Fri, 6-6-25

Week 9 Tutorial

- **Theory**:
  - If X and Y are independent, $Cov(X,Y) = 0$

    But the converse is NOT true! 1
  - Conditional distribution for discrete r.v.:

    X and Y have a joint distribution with pmf $f(x,y)$ on the space S, then:

    The conditional pmf of X, given $Y = x$:

    $$h(x|y) = \frac{f(x,y)}{f_2(y)}, \quad \text{provided that } f_2(y) > 0$$

    The conditional pmf of Y, given $X = x$:

    $$g(x|y) = \frac{f(x,y)}{f_1(x)}$$

  - The conditional expectation: $E[u(X) \mid Y = y] = \sum_{x \in S_1} u(x) g(x|y)$

    $$E[v(x) \mid X = x] = \sum_{y \in S_2} v(y) h(y|x)$$

  - Conditional distributions for continuous r.v.:

    pmf $\rightarrow$ pdf

    $\Sigma \rightarrow \int$

- **Questions**:

1. $f(x,y) = \frac{1}{4}$, $(x,y) \in S = \{(0,0), (1,1), (1,-1), (2,0)\}$

(a) Joint pmf table

$y \backslash x$	0	1	1	2
0	1/4	1/4	1/4	1/4
1	1/4	1/4	1/4	1/4
-1	1/4	1/4	1/4	1/4
0	1/4	1/4	1/4	1/4

(a) Represent the joint pmf by a table.

	Y			
X	-1	0	1	$f_X(x)$
0		$\frac{1}{4}$		$\frac{1}{4}$
1	$\frac{1}{4}$		$\frac{1}{4}$	$\frac{1}{2}$
2		$\frac{1}{4}$		$\frac{1}{4}$
$f_Y(y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

(b) $f_1(x) = \frac{1}{4}$; $f_2(y) = \frac{1}{4}$

$f_1(x) * f_2(y) = \frac{1}{16} \neq f(x,y)$ $\therefore$ X and Y are NOT independent

(c) $Cov(X,Y) = E(XY) - \mu_1 \mu_2$

$(x,y) \in S = \{(0,0), (1,1), (1,-1), (2,0)\}$

$E(XY) = \sum_{x=S_X}\sum_{y=S_Y} \frac{1}{4}xy = \sum_{x=S_X} \frac{1}{4}x \cdot 0 + \frac{1}{4}x \cdot 1 + \frac{1}{4}x \cdot (-1) + \frac{1}{4}x \cdot 0$
$= \sum_{x \in S_X} 0 = 0$

$\mu_1 = E(X) = \sum_{x=S_X} x\frac{1}{4} = 0\frac{1}{4} + 1\frac{1}{4} + 1\frac{1}{4} + 2\cdot\frac{1}{4}$
$= \frac{1}{4} + \frac{1}{4} + \frac{2}{4} = 1$

$\mu_2 = E(Y) = \sum_{y=S_Y} y\frac{1}{4} = \frac{1}{4}0 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot(-1) + \frac{1}{4}\cdot 0$
$= 0$

So $Cov(X,Y) = 0$ $\therefore$ suggest that the inverse is not true 1

$\rho = \frac{0}{\sigma_1 \sigma_2} = 0$

(b) y bounds: Must be a function of the other variable (x) (code)

$y > 2x$ (b)

$\boxed{0 < y < 1-x}$

Combined info:

