Tutorial 10 - Module 4

Week 10 Tutorial

The theory related to some questions in this tutorial might not have been entirely covered in the lectures before the tutorial takes place. The main results necessary to solve these questions are summarised below.

Assume that $X_1, X_2, \ldots, X_n$ are independent of each other.

- Let the pmf/pdf of X_i be $f_i(x_i)$ with space S_i. Then the joint pmf/pdf of $X_1, X_2, \ldots, X_n$ under the independence assumption is

$$f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\cdots f(x_n), \quad x_1 \in S_1, x_2 \in S_2, \ldots, x_n \in S_n.$$

- If $E[u(X_i)]$ exists for $i = 1, 2, \ldots, n$, then

$$E[u_1(X_1)u_2(X_2)\cdots u_n(X_n)] = E[u_1(X_1)]E[u_2(X_2)]\cdots E[u_n(X_n)].$$

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1. (*Q4.4-1*). Let X_1 and X_2 denote two independent random variables, each with a $\chi^2(2)$ distribution. Find the joint pdf of $Y_1 = X_1$ and $Y_2 = X_1 + X_2$. Here note that the support of Y_1 and Y_2 is $0 < y_1 < y_2 < \infty$. Also find the marginal pdf of Y_1 and of Y_2. Are Y_1 and Y_2 independent?

Reference Example 21:

Given: $X_1 \overset{d}{=} \chi^2(2)$

$X_2 \overset{d}{=} \chi^2(2)$

$X_1 \perp X_2$ (independent of)

Define: $Y_1 = X_1$ (1)

$Y_2 = X_1 + X_2$

By independence of 2 $\chi^2(2)$ variables (each a Gamma(1,2)):

$$f_{X_1X_2}(x_1,x_2) = \left(\tfrac{1}{2}e^{-x_1/2}\right)\left(\tfrac{1}{2}e^{-x_2/2}\right)$$

$$= \tfrac{1}{4}e^{-(x_1+x_2)/2}, \quad x_1 > 0, x_2 > 0 \quad (2)$$

Using change-of-variable technique (p. 97):

① Find the inverse transformation:

$x_1 = y_1$ (Inverse from (1))

$x_2 = y_2 - y_1$

② The Jacobian / determinant:

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

③ Support: From $x_1 > 0, x_2 > 0$ (2)

$\therefore \quad 0 < y_1 < y_2 < \infty$

④ New joint pdf:

$$g(y_1, y_2) = |J| \, f_{X_1 X_2}(y_1, y_2 - y_1)$$

$$= \frac{1}{4} e^{-y_2/2}, \quad 0 < y_1 < y_2 < \infty$$

Marginal distribution :

Y_1: $$f_{Y_1}(y_1) = \int_{y_2=y_1}^{\infty} \frac{1}{4} e^{-y_2/2} \, dy_2$$

$$= \frac{1}{4}\left[-2e^{-y_2/2}\right]_{y_1}^{\infty}$$

$$= \frac{1}{2} e^{-y_1/2}, \quad y_1 > 0$$

So, $Y_1 \overset{d}{=} \chi^2(2)$ or $\Gamma(1,2)$

Y_2:

$$f_{Y_2}(y_2) = \int_{y_1=0}^{y_2} \frac{1}{4} e^{-y_2/2} \, dy_1$$

$$= \frac{1}{4} y_2 e^{-y_2/2}, \quad y_2 > 0$$

So, $Y_2 \overset{d}{=} \chi^2(4)$ or $\Gamma(2,2)$ by Theorem ~~8~~ 6 p.144

Theorem 6

Let $X_1, X_2, \ldots, X_n$ be independent chi-square random variables with $r_1, r_2, \cdots, r_n$ degrees of freedom, respectively. Then $Y = X_1 + X_2 + \cdots + X_n$ is $\chi^2(r_1 + r_2 + \cdots + r_n)$.

Since $g_{Y_1Y_2}(y_1, y_2) = \frac{1}{4}e^{-y_2/2}$

$\neq f_{Y_1}(y_1) f_{Y_2}(y_2) = (\frac{1}{2}e^{-y_1/2})(\frac{y_2}{4}e^{-y_2/2})$

So Y_1 and Y_2 are NOT independent

Q2:

2. We have learned the following in lectures:

- If X_1 and X_2 are independent gamma random variables with parameters (α, θ) and (β, θ) respectively, then the random variable $\frac{X_1}{X_1+X_2}$ has a beta(α, β) distribution.
- If X_1 and X_2 are independent chi-square random variables with degrees of freedom being r_1 and r_2 respectively, then the random variable $\frac{X_1/r_1}{X_2/r_2}$ has an $F(r_1, r_2)$ distribution.

Using these results, name the distributions of the following random variables defined from X and Y, where X and Y are independent random variables both having an exponential $(\theta = 2)$ distribution.

(a) $U = \frac{X}{X+Y}$;

(b) $V = \frac{Y}{X+Y}$;

(c) $W = \frac{X}{Y}$;

(d) $Z = \frac{Y}{X}$.

