Week 6 Tutorial Module 2 + 3

The theory related to some questions in this tutorial might not have been entirely covered in the lectures before the tutorial takes place. The main results necessary to solve these questions are summarised below.

Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$;

- the *expectation* or *mean* of X is $\mu = E(X) = \int_{-\infty}^{+\infty} x\, f(x)dx$;
- the *variance* of X is $\sigma^2 = Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{+\infty}(x-\mu)^2 f(x)dx$; equivalently, $\sigma^2 = E[X^2] - [E(X)]^2$.
- the *moment-generating function* (mgf) of X, if exists, is $M(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f(x)dx$, $-h < t < h$;
- the $(100p)$*th percentile* of X is a number π_p such that $p = \int_{-\infty}^{\pi_p} f(x)dx = F(\pi_p)$. The 50th percentile is usually called the *median.*

1. (*Q2.6-2*). Let X have a Poisson distribution with a variance of 3. Find $P(X = 2)$.

* Exclusive for Poisson distribution:

$\sigma^2 = Var(X) = E(X^2) = \lambda = 3$

so $X \sim Poi(\lambda = 3)$

$$P(X=2) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{3^2 e^{-3}}{2!} = 0.224 \text{ (3 d.p.)} \checkmark$$

2. (*Q2.6-5*). Flaws in a certain type of drapery material appear on the average of one in 150 square feet. If we assume the Poisson distribution, find the probability of at most one flaw in 225 square feet. Discrete!

$\lambda = \frac{150}{225}$

$X \sim Poi(\lambda = 150/225)$

$$P(X \le 1) = \sum_{k=0}^{1} \frac{150/225\, e^{-1}}{1!} = P(X=0) + P(X=1)$$

$$= \frac{150/225\, e^{-0}}{0!} + \frac{150/225\, e^{-1}}{1!} = 0.912 \text{ (3 d.p.)}$$

$\lambda = \frac{225}{150}$

$X \sim Poi(\lambda = 225/150)$

$P(X \le 1) = P(X=0) + P(X=1) = \frac{\frac{225}{150}^0 e^{-\frac{225}{150}}}{0!} + \frac{\frac{225}{150}^1 e^{-\frac{225}{150}}}{1!}$

Discrete

$= 0.5578$ (4 d.p.)

3. (*Q2.6-8*). Suppose that the probability of suffering a side effect from a certain flu vaccine is 0.005. Also suppose 1000 persons are inoculated.

   (a) Find the exact probability that at most 1 person suffers a side effect, using a binomial distribution.

   (b) Find approximately the probability that at most 1 person suffers a side effect, using a Poisson distribution.

(a)

bnom: independent, p const

$n = 1{,}000$

$p = 0.005$

$X \sim b(n = 1{,}000,\ p = 0.005)$

$P(X \le 1) = P(X=0) + P(X=1)$

$= \binom{1000}{0} 0.005^0 (1-0.005)^{1000} + \binom{1000}{1} 0.005^1 (1-0.005)^{999}$

$= 0.0401$ (4 d.p.)

(b) $\lambda = 0.005 \times 1{,}000 = 5$

$X \sim Poi(\lambda = 5)$

$P(X \le 1) = P(X=0) + P(X=1)$

$= \frac{5^0 e^{-5}}{0!} + \frac{5^1 e^{-5}}{1!} = 0.0404$ (4 d.p.)

$2\int_0^x 1-t \, dt = 2\left[t - \frac{t^2}{2}\right]_0^x$

$2(x - \frac{x^2}{2})$

$\frac{x}{2} - x^2$

4. Let the random variable X have the pdf $f(x) = 2(1-x)$, $0 \le x \le 1$, 0 elsewhere.

   (a) Sketch the graph of this pdf.

   (b) Determine and sketch the graph of the distribution function of X.

