1 x + \frac{-6c_0-3c_1}{2}x^2 | c_0, c_1 \in R\}$
(b)

$$\{\beta_1, \beta_2, \beta_3, \beta_4\} = \{1, x, x^2, x^3\}$$

Compute $\alpha_1$

$$\alpha_1 = \beta_1 = 1$$

$$(\alpha_1|\alpha_1) = (1|1) = \int_0^1 1 * 1dt = t|_0^1 = 1$$

Compute $\alpha_2$

$$(\beta_2|\alpha_1) = (x|1) = \int_0^1 t * 1dt = \frac{t^2}{2}|_0^1 = \frac{1}{2}$$

$$\alpha_2 = \beta_2 - \frac{(\beta_2|\alpha_1)}{(\beta_1|\beta_1)}\beta_1 = x - \frac{1}{1}(x|1) = x - \frac{1}{2}$$

$$(\alpha_2|\alpha_2) = (x-\frac{1}{2}|x-\frac{1}{2}) = \int_0^1 (t-\frac{1}{2})(t-\frac{1}{2})dt = \int_0^1 (t^2-t+\frac{1}{4})dt = (\frac{1}{3}t^3-\frac{1}{2}t^2+\frac{1}{4}t)|_0 1 = \frac{1}{3}-\frac{1}{2}+\frac{1}{4} = \frac{1}{12}$$

Compute $\alpha_3$

$$(\beta_3|\alpha_1) = (x^2|1) = \int_0^1 t^2 * 1dt = \frac{t^3}{3}|_0^1 = \frac{1}{3}$$

$$(\beta_3|\alpha_2) = (x^2|x - \frac{1}{2}) = \int_0^1 t^2(t - \frac{1}{2})dt = \int_0^1 (t^3 - \frac{1}{2}t^2)dt = (\frac{t^4}{4} - \frac{t^3}{6})|_0^1 = \frac{1}{12}$$

$$\alpha_3 = \beta_3 - \frac{(\beta_3|\alpha_1)}{(\alpha_1|\alpha_1)}\alpha_1 - \frac{(\beta_3|\alpha_2)}{(\alpha_2|\alpha_2)}\alpha_2 = x^2 - \frac{1}{3}\frac{1}{1}(1) - \frac{1}{12} * 12(x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$$

$$(\alpha_3|\alpha_3) = (x^2 - x + \frac{1}{6}|x^2 - x + \frac{1}{6}) = \int_0^1 (t^4 - 2t^3 + \frac{4}{3}t^2 - \frac{1}{3}t + \frac{1}{36})dt$$
$$= (\frac{t^5}{5} - \frac{1}{2}t^4 + \frac{4}{9}t^3 - \frac{1}{6}t^2 + \frac{1}{36}t)|_0^1 = \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36} = \frac{1}{180}$$

Compute $\alpha_4$

$$(\beta_4|\alpha_1) = (x^3|1) = \int_0^1 t^3 * 1dt = \frac{t^4}{4}|_0^1 = \frac{1}{4}$$

$$(\beta_4|\alpha_2) = (x^3|x - \frac{1}{2}) = \int_0^1 t^3(t - \frac{1}{2})dt = \int_0^1 (t^4 - \frac{1}{2}t^3)dt = (\frac{t^5}{5} - \frac{t^4}{8})|_0^1 = \frac{3}{40}$$

$$(\beta_4|\alpha_3) = (x^3|x^2 - x + \frac{1}{6}) = \int_0^1 t^3(t^2 - t + \frac{1}{6})dt = \int_0^1 (t^5 - t^4 + \frac{1}{6}t^3)dt = (\frac{t^6}{6} - \frac{t^5}{5} + \frac{t^4}{24})|_0^1 = \frac{1}{6} - \frac{1}{5} + \frac{1}{24} = \frac{1}{120}$$

$$\alpha_4 = \beta_4 - \frac{(\beta_4|\alpha_1)}{(\alpha_1|\alpha_1)}\alpha_1 - \frac{(\beta_4|\alpha_2)}{(\alpha_2|\alpha_2)}\alpha_2 - \frac{(\beta_4|\alpha_3)}{(\alpha_3|\alpha_3)}\alpha_3$$
$$= x^3 - \frac{1}{4}\frac{1}{1}(1) - \frac{3}{40} * 12(x - \frac{1}{2}) - \frac{1}{120} * 180(x^2 - x + \frac{1}{6}) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$$

So the basis obatined by Gram-Schmidt is

$$\{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}, x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\}$$

2. Let $W$ be a finite-dimensional subspace of an inner product space $V$ , and let $E$ be the orthogonal projection of $V$ on $W$. Prove that $(E\alpha|\beta) = (\alpha|E\beta)$ for all $\alpha$ and $\beta$ in $V$.

Answer :

Since $E$ is the orthogonal projection of $V$ on $W$, we have

$$E\alpha \in W, E\beta \in W, \alpha - E\alpha \in W^{\perp}, \beta - E\beta \in W^{\perp}$$

So

$$(E\alpha|\beta - E\beta) = 0, (\alpha - E\alpha|E\beta) = 0$$

And thus

$$(E\alpha|\beta) = (E\alpha|E\beta + (\beta - E\beta)) = (E\alpha|E\beta) + (E\alpha|\beta - E\beta) = (E\alpha|E\beta)$$

$$(\alpha|E\beta) = (E\alpha + (\alpha - E\alpha)|E\beta) = (E\alpha|E\beta) + (\alpha - E\alpha|E\beta) = (E\alpha|E\beta)$$

So

$$(E\alpha|\beta) = (\alpha|E\beta)$$