Thu, 5.6.25

Week 8 Tutorial

- Some theories: $E[u(X_1, X_2)] = \sum_{(x_1, x_2) \in S} u(x_1, x_2) f(x_1, x_2)$

$P[(X, Y) \in A] = \iint_A f(x, y)\, dx\, dy$, $\{(X, Y) \in A\}$ is an event (subset) defined in the plane

Joint cdf of X and Y: $F(x, y) = P[X \le x, Y \le y]$

$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(s, t)\, dt\, ds = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s, t)\, ds\, dt$

(orders don't matter) Differentiate distribution-function vs change of var.

Marginal pdf: $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$, $\boxed{x \in S_X}$

$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$, $\boxed{y \in S_Y}$

X and Y are independent $\iff$ $f(x, y) = f_X(x) f_Y(y)$

$\iff$ $F(x, y) = F_X(x) F_Y(y)$, $\forall (x, y) \in S$

For $\forall\ u(X_1, X_2)$: $E[u(X_1, X_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x_1, x_2) f(x_1, x_2)\, dx_1\, dx_2$

- Tutorial Questions:

1. Given $f_X(x) = 4x^3$, $0 < x < 1$. Find pdf of $Y = X^2$

Use the $\boxed{\text{Change of Variable technique}}$ (Doesn't apply for Discrete):

Now $Y = X^2$ $\therefore$ support of Y: $0 < y < 1$

$Y = u(X) = X^2$ is a continuous increasing function of X with inverse $X = \pm\sqrt{Y} = \sqrt{Y}$
$(0 < x < 1)$

For Y in the support, cdf of Y is: $G(y) = \int_0^{\sqrt{y}} 4x^3\,dx$

$= [x^4]_0^{\sqrt{y}} = (\sqrt{y})^4 = y^2$ ✓

pdf of Y: $g(y) = G'(y) = 2y$, $\boxed{0 < y < 1}$

↑ don't forget this!

(2) Given $X \sim U(-1, 3)$. Find pdf of $Y = X^2$

Since $X \sim U(-1,3)$, pdf of X is $\frac{1}{3+1} = \frac{1}{4}$, $-1 \leq x \leq 3$

! classic mistake: $(c_1, c_2) : (d_1, d_2)$ is NOT 1-to-1 $\therefore$ can't use change of var

$\therefore$ support of Y: $0 \leq y < 9$ $\boxed{\text{NOT } 1 \leq y \leq 9}$

Support $\left[\begin{array}{l}\text{of } X: [-1, 3] \\ \text{of } Y: [0, 9)\end{array}\right.$

Use the distribution-function technique:

For each y in the support, cdf of Y is:

$G(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y})$ $(Y \in [0, 9))$

$= F(\sqrt{y}) = \int_{-1}^{\sqrt{y}} \frac{1}{4}\,dx = [\frac{1}{4}x]_{-1}^{\sqrt{y}} = \frac{1}{4}\sqrt{y} + \frac{1}{4}$, $0 \leq y < 9$

$\therefore$ pdf of Y: $\frac{1}{8\sqrt{y}}$

when $0 \le y < 1$, $F_Y(y) = P(Y \le y) = P(X^2 \le y) =$
$P(\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{4}\,dx = \frac{\sqrt{y}}{2}$

when $1 \le y \le 9$, $F_Y(y) = P(-1 \le X \le \sqrt{y}) = \int_{-1}^{\sqrt{y}} \frac{1}{4}\,dx = \frac{\sqrt{y}+1}{4}$

So pdf $f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{4\sqrt{y}}, & 0 \le y < 1 \\ \frac{1}{8\sqrt{y}}, & 1 \le y < 9 \end{cases}$

③ $f(x) = \theta x^{\theta - 1}$, $0 < x < 1$, $0 < \theta < \infty$

Let $Y = -2\theta \ln X$. Find the distribution of Y?

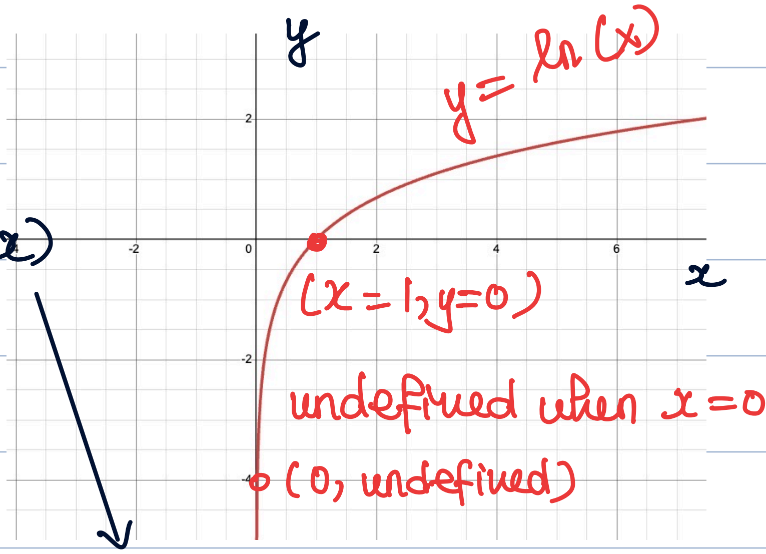
Solution:

