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Subject Code: MAST 20006

Subject Name: Probabilities for Stats

Assignment 4 - Multivariate distributions

Q1: Given: $P(X=x) = e^{-1} \cdot \frac{1^x}{x!}$

By definition, the joint pmf $f(x,y)$ is positive on

$S = \{(x,y) : x = 0, 1, 2, \cdots ; y = 0, 1, \cdots, x\}$

and 0 elsewhere

(a) By definition of a conditional pmf for discrete r.v:

$h(y|x) = P(Y=y \mid X=x) = \binom{x}{y} p^y (1-p)^{x-y}$,

for $y = 0, 1, \cdots, x$

(b) By the binomial mean formula:

$E(Y \mid X=x) = xp$

By law of total expectation:

$E(Y) = E[E(Y|X)] = E[Xp] = pE[X]$

$= p \cdot 1 = p$

(c) $f(x,y) = P(X=x, Y=y) = P(X=x \mid Y=y)P(X=x)$

$= \binom{x}{y} p^y (1-p)^{x-y} e^{-1} \frac{1^x}{x!}$

for $x = 0, 1, 2, \ldots$

$y = 0, 1, \ldots, x$

(d) $f_Y(y) = \sum_{x=y}^{\infty} f(x,y) = \sum_{x=y}^{\infty} \binom{x}{y} p^y (1-p)^{x-y} e^{-1} \frac{1^x}{x!}$

$$= \sum_{x=y}^{\infty} \frac{x!}{y!(x-y)!} p^y (1-p)^{x-y} e^{-1} \frac{1^x}{x!}$$

$$= \frac{e^{-1} p^y}{y!} \sum_{x=y}^{\infty} \frac{1}{(x-y)!} (1-p)^{x-y}$$

since $\frac{e^{-1} p^y}{y!}$ does NOT depend on x

Let $m = x-y$, as x runs from y to ∞, m runs from 0 to ∞, then:

$$\sum_{x=y}^{\infty} \frac{(1-p)^{x-y}}{(x-y)!} = \sum_{m=0}^{\infty} \frac{(1-p)^m}{m!}$$

By the Maclaurin series for exponential function:

$$\sum_{m=0}^{\infty} \frac{a^m}{m!} = e^a, \text{ so:}$$

$$\sum_{m=0}^{\infty} \frac{(1-p)^m}{m!} = e^{1-p}$$

So, $f_Y(y) = \frac{e^{-1} p^y}{y!} \sum_{x=y}^{\infty} \frac{1}{(1-y)!} (1-p)^{x-y}$

$$= \frac{e^{-1} p^y}{y!} e^{1-p} = \frac{p^y}{y!} e^{-p}$$

which is the pmf of $Y \stackrel{d}{=} \text{Poi}(p)$

(e)

By definition 4 of the conditional pmf:

$$g(x|y) = \frac{f(x,y)}{f_Y(y)}$$

and from (c): $f(x,y) = \binom{x}{y} p^y (1-p)^{x-y} e^{-1} \frac{1^x}{x!}$

and from (d): $f_Y(y) = \frac{p^y e^{-p}}{y!}$

So, $g(x|y) = \dfrac{\binom{x}{y} p^y (1-p)^{x-y} e^{-1} \frac{1^x}{x!}}{\frac{p^y e^{-p}}{y!}}$

$$= \frac{y!}{x!} \frac{\binom{x}{y}}{1} \frac{p^y}{p^y} \frac{e^{-1}}{e^{-p}} 1^x (1-p)^{x-y}$$

$$= e^{-(1-p)} \frac{(1-p)^{x-y}}{(x-y)!} \quad \text{for } x = y, y+1, y+2, \ldots$$

Q2 :