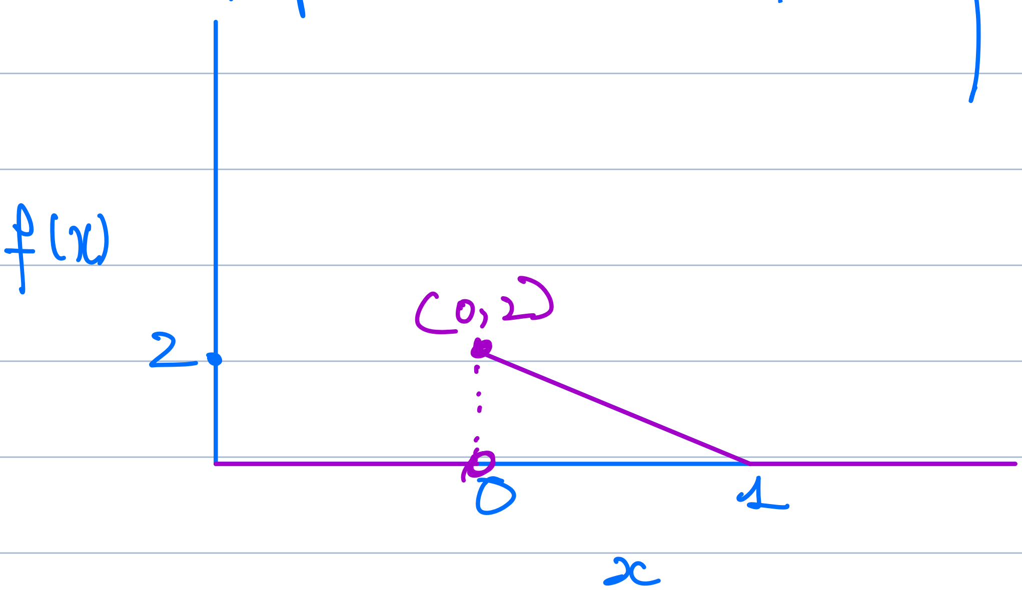
   (c) Find

      i. $P(0 \le X \le 1/2)$,
      ii. $P(1/4 \le X \le 3/4)$,
      iii. $P(1/4 \le X \le 5/4)$,
      iv. $P(X = 3/4)$,

      v. $P(X \ge 3/4)$,
      vi. the value of μ,
      vii. the value of σ^2, and
      viii. the 36th percentile $\pi_{0.36}$ of X.

4(a) pdf:

$f(x) = \begin{cases} 2(1-x), & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$



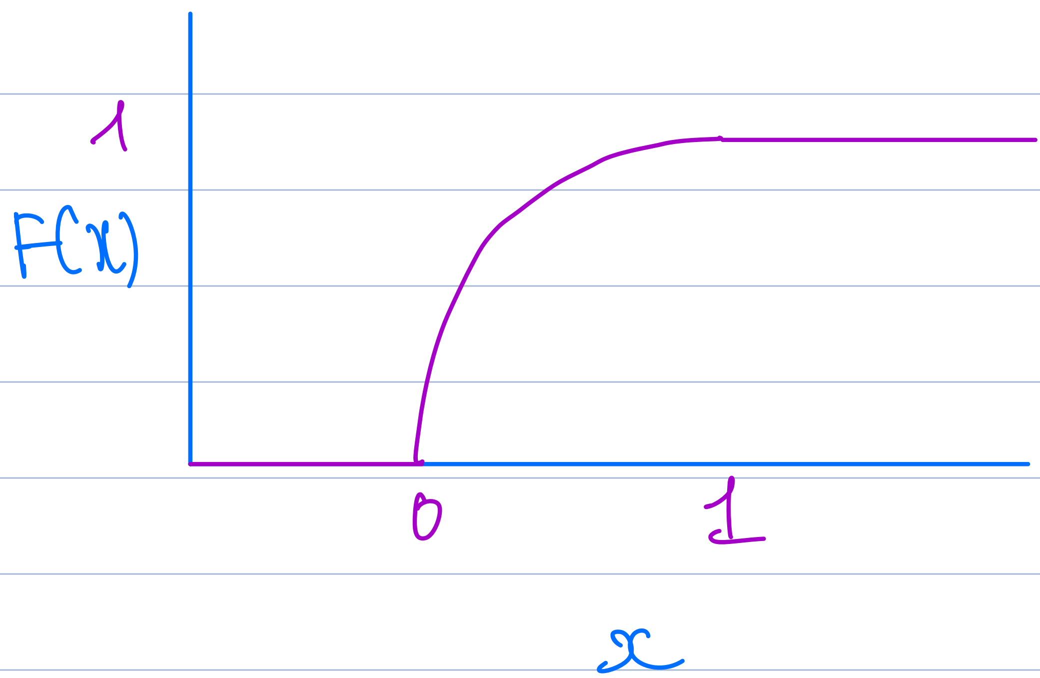
(b) Distribution function/ cdf:

