Homework 1

ECE 269

Due : 11:59pm PT on Jan 17

1. Show that the space $V = \{(x_1, x_2, x_3) \in F^3 | x_1 + 2x_2 + 2x_3 = 0\}$ forms a vector space.

Answer :
For any $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in V, c \in F$ we define vector addition $u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$ and scalar multiplication $cu = (cu_1, cu_2, cu_3)$. We prove that V forms a vector space.

(0.0) Closed under vector addition: if $u = (u_1, u_2, u_3) \in V$ and $v = (v_1, v_2, v_3) \in V$ then $u + v \in V$

Since u and v are in V we have $u_1 + 2u_2 + 2u_3 = 0$ and $v_1 + 2v_2 + 2v_3 = 0$ so $(u_1 + v_1) + 2(u_2 + v_2) + 2(u_3 + v_3) = (u_1 + 2u_2 + 2u_3) + (v_1 + 2v_2 + 2v_3) = 0$ and hence $u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$ is in V.

(0.1) Closed under scalar multiplication : $cu \in V$ for all $c \in F$

Define $cu = (cu_1, cu_2, cu_3)$. Since we have $u_1 + 2u_2 + 2u_3 = 0$ so $cu_1 + 2cu_2 + 2cu_3 = 0 = c(u_1 + 2u_2 + 2u_3) = 0$ thus $cu \in V$.

(1) Commutativity: $u + v = v + u$ for all $u, v \in V$

$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3) = (v_1 + u_1, v_2 + u_2, v_3 + u_3) = v + u$ for all $u, v \in V$.

(2) Associativity of vector addition: $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$

$u + (v + w) = (u_1, u_2, u_3) + (v_1 + w_1, v_2 + w_2, v_3 + w_3) = (u_1 + v_1 + w_1, u_2 + v_2 + w_2, u_3 + v_3 + w_3)$ and $(u + v) + w = (v_1 + u_1, v_2 + u_2, v_3 + u_3) + (w_1, w_2, w_3) = (u_1 + v_1 + w_1, u_2 + v_2 + w_2, u_3 + v_3 + w_3)$ so $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$.

(3) Additive identity: $\exists -u$ such that $u + (-u) = 0$, for all $u \in V$:

$(u_1, u_2, u_3) + (-u_1, -u_2, -u_3) = (u_1 - u_1, u_2 - u_2, u_3 - u_3) = (0, 0, 0) = 0$, and $(-u_1, -u_2, -u_3) \in V$ since $-u_1 + 2(-u_2) + 2(-u_3) = -(u_1 + 2u_2 + 2u_3) = 0$ so $(-u_1, -u_2, -u_3)$ is $-u$ for any $u = (u_1, u_2, u_3)$ in V.

(4) Existence of additive inverse: $\exists 0$ such that $u + 0 = u$ for all $u \in V$

$(0, 0, 0) + (u_1, u_2, u_3) = (0 + u_1, 0 + u_2, 0 + u_3) = (u_1, u_2, u_3)$ for any u in V, and $(0, 0, 0) \in V$ since $0 + 2 * 0 + 2 * 0 = 0$ so $(0, 0, 0)$ is the zero in V.

(5) Associativity of scalar multiplication: $c_1(c_2 u) = (c_1 c_2)u$ for all $c_1, c_2 \in F, u \in V$

$c_1(c_2 u) = c_1(c_2 u_1, c_2 u_2, c_2 u_3) = (c_1(c_2 u_1), c_1(c_2 u_2), c_1(c_2 u_3)) = ((c_1 c_2)u_1, (c_1 c_2)u_2, (c_1 c_2)u_3) = (c_1 c_2)(u_1, u_2, u_3) = (c_1 c_2)u$ The second = is according to the associativity of field.

(6) Distributivity of scalar sums: $(c_1 + c_2)(u) = c_1 u + c_2 u$ for all $c_1, c_2 \in F, u \in V$

$(c_1 + c_2)(u) = (c_1 + c_2)(u_1, u_2, u_3) = ((c_1 + c_2)u_1, (c_1 + c_2)u_2, (c_1 + c_2)u_3) = (c_1 u_1 + c_2 u_1, c_1 u_2 + c_2 u_2, c_1 u_3 + c_2 u_3)$ The second = is according to the distributivity of field.And $(c_1 u_1 + c_2 u_1, c_1 u_2 + c_2 u_2, c_1 u_3 + c_2 u_3) = (c_1 u_1, c_1 u_2, c_1 u_3) + (c_2 u_1, c_2 u_2, c_2 u_3) = c_1(u_1, u_2, u_3) + c_2(u_1, u_2, u_3) = c_1 u + c_2 u$ so $(c_1 + c_2)(u) = c_1 u + c_2 u$

(7) Distributivity of vector sums: $c(u + v) = cu + cv$ for all $c \in F, u, v \in V$

$c(u + v) = c(u_1 + v_1, u_2 + v_2, u_3 + v_3) = (cu_1 + cv_1, cu_2 + cv_2, cu_3 + cv_3) = (cu_1, cu_2, cu_3) + (cv_1, cv_2, cv_3) = c(u_1, u_2, u_3) + c(v_1, v_2, v_3) = cu + cv$. for all $c \in F$, $u, v \in V$

(8) Scalar multiplication identity: $\exists 1 \in F$ such that $1u = u$ for all $u \in V$

F is a field so there exists $1 \in F$ such that $1x = x \forall x \in F$. So $1u = 1(u_1, u_2, u_3) = (1u_1, 1u_2, 1u_3) = (u_1, u_2, u_3) = u$ since $u_1, u_2, u_3 \in F$ for all $u \in V$

V satisfies condition (0)-(8) for vector space so V is a vector space.

