Week 7 Tutorial

The theory related to some questions in this tutorial might not have been entirely covered in the lectures before the tutorial takes place. The main results necessary to solve these questions are summarised below.

Let X be a gamma(α, θ) random variable;

- the *mean* of X is $\mu = \alpha\theta$;
- the *variance* of X is $\sigma^2 = \alpha\theta^2$.
- the *moment-generating function* (mgf) of X is $M(t) = (1 - t\theta)^{-\alpha}$, for $t < 1/\theta$;
- when $\theta = 2$ and $\alpha = r/2$ (with r being a positive integer), X is said to have a *chi-square distribution with r degrees of freedom*, i.e., $X \stackrel{d}{=} \chi^2(r)$.

♢ ♢ ♢

1. (*Q3.3-3*). Customers arrive randomly at a bank teller's window. Given that one customer arrived during a particular 10-minute period, let X equal the time within the 10 minute-period that the customer arrived. If X is $U(0, 10)$, find:
   (a) the pdf of X,
   (b) $P(X \geq 8)$,
   (c) $P(2 \leq X < 8)$,
   (d) $E(X)$, and
   (e) $\text{Var}(X)$.

2. (*Q3.3-6*). Let X have an exponential distribution with a mean of $\theta = 20$. Compute
   (a) $P(10 < X < 30)$,
   (b) $P(X > 30)$.
   (c) $P(X > 40|X > 10)$.
   (d) What are the variance and the mgf of X?
   (e) Find the 80th percentile of X.

3. (*Q3.3-9*). What are the pdf, the mean, and the variance of X if the mgf of X is given by the following?
   (a) $M(t) = (1 - 3t)^{-1}$, $t < 1/3$.
   (b) $M(t) = \frac{3}{3-t}$, $t < 3$.

4. Let random variable X have the pdf

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty.$$

(The distribution of such X is known as the *logistic distribution*.)

(a) Write down the cdf of X.

(b) Find the mean and variance of X.

(c) Find $P(3 < X < 5)$.

(d) Find the 85-th percentile of X.

(e) Let $Y = \frac{1}{1+e^{-X}}$. Find the cdf of Y.
Can you tell the name of the distribution of Y?

5. (*Q3.4-1*) Telephone calls enter a college switchboard at a mean rate of 2/3 call per minute according to a Poisson process. Let X denote the waiting time until the 10th call arrives.

(a) What is the pdf of X?

(b) What are the mgf, mean and variance of X?

6. (*Q3.4-2*) If X has a gamma distribution with $\theta = 4$ and $\alpha = 2$, find $P(X < 5)$.

7. (*Q3.4-4*) Use the moment-generating function of a gamma distribution to show that $E(X) = \alpha\theta$ and $\mathrm{Var}(X) = \alpha\theta^2$.

8. Let X have a $\chi^2(2)$ distribution. Find constants a and b such that

$$P(a < X < b) = 0.90, \quad \text{and} \quad P(X < a) = 0.05.$$

1. (*Q3.3-3*). Customers arrive randomly at a bank teller's window. Given that one customer arrived during a particular 10-minute period, let X equal the time within the 10 minute-period that the customer arrived. If X is $U(0, 10)$, find:

   (a) the pdf of X,

   (b) $P(X \geq 8)$,

   (c) $P(2 \leq X < 8)$,

   (d) $E(X)$, and

   (e) $\text{Var}(X)$.

(a) For a uniform r.v. on $[a, b]$, pdf is:

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

so, $f(x) = \frac{1}{10-0} = \frac{1}{10}, \; 0 \leq x \leq 10$

and $f(x) = 0$ elsewhere.

(b) $P(X \geq 8) = 1 - F(8) = 1 - \frac{8-0}{10} = 1 - 0.8 = 0.2$

(c) $P(2 \leq X < 8) = F(8) - F(2) = \frac{8}{10} - \frac{2}{10}$

$= 0.8 - 0.2 = 0.6$

(d) $E(X) = \frac{a+b}{2} = \frac{0+10}{2} = 5$

(e) $\text{Var}(X) = \frac{(b-a)^2}{12} = \frac{(10-0)^2}{12} = \frac{100}{12} \approx 8.33$

2. (*Q3.3-6*). Let X have an exponential distribution with a mean of $\theta = 20$. Compute

(a) $P(10 < X < 30)$,

(b) $P(X > 30)$.