$$F(x) = P(X \le x) = \begin{cases} \int_{-\infty}^{x} 0\,dt = 0, & x < 0 \\ \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} 2(1-t)\,dt, & 0 \le x \le 1 \\ \int_{-\infty}^{0} 0\,dt + \int_{0}^{1} 2(1-t)\,dt + \int_{1}^{x} 0\,dt, & x > 1 \end{cases}$$

$$= \begin{cases} 0, & x < 0 \\ 0 + 2\left[t - \frac{t^2}{2}\right]_0^x, & 0 \le x \le 1 \\ 0 + 2\left[t - \frac{t^2}{2}\right]_0^1 + 0, & x > 1 \end{cases}$$

$$2\left(1 - \frac{1}{2}\right) = 2 \cdot \frac{1}{2} = 1$$

$$= \begin{cases} 0, & x < 0 \\ 2x - x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$ ✓



(c) Find

i. $P(0 \le X \le 1/2)$,
ii. $P(1/4 \le X \le 3/4)$,
iii. $P(1/4 \le X \le 5/4)$,
iv. $P(X = 3/4)$,

$$\boxed{2x - x^2}$$

i. $P(0 \le X \le \frac{1}{2}) = F(\frac{1}{2}) - F(0)$

$$= 2\cdot\frac{1}{2} - \left(\frac{1}{2}\right)^2$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$ ✓

ii. $P(\frac{1}{4} \le X \le \frac{3}{4}) = F(\frac{3}{4}) - F(\frac{1}{4})$

$$= \left(2\cdot\frac{3}{4} - \left(\frac{3}{4}\right)^2\right) - \left(2\cdot\frac{1}{4} - \left(\frac{1}{4}\right)^2\right)$$

$$= \left(\frac{3}{2} - \frac{9}{16}\right) - \left(\frac{1}{2} - \frac{1}{16}\right)$$

$$= \frac{15}{16} - \frac{7}{16} = \frac{1}{2}$$ ✓

iii. same ↑

continuous : $P(X=x)=0$

iv. $P(X = \frac{3}{4}) = 0$ ~~$F(\frac{3}{4}) = 2\cdot\frac{3}{4} - \frac{9}{16}$~~

This is a trap

TT_TT

~~$= \frac{3}{2} - \frac{9}{16} = \frac{24}{16} - \frac{9}{16}$~~

$= \cancel{\frac{15}{16}}$

v. $P(X \geq 3/4)$,

vi. the value of μ,

vii. the value of σ^2, and

viii. the 36th percentile $\pi_{0.36}$ of X.

v. $P(X \geq \frac{3}{4}) = 1 - P(X < \frac{3}{4})$

$= 1 - F(\frac{3}{4})$

$= 1 - \frac{15}{16}$

$= \frac{1}{16}$ ✓

(Not t)

$\mu = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{\infty} x\, 2(1-x)\, dx$

$= \int_{-\infty}^{\infty} 2x - 2x^2\, dx = \left[x^2 - \frac{2x^3}{3} \right]_{-\infty}^{\infty}$

$= \left[x^2 - \frac{2x^3}{3} \right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$ ✓

vii. $\sigma^2 = Var(X) = E(X^2) - (E(X))^2$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^1 2x^2 (1-x)\, dx$

$= \left[\frac{2x^3}{3} - \frac{2x^4}{4} \right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$ ✓

So $Var(X) = \sigma^2 = E(X^2) - (E(X))^2 = \frac{1}{6} - \frac{1}{9} = \frac{3}{18} - \frac{2}{18}$