(a) · By definition, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (x+y)\,dy$

$$= \int_0^1 x\,dy + \int_0^1 y\,dy = x[y]_0^1 + \left[\frac{1}{2}y^2\right]_0^1$$

$$= x\cdot(1-0) + \frac{1}{2}(1^2-0^2) \boxed{= x+\frac{1}{2},\ 0<x<1}$$

· $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (x+y)\,dx$

$$= \int_0^1 x\,dx + \int_0^1 y\,dx = \left[\frac{1}{2}x^2\right]_0^1 + y[x]_0^1$$

$$\boxed{= \frac{1}{2}+y,\ 0<y<1}$$

Now, $f_X(x)f_Y(y) = (x+\frac{1}{2})(\frac{1}{2}+y) = \frac{1}{2}x + xy + \frac{1}{4} + \frac{1}{2}y$

$\neq f(x,y) = x+y$ for $0<x<1,\ 0<y<1$

so X and Y are NOT independent for $\forall (x,y) \in S$

(b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^1 4xy\, dy = 4x\left[\frac{y^2}{2}\right]_0^1$

$= 2x, 0<x<1$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^1 4xy\, dx = 4y\left[\frac{x^2}{2}\right]_0^1$

$= 2y, 0<y<1$

Now, $f_X(x)\, f_Y(y) = 2x\,2y = 4xy = f(x,y)$

so X and Y are independent $\forall (x,y) \in S$

Q3 :

(a) $$P(X<Y) = \int_{y=0}^{\infty} \left(\int_{x=0}^{y} 2e^{-x} e^{-2y} \, dx \right) dy$$

$$= \int_{y=0}^{\infty} \left(2e^{-2y} \int_0^y e^{-x} dx \right) dy = \int_{y=0}^{\infty} 2e^{-2y} \left[-e^{-x} \right]_0^y \, dy$$

$$= \int_{y=0}^{\infty} 2e^{-2y} (-e^{-y} + 1) \, dy = \int_{y=0}^{\infty} - 2e^{-3y} + 2e^{-2y} \, dy$$

$$= 2\int_0^{\infty} e^{-2y} dy - 2\int_0^{\infty} e^{-3y} dy$$

$$= 2\left[-\frac{1}{2} e^{-2y} \right]_0^{\infty} - 2\left[-\frac{1}{3} e^{-3y} \right]_0^{\infty}$$

$$= 2\left(\frac{1}{2}\right) - 2\left(\frac{1}{3}\right) = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$$

since $\lim_{y \to \infty} e^{-y} = 0$

(b) $P(X>1, Y<1) = \int_{y=0}^{1}\left(\int_{x=1}^{\infty} 2e^{-x}e^{-2y}\,dx\right)dy$

$$= \int_{y=0}^{1} 2e^{-2y}\left(\int_{1}^{\infty} e^{-x}\,dx\right)dy = \int_{0}^{1} 2e^{-2y}\left[-e^{-x}\right]_{1}^{\infty} dy$$

$$= \int_{0}^{1} 2e^{-2y}(e^{-1})\,dy = 2e^{-1}\int_{0}^{1} e^{-2y}\,dy = 2e^{-1}\left[\frac{e^{-2y}}{-2}\right]_{0}^{1}$$

$$= 2e^{-1}\left(\frac{e^{-2}}{-2} + \frac{1}{2}\right) = \boxed{-e^{-3} + e^{-1}}$$

(c) Use the change-of-variable method:

① Inverse mapping:

$x = e^{u}$

$y = x - v = e^{u} - v$

② $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^{u} & 0 \\ e^{u} & -1 \end{vmatrix} = -e^{u}$

$|J| = e^{u}$

③ Original support: $0 < x < \infty$, $0 < y < \infty$

So, $x = e^{u} > 0$ for $\forall u \in \mathbb{R}$

$y = e^{u} - v \iff v < e^{u}$

So new support is: $\{(u, v)\}: u \in \mathbb{R}, v < e^{u}$

④ New joint pdf:

$$g(u,v) = f(x(u,v), y(u,v))\,|J| = 2e^{-e^{u}} e^{-2(e^{u}-v)} e$$

$$= \underline{2e^{u} e^{-3e^{u}+2v}, u \in \mathbb{R}, v < e^{u}.}$$