(c) $P(X > 40 | X > 10)$.

(d) What are the variance and the mgf of X?

(e) Find the 80th percentile of X.

Exponent r.v. with mean $\theta = 20$

pdf: $f(x) = \frac{1}{20} e^{-x/20}, x \geq 0$

cdf: $F(x) = 1 - e^{-x/20}$

mgf: $M(t) = \frac{1}{1-20t}, t < \frac{1}{20}$

(a) $P(10 < X < 30) = F(30) - F(10)$

$= (1 - e^{-30/20}) - (1 - e^{-10/20})$

(via the cdf) $= e^{-10/20} - e^{-30/20} = e^{-0.5} - e^{-1.5}$

(b) Using the survival function:

$P(X > 30) = 1 - F(30) = e^{-30/20} = e^{-1.5}$

(c) $P(X > 40 | X > 10)$

The exponential distribution has memoryless property:

$P(X > 40 | X > 10) = P(X > 40 - 10) = P(X > 30)$

$= e^{-30/20} = e^{-1.5}$

(d) For exponential r.v. with mean θ:

$$Var(X) = \theta^2 = 20^2 = 400$$

$$M(t) = \frac{1}{1-20t} \text{ for } t < \frac{1}{20}$$

(e) Solve $F(x_{0.80}) = 0.80 \Rightarrow 1 - e^{-x_{0.80}/20} = 0.8$

So, $e^{-x_{0.80}/20} = 0.20 \Rightarrow x_{0.8} = -20 \ln(0.20) = 20 \ln(5)$

3. (*Q3.3-9*). What are the pdf, the mean, and the variance of X if the mgf of X is given by the following?

(a) $M(t) = (1-3t)^{-1}$, $t < 1/3$.

(b) $M(t) = \frac{3}{3-t}$, $t < 3$.

(a) when $M(t) = (1-3t)^{-1}$, $t < \frac{1}{3}$

This mgf is of the form of an exponential distibution with scale parameter θ: $M(t) = \frac{1}{1-\theta t}$

Here $\theta = 3$, so:

pdf: $f(x) = \frac{1}{3} e^{-x/3}$, $x \geq 0$

Mean: $E(X) = \theta = 3$

Var: $Var(X) = \theta^2 = 9$

Derivation via differentiation:

$$M'(t) = \frac{3}{(1-3t)^2}, \quad M'(0) = 3$$

$$M''(t) = \frac{18}{(1-3t)^3}, \quad M''(0) = 18$$

So, $Var(X) = M''(0) - (M'(0))^2 = 18 - 3^2 = 9$

(b) Now, $M(t) = \dfrac{3}{3-t} = \dfrac{1}{1-t/3}$ which is

the mgf of an exponential distribution with $\theta = \frac{1}{3}$

So, pdf: $f(x) = \dfrac{1}{1/3} e^{-3x} = 3e^{-3x}, \ x \geq$

Mean: $E(X) = \dfrac{1}{3}$

Var : $Var(X) = \dfrac{1}{9}$

Write $M(t) = (1 - \frac{t}{3})^{-1}$ then:

$$M'(t) = \frac{1/3}{(1-t/3)^2}, \quad M'(0) = \frac{1}{3}$$

$$M''(t) = \frac{2/9}{(1-\frac{t}{3})^3}, \quad M''(0) = \frac{2}{9}$$

So, $Var(X) = \frac{2}{9} - (\frac{1}{3})^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$

4. Let random variable X have the pdf

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty.$$

(The distribution of such X is known as the *logistic distribution.*)

(a) Write down the cdf of X.

(b) Find the mean and variance of X.

(c) Find $P(3 < X < 5)$.

(d) Find the 85-th percentile of X.

(e) Let $Y = \frac{1}{1+e^{-X}}$. Find the cdf of Y.
Can you tell the name of the distribution of Y?