$= \frac{1}{18}$

viii. $\pi_{0.36}$

$$0.36 = \int_{-\infty}^{\pi_{0.36}} f(x)\,dx$$

$$\therefore \int_0^{\pi_{0.36}} 2(1-x)\,dx = \left[2x - x^2\right]_0^{\pi_{0.36}}$$

$$= 2\pi_{0.36} - (\pi_{0.36})^2 = 0.36$$

$\Delta = b^2 - 4ac$

$x = \frac{-b \pm \sqrt{\Delta}}{2a}$

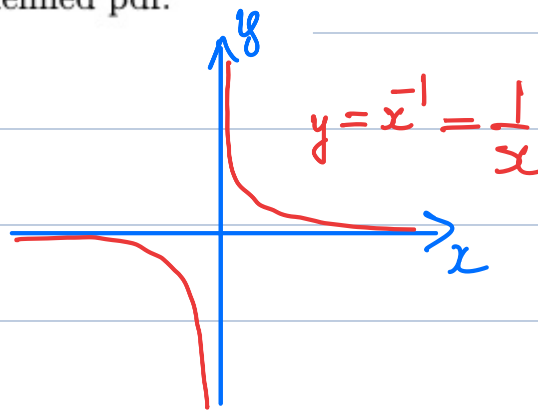
$$-(\pi_{0.36})^2 + 2\pi_{0.36} - 0.36 = 0$$

$\pi_{0.36} = 0.2$ ~~or 1.8~~

5. The pdf of X is $f(x) = c/x^2$, $1 < x < \infty$.

(a) Find the value of c so that $f(x)$ is a well defined pdf.

(b) Show that $E(X)$ is not finite.



$$\lim_{x \to \infty} \frac{1}{x} = 0$$

(a) f(x) is a well defined pdf $\Leftrightarrow \int_S \frac{c}{x^2}\,dx = 1$

$$\Leftrightarrow c\int_1^\infty x^{-2}\,dx = c\left[\frac{1}{-1}x^{-1}\right]_1^\infty = c(0+1) = 1$$