2. Let V be a vector space, and $v_1, ..., v_n$ be a set of vectors that spans V. Show that the set $v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n$ also spans V.

Answer :

We prove that $Span(v_1, v_2, ..., v_n) = Span(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$:

For any u in $Span(v_1, v_2, ..., v_n)$ we can write u as $u = c_1(v_1) + c_2(v_2) + ... + c_{n-1}(v_n)$, which is also $c_1(v_1 - v_2) + (c_1 + c_2)(v_2 - v_3) + ... + (c_{n-1} + c_n)(v_{n-1} - v_n) + 2c_n(v_n)$ so $Span(v_1, v_2, ..., v_n) \subseteq Span(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$. On the other hand, for any u in $Span(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$ we can write u as $c_1(v_1 - v_2) + c_2(v_2 - v_3) + ... + c_{n-1}(v_{n-1} - v_n) + c_n(v_n)$, which is also $c_1(v_1) + (-c_1 + c_2)(v_2) + (-c_2 + c_3)(v_3) + ... + (-c_{n-1} + c_n)(v_n)$ so $Span(v_1, v_2, ..., v_n) \supseteq Span(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$. So we have $Span(v_1, v_2, ..., v_n) \subseteq Span(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$ and $Span(v_1, v_2, ..., v_n) \supseteq Span(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$ and thus $Span(v_1, v_2, ..., v_n) = Span(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$.

3. Let vectors u, v, w be a basis in V. Show that $u + v + w, v + w, w$ is also a basis in V.

Answer :

(1) $u + v + w, v + w, w$ is linearly independent :

We prove by contradiction. If $u + v + w, v + w, w$ is not linearly independent, there exist $(c_1, c_2, ..., c_n)$ not all zero such that $c_1(u + v + w) + c_2(v + w) + ... + c_3(w) = 0$. If that's the case, we will have $c_1(u) + (c_1 + c_2)(v) + (c_1 + c_2 + c_3)(w) = 0$. If $(c_1, c_1 + c_2, c_1 + c_2 + c_3)$ is all zero, we'll have $c_1 = 0$ and $c_2 = 0 - c_1 = 0, c_3 = 0 - c_1 - c_2 = 0$ and thus (c_1, c_2, c_3) would be all zero too, which contradicts to our assumption. So $(c_1, c_1 + c_2, c_1 + c_2 + c_3)$ is also not all zero. But then we can conclude that $c_1(u) + (c_1 + c_2)(v) + (c_1 + c_2 + c_3)(w) = 0$ with $(c_1, c_1 + c_2, c_1 + c_2 + c_3)$ being not all zero, which indicates that u, v, w is not linearly independent and this is a contradiction to the fact that u, v, w is a basis (a linearly independent set that spans V), which is a contradiction to the fact that $v_1, v_2, ..., v_{n-1}, v_n$ is linearly independent, so the assumption that there exist $(c_1, c_2, ..., c_n)$ not all zero such that $c_1(v_1 - v_2) + c_2(v_2 - v_3) + ... + c_{n-1}(v_{n-1} - v_n) + c_n(v_n)) = 0$ is false. This yields our conclusion that $v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n$ is linearly independent.

(2) $Span(u, v, w) = Span(u + v + w, v + w, w)$: For any u in $Span(u, v, w)$ we can write u as $u = c_1(u) + c_2(v) + c_3(w)$, which is also $(c_1)(u + v + w) + (c_2 - c_1)(v + w) + ... + (c_3 - c_2 - c_1)(w)$ so $Span(u, v, w) \subseteq Span(u + v + w, v + w, w)$. On the other hand, for any u in $Span(u + v + w, v + w, w)$ we can write u as $c_1(u + v + w) + c_2(v + w) + c_3(w)$, which is also $c_1(u) + (c_1 + c_2)(v) + (c_1 + c_2 + c_3)(w))$ so $Span(u, v, w) \supseteq Span(u + v + w, v + w, w)$. So we have $Span(u, v, w) \subseteq Span(u + v + w, v + w, w)$ and $Span(u, v, w) \supseteq Span(u + v + w, v + w, w)$ and thus $Span(u, v, w) = Span(u + v + w, v + w, w)$.

4. Suppose that V is finite dimensional and U is a subspace of V such that $dimU = dimV$. Prove that $U = V$.

Answer :

Lemma. Let S be a linearly independent subset of a vector space V. Suppose β is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining β to S is linearly independent.

If $U \subset V$ and $U \neq V$, there exist vector $\beta \in V$ but β not in W. According to lemma, β along with a basis $\alpha_1, \alpha_2, ..., \alpha_k$ of W is linearly independent, and $Span(\alpha_1, \alpha_2, ..., \alpha_k, \beta) \subset V$, so

$dim(V) >= dim(Span(\alpha_1, \alpha_2, ..., \alpha_k, \beta) \subset V) = k + 1$ which is a contradiction to $dimU = dimV$. So the assumption that U is a proper subspace of V is false, this means $U = V$