(a) Let X have the logistic distribution with pdf:

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty$$

cdf: $F(x) = \frac{1}{1+e^{-x}}$

(b) For a standard logistic distribution:

$E(X) = 0$

$Var(X) = \frac{\pi^2}{3}$

(c) Using the cdf, $P(3 < X < 5) = F(5) - F(3)$

$$= \frac{1}{1+e^{-5}} - \frac{1}{1+e^{-3}}$$

(d) Solve for $x_{0.85}$ s.t.

$$F(x_{0.85}) = \frac{1}{1+e^{-x_{0.85}}} = 0.85$$

$\therefore \; 1 + e^{-x_{0.85}} = \frac{1}{0.85}$ $\quad \therefore \; e^{-x_{0.85}} = \frac{1}{0.85} - 1$

$$= \frac{0.15}{0.85} \simeq 0.1765 \; (4 \text{ d.p.})$$

$\therefore \; x_{0.85} = -\ln(0.1765) \simeq 1.73 \; (2 \text{ d.p.})$

(e) Notice that $Y = F_X(X)$ and by the probability integral transform, if X has a continuous CDF then

$$Y \stackrel{d}{=} U(0,1)$$

So, cdf of Y: $F(y) = y$, $0 \leq y \leq 1$

5. (*Q3.4-1*) Telephone calls enter a college switchboard at a mean rate of 2/3 call per minute according to a Poisson process. Let X denote the waiting time until the 10th call arrives.

   (a) What is the pdf of X?

   (b) What are the mgf, mean and variance of X?

6. (*Q3.4-2*) If X has a gamma distribution with $\theta = 4$ and $\alpha = 2$, find $P(X < 5)$.

7. (*Q3.4-4*) Use the moment-generating function of a gamma distribution to show that $E(X) = \alpha\theta$ and $\text{Var}(X) = \alpha\theta^2$.

8. Let X have a $\chi^2(2)$ distribution. Find constants a and b such that

$$P(a < X < b) = 0.90, \quad \text{and} \quad P(X < a) = 0.05.$$

Q5:

For a Poisson process, the waiting time for a n^{th} event is gamma distributed w/ parameters:

- Shape: $\alpha = n$
- Scale: $\theta = \frac{1}{\lambda}$

Here, $n = 10$ and $\theta = \frac{1}{\lambda} = \frac{1}{2/3} = \frac{3}{2}$

a) Gamma pdf: $f(x) = \frac{1}{\Gamma(10)\,\theta^{10}} x^{10-1} e^{-x/\theta}, x \geq 0$

gamma

Sub $\theta = \frac{3}{2}$ and $\Gamma(10) = 9!$

$$f(x) = \frac{(2/3)^{10} x^9 e^{-x/(3/2)}}{9!}, x \geq 0$$

$$= \frac{(2/3)^{10} x^9 e^{-\frac{2}{3}x}}{9!}, x \geq 0$$

(b) $M(t) = \left(\frac{1}{1-\theta t}\right)^{\alpha}, \ t < \frac{1}{\theta}$

$= \left(\frac{1}{1-\frac{3}{2}t}\right)^{10}, \ t < \frac{2}{3}$

$E(X) = \alpha\theta = 10 \cdot \frac{3}{2} = 15$ mins

$Var(X) = \alpha\theta^2 = 10 \cdot \frac{9}{4} = 22.5$

Gamma dis

Q6: $\alpha = 2$

$\theta = 4$

pdf : $f(x) = \frac{1}{\Gamma(2)\, 4^2} x^{2-1} e^{-x/4} = \frac{x e^{-x/4}}{16}, \ x \geq 0$

$P(X < 5)$

For $\alpha = 2$: CDF $= F(x) = 1 - e^{-x/4}\left(1 + \frac{x}{4}\right)$

So $P(X<5) = 1 - e^{-5/4}\left(1 + \frac{5}{4}\right)$

$= 1 - e^{-1.25}(2.25)$

$= 1 - 0.6446$

$= 0.3554$

Q7 :