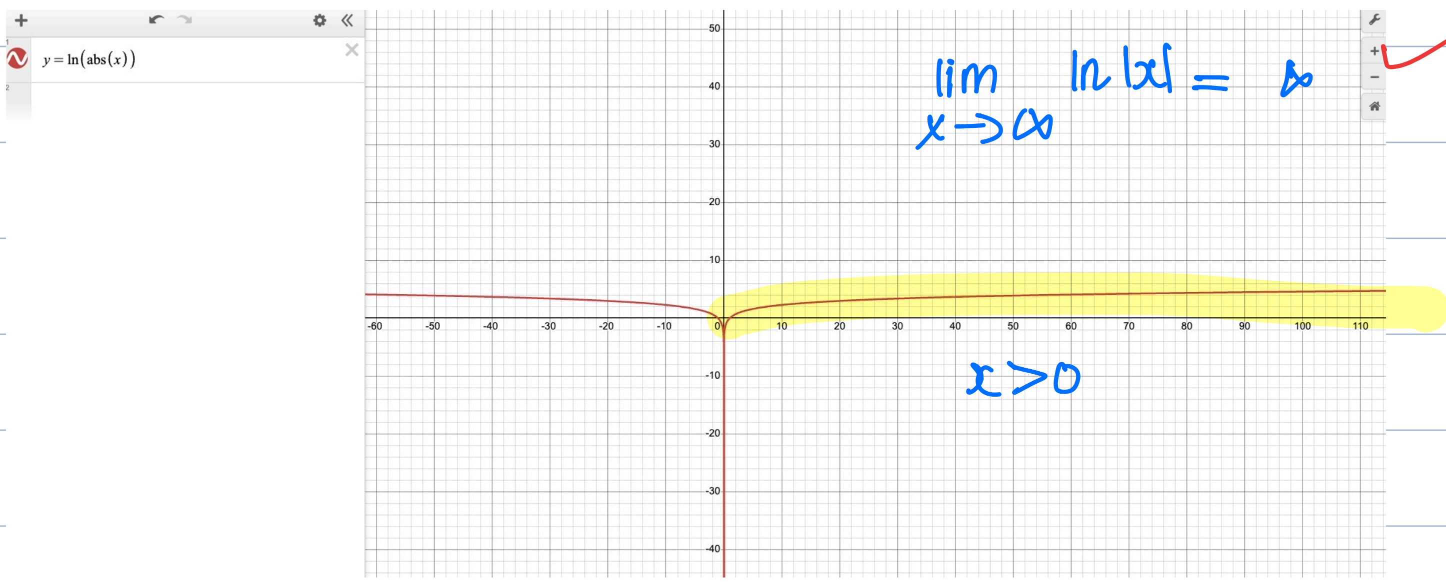
$$\therefore c = 1$$

So pdf of X is $f(x) = \frac{1}{x^2}$, $1 < x < \infty$

(b) $E(X) = \int_1^\infty x\frac{1}{x^2}\,dx = \int_1^\infty \frac{1}{x}\,dx$

$$= [\ln|x|]_1^\infty$$

$$= \infty - \ln|1| = \infty$$



$\lim_{x \to \infty} \ln|x| = \infty$

$x > 0$

6. Let $f(x) = 1/2$, $0 < x < 1$ or $2 < x < 3$, 0 elsewhere, be the pdf of X.

(a) Sketch the graph of this pdf.

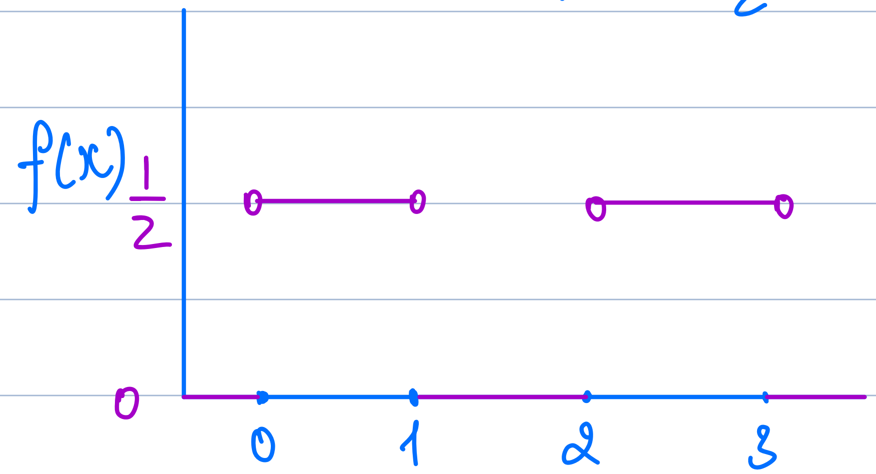
(b) Define cdf of X and sketch its graph.

(c) Find $q_1 = \pi_{0.25}$.

(d) Find the median $m = \pi_{0.50}$. Is it unique?

(e) Find the value of $E(X)$.

(a) Graph of pdf $f(x) = \frac{1}{2}$, $0 < x < 1$ or $2 < x < 3$



(b) cdf of X:

~~Ranges~~ Domains: $x > 0$: 0

$0 < x < 1$: $0 + \int_0^x \frac{1}{2}\,dt = \left[\frac{1}{2}t\right]_0^x = \frac{1}{2}x$

$1 < x < 2$: $\left[\frac{1}{2}t\right]_0^1 + 0 = \frac{1}{2}$

$2 < x < 3$: $\frac{1}{2} + \left[\frac{1}{2}t\right]_2^x = \frac{1}{2} + \left(\frac{x}{2} - 1\right)$

