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Subject Name: Probability for Statistics

Assignment 2

Q1: H: head; T: tail

(a) Let X be the number of heads obtained

Tossing a coin twice, so the possible values of X is 0, 1, 2.

Identifying the cases:

- If a 2-headed coin is chosen, then every toss gives a head, and thus $P(HH) = \frac{2}{7}$.
- If a normal coin is chosen, then each toss is fair with $P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$. When tossed twice, the possible outcomes are:
  - $P(TT) = (\frac{1}{2})^2 = \frac{1}{4}$ (0 head)
  - $P(HT) = P(TH) = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$ (1 head)
  - $P(HH) = (\frac{1}{2})^2 = \frac{1}{4}$ (2 heads)

For $X = 0$: Only possible when a normal coin is chosen AND both tosses are tails:

$$P(X=0) = \frac{5}{7} \times \frac{1}{4} = \frac{5}{28}$$

For $X = 1$: Only possible with a normal coin AND one head is tossed:

$$P(X=1) = \frac{5}{7} \times \frac{1}{2} = \frac{5}{14}$$

For $X=2$: This is possible in 2 ways:

1. A normal coin produces 2 heads: $\frac{5}{7} \times \frac{1}{4} = \frac{5}{28}$, or
2. A 2-headed coin (always 2 heads): $\frac{2}{7}$

$$P(X=2) = \frac{5}{28} + \frac{2}{7} = \frac{13}{28}$$

Therefore, the pmf of X can be given by the table:

x	0	1	2
$f(x) = P(X=x)$	$5/28$	$5/14$	$13/28$

(b) $\mu = E(X) = \sum_x x\, P(X=x) = 0 \times \frac{5}{28} + 1 \times \frac{5}{14} + 2 \times \frac{13}{28}$

$$= \frac{9}{7} = 1.2857 \text{ (4 d.p.)}$$

(c) $E(X^2) = 0^2 \times \frac{5}{28} + 1^2 \times \frac{5}{14} + 2^2 \times \frac{13}{28} = \frac{31}{14}$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{31}{14} - \left(\frac{9}{7}\right)^2 = \frac{55}{98}$$

$$= 0.5612 \text{ (4 d.p.)}$$

(d) mgf of X:

$$M(t) = e^{0t} \times \frac{5}{28} + e^{1t} \times \frac{5}{14} + e^{2t} \times \frac{13}{28} = \frac{1}{28}(5 + 10e^t + 13e^{2t})$$

(e) Pay \$1 to play, receive \$2 per head means:

$$\text{profit} = -1 + 2X$$

Thus, expected profit is: $E(-1 + 2X) = -1 + E(2X)$

$$= -1 + 2E(X)$$

$$= -1 + 2 \times \frac{9}{7} = \frac{11}{7} = \$1.5714$$

By linearity of expectation (4 d.p.)

Q_2: $3\% = 0.03$

(a) For X: X counts the number of test until the 1^{st} positive so X is geometrically distributed:

$$X \stackrel{d}{=} Geo(p = 0.03)$$

For Y: Y counts the number of tests required to find 3 positive cases so Y has a negative binonomial distribution: $Y \stackrel{d}{=} NB(r=3, p=0.03)$

(b) The probability that at least 5 players have to be tested to find the 1^{st} positive is:

$$P(X \geq 5) = P(X > 4) = (1-p)^4 = 0.97^4$$
$$= 0.8853 \text{ (4 d.p.)}$$

(For a geometric random variable with support $\{1, 2, 3, \ldots\}$)

(c) For a negative binomial distribution, the expected number of trials is: $E(Y) = \frac{r}{p} = \frac{3}{0.03} = 100$ trials

Hence, it takes 100 players to test to find 3 positive cases.

(d) Now, $P(\text{Next 5 no positive} \mid \text{first 5 no positive})$

$$= \frac{P(\text{Next 5 no positive} \cap \text{First 5 no positive})}{P(\text{First 5 No positive})}$$

By definition of conditional probability

The test outcomes are independent, so

$P(\text{Next 5 No positive} \mid \text{First 5 No positive})$

$= P(\text{Next 5 No positive}) = (0.97)^5 = 0.8587$ (4 d.p.)

By definition of independent events

(e)

① Binomial model: If $X \overset{d}{=} \text{Bin}(n = 100, p = 0.03)$ then

$$P(X=4) = \binom{100}{4}(0.03)^4(0.97)^{96}$$

$$= 0.1706 \text{ (4 d.p.)}$$

② Poisson Approximation:

For a large n and small p with $\lambda = np = 100 \times 0.03$,
$= 3$

the probability is approximated by:

$$P(X=4) = \frac{3^4 e^{-3}}{4!} = 0.1680 \text{ (4 d.p.)}$$

Difference between 2 distributions:

$$|0.1706 - 0.1680| = 0.0026$$

which is 0.26% point difference.

$\underline{Q_3}$:

(a) Given $E(X^r) = 0.6$ for $r = 1, 2, 3, \dots$

Since for a Bernoulli random variable we have $E(X^r) = p$ for $\forall r$, X must be a Bernoulli random variable : $X \stackrel{d}{=} \text{Bernoulli}(0.6)$

(b) $Var(X) = p(1-p) = 0.6 \times 0.4 = 0.24$

——— end ———