Let $X, Y \stackrel{d}{=} \text{Exp}(\theta = 2)$ or $\Gamma(1, 2)$:

(a) Beta:

$$U = \frac{X}{X+Y} \stackrel{d}{=} B(1,1) \text{ or } U(0,1)$$

(b) Beta:

$$V = \frac{Y}{X+Y} \stackrel{d}{=} B(1, 1) \sim U(0,1)$$

(c) F

$$W = \frac{X}{Y} = \frac{X/2}{Y/2} \stackrel{d}{=} F(2,2)$$

(d) F

$$Z = \frac{Y}{X} = \frac{Y/2}{X/2} \stackrel{d}{=} F(2,2)$$

3. (*Q4.5-2*). Let X_1 and X_2 be independent random variables with respective binomial distributions $b(3, 1/2)$ and $b(5, 1/2)$. Determine

(a) $P(X_1 = 2, X_2 = 4)$.

(b) $P(X_1 + X_2 = 7)$.

Let $X_1 \stackrel{d}{=} b(3, \frac{1}{2})$; $X_2 \stackrel{d}{=} b(5, \frac{1}{2})$ independent

By definition 3: Marginal / Independence

Marginal pmf

Definition 3

Let X and Y have the joint pmf $f(x, y)$ with support (or space) S. The pmf of X alone, called the ***marginal pmf*** *of X, is defined by*

$$f_1(x) \equiv f_X(x) = \sum_y f(x, y) = P(X = x), \quad x \in S_1 \equiv S_X,$$

where the sum is taken over all possible y values for each given x in S_1 or S_X.
Similarly, the ***marginal pmf*** *of Y is defined by*

$$f_2(y) \equiv f_Y(y) = \sum_x f(x, y) = P(Y = y), \quad y \in S_2 \equiv S_Y,$$

where the sum is taken over all possible x values for each given y in S_2 or S_Y.

Independence and dependence

The random variables X and Y are said to be ***independent*** *if and only if*

$$P(X = x, Y = y) \equiv P(X = x)P(Y = y),$$

that is,

$$f(x, y) = f_1(x)f_2(y) \qquad \textbf{for any} \qquad x \in S_1, y \in S_2.$$

(a) $P(X_1 = 2, X_2 = 4) = P(X_1 = 2)\, P(X_2 = 4)$

$= \binom{3}{2}\left(\frac{1}{2}\right)^3 \times \binom{5}{4}\left(\frac{1}{2}\right)^5$

$= \frac{3}{8} \times \frac{5}{32} = \frac{15}{256}$

(b) $P(X_1 + X_2 = 7) = P(X_1 = 2, X_2 = 5) + P(X_1 = 3, X_2 = 4)$

$= \frac{3}{8} \cdot \frac{1}{32} + \frac{1}{8} \cdot \frac{5}{32} = \frac{1}{32}.$

4. (*Q4.5-3*). Let X_1 and X_2 be independent random variables with probability density functions $f_1(x_1) = 2x_1$, $0 < x_1 < 1$, and $f_2(x_2) = 4x_2^3$, $0 < x_2 < 1$, respectively. Compute

   (a) $P(0.5 < X_1 < 1.0, 0.4 < X_2 < 0.8)$.

   (b) $E(X_1^2 X_2^3)$.

$f_1(x_1) = 2x_1, \; 0 < x_1 < 1$

$f_2(x_2) = 4x_2^3, \; 0 < x_2 < 1$

By independence:

(a) $P(0.5 < X_1 < 1.0,\ 0.4 < X_2 < 0.8)$

$$= \left[\int_{0.5}^{1} 2x\,dx\right]\left[\int_{0.4}^{0.8} 4x^3\,dx\right]$$

$$= 0.75 \times 0.384 = 0.288$$

(b) $E(X_1^2 X_2^3) = E(X_1^2)\,E(X_2^3)$

$$= \int_0^1 x_1^2(2x_1)\,dx_1 \times \int_0^1 x_2^3(4x_2^3)\,dx_2$$

$$= \frac{1}{2}\cdot\frac{4}{7} = \frac{2}{7}$$

5. (*Q4.5-13*) Flip $n = 8$ fair coins and remove all that came up heads. Flip the other (tails) coins and remove the heads. Continuing flipping the remaining coins until each has come up heads. We shall find the pmf of Y, the number of trials needed to finish off all coins. Let X_i equal the number of flips required to observe heads on coin i, $i = 1, 2, \cdots, 8$. Then $Y = \max\{X_1, X_2, \cdots, X_8\}$.

(a) Show that $P(Y \le y) = [1 - (1/2)^y]^8, \quad y = 1, 2, \cdots$.

(b) Show that $P(Y = y) = [1 - (1/2)^y]^8 - [1 - (1/2)^{y-1}]^8, \quad y = 1, 2, \cdots$.

(c) (*To be done in the lab.*) Use a computer software such as R to show that $E(Y) = 4.421$.

(d) (*To be done in the lab.*) What happens to the expected value of Y if the number of coins is doubled?

Flip 8 independent fair coins

Let X_i = # of flips to first H on coin i

so $X_i \stackrel{d}{=} \text{Geo}(p = \frac{1}{2})$

set $Y = \max_i X_i$

By independence, $P(Y \le y) = \prod_{i=1}^{8} P(X_i \le y)$

$$= [1 - (\tfrac{1}{2})^y]^8, \quad y = 1, 2, 3, \ldots$$

$$P(Y = y) = P(Y \le y) - P(Y \le y-1)$$

$$= [1 - (\tfrac{1}{2})^y]^8 - [1 - (\tfrac{1}{2})^{y-1}]^8.$$