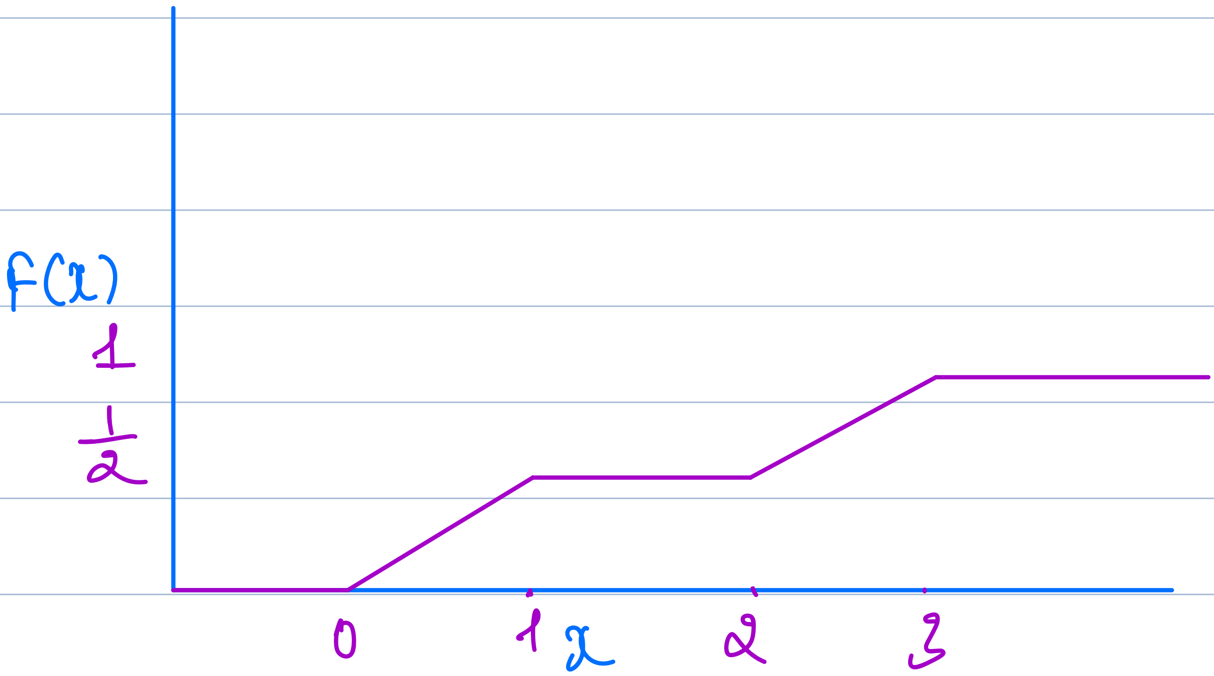
$= -\frac{1}{2} + \frac{x}{2}$

$x < 3$: $\frac{1}{2} + \left[\frac{1}{2}t\right]_2^3 = 1$

So: $F(x) = \begin{cases} 0 & , x > 0 \\ \frac{1}{2}x & , 0 < x < 1 \\ \frac{1}{2} & , 1 < x < 2 \end{cases}$

$$\begin{cases} \ldots \\ -\frac{1}{2}+\frac{x}{2}, & 2<x<3 \\ 1, & x>3 \end{cases}$$

Graph:



(c) $q_1 = \pi_{0.25} = ?$

$$0.25 = \int_{-\infty}^{\pi_{0.25}} \frac{1}{2}\, dx$$

$\therefore F(\pi_{0.25}) - F(0) = 0.25$

$\therefore \frac{1}{2}\pi_{0.25} = 0.25$

$q_1 = \pi_{0.25} = 0.5$ ✓

Median:

(d) $0.50 = \int_{-\infty}^{\pi_{0.5}} f(x)\,dx = F(\pi_{0.5})$

$\therefore \frac{1}{2}\pi_{0.5} = 0.5$

$\pi_{0.5} = 1$

Not unique bc $F(x) = 0.5 = \frac{1}{2}$ when $1<x<2$

(e) $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x \frac{1}{2} dx + \int_2^3 x \frac{1}{2} dx$

$= [\frac{1}{4}x^2]_0^1 + [\frac{1}{4}x^2]_2^3$

$= \frac{1}{4} + \frac{5}{4} = \frac{3}{2}$.

7. Let $F(x) = 1 - (\frac{1}{2}x^2 + x + 1)e^{-x}$, $0 < x < \infty$ be the cdf of X.

(a) Find the mgf $M(t)$ of X.

(b) Find the values of μ and σ^2.

(a) Given $F(x) = 1 - (\frac{1}{2}x^2 + x + 1)e^{-x}$

product Rule

$(uv)' = u'v + uv'$

So pdf $f(x) = F'(x)$

$= 0 - [(x+1+0)e^{-x} - (\frac{1}{2}x^2 + x + 1)e^{-x}]$

Quotient: $(\frac{u}{v})' = \frac{u'v - uv'}{v^2}$

chain: if $y = f(g(x))$

they $\frac{dy}{dx} = f'(g(x)) \times g'(x)$

$= (-x - 1 + \frac{1}{2}x^2 + x + 1)e^{-x}$

$= \frac{1}{2}x^2 e^{-x}$, $0 < x < \infty$

$M(t) = E(e^{tX}) = \int_0^\infty e^{tx} \frac{1}{2} x^2 e^{-x}\, dx$

$\int u\,dv = uv - \int v\,du$

$= \frac{1}{2}\int_0^\infty x^2 e^{(t-1)x}\, dx$

$u = x^2 \Rightarrow du = 2x$

$dv = e^{(t-1)x} \Rightarrow v = \frac{e^{(t-1)x}}{t-1}$

$$\frac{1}{2}\int_0^\infty x^2 e^{(t-1)x}\,dx = x^2 \cdot \frac{e^{(t-1)x}}{t-1}\Big|_0^\infty - \int_0^\infty \frac{e^{(t-1)x}}{t-1}\,2x\,dx$$

