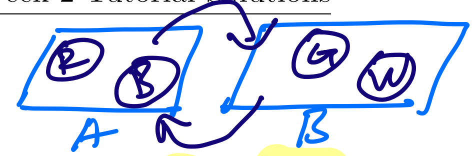
17.3.25



Week 2 Tutorial Solutions

1. Box A contains one red and one black marble. Box B contains one green and one white marble. Consider the experiment of drawing a marble at random from Box A, transferring it to Box B and then drawing a marble from Box B at random.

   (a) Write down the associated sample space.

   First letter: Marble A → B
   Second: Marble B → A

   - $S = \{RR, RG, RW, BB, BG, BW\}$.

   (b) List the sample point(s) that comprise the event that a red ball is selected from Box B.

   - $\{RR\}$.

   (c) List the sample point(s) that comprise the event that a white ball is selected from Box B.

   - $\{RW, BW\}$.

   (d) List the sample points that comprise the event that a green ball is not selected from Box B.

   - $\{RR, RW, BB, BW\}$.

2. Let $A_n = [0, 1 + \frac{1}{n})$, $n = 1, 2, \cdots$, be a sequence of events (i.e. intervals). Find $A_1 \cap A_2$, $A_1 \cap A_2 \cap A_3$ and $A_1 \cap A_2 \cap \cdots \cap A_{100}$. Then guess the result of $\cap_{n=1}^{\infty} A_n$.

   - $A_1 \cap A_2 = [0, 2) \cap [0, 1\frac{1}{2}) = [0, 1\frac{1}{2}) = A_2$.
   - $A_1 \cap A_2 \cap A_3 = [0, 1\frac{1}{3}) = A_3$. $= [0, 1\frac{1}{2}) \cap [0, 1\frac{1}{3})$
   - $A_1 \cap A_2 \cap \cdots \cap A_{100} = [0, 1\frac{1}{100}) = A_{100}$.
   - $\cap_{n=1}^{\infty} A_n = \lim_{m\to\infty} \cap_{n=1}^{m} A_n = \lim_{m\to\infty}[0, 1 + \frac{1}{m}) = [0, 1]$.

3. Show why $(A \cap B') \cap (C \cap D') = (A \cap C) \cap (B \cup D)'$ is true.

   - $(A \cap B') \cap (C \cap D') = (A \cap C) \cap (B' \cap D') = (A \cap C) \cap (B \cup D)'$.

   Associative law De Morgan Law

4. A six-sided die is to be rolled. However, it is known that the die is loaded so that a 1 and 6 are equally likely and are three times as likely to occur as any of the other sides. What are the probabilities for each of the six sides to be observed?

   - *Let $p_1, p_2, \cdots, p_6$ be the probabilities for the six sides of the dice.*
   - *Then $p_1 = p_6 = 3p_2 = 3p_3 = 3p_4 = 3p_5$.*
   - *Since $\sum_{i=1}^{6} p_i = 1$, we have $p_1 = p_6 = 0.3$ and $p_2 = p_3 = p_4 = p_5 = 0.1$.*

5. On a given day, a machine produces 100 items of a certain product. Assume that 10 of these items are defective, what is the probability that a random sample of 5 items to be selected from the outputs will contain 3 defective items?

- *Let A be the event that the random sample will contain 3 defectives.*
- *Then* $P(A) = \frac{\binom{10}{3}\binom{100-10}{5-3}}{\binom{100}{5}} = \frac{120\times4005}{75287520} \approx 0.00638.$

6.* (*Q1.1-13 in textbook*) If we had a choice of two airlines, we would possibly choose the airline with the better "on time performance". So consider Alaska and America West using data reported by Arnold Barnett, "How Numbers Can Trick You," in *Technology Review*, 1994.

Airline	Alaska Airlines	America West
	Relative Frequency	Relative Frequency
Destination	On Time	On Time
Los Angeles	$\frac{497}{559} = 0.889$	$\frac{694}{811} = 0.856$
Phoenix	$\frac{221}{233} = 0.948$	$\frac{4840}{5255} = 0.921$
San Diego	$\frac{212}{232} = 0.914$	$\frac{383}{448} = 0.855$
San Francisco	$\frac{503}{605} = 0.831$	$\frac{320}{449} = 0.713$
Seattle	$\frac{1841}{2146} = 0.858$	$\frac{201}{262} = 0.767$
Five-city Total	$\frac{3274}{3775} = 0.867$	$\frac{6438}{7225} = 0.891$

(a) For each of the five cities listed, which airline has the better on time performance?

- *Alaska Airlines has the better performance for each city.*

(b) Combining the results, which airline has the better on time performance?

- *America West has the better performance when the factor "Destination" is dropped.*

(c) Interpret your results.

- *This is an example of Simpson's Paradox.*

7.* (*Q1.2-15 in textbook*) Divide a line segment into two parts by selecting a point at random. Use your intuition to assign a probability to the event that the longer segment is at least two times as long as the shorter segment.