$0 < x < 1$

$\therefore \; -\infty < \ln x < 0$ (range of $\ln x$)

$\therefore \; 0 < -2\theta \ln x < +\infty$

$\therefore \; Y \in (0, +\infty)$



$y = \ln(x)$ $\begin{cases} \text{range: } y \in (-\infty, \infty) \\ \text{domain: } x \in (0, \infty) \end{cases}$

Use the distribution-function technique:

cdf of Y: $G(y) = P(Y \le y)$

$= P(-2\theta \ln(X) \le y) = P\left(\ln(X) \le \frac{y}{-2\theta}\right) = P(X \le e^{-y/2\theta})$

$= P(e^{-y/2\theta})$

$\therefore G(y) = \int_0^{e^{-y/2\theta}} \theta x^{\theta-1}\,dx = \left[\, x^{\theta} \,\right]_0^{e^{-y/2\theta}} = \left(e^{-y/2\theta}\right)^{\theta}$

$$= e^{-\frac{y}{2\theta}\cdot\theta} = e^{-y/2}$$

$$\therefore g(y) = G'(y) = -\frac{1}{2}e^{-\frac{y}{2}}$$

So $Y \sim \exp(\theta = 2)$

can also use change of variable

④ Given joint pmf of X and Y: $f(x,y) = \dfrac{x+y}{32}$

$x = 1,2,\ y = 1,2,3,4 \rightarrow$ Discrete

(a) $$f_1(x) = \sum_{y=1}^{4} \frac{x+y}{32} = \frac{x+1}{32} + \frac{x+2}{32} + \frac{x+3}{32} + \frac{x+4}{32}$$

$$= \frac{4x+10}{32} = \frac{2x+5}{16},\ x = 1,2$$

(b) $$f_2(y) = \sum_{x=1}^{2} \frac{x+y}{32} = \frac{1+y}{32} + \frac{2+y}{32} = \frac{3+2y}{32},\ y = 1,2,3,4$$

(c) $P(X>Y) = P(X>Y, Y=y)$

$$= \sum_{x=1}^{2}\sum_{y=1}^{1} \frac{x+y}{32} = \sum_{x=1}^{2} \frac{x+1}{32} = \frac{2}{32} + \frac{3}{32} = \frac{5}{32}$$

(d) $$P(Y=2X) = \sum_{x=1,2}\ \sum_{y=2,4} \frac{x+y}{32} = \sum_{x=1,2} \frac{2x+6}{32} = \frac{8}{32} + \frac{10}{32} = \frac{18}{32} = \frac{9}{16}$$

(e) $$P(X+Y=3) = \sum_{x=1,2}\ \sum_{y=2,1} \frac{x+y}{32} = \sum_{x=1,2} \frac{2x+8}{32}$$

$$= \frac{2+8}{32} + \frac{4+3}{32} = \frac{12}{32} = \frac{6}{16}$$

(f) $P(X \le 3-Y) = \sum_{x=1,1} \sum_{y=1,2} \frac{x+y}{32} = \sum_{x=1,1} \frac{2x+3}{32}$

$= \frac{10}{32} = \frac{5}{16}$

(c) $P(X > Y) = P(\{X=2, Y=1\})$

$= \frac{2+1}{32} = \frac{3}{32}$

(d) $P(Y = 2X) = P(\{X=1, Y=2\} \overset{\cup}{\cancel{\cap}} \{X=2, Y=4\})$

$= \frac{1+2}{32} + \frac{2+4}{32} = \frac{3}{32} + \frac{6}{32} = \frac{9}{32}$

(e) $P(X+Y=3) = P(\{X=1, Y=2\} \cup \{X=2, Y=1\})$

$= \frac{1+2}{32} + \frac{2+1}{32} = \frac{6}{32}$

(f) $P(X \le 3-Y) = P(\{X=1, Y=1\} \cup \{X=1, Y=2\} \cup \{X=2, Y=1\})$

$= \frac{1+1}{32} + \frac{1+2}{32} + \frac{2+1}{32} = \frac{8}{32}$

(g) X and Y are independent $\Leftrightarrow f(x,y) = f_1(x) f_2(y)$

$f_1(x) f_2(y) = \frac{2x+5}{32} \cdot \frac{3+2y}{32} = \frac{6x + 4x^2 + 15 + 10y}{1024}$

$\ne f(x,y) = \frac{x+y}{32}$

So X and Y are NOT independent.