$\boxed{2x < y < 1-x}$

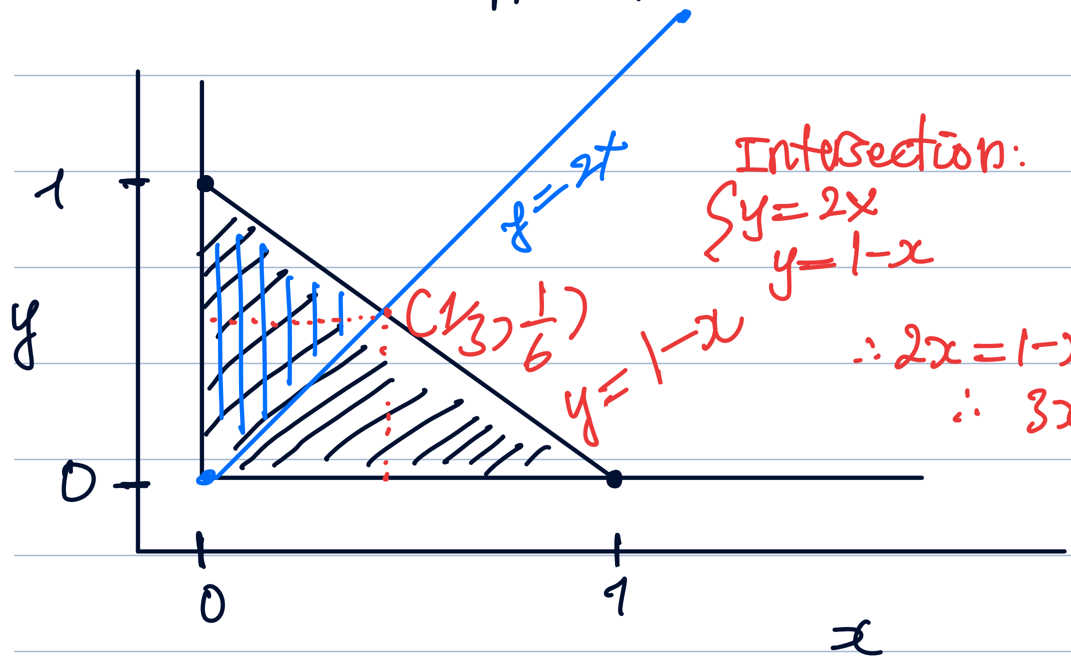
x must be numeric

(pdf) $\qquad x \geq 1-y$

② $f(x,y) = 24xy \; , x>0, y>0, x+y<1 \quad \therefore \begin{cases} y < 1-x \end{cases}$

(a) Support of X and Y $\qquad (1,0)$

$(0,1)$



Intersection:

$\begin{cases} y = 2x \\ y = 1-x \end{cases}$

$\therefore 2x = 1-x$

$\therefore 3x = 1 \therefore x = \frac{1}{3}$

$Y = 2X$

$X = 0, Y = 0$

$X = 1, Y = 2$

(b) $P(Y > 2X)$ $\qquad$ Range: $x \in (0, \frac{1}{3}), y \in$ ~~$(0,1)$~~ ~~$(0, 1-x)$~~ $(2x, 1-x)$

Ranges: $\quad x \in (0, \frac{y}{2}) \; , y \in (0,1)$

$P(Y > 2X) = \int_0^{y/2} \int_0^1 24xy \, dy \, dx = \int_0^{y/2} \left[12xy^2 \right]_0^1 dx$

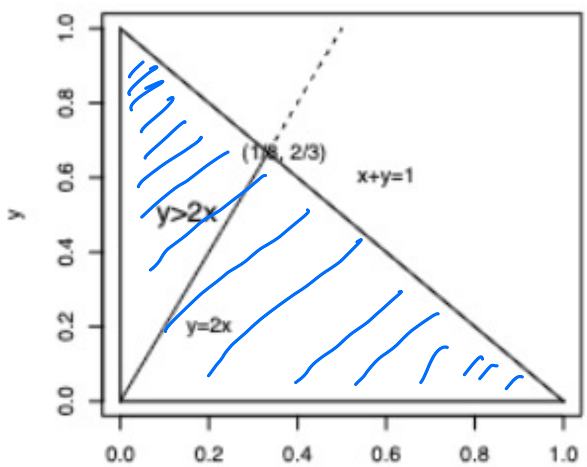
$= \int_0^{y/2} 12x \, dx = \left[\frac{12x^2}{2} \right]_0^{y/2} = 6 \left(\frac{y}{2}\right)^2 = 6 \frac{y^2}{4} = \frac{3}{2} y^2 .$

$x \in S_X, y \in S_Y$ ✗

In general: **whenever your event is given by a curve or line that tilts "off the vertical or horizontal," you must let your inner integration limit be a function of the outer variable.** Only if the event region and support are literally a rectangle (or a region whose boundaries are purely vertical or horizontal lines) will the y-bounds (or x-bounds) be constant.

$P(Y > 2X) = \int_0^{1/3} \int_{2x}^{1-x} 24xy \, dy \, dx = \cdots = \frac{2}{\ldots}$

(c) $f_1(x) = \int_0^{1-x} 24xy \, dy = \left[12xy^2 \right]_0^{1-x} = 12x(1-x)^2$
$0 < x < 1$



$x \in (0,1)$
$y \in (0,1)$

is the support of X and Y both (0,1) for this question

Thought for 10 seconds ›

Not quite – the joint support isn't the whole unit square but the triangle in the first quadrant defined by

$$x > 0, \quad y > 0, \quad x + y < 1.$$

However, if you look at the marginals in isolation, you'll find that each of X and Y does indeed take values only between 0 and 1 (since outside that range there's no y or x to keep $x + y < 1$). In other words, the support of (X, Y) is the triangle with vertices $(0,0), (1,0), (0,1)$, but marginally $X \in (0,1)$ and $Y \in (0,1)$.