$$= 0 - \frac{1}{t-1}\int_0^\infty e^{(t-1)x}\,2x\,dx$$

$$= \frac{1}{1-t}\int_0^\infty e^{(t-1)x}\,2x\,dx$$

Set $2x = u_2 \Rightarrow du_2 = 2$

$dv_2 = e^{(t-1)x} \Rightarrow v_2 = \frac{e^{(t-1)x}}{t-1}$

Then $\int_0^\infty e^{(t-1)x}\,2x\,dx = \left(2x \cdot \frac{e^{(t-1)x}}{t-1}\right)\Big|_0^\infty - \int \frac{e^{(t-1)x}}{t-1}\,2\,dx$

$=$

7. Let $F(x) = 1 - (\frac{1}{2}x^2 + x + 1)e^{-x}$, $0 < x < \infty$ be the cdf of X.

(a) Find the mgf $M(t)$ of X.

- *First find the pdf of X: $f(x) = F'(x) = \frac{1}{2}x^2 e^{-x}$, $0 < x < \infty$ and 0 elsewhere.*
- *Then $M(t) = E(e^{tX}) = \int_0^\infty e^{tx} \cdot \frac{1}{2}x^2 e^{-x}dx = \int_0^\infty \frac{1}{2}x^2 e^{(t-1)x}dx$ $= \frac{1}{(1-t)^3}\frac{1}{2}\int_0^\infty y^2 e^{-y}dy$ if we denote $y = (1-t)x$ and assume $t < 1$.*
- *By integration by parts it can be found that $\int_0^\infty y^2 e^{-y}dy = 2! = 2$. Therefore $M(t) = \frac{1}{(1-t)^3}$, $t < 1$.*

(b) Find the values of μ and σ^2.

- $E(X) = \int_{-\infty}^\infty xf(x)dx = \int_0^\infty \frac{1}{2}x^3 e^{-x}dx = (-\frac{1}{2}x^3 e^{-x})|_0^\infty + \int_0^\infty \frac{3}{2}x^2 e^{-x}dx$

  $= (-\frac{3}{2}x^2 e^{-x})|_0^\infty + \int_0^\infty \frac{6}{2}xe^{-x}dx = (-\frac{6}{2}xe^{-x})|_0^\infty + \int_0^\infty \frac{6}{2}e^{-x}dx = (-3e^{-x})|_0^\infty = 3.$
- $E(X^2) = \int_{-\infty}^\infty x^2 f(x)dx = \int_0^\infty \frac{1}{2}x^4 e^{-x}dx = 12$ *by integration by parts.*
- *So $\sigma^2 = 12 - 3^2 = 3$.*

8. (*Q3.2-25*) The life X in years of a voltage regulator of a car has the pdf

$$f(x) = \frac{3x^2}{7^3}e^{-(x/7)^3}, \quad 0 < x < \infty.$$

(a) What is the probability that this regulator will last at least 7 years?

- $P(X \geq 7) = \int_7^\infty \frac{3x^2}{7^3}e^{-(x/7)^3}dx = -e^{-(x/7)^3}|_7^\infty = e^{-1}$.

(b) Given that it has lasted at least 7 years, what is the conditional probability it will last at least another 3.5 years?

- $P(X \geq 10.5|X \geq 7) = \frac{P(\{X\geq 10.5\}\cap\{X\geq 7\})}{P(X\geq 7)} = \frac{P(X\geq 10.5)}{P(X\geq 7)}$ $= \frac{\int_{10.5}^\infty \frac{3x^2}{7^3}e^{-(x/7)^3}dx}{\int_7^\infty \frac{3x^2}{7^3}e^{-(x/7)^3}dx} = \frac{e^{-1.5^3}}{e^{-1}} = e^{-2.375}$.