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Subject Code: MAST20006

Subject Name: Probabilities for Statistics

Assignment 5 — The Normal Distribution

Q1:

(a) Since $z = xy$, $u = x$

$$\text{so } x = u, \quad y = \frac{z}{u}$$

(b) Originally $0 < x < 1$ and $0 < y < 1$

Under the inverse map,

$$0 < u < 1, \quad 0 < \frac{z}{u} < 1$$

$$\therefore 0 < z < u < 1$$

(c) The Jacobian of the inverse transformation:

$$J = \frac{\partial(x, y)}{\partial(z, u)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{u} & -\frac{z}{u^2} \end{vmatrix} = -\frac{1}{u}$$

$$\therefore |J| = \frac{1}{u}$$

(d) Joint pdf of (z, u)

By the 2 dimensional change-of-variable formula:

$$\begin{aligned} f_{z, u}(z, u) &= f_{x, y}(x=u, y=\frac{z}{u}) \times |J| \\ &= f_{x, y}(u, \frac{z}{u}) \times \frac{1}{u} \end{aligned}$$

for $0 < z < u < 1$ and 0 elsewhere

$$\text{So, } f_{z, u}(z, u) = \left(u + \frac{z}{u}\right) \frac{1}{u}, \quad \text{for } 0 < z < u < 1$$

(e) Marginal pdf of z :

$$f_z(z) = \int_{u=z}^1 \left(u + \frac{z}{u}\right) \frac{1}{u} du = \int_{u=z}^1 \left(1 + \frac{z}{u^2}\right) du$$

$$= \left[u \right]_z^1 + \left[-\frac{z}{u} \right]_z^1 = 1 - z - \frac{z}{1} + \frac{z}{z}$$

$$= 1 - z + 1 - z = 2 - 2z, \text{ for } 0 < z < 1$$

and $f_z(z) = 0$ elsewhere.

Q2:

(a) mgf of Y_n

for each X_i :

$$M_{X_i}(t) = E(e^{tX_i}) = \sum_{k=0}^{\infty} e^{tk} \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

$$= e^{\lambda(e^t - 1)}$$

By independence, $M_{Y_n}(t) = \prod_{i=1}^n M_{X_i}(t) = \left[e^{\lambda(e^t - 1)} \right]^n$

$$= e^{n\lambda(e^t - 1)}, \text{ which is exactly}$$

the mgf of $\text{Poi}(n\lambda)$ so:

$$Y_n \sim \text{Poi}(n\lambda)$$

(b) Given: $\hat{Y}_n := \frac{1}{n} Y_n$, $Y_n = \sum_{i=1}^n X_i$ where each $X_i \sim \text{Poi}(\lambda)$, so:

$$\begin{aligned} M_{\hat{Y}_n}(t) &= E[e^{t(Y_n/n)}] = M_{Y_n}\left(\frac{t}{n}\right) \\ &= e^{n\lambda(e^{t/n} - 1)} \end{aligned}$$

(c) Use the limit-mgf technique:

$$M_{\hat{Y}_n}(t) = e^{n\lambda(e^{t/n} - 1)}$$

Define: $L_n = \ln M_{\hat{Y}_n}(t) = n\lambda(e^{t/n} - 1)$

Let $u = \frac{t}{n}$, as $n \rightarrow \infty$, $u = \frac{t}{n} \rightarrow 0$:

$$L_n = n\lambda(e^u - 1)$$

$$\text{Now, } n(e^u - 1) = \frac{t}{u}(e^u - 1) \quad \text{as } n = \frac{t}{u}$$

$$\text{So, } L_n = \lambda \frac{t}{u}(e^u - 1) = \lambda t \frac{e^u - 1}{u}$$

$$\text{As } \lim_{u \rightarrow 0} \frac{e^u - 1}{u} = 1, \quad \lim_{n \rightarrow \infty} L_n = \lim_{u \rightarrow 0} \lambda t \frac{e^u - 1}{u} = \lambda t \times 1 = \lambda t$$

$$\text{Since } L_n = \ln M_{\hat{Y}_n}(t), \quad \lim_{n \rightarrow \infty} M_{\hat{Y}_n}(t) = e^{\lim_{n \rightarrow \infty} L_n} = e^{\lambda t}$$

Now, $e^{\lambda t}$ is exactly the mgf of a degenerate (point-mass) r.v. at λ , so:

$$\hat{Y}_n = \frac{Y_n}{n} \xrightarrow{d} \lambda$$

Since $\lim_{n \rightarrow \infty} M_{\hat{Y}_n}(t) = e^{\lambda t}$ for $\forall t$, the distribution of \hat{Y}_n

converges to the distribution with mgf $= e^{\lambda t}$. Hence, \hat{Y}_n converges in distribution to the degenerate distribution concentrated at λ , by Theorem 8 (Limiting mgf)

Q3:

Let T_i be the grading time for exam i , so that T_i are i.i.d with $E[T_i] = \mu = 20$, $\text{Var}(T_i) = \sigma^2 = 4^2 = 16$

Set: $S_{25} = \sum_{i=1}^{25} T_i$, so "grading at least 25 exams in 450 min" is the event $S_{25} \leq 450$.

By linearity and independence:

$$E[S_{25}] = 25\mu = 25 \times 20 = 500$$

$$\text{Var}(S_{25}) = 25\sigma^2 = 25 \times 16 = 400$$

Define the standardised variable:

$$Z = \frac{S_{25} - 500}{\sqrt{400}} = \frac{S_{25} - 500}{20}$$

$$\text{Then, } S_{25} \leq 450 \Leftrightarrow \frac{S_{25} - 500}{20} \leq \frac{450 - 500}{20}$$

$$\Leftrightarrow Z \leq -2.5$$

By theorem 7, for $n=25$ (sufficiently large):

$$Z = \frac{S_{25} - 25\mu}{\sigma\sqrt{25}} \approx N(0,1)$$

$$\text{So, } P(S_{25} \leq 450) = P(Z \leq -2.5) = \Phi(-2.5) \\ \approx 0.0062$$

(From the standard normal table)

Thus, there's a 0.62% chance that at least 25 exams are graded within 250 minutes.