(d) $E(X) = \int_0^1 x \, 12x(1-x)^2 dx = \int_0^1 12x^2(1^2 - 2x + x^2) dx$

$= \int_0^1 12x^2 - 24x^3 + 12x^4 dx = \left[4x^3 - 6x^4 + \frac{12}{5}x^5 \right]_0^1$

$= 4 - 6 + \frac{12}{5} = \frac{2}{5}, \quad 0 < x < 1$ ✓

(e) $Var(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \int_0^1 x^2\, 12x(1-x)dx = \int_0^1 12x^3 - 24x^4 + 12x^5\, dx$$

$$= \left[3x^4 - \frac{24}{5}x^5 + 2x^6 \right]_0^1$$

$$= 3 - \frac{24}{5} + 2 = \frac{1}{5}$$

$$\therefore\ var(X) = \frac{1}{5} - \left(\frac{2}{5}\right)^2 = \frac{1}{5} - \frac{4}{25} = \frac{1}{25}$$

(f) ~~$Cov(X, Y) = E(XY) - \mu_X \mu_Y$~~

$$f_2(y) = \int_{x=0}^{1-y} 24xy\, dx = \left[12yx^2\right]_0^{1-y} = 12y(1-y)^2,\ y \in S_Y$$

$$E(Y) = \int_0^1 y\, 12y(1-y)^2 dy = \int_0^1 12y^2(1-2y+y^2)dy$$

$$= \int_0^1 12y^2 - 24y^3 + 12y^4\, dy = \left[\frac{12x^3}{3} - \frac{24}{4}y^4 + \frac{12}{5}y^5\right]_0^1$$

$$= \frac{2}{5},\ 0 < y < 1$$

$$cov(x, y) = \int_0^1 \int_0^{1-y} \left(x - \frac{2}{5}\right)\left(y - \frac{2}{5}\right) 24xy\, dx\, dy = -\frac{2}{75}$$

(i) $$P\left(X \le \frac{1}{3}(1-X) \mid X = x\right) = \int_0^{(1-x)/3} h(x|y)\, dy$$

③ $Cov(aX+b, cX+d) = ac\,Var(X)$

$RHS = Cov(aX+b, cX+d) = E(aX+b-E(aX+b))(cX+d-E(cX+d))$

$= E(aX+\cancel{b} - aE(X) - \cancel{b})(cX+\cancel{d} - cE(X) - \cancel{d})$

$= E(aX-aE(X))(cX-cE(X))$

$= E(ac(X-E(X))(X-E(X))) = acE(X-E(X))^2)$

$= ac\,Var(X) = LHS$ ✓

Given:

④ $f_1(x) = \frac{1}{10}, x = 0, 1, \ldots, 9$

$h(y|x) = \frac{1}{10-x}, y = x, x+1, \ldots, 9$

(a) Joint pmf: $f(x,y) = ?$

Now $h(y|x) = \frac{f(x,y)}{f_1(x)}$ $\therefore f(x,y) = \frac{1}{10} \times \frac{1}{10-x}$

$= \frac{1}{10(10-x)}, \begin{cases} x = 0, 1, \ldots, 9 \\ y = x, x+1, \ldots, 9 \end{cases}$

(b) $f_2(y) = \sum_x f(x,y) = \sum_{x=0}^{9} \frac{1}{10(10-x)} = \frac{1}{100} + \frac{1}{90} + \frac{1}{80} + \frac{1}{70}$

$+ \frac{1}{60} + \frac{1}{50} + \frac{1}{40} + \frac{1}{30} + \frac{1}{20} + \frac{1}{10}, y = 0, 1, \ldots, 9$ ✓

(c) $E(Y|x) = \sum_{y=x}^{9} y\,h(y|x) = \sum_{y=x}^{9} \frac{y}{10-x} = \ldots = \frac{x+9}{2}$

⑤ Marginal $X \sim U(0,1)$ $\therefore f_1(x) = \frac{1}{1} = 1$

(a) $P(Y|X=x) \sim U(0, e^x)$ $\therefore h(y|x) = \frac{1}{e^x}$

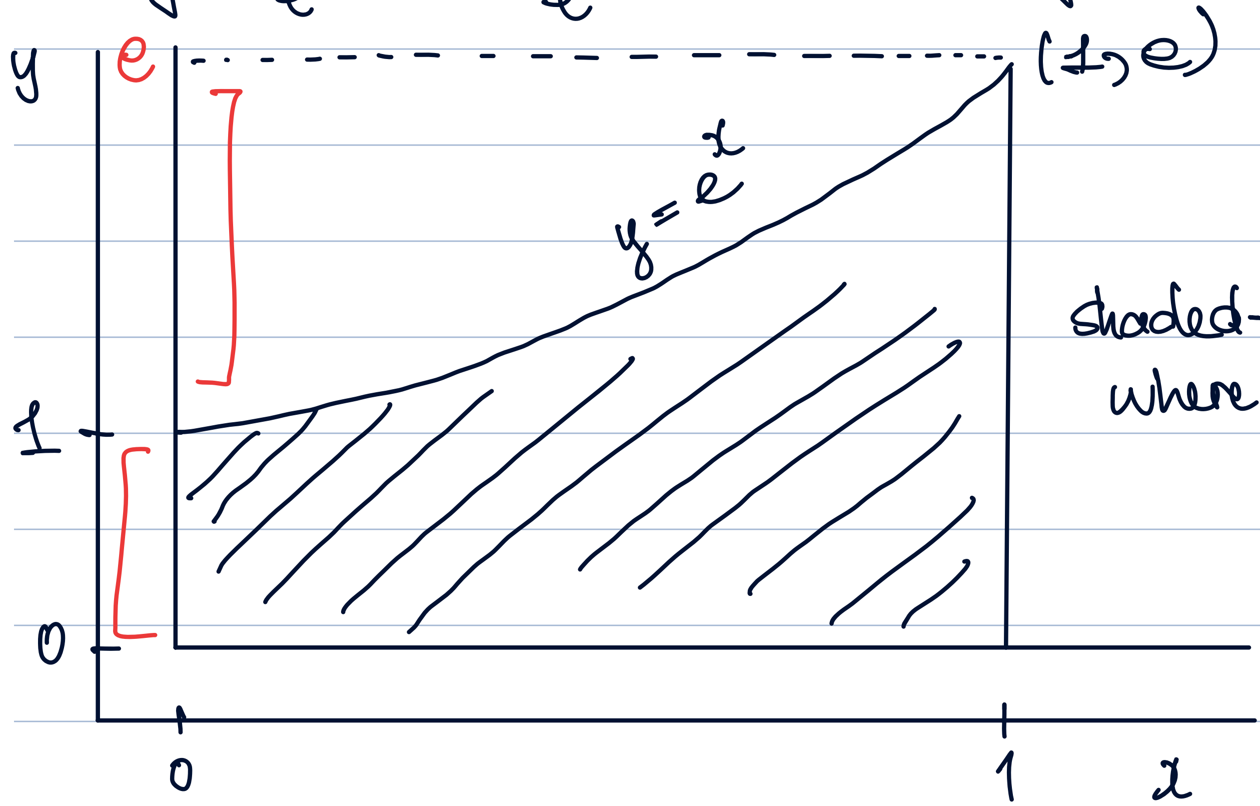
(b) $E(Y|x) = \int_0^{e^x} y\,h(y|x)\,dy$

$$= \int_0^{e^x} \frac{y}{e^x}\,dy = \frac{1}{e^x}\left[\frac{y^2}{2}\right]_0^{e^x} = \frac{1}{\cancel{e^x}} \cdot \frac{\cancel{e^x} e^x}{2} = \frac{e^x}{2},$$

$0 < x < 1$

(c) Joint pdf of X and Y? $P(Y|X) = \dfrac{f_{X,Y}}{f_1(x)}$

$$\therefore f_{X,Y} = \frac{1}{e^x} \cdot 1 = \frac{1}{e^x} \quad (0 < x < 1,\ 0 < y < e^x)$$



shaded → region where $f(x, y) > 0$

(d) $f_2(y)$?

$$f_2(y) = \begin{cases} \displaystyle\int_0^1 \frac{1}{e^x}\,dx = \left[-e^{-x}\right]_0^1 = 1 + e^{-1} = 1 + \frac{1}{e}, & x \in (0,1) \\[2ex] \displaystyle\int_{x=\ln(y)}^1 e^{-x}\,dx = y^{-1} - e^{-1}, & \underline{1 < y < e} \end{cases}$$

$\downarrow$

(e) Find $g(x|y)$, the conditional pdf of X, given $Y = y$.

- $g(x|y) = \frac{f(x,y)}{f_2(y)} = \begin{cases} \frac{e^{-x}}{1-e^{-1}}, & 0 < x < 1, & if\, 0 < y \leq 1, \\ \frac{e^{-x}}{y^{-1}-e^{-1}}, & \ln(y) < x < 1, & if\, 1 < y < e. \end{cases}$