0 short long (x2) 1

- *Consider partitioning the segment into 3 equal sub-segments. Then whenever a randomly selected point falls into the two end sub-segments, the longer part created by the selected point will be at least two times as long as the short part. Thus the probability for this event to occur is 2/3.*

8.* (*Q1.3-7*) In a state lottery four digits are drawn at random (one digit at a time) with replacement from 0 to 9. Suppose that you win if any permutation of your selected integers is drawn. Give the probability of winning if you select

Q7: Set up the interval: [0,1] – segment endpoints

Condition for longer at least twice the shorter:

- If point X lies near the left end: $0 \le X \le \frac{1}{3}$

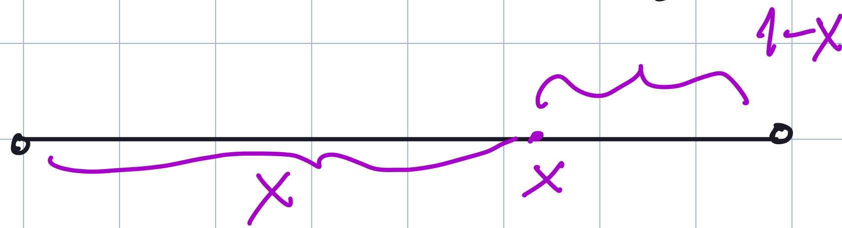
Then:

$\Rightarrow 1-X > x \geq 2x$

$X \le \frac{1}{3}$ (1)

- If point X lies near the right end: $\frac{2}{3} \le X \le 1$

Then:



$\Rightarrow 2(1-X) \le X$

$2-2X \le X$

$2 \le 3X$

$X \geq \frac{2}{3}$ (2)

U − OR

∩ − and

(1), (2) $\Rightarrow$ $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

length $= 1/3$

length $= \frac{1}{3}$

So $P = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

(a) 6, 7, 8, 9.

- $P(\{6,7,8,9\}) =$
  $\frac{\textit{number of ways that the 4 numbers drawn are any permutations of 6,7,8,9}}{\textit{number of ways of drawing 4 numbers from 0,1,...9}}$ $= \frac{4!}{10^4}$
  $= \frac{4\times3\times2\times1}{10\times10\times10\times10} = 0.0024.$

(b) 6, 7, 8, 8.

- $P(\{6,7,8,8\}) =$
  $\frac{\textit{number of ways that the 4 numbers drawn are any permutations of 6,7,8,8}}{\textit{number of ways of drawing 4 numbers from 0,1,...9}}$ $= \frac{4!/2!}{10^4}$
  $= \frac{\binom{4}{2}\times2\times1}{10\times10\times10\times10} = 0.0012.$

(c) 7, 7, 8, 8.

- $P(\{7,7,8,8\}) =$
  $\frac{\textit{number of ways that the 4 numbers drawn are any permutations of 7,7,8,8}}{\textit{number of ways of drawing 4 numbers from 0,1,...9}}$ $= \frac{4!/2!2!}{10^4}$
  $= \frac{\binom{4}{2}\times1}{10\times10\times10\times10} = 0.0006.$

(d) 7, 8, 8, 8.

- $P(\{7,8,8,8\}) =$
  $\frac{\textit{number of ways that the 4 numbers drawn are any permutations of 7,8,8,8}}{\textit{number of ways of drawing 4 numbers from 0,1,...9}}$ $= \frac{4/3!}{10^4}$
  $= \frac{\binom{4}{1}\times1}{10\times10\times10\times10} = 0.0004.$

9.* (a) Calculate $\sum_{r=0}^{n}\binom{n}{r}$ for $n = 0, 1$ and 2. Then guess the result of $\sum_{r=0}^{n}\binom{n}{r}$ for general $n \geq 0$ and prove it.

- *From the binomial expansion* $(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}$ *it follows that*
  $\sum_{r=0}^{n}\binom{n}{r} = (1+1)^n = 2^n$
  by taking $a = b = 1$.

(b) Calculate $\sum_{r=0}^{n}(-1)^r\binom{n}{r}$ for $n = 1, 2$ and 3. Then guess the result of $\sum_{r=0}^{n}(-1)^r\binom{n}{r}$ for general $n \geq 1$ and prove it.

- *From the binomial expansion* $(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}$ *it follows that*
  $0 = (-1+1)^n = \sum_{r=0}^{n}(-1)^r\binom{n}{r}$
  by taking $a = -1$ *and* $b = 1$ *(note that if* $n = 0$ *then we are in the undefined case* 0^0*, to which we may assign value 1).*

Some useful counting formulas

Combinations: The number of combinations of n distinct objects chosen r at a time is

$$ {}_nC_r \equiv C_{n,r} = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}, $$

which can also be regarded as the number of ways of taking r objects from n distinct ones and putting them into a "box" or "cell". So *the order does not matter here.*

Remark: $\binom{n}{r}$ is frequently called a **binomial coefficient** because it appears in the binomial expansion:

$$ (a+b)^n = \sum_{r=0}^{n} \binom{n}{r} b^r a^{n-r}. $$