(h) $E(X) = \sum_{x=1}^{2} x \frac{2x+5}{16} = \sum_{x=1}^{2} \frac{2x^2+5x}{16}$

$= \frac{2+5}{16} + \frac{8+10}{16} = \frac{25}{16}$

(i) $E(X+Y) = \sum_{x=1}^{2} \sum_{y=1}^{4} \frac{x+y}{32} (x+y)$

(i) Find $E(X+Y)$.

- $E(X+Y) = \sum_{x=1}^{2} \sum_{y=1}^{4} (x+y) \frac{x+y}{32}$
$= \frac{(1+1)^2+(1+2)^2+(1+3)^2+(1+4)^2+(2+1)^2+(2+2)^2+(2+3)^2+(2+4)^2}{32} = \frac{140}{32}$.

5 Joint pmf: $f(x,y) \sim U(a,b)$

Support: $2 < x < 2.5$; $2 < y < 2.3$

If X and Y are within 0.1 of each other → rebid

else the lower bidder gets the contract

P(rebid)? → Just rewrite this in math!

isolate

$P(\text{rebid}) = P(|X-Y| < 0.1) = P(-0.1 < X-Y < 0.1)$

$= P(-0.1+Y < X < 0.1+Y)$

$= \int_{-0.1+Y}^{0.1+Y} \int_{2}^{2.5}$

Dimensions & Uniform distribution:

- In one-dimension: $X \sim U(a,b)$

pdf: $f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$

cdf: $P(c \le X \le d) = \int_c^d \frac{1}{b-a}\,dx = \frac{d-c}{b-a}$ ∴ ratio of the subinterval length $(d-c)$ to total length $(b-a)$

- In 2 dimension:

Joint pdf/density $f_{X,Y}(x,y)$ is constant over some planar set $R \subset \mathbb{R}^2$ and 0 outside

pdf: $f_{X,Y}(x,y) = \begin{cases} \frac{1}{Area(R)} = \frac{1}{(b-a)(d-c)}, & a \le x \le b,\ c \le y \le d \\ 0, & \text{otherwise} \end{cases}$

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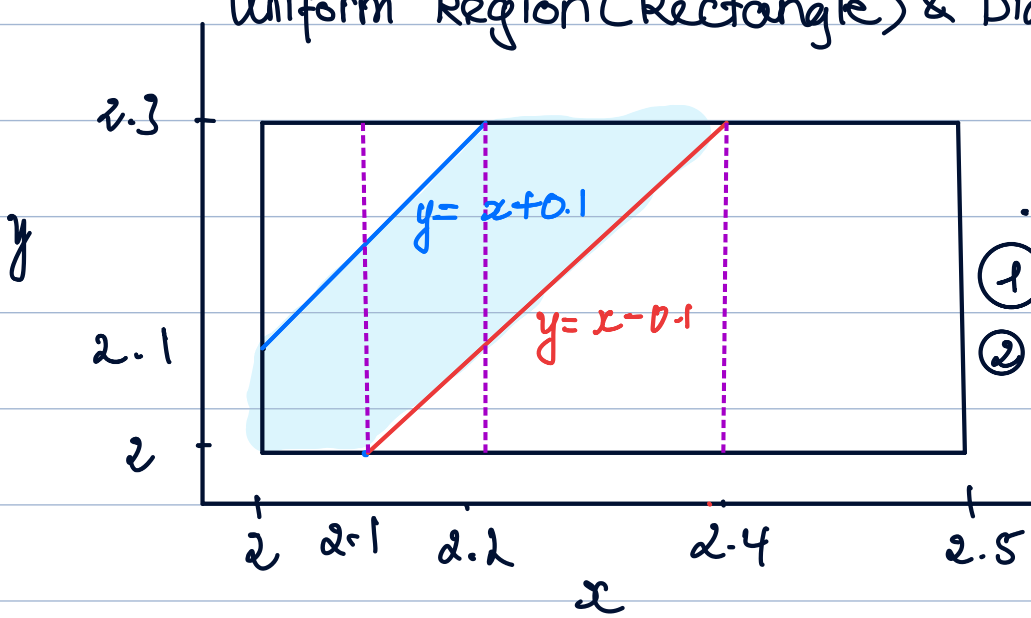
(5) $2 < x < 2.5$; $2 < y < 2.3$ ∴ $f_{X,Y}(x,y) = \frac{1}{0.5 \times 0.3} = \frac{1}{0.15}$

$P(|X-Y| < 0.1) = P(-0.1 < X - Y < 0.1) = P(-0.1 + Y < X < 0.1 + Y)$

Uniform Region (Rectangle) & Diagonal band $|x-y| < 0.1$



∴ split into 3 x-ranges:

1. $x \in (2, 2.1)$, $y \in (2, x+0.1)$
2. $x \in [2.1, 2.2]$, ~~$y \in (x+0.1, x-0.1)$~~ $y \in (x-0.1, x+0.1)$
3. $x \in (2.2, 2.4)$, $y \in (x-0.1, 2.3)$

$\therefore$ P(rebid) = Area of the shaded region

$$= \int_{2.0}^{2.1}\int_{2.0}^{x+0.1} \frac{1}{0.15}\,dy\,dx + \int_{2.1}^{2.2}\int_{x-0.1}^{x+0.1} \frac{1}{0.15}\,dy\,dx$$

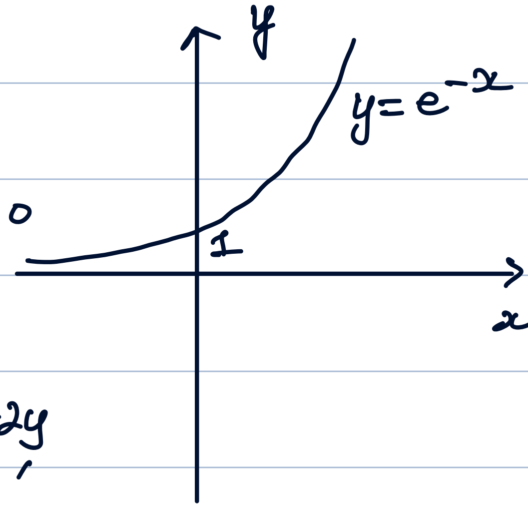
$$+ \int_{2.2}^{2.4}\int_{x-0.1}^{2.3} \frac{1}{0.15}\,dy\,dx = \cdots = \frac{11}{30}$$

The probability that they will be asked to rebid is $\frac{11}{30}$

⑥ $f(x,y) = 2e^{-x-y},\ 0 \le x < y < \infty$

(a) $f_1(x) = \int_x^{\infty} 2e^{-x-y}\,dy = \int_x^{\infty} 2e^{-x}e^{-y}\,dy = \left[-2e^{-x}e^{-y}\right]_x^{\infty}$

$= 2e^{-x}e^{-x} = 2e^{-2x},\ \boxed{0 \le x < \infty}$



(b) $f_2(y) = \int_0^{y} 2e^{-x-y}\,dx = \int_0^{y} 2e^{-x}e^{-y}\,dx \leftarrow 0$

$= \left[-2e^{-y}e^{-x}\right]_0^{y} = -2e^{-y}e^{-y} = -2e^{-2y},$

$\boxed{x \le y < \infty}$

(c) $E(X) = \int_0^{\infty} x f_1(x)\,dx = \int_0^{\infty} x\,2e^{-2x}\,dx = 2\int_0^{\infty} xe^{-2x}\,dx$

Integrate by parts : $u = x \therefore \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-2x} \therefore v = \frac{e^{-2x}}{-2}$

L - log
I - inverse
A - algebraic
T - trig
E - exp

$$\int_0^\infty xe^{-2x}\,dx = \frac{xe^{-2x}}{-2}\Big|_0^\infty - \int_0^\infty \frac{e^{-2x}}{-2}\,1\,dx = -\frac{xe^{-2x}}{2}\Big|_0^\infty + \frac{1}{2}\int_0^\infty e^{-2x}\,dx \rightarrow u = x\ (A)$$

$$= \frac{-xe^{-2x}}{2}\Big|_0^\infty + \frac{1}{2}\left[\frac{e^{-2x}}{-2}\right]_0^\infty = \frac{-xe^{-2x}}{2}\Big|_0^\infty + \frac{1}{2}\left(-\frac{1}{2}\right)$$

$$= \frac{-xe^{-2x}}{2}\Big|_0^\infty - \frac{1}{4} = 0 - \frac{1}{4} = -\frac{1}{4}$$

$$\therefore E(X) = -\frac{1}{4} \times 2 = -\frac{1}{2}$$

$$E(X) = \frac{1}{2}$$