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Subject Code: MAST20006

Subject Name: Probabilities for Statistics

Assignment 5 — The Normal Distribution

Q1:

(a) Since $Z = XY$, $U = X$

so $X = U$, $Y = \frac{Z}{U}$

(b) Originally $0 < X < 1$ and $0 < Y < 1$

Under the inverse map,

$$0 < U < 1, \quad 0 < \frac{Z}{U} < 1$$

$$\therefore \quad 0 < Z < U < 1$$

(c) The Jacobian of the inverse transformation:

$$J = \frac{\partial(x,y)}{\partial(z,u)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{u} & -\frac{z}{u^2} \end{vmatrix} = -\frac{1}{u}$$

$$\therefore \ |J| = \frac{1}{u}$$

(d) Joint pdf of (Z,U)

By the 2 dimensional change-of-variable formula:

$$f_{Z,U}(z,u) = f_{X,Y}\left(x = u, y = \frac{z}{u}\right) \times |J|$$

$$= f_{X,Y}\left(u, \frac{z}{u}\right) \times \frac{1}{u}$$

for $0 < z < u < 1$, and 0 elsewhere

So, $\quad f_{Z,U}(z,u) = \left(u + \frac{z}{u}\right)\frac{1}{u}, \quad$ for $0 < z < u < 1$

(e) Marginal pdf of Z:

$$f_Z(z) = \int_{u=z}^{1} \left(u + \frac{z}{u}\right)\frac{1}{u}\,du = \int_{u=z}^{1} \left(1 + \frac{z}{u^2}\right)du$$

$$= \Big[u\Big]_z^1 + \left[-\frac{z}{u}\right]_z^1 = 1 - z - \frac{z}{1} + \frac{z}{z}$$

$$= 1 - z + 1 - z = 2 - 2z \quad , \text{ for } 0 < z < 1$$

and $f_Z(z) = 0$ elsewhere.

Q2:

(a) mgf of Y_n

for each X_i:

$$M_{X_i}(t) = E(e^{tX_i}) = \sum_{k=0}^{\infty} e^{tk} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

$$= e^{\lambda(e^t - 1)}$$

By independence, $M_{Y_n}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \left[e^{\lambda(e^t-1)} \right]^n$

$= e^{n\lambda(e^t-1)}$, which is exactly the mgf of Poi$(n\lambda)$ so:

$$Y_n \sim \text{Poi}(n\lambda)$$

(b) Given: $\hat{Y}_n := \frac{1}{n} Y_n$, $Y_n = \sum_{i=1}^{n} X_i$ where each $X_i \sim \text{Poi}(\lambda)$, so:

$$M_{\hat{Y}_n}(t) = E[e^{t(Y_n/n)}] = M_{Y_n}\left(\frac{t}{n}\right)$$

$$= e^{n\lambda(e^{t/n} - 1)}$$

(c) Use the limit-mgf technique:

$$M_{\hat{Y}_n}(t) = e^{n\lambda(e^{t/n} - 1)}$$

Define: $L_n = \ln M_{\hat{Y}_n}(t) = n\lambda(e^{t/n} - 1)$

Let $u = \frac{t}{n}$, as $n \to \infty$, $u = \frac{t}{n} \to 0$:

$$L_n = n\lambda(e^u - 1)$$

Now, $n(e^u - 1) = \frac{t}{u}(e^u - 1)$ as $n = \frac{t}{u}$

So, $L_n = \lambda \frac{t}{u}(e^u - 1) = \lambda t \frac{e^u - 1}{u}$

As $\lim_{u \to 0} \frac{e^u - 1}{u} = 1$, $\lim_{n \to \infty} L_n = \lim_{u \to 0} \lambda t \frac{e^u - 1}{u} = \lambda t \times 1$
$= \lambda t$

Since $L_n = \ln M_{\hat{Y}_n}(t)$, $\lim_{n \to \infty} M_{\hat{Y}_n}(t) = e^{\lim_{n \to \infty} L_n} = e^{\lambda t}$

Now, $e^{\lambda t}$ is exactly the mgf of a degenerate (point-mass) r.v. at λ, so:

$$\hat{Y}_n = \frac{Y_n}{n} \xrightarrow{d} \lambda$$

Since $\lim_{n \to \infty} M_{\hat{Y}_n}(t) = e^{\lambda t}$ for $\forall t$, the distribution of $\hat{Y}_n$ converges to the distribution with mgf $= e^{\lambda t}$. Hence, $\hat{Y}_n$ converges in distribution to the degenerate distribution concentrated at λ, by Theorem 8 (Limiting mgf)

Q3 :

Let T_i be the grading time for exam i, so that T_i are i.i.d with $E[T_i] = \mu = 20$, $Var(T_i) = \sigma^2 = 4^2 = 16$

Set : $S_{25} = \sum_{i=1}^{25} T_i$, so "grading at least 25 exams in 450 min" is the event $S_{25} \le 450$.

By linearity and independence:

$$E[S_{25}] = 25\mu = 25 \times 20 = 500$$

$$Var(S_{25}) = 25\sigma^2 = 25 \times 16 = 400$$

Define the standardised variable:

$$Z = \frac{S_{25} - 500}{\sqrt{400}} = \frac{S_{25} - 500}{20}$$

Then, $S_{25} \le 450 \iff \frac{S_{25} - 500}{20} \le \frac{450 - 500}{20}$

$\iff Z \le -2.5$

By theorem 7, for $n = 25$ (sufficiently large):

$$Z = \frac{S_{25} - 25\mu}{6\sqrt{25}} \simeq N(0, 1)$$

So, $P(S_{25} \leq 450) = P(Z \leq -2.5) = \Phi(-2.5)$

$\simeq 0.0062$

(From the standard normal table)

Thus, there's a 0.62 % chance that at least 25 exams are graded